Untitled

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$$p(\boldsymbol{y},\boldsymbol{s}|\mu_1,...,\mu_k,\sigma_1^2,...,\sigma_K^2,\xi) = p(\boldsymbol{y}|\boldsymbol{s},\mu_1,...,\mu_k,\sigma_1^2,...,\sigma_K^2)p(\boldsymbol{s}|\xi)$$

where

$$\begin{aligned}
p(\boldsymbol{y}|\boldsymbol{s},\mu_1,...,\mu_k,\sigma_1^2,...,\sigma_K^2) = p(y_1|s_1 = s_1,\mu_{s_1},\sigma_{s_1}^2) \times p(y_2|s_2 = s_2,\mu_{s_2},\sigma_{s_2}^2) \times ... \times p(y_T|s_T,\mu_{s_T},\sigma_{s_T}^2) \\
= \prod_{k=1}^{K}\prod_{t:s_t=k} p(y_t|s_t = k,\mu_k,\sigma_k^2) \\
= \prod_{k=1}^{K}\prod_{t:s_t=k} \frac{1}{\sqrt{2\pi\sigma_k^2}} exp\left\{-\frac{1}{2\sigma_k^2}(y_t-\mu_k)^2\right\} \\
\propto \prod_{k=1}^{K} {\sigma_k^2}^{\left(-\frac{N_k}{2}\right)} exp\left\{-\frac{1}{2\sigma_k^2}\sum_{t:s_t=k}(y_t-\mu_k)^2\right\}
\end{aligned}$$

where N_k is the number of observations in state k.

The full conditional distribution for $\mu_1,...,\mu_k$ is therefore:

$$\begin{aligned}
\pi(\mu_1,...,\mu_K|\boldsymbol{y},\boldsymbol{s},\sigma_1^2,...,\sigma_K^2,\xi) \propto p(\boldsymbol{y},\boldsymbol{s}|\mu_1,...,\mu_k,\sigma_1^2,...,\sigma_K^2,\xi)\pi(\mu_1,...,\mu_K) \\
= p(\boldsymbol{y}|\boldsymbol{s},\mu_1,...,\mu_k,\sigma_1^2,...,\sigma_K^2)p(\boldsymbol{s}|\xi)\pi(\mu_1,...,\mu_K) \\
\propto p(\boldsymbol{y}|\boldsymbol{s},\mu_1,...,\mu_k,\sigma_1^2,...,\sigma_K^2)\pi(\mu_1,...,\mu_K) \\
= p(\boldsymbol{y}|\boldsymbol{s},\mu_1,...,\mu_k,\sigma_1^2,...,\sigma_K^2) \times \prod_{k=1}^{K}\pi(\mu_k)
\end{aligned}$$

$$
\begin{aligned}
&\propto \prod_{k=1}^{K} {\sigma_k^2}^{\left(-\frac{N_k}{2}\right)} exp\left\{-\frac{1}{2\sigma_k^2}\sum_{t:s_t=k}(y_t-\mu_k)^2\right\} \times \prod_{k=1}^{K} exp\left\{-\frac{1}{2v}(\mu_k-m)^2\right\} \\
&\propto \prod_{k=1}^{K} exp\left\{-\frac{1}{2\sigma_k^2}\left(\sum_{t:s_t=k}(y_t^2-2y_t\mu_k+\mu_k^2)\right)-\frac{1}{2v}(\mu_k-m)^2\right\} \\
&= \prod_{k=1}^{K} exp\left\{-\frac{1}{2\sigma_k^2}\left(\sum_{t:s_t=k}y_t^2-2\mu_k\sum_{t:s_t=k}y_t+\mu_k^2\sum_{t:s_t=k}1\right)-\frac{1}{2v}(\mu_k^2-2\mu_k m+m^2)\right\} \\
&= \prod_{k=1}^{K} exp\left\{-\frac{1}{2\sigma_k^2}\left(\sum_{t:s_t=k}y_t^2-2\mu_k N_k\bar{y}_k+N_k\mu_k^2\right)-\frac{1}{2v}(\mu_k^2-2\mu_k m+m^2)\right\} \\
&\propto \prod_{k=1}^{K} exp\left\{-\frac{1}{2\sigma_k^2}(-2\mu_k N_k\bar{y}_k+N_k\mu_k^2)-\frac{1}{2v}(\mu_k^2-2\mu_k m)\right\} \\
&= \prod_{k=1}^{K} exp\left\{-\frac{1}{2\sigma_k^2 v}(v[-2\mu_k N_k\bar{y}_k+N_k\mu_k^2])+\sigma_k^2(\mu_k^2-2\mu_k m)\right\} \\
&= \prod_{k=1}^{K} exp\left\{-\frac{1}{2\sigma_k^2 v}([vN_k+\sigma_k^2]\mu_k^2-2\mu_k N_k\bar{y}_k v-2\mu_k m\sigma_k^2)\right\} \\
&= \prod_{k=1}^{K} exp\left\{-\frac{1}{\frac{2\sigma_k^2 v}{vN_k+\sigma_k^2}}\left[\mu_k^2-2\left(\frac{vN_k\bar{y}_k+\sigma_k^2 m}{vN_k+\sigma_k^2}\right)\mu_k\right]\right\} \\
&= \prod_{k=1}^{K} exp\left\{-\frac{1}{\frac{2\sigma_k^2 v}{vN_k+\sigma_k^2}}\left[\left(\mu_k-\frac{vN_k\bar{y}_k+\sigma_k^2 m}{vN_k+\sigma_k^2}\right)^2-\left(\frac{vN_k\bar{y}_k+\sigma_k^2 m}{vN_k+\sigma_k^2}\right)^2\right]\right\} \\
&\propto \prod_{k=1}^{K} exp\left\{-\frac{vN_k+\sigma_k^2}{2\sigma_k^2 v}\left[\left(\mu_k-\frac{vN_k\bar{y}_k+\sigma_k^2 m}{vN_k+\sigma_k^2}\right)^2\right]\right\}
\end{aligned}
$$

