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FRM®
Exam Prep

Valuation and Risk Models

eBook 4

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Part I FRM® Exam

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FRM Part I Book 4: Valuation and Risk Models

FRM 2017 PART I BOOK 4: VALUATION AND RISK MODELS

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Reading Assignments and Learning Objectives

The following material is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by the Global Association of Risk Professionals.

Reading Assignments

Linda Allen, Jacob Boudoukh, and Anthony Saunders, *Understanding Market, Credit and Operational Risk: The Value at Risk Approach* (New York: Wiley-Blackwell, 2004).

Kevin Dowd, *Measuring Market Risk, 2nd Edition* (West Sussex, England: John Wiley & Sons, 2005).

John Hull, *Options, Futures, and Other Derivatives, 9th Edition* (New York: Pearson, 2014).

Bruce Tuckman, *Fixed Income Securities, 3rd Edition* (Hoboken, NJ: John Wiley & Sons, 2011).

Arnaud de Servigny and Olivier Renault, *Measuring and Managing Credit Risk* (New York: McGraw-Hill, 2004).

Learning Objectives

52. Quantifying Volatility in VaR Models

After completing this reading, you should be able to:
1. Explain how asset return distributions tend to deviate from the normal distribution. (page 12)
2. Explain reasons for fat tails in a return distribution and describe their implications. (page 12)
3. Distinguish between conditional and unconditional distributions. (page 12)
4. Describe the implications of regime switching on quantifying volatility. (page 14)
5. Explain the various approaches for estimating VaR. (page 15)
6. Compare and contrast different parametric and non-parametric approaches for estimating conditional volatility. (page 15)
7. Calculate conditional volatility using parametric and non-parametric approaches. (page 15)
8. Explain the process of return aggregation in the context of volatility forecasting methods. (page 25)
9. Evaluate implied volatility as a predictor of future volatility and its shortcomings. (page 25)
10. Explain long horizon volatility/VaR and the process of mean reversion according to an AR(1) model. (page 26)
11. Calculate conditional volatility with and without mean reversion. (page 26)
12. Describe the impact of mean reversion on long horizon conditional volatility estimation. (page 26)

53. Putting VaR to Work

After completing this reading, you should be able to:
1. Explain and give examples of linear and non-linear derivatives. (page 34)
2. Describe and calculate VaR for linear derivatives. (page 36)
3. Describe the delta-normal approach for calculating VaR for non-linear derivatives. (page 36)
4. Describe the limitations of the delta-normal method. (page 36)
5. Explain the full revaluation method for computing VaR. (page 40)
6. Compare delta-normal and full revaluation approaches for computing VaR. (page 40)
7. Explain structured Monte Carlo, stress testing, and scenario analysis methods for computing VaR, and identify strengths and weaknesses of each approach. (page 40)
8. Describe the implications of correlation breakdown for scenario analysis. (page 40)
9. Describe worst-case scenario (WCS) analysis and compare WCS to VaR. (page 42)

54. Measures of Financial Risk

After completing this reading, you should be able to:
1. Describe the mean-variance framework and the efficient frontier. (page 48)
2. Explain the limitations of the mean-variance framework with respect to assumptions about the return distributions. (page 50)
3. Define the Value-at-Risk (VaR) measure of risk, describe assumptions about return distributions and holding period, and explain the limitations of VaR. (page 51)
4. Define the properties of a coherent risk measure and explain the meaning of each property. (page 52)
5. Explain why VaR is not a coherent risk measure. (page 53)

6. Explain and calculate expected shortfall (ES), and compare and contrast VaR and ES. (page 53)
7. Describe spectral risk measures, and explain how VaR and ES are special cases of spectral risk measures. (page 54)
8. Describe how the results of scenario analysis can be interpreted as coherent risk measures. (page 54)

55. Binomial Trees

After completing this reading, you should be able to:
1. Calculate the value of an American and a European call or put option using a one-step and two-step binomial model. (page 60)
2. Describe how volatility is captured in the binomial model. (page 67)
3. Describe how the value calculated using a binomial model converges as time periods are added. (page 70)
4. Explain how the binomial model can be altered to price options on: stocks with dividends, stock indices, currencies, and futures. (page 67)

