

Spatial Reading Group

Optional Subtitle

February 13, 2017

Outline

First Main Section

First Subsection

Second Subsection

Extension - Preferential Sampling

Outline

First Main Section

First Subsection

Second Subsection

Extension - Preferential Sampling

First Slide Title

Optional Subtitle

- ▶ My first point.
- ▶ My second point.

Outline

First Main Section

First Subsection

Second Subsection

Extension - Preferential Sampling

Second Slide Title

- ▶ First item.

Second Slide Title

- ▶ First item.
- ▶ Second item.

Second Slide Title

- ▶ First item.
- ▶ Second item.
- ▶ Third item.

Second Slide Title

- ▶ First item.
- ▶ Second item.
- ▶ Third item.
- ▶ Fourth item.

Second Slide Title

- ▶ First item.
- ▶ Second item.
- ▶ Third item.
- ▶ Fourth item.
- ▶ Fifth item.

Second Slide Title

- ▶ First item.
- ▶ Second item.
- ▶ Third item.
- ▶ Fourth item.
- ▶ Fifth item. Extra text in the fifth item.

Preferential Sampling

The Problem

- ▶ So far we have assumed the sampling locations X are fixed, or assumed known.
- ▶ What if the sampling locations depend on the underlying field S ?

Example

- ▶ Pollution data from measuring stations
- ▶ Ocean temperature data from marine mammals
- ▶ Lead concentration in Galicia (to be shown)

Preferential Sampling

Figure: Example of a single realisation of S and corresponding 100 sampling locations selected using a spatial Poisson Process with intensity $\lambda(x) = \exp(\beta S(x))$.

Kriging

- ▶ Suppose our objective is to predict the value of the signal at an arbitrary location $S(x)$.
- ▶ Note that $(S(x), Y)$ is multivariate Gaussian with mean vector $\mu \mathbf{1}$ and covariance matrix

$$\begin{pmatrix} \sigma^2 & \sigma^2 r^T \\ \sigma^2 r & \sigma^2 V \end{pmatrix},$$

where r is a vector with elements $r_i = \rho(\|x - x_i\|)$ and $V = \sigma^2 R + \tau^2 I$.

Kriging

- ▶ Conditional mean and variance:

$$E(S(x)|Y) = \mu + r^T V^{-1}(Y - \mu \mathbf{1}),$$

$$\text{Var}(S(x)|Y) = \sigma^2(1 - r^T V^{-1}r).$$

- ▶ Two types:

- ▶ Ordinary kriging: replace μ by its weighted least squares estimator

$$\hat{\mu} = (\mathbf{1}^T V^{-1} \mathbf{1})^{-1} \mathbf{1}^T V^{-1} Y.$$

- ▶ Simple kriging: replace μ by $\hat{\mu} = \bar{y}$.
- ▶ Both kriging predictors can be expressed as a linear combination: $\hat{S}(x) = \sum_{i=1}^n a_i(x) Y_i$, but $\sum_{i=1}^n a_i(x) = 1$ only for ordinary kriging.

Maximum Likelihood Estimation

- ▶ Gaussian model

$$Y \sim N(D\beta, \sigma^2 R(\phi) + \tau^2 I)$$

with covariates matrix $D_{n \times p}$, regression coefficients β , covariance of a parametric model for $S(x)$, and nugget variance τ^2 .

- ▶ The log-likelihood function is

$$\begin{aligned} L(\beta, \tau^2, \sigma^2, \phi) = & -0.5 \{ n \log(2\pi) + \log\{ |(\sigma^2 R(\phi) + \tau^2 I)| \} \\ & + (y - D\beta)^T (\sigma^2 R(\phi) + \tau^2 I)^{-1} (y - D\beta) \} \end{aligned}$$

- ▶ Let $\nu^2 = (\tau^2/\sigma^2)$, $V = R(\phi) + \nu I$, then $L(\beta, \tau^2, \sigma^2, \phi)$ is maximized at

$$\hat{\beta}(V) = (D^T V^{-1} D)^{-1} D^T V^{-1} y \quad (1)$$

$$\hat{\sigma}^2(V) = n^{-1} \{ y - D\hat{\beta}(V) \}^T V^{-1} \{ y - D\hat{\beta}(V) \} \quad (2)$$