So, because this factorises as a product of densities for $\mu_1, ..., \mu_K$, the μ_k are independent in the full conditional distribution with

$$\mu_k|\boldsymbol{y},\boldsymbol{s},\sigma_k^2,\xi \sim N\left(\frac{vN_k\bar{y}_k+\sigma_k^2 m}{vN_k+\sigma_k^2}, \frac{\sigma_k^2 v}{vN_k+\sigma_k^2}\right)$$

where $\bar{y}_k = \frac{1}{N_k}\sum_{t:s_t=k} y_t$ for $k = 1, ..., K$.

The full conditional distribution for $\sigma_1^2, ..., \sigma_K^2$ is:

$$
\begin{aligned}
\pi(\sigma_1^2,...,\sigma_K^2|\boldsymbol{y},\boldsymbol{s},\mu_1,...,\mu_K,\xi) &\propto p(\boldsymbol{y},\boldsymbol{s}|\mu_1,...,\mu_k,\sigma_1^2,...,\sigma_K^2,\xi)\pi(\sigma_1^2,...,\sigma_K^2) \\
&= p(\boldsymbol{y}|\boldsymbol{s},\mu_1,...,\mu_k,\sigma_1^2,...,\sigma_K^2)p(\boldsymbol{s}|\xi)\pi(\sigma_1^2,...,\sigma_K^2) \\
&\propto p(\boldsymbol{y}|\boldsymbol{s},\mu_1,...,\mu_k,\sigma_1^2,...,\sigma_K^2)\pi(\sigma_1^2,...,\sigma_K^2) \\
&= p(\boldsymbol{y}|\boldsymbol{s},\mu_1,...,\mu_k,\sigma_1^2,...,\sigma_K^2) \times \prod_{k=1}^{K}\pi(\sigma_k^2)
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{k=1}^{K} \sigma_k^{2\left(-\frac{N_k}{2}\right)} exp\left\{-\frac{1}{2\sigma_k^2}\sum_{t:s_t=k}(y_t-\mu_k)^2\right\} \times \prod_{k=1}^{K} \frac{\beta}{\Gamma(\alpha)}\sigma_k^{2(-\alpha-1)} exp\left\{-\frac{\beta}{\sigma_k^2}\right\} \\
&\propto \prod_{k=1}^{K} \sigma_k^{2\left(-\alpha-\frac{N_k}{2}-1\right)} exp\left\{-\frac{1}{2\sigma_k^2}\left(\sum_{t:s_t=k}(y_t-\mu_k)^2\right)-\frac{\beta}{\sigma_k^2}\right\} \\
&= \prod_{k=1}^{K} \sigma_k^{2\left(-\alpha-\frac{N_k}{2}-1\right)} exp\left\{-\frac{1}{\sigma_k^2}\left(\beta+\sum_{t:s_t=k}\frac{(y_t-\mu_k)^2}{2}\right)\right\}
\end{aligned}
$$

So, we can see that σ_k^2 is independent in its full conditional distributions and that

$$
\sigma_k^2|\boldsymbol{y},\boldsymbol{s},\mu_k,\xi \sim invgamma\left(\alpha+\frac{N_k}{2},\beta+\sum_{t:s_t=k}\frac{(y_t-\mu_k)^2}{2}\right)
$$