56. The Black-Scholes-Merton Model

After completing this reading, you should be able to:
1. Explain the lognormal property of stock prices, the distribution of rates of return, and the calculation of expected return. (page 77)
2. Compute the realized return and historical volatility of a stock. (page 77)
3. Describe the assumptions underlying the Black-Scholes-Merton option pricing model. (page 80)
4. Compute the value of a European option using the Black-Scholes-Merton model on a non-dividend-paying stock. (page 81)
5. Compute the value of a warrant and identify the complications involving the valuation of warrants. (page 87)
6. Define implied volatilities and describe how to compute implied volatilities from market prices of options using the Black-Scholes-Merton model. (page 88)
7. Explain how dividends affect the decision to exercise early for American call and put options. (page 86)
8. Compute the value of a European option using the Black-Scholes-Merton model on a dividend-paying stock. (page 83)

57. Greek Letters

After completing this reading, you should be able to:
1. Describe and assess the risks associated with naked and covered option positions. (page 95)
2. Explain how naked and covered option positions generate a stop loss trading strategy. (page 96)
3. Describe delta hedging for an option, forward, and futures contracts. (page 96)
4. Compute the delta of an option. (page 96)
5. Describe the dynamic aspects of delta hedging and distinguish between dynamic hedging and hedge-and-forget strategy. (page 99)
6. Define the delta of a portfolio. (page 102)
7. Define and describe theta, gamma, vega, and rho for option positions. (page 103)
8. Explain how to implement and maintain a delta-neutral and a gamma-neutral position. (page 103)
9. Describe the relationship between delta, theta, gamma, and vega. (page 103)

10. Describe how hedging activities take place in practice, and describe how scenario analysis can be used to formulate expected gains and losses with option positions. (page 109)
11. Describe how portfolio insurance can be created through option instruments and stock index futures. (page 110)

58. Prices, Discount Factors, and Arbitrage

After completing this reading, you should be able to:

1. Define discount factor and use a discount function to compute present and future values. (page 118)
2. Define the "law of one price," explain it using an arbitrage argument, and describe how it can be applied to bond pricing. (page 120)
3. Identify the components of a U.S. Treasury coupon bond, and compare and contrast the structure to Treasury STRIPS, including the difference between P-STRIPS and C-STRIPS. (page 122)
4. Construct a replicating portfolio using multiple fixed income securities to match the cash flows of a given fixed income security. (page 123)
5. Identify arbitrage opportunities for fixed income securities with certain cash flows. (page 120)
6. Differentiate between "clean" and "dirty" bond pricing and explain the implications of accrued interest with respect to bond pricing. (page 124)
7. Describe the common day-count conventions used in bond pricing. (page 124)

59. Spot, Forward, and Par Rates

After completing this reading, you should be able to:

1. Calculate and interpret the impact of different compounding frequencies on a bond's value. (page 131)
2. Calculate discount factors given interest rate swap rates. (page 132)
3. Compute spot rates given discount factors. (page 134)
4. Interpret the forward rate, and compute forward rates given spot rates. (page 136)
5. Define par rate and describe the equation for the par rate of a bond. (page 138)
6. Interpret the relationship between spot, forward and par rates. (page 139)
7. Assess the impact of maturity on the price of a bond and the returns generated by bonds. (page 141)
8. Define the "flattening" and "steepening" of rate curves and describe a trade to reflect expectations that a curve will flatten or steepen. (page 141)

60. Returns, Spreads, and Yields

After completing this reading, you should be able to:

1. Distinguish between gross and net realized returns, and calculate the realized return for a bond over a holding period including reinvestments. (page 149)
2. Define and interpret the spread of a bond, and explain how a spread is derived from a bond price and a term structure of rates. (page 151)
3. Define, interpret, and apply a bond's yield-to-maturity (YTM) to bond pricing. (page 151)
4. Compute a bond's YTM given a bond structure and price. (page 151)
5. Calculate the price of an annuity and a perpetuity. (page 155)
6. Explain the relationship between spot rates and YTM. (page 156)
7. Define the coupon effect and explain the relationship between coupon rate, YTM, and bond prices. (page 157)

8. Explain the decomposition of P&L for a bond into separate factors including carry roll-down, rate change, and spread change effects. (page 158)
9. Identify the most common assumptions in carry roll-down scenarios, including realized forwards, unchanged term structure, and unchanged yields. (page 159)