Maximum Likelihood Estimation

- ▶ Plug (1) and (2) into $L(\beta, \tau^2, \sigma^2, \phi)$ and obtain the **concentrated log-likelihood**:
$$L_0(\nu^2, \phi) = -0.5\{n \log(2\pi) + n \log \hat{\sigma}^2(V) + \log|V| + n\}$$
- ▶ Optimize $L_0(\nu^2, \phi)$ numerically with respect to ν and ϕ ; back substitution to obtain $\hat{\sigma}^2$ and $\hat{\beta}$
- ▶ Re-parameterisation of V can be used to obtain more stable estimation, e.g the ratio σ^2/ϕ is more stable than σ^2 and ϕ
- ▶ Computational tool: **profile log-likelihood**:
Assume a model with parameters (α, ψ) ,

$$L_p(\alpha) = L(\alpha, \hat{\psi}(\alpha)) = \max_{\psi} (L(\alpha, \psi))$$

Maximum Likelihood Estimation

- ▶ Non-Gaussian data:
 - (1): transformation to Gaussian (2) generalized linear model
- ▶ (1) E.g. Box-Cox transformation; denote the transformed responses $Y^* = (Y_1^*, \dots, Y_n^*)$, and fit a Gaussian model

$$Y^* \sim N(D\beta, \sigma^2\{R(\phi) + \tau^2 I\})$$

Computationally demanding, transformation may impede scientific interpretation

- ▶ (2) Generalized linear model

$$L(\theta|S) = \prod_{i=1}^n f_i(y_i|S, \theta)$$

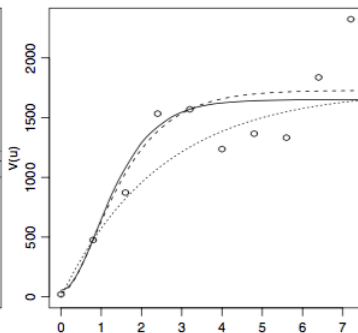
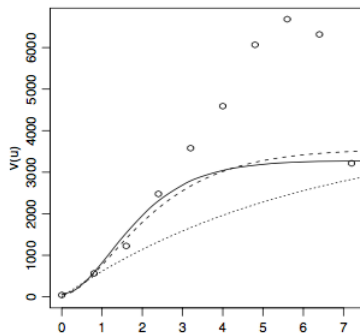
$$L(\theta, \phi) = \int_S \prod_{i=1}^n f_i(y_i|s, \theta) g(s|\phi) ds$$

Involve high dimensional integration; need MCMC/Hierarchical likelihood/Generalized estimating equations

Maximum Likelihood Estimation (An Example)

Model with constant mean							
Model	$\hat{\mu}$	$\hat{\sigma}^2$		$\hat{\phi}$	$\hat{\tau}^2$	logL	
$\kappa = 0.5$	863.71	4087.6		6.12	0	-244.6	
$\kappa = 1.5$	848.32	3510.1		1.2	48.16	-242.1	
$\kappa = 2.5$	844.63	3206.9		0.74	70.82	-242.33	

Model with linear trend							
Model	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\sigma}^2$	$\hat{\phi}$	$\hat{\tau}^2$	logL
$\kappa = 0.5$	919.1	-5.58	-15.52	1731.8	2.49	0	-242.71
$\kappa = 1.5$	912.49	-4.99	-16.46	1693.1	0.81	34.9	-240.08
$\kappa = 2.5$	912.14	-4.81	-17.11	1595.1	0.54	54.72	-239.75



Preferential Sampling

Solution

- ▶ We must account for the dependence between X and S .

$$L(\theta) = \int [X, Y, S] dS. \quad (3)$$

- ▶ Diggle et al. 2010 - Monte Carlo
- ▶ Integrated Nested Laplace Approximation (INLA) - Joe
- ▶ Template Model Builder - Danny

Preferential Sampling

Results

Model	Parameter	Standard MLE	<i>TMB</i>
Preferential	Bias	(0.77, 1.36)	(0.41, 0.94)
Preferential	Root-mean-square error	(0.86, 1.40)	(0.60, 1.05)

Table: Comparison of approximate 95% confidence intervals for the root-mean-square errors and bias between standard MLE and *TMB* over 50 independent simulations for preferential ($\beta = 2$) at location $x_0 = (0.49, 0.49)$.

Summary

- ▶ The **first main message** of your talk in one or two lines.
- ▶ The **second main message** of your talk in one or two lines.
- ▶ Perhaps a **third message**, but not more than that.
- ▶ Outlook
 - ▶ Something you haven't solved.
 - ▶ Something else you haven't solved.

For Further Reading I



A. Author.

Handbook of Everything.

Some Press, 1990.



S. Someone.

On this and that.

Journal of This and That, 2(1):50–100, 2000.