61. One-Factor Risk Metrics and Hedges

After completing this reading, you should be able to:
1. Describe an interest rate factor and identify common examples of interest rate factors. (page 165)
2. Define and compute the DV01 of a fixed income security given a change in yield and the resulting change in price. (page 166)
3. Calculate the face amount of bonds required to hedge an option position given the DV01 of each. (page 166)
4. Define, compute, and interpret the effective duration of a fixed income security given a change in yield and the resulting change in price. (page 168)
5. Compare and contrast DV01 and effective duration as measures of price sensitivity. (page 170)
6. Define, compute, and interpret the convexity of a fixed income security given a change in yield and the resulting change in price. (page 171)
7. Explain the process of calculating the effective duration and convexity of a portfolio of fixed income securities. (page 173)
8. Explain the impact of negative convexity on the hedging of fixed income securities. (page 174)
9. Construct a barbell portfolio to match the cost and duration of a given bullet investment, and explain the advantages and disadvantages of bullet versus barbell portfolios. (page 175)

62. Multi-Factor Risk Metrics and Hedges

After completing this reading, you should be able to:
1. Describe and assess the major weakness attributable to single-factor approaches when hedging portfolios or implementing asset liability techniques. (page 182)
2. Define key rate exposures and know the characteristics of key rate exposure factors including partial '01s and forward-bucket '01s. (page 183)
3. Describe key-rate shift analysis. (page 183)
4. Define, calculate, and interpret key rate '01 and key rate duration. (page 184)
5. Describe the key rate exposure technique in multi-factor hedging applications; summarize its advantages and disadvantages. (page 185)
6. Calculate the key rate exposures for a given security, and compute the appropriate hedging positions given a specific key rate exposure profile. (page 185)
7. Relate key rates, partial '01s and forward-bucket '01s, and calculate the forward bucket '01 for a shift in rates in one or more buckets. (page 187)
8. Construct an appropriate hedge for a position across its entire range of forward bucket exposures. (page 188)
9. Apply key rate and multi-factor analysis to estimating portfolio volatility. (page 189)

63. Country Risk: Determinants, Measures and Implications

After completing this reading, you should be able to:
1. Identify sources of country risk (page 195)
2. Explain how a country's position in the economic growth life cycle, political risk, legal risk, and economic structure affect its risk exposure. (page 196)

3. Evaluate composite measures of risk that incorporate all types of country risk and explain limitations of the risk services. (page 198)
4. Compare instances of sovereign default in both foreign currency debt and local currency debt, and explain common causes of sovereign defaults. (page 198)
5. Describe the consequences of sovereign default. (page 200)
6. Describe factors that influence the level of sovereign default risk; explain and assess how rating agencies measure sovereign default risks. (page 201)
7. Describe the advantages and disadvantages of using the sovereign default spread as a predictor of defaults. (page 206)

64. External and Internal Ratings

After completing this reading, you should be able to:

1. Describe external rating scales, the rating process, and the link between ratings and default. (page 214)
2. Describe the impact of time horizon, economic cycle, industry, and geography on external ratings. (page 216)
3. Explain the potential impact of ratings changes on bond and stock prices. (page 217)
4. Compare external and internal ratings approaches. (page 217)
5. Explain and compare the through-the-cycle and at-the-point internal ratings approaches. (page 218)
6. Describe a ratings transition matrix and explain its uses. (page 215)
7. Describe the process for and issues with building, calibrating and backtesting an internal rating system. (page 218)
8. Identify and describe the biases that may affect a rating system. (page 219)

65. Capital Structure in Banks

After completing this reading, you should be able to:

1. Evaluate a bank's economic capital relative to its level of credit risk. (page 230)
2. Identify and describe important factors used to calculate economic capital for credit risk: probability of default, exposure, and loss rate. (page 224)
3. Define and calculate expected loss (EL). (page 225)
4. Define and calculate unexpected loss (UL). (page 225)
5. Estimate the variance of default probability assuming a binomial distribution. (page 225)
6. Calculate UL for a portfolio and the risk contribution of each asset. (page 227)
7. Describe how economic capital is derived. (page 230)
8. Explain how the credit loss distribution is modeled. (page 231)
9. Describe challenges to quantifying credit risk. (page 231)

66. Operational Risk

After completing this reading, you should be able to:

1. Compare three approaches for calculating regulatory capital. (page 237)
2. Describe the Basel Committee's seven categories of operational risk. (page 238)
3. Derive a loss distribution from the loss frequency distribution and loss severity distribution using Monte Carlo simulations. (page 239)
4. Describe the common data issues that can introduce inaccuracies and biases in the estimation of loss frequency and severity distributions. (page 240)
5. Describe how to use scenario analysis in instances when data is scarce. (page 241)

6. Describe how to identify causal relationships and how to use risk and control self assessment (RCSA) and key risk indicators (KRIs) to measure and manage operational risks. (page 241)
7. Describe the allocation of operational risk capital to business units. (page 242)
8. Explain how to use the power law to measure operational risk. (page 243)
9. Explain the risks of moral hazard and adverse selection when using insurance to mitigate operational risks. (page 243)

67. Governance Over Stress Testing

After completing this reading, you should be able to:
1. Describe the key elements of effective governance over stress testing. (page 249)
2. Describe the responsibilities of the board of directors and senior management in stress testing activities. (page 249)
3. Identify elements of clear and comprehensive policies, procedures, and documentations on stress testing. (page 251)
4. Identify areas of validation and independent review for stress tests that require attention from a governance perspective. (page 252)
5. Describe the important role of the internal audit in stress testing governance and control. (page 252)
6. Identify key aspects of stress testing governance, including stress testing coverage, stress testing types and approaches, and capital and liquidity stress testing. (page 253)

68. Stress Testing and Other Risk Management Tools

After completing this reading, you should be able to:
1. Describe the relationship between stress testing and other risk measures, particularly in enterprise-wide stress testing. (page 260)
2. Describe the various approaches to using VaR models in stress tests. (page 261)
3. Explain the importance of stressed inputs and their importance in stressed VaR. (page 261)
4. Identify the advantages and disadvantages of stressed risk metrics. (page 262)

69. Principles for Sound Stress Testing Practices and Supervision

After completing this reading, you should be able to:
1. Describe the rationale for the use of stress testing as a risk management tool. (page 266)
2. Describe weaknesses identified and recommendations for improvement in:
   - The use of stress testing and integration in risk governance
   - Stress testing methodologies
   - Stress testing scenarios
   - Stress testing handling of specific risks and products (page 267)
3. Describe stress testing principles for banks regarding the use of stress testing and integration in risk governance, stress testing methodology and scenario selection, and principles for supervisors. (page 267)

VaR Methods

Exam Focus

Value at risk (VaR) was developed as an efficient, inexpensive method to determine economic risk exposure of banks with complex diversified asset holdings. In this reading, we define VaR, demonstrate its calculation, discuss how VaR can be converted to longer time periods, and examine the advantages and disadvantages of the three main VaR estimation methods. For the exam, be sure you know when to apply each VaR method and how to calculate VaR using each method. VaR is one of GARP's favorite testing topics and it appears in many assigned readings throughout the FRM Part I and Part II curricula.

Defining VaR

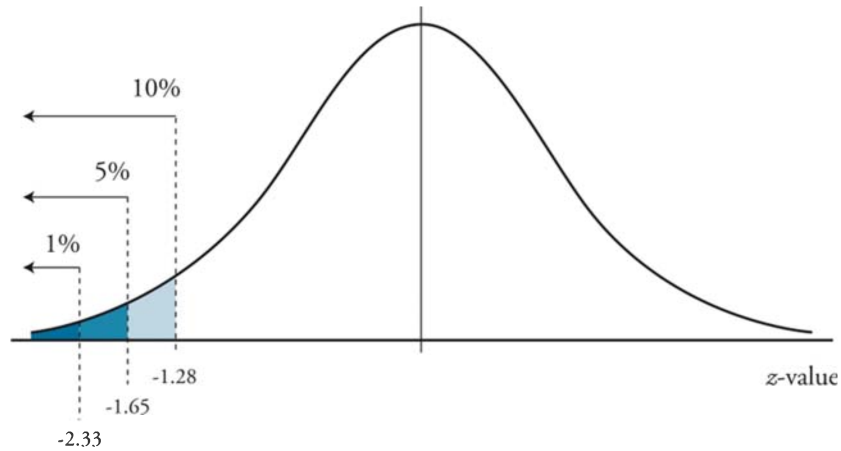
Value at risk (VaR) is a probabilistic method of measuring the potential loss in portfolio value over a given time period and for a given distribution of historical returns. VaR is the dollar or percentage loss in portfolio (asset) value that will be equaled or exceeded only *X* percent of the time. In other words, there is an *X* percent probability that the loss in portfolio value will be equal to or greater than the VaR measure. VaR can be calculated for any percentage probability of loss and over any time period. A 1%, 5%, and 10% VaR would be denoted as VaR(1%), VaR(5%), and VaR(10%), respectively. The risk manager selects the *X* percent probability of interest and the time period over which VaR will be measured. Generally, the time period selected (and the one we will use) is one day.

A brief example will help solidify the VaR concept. Assume a risk manager calculates the daily 5% VaR as $10,000. The VaR(5%) of $10,000 indicates that there is a 5% chance that on any given day, the portfolio will experience a loss of $10,000 or more. We could also say that there is a 95% chance that on any given day the portfolio will experience either a loss less than $10,000 or a gain. If we further assume that the $10,000 loss represents 8% of the portfolio value, then on any given day there is a 5% chance that the portfolio will experience a loss of 8% or greater, but there is a 95% chance that the loss will be less than 8% or a percentage gain greater than zero.

Calculating VaR

Calculating delta-normal VaR is a simple matter but requires assuming that asset returns conform to a standard normal distribution. Recall that a standard normal distribution is defined by two parameters, its mean ($\mu = 0$) and standard deviation ($\sigma = 1$), and is perfectly symmetric with 50% of the distribution lying to the right of the mean and 50% lying to the left of the mean. Figure 1 illustrates the standard normal distribution and the cumulative probabilities under the curve.

Figure 1: Standard Normal Distribution and Cumulative Probabilities



From Figure 1, we observe the following: the probability of observing a value more than 1.28 standard deviations below the mean is 10%; the probability of observing a value more than 1.65 standard deviations below the mean is 5%; and the probability of observing a value more than 2.33 standard deviations below the mean is 1%. Thus, we have critical *z*-values of –1.28, –1.65, and –2.33 for 10%, 5%, and 1% lower tail probabilities, respectively. We can now define percent VaR mathematically as:

$$\text{VaR (X\%)} = z_{X\%}\sigma$$

where:
VaR (X%) = the X% probability value at risk
$z_{X\%}$ = the critical *z*-value based on the normal distribution and the selected X% probability
σ = the standard deviation of daily returns on a percentage basis

Professor's Note: VaR is a one-tailed test, so the level of significance is entirely in one tail of the distribution. As a result, the critical values will be different than a two-tailed test that uses the same significance level.

In order to calculate VaR(5%) using this formula, we would use a critical *z*-value of –1.65 and multiply by the standard deviation of percent returns. The resulting VaR estimate would be the percentage loss in asset value that would only be exceeded 5% of the time. VaR can also be estimated on a dollar rather than a percentage basis. To calculate VaR on a dollar basis, we simply multiply the percent VaR by the asset value as follows:

$$\text{VaR (X\%)}_{\text{dollar basis}} = \text{VaR (X\%)}_{\text{decimal basis}} \times \text{asset value}$$

$$= (z_{X\%}\sigma) \times \text{asset value}$$

To calculate VaR(5%) using this formula, we multiply VaR(5%) on a percentage basis by the current value of the asset in question. This is equivalent to taking the product of the critical *z*-value, the standard deviation of percent returns, and the current asset value. An estimate of VaR(5%) on a dollar basis is interpreted as the dollar loss in asset value that will only be exceeded 5% of the time.

Example: Calculating percentage and dollar VaR

A risk management officer at a bank is interested in calculating the VaR of an asset that he is considering adding to the bank's portfolio. If the asset has a daily standard deviation of returns equal to 1.4% and the asset has a current value of $5.3 million, **calculate** the VaR (5%) on both a percentage and dollar basis.

Answer:

The appropriate critical *z*-value for a VaR (5%) is –1.65. Using this critical value and the asset's standard deviation of returns, the VaR (5%) on a percentage basis is calculated as follows:

$$\text{VaR (5\%)} = z_{5\%}\sigma = -1.65(0.014) = -0.0231 = -2.31\%$$

The VaR(5%) on a dollar basis is calculated as follows:

$$\begin{aligned}\text{VaR (5\%)}_{\text{dollar basis}} &= \text{VaR (5\%)}_{\text{decimal basis}} \times \text{asset value} = -0.0231 \times \$5{,}300{,}000 \\ &= -\$122{,}430\end{aligned}$$

Thus, there is a 5% probability that, on any given day, the loss in value on this particular asset will equal or exceed 2.31%, or $122,430.

If an expected return other than zero is given, VaR becomes the expected return minus the quantity of the critical value multiplied by the standard deviation.

$$\text{VaR} = [E(R) - z\sigma]$$

In the example above, the expected return value is zero and thus ignored. The following example demonstrates how to apply an expected return to a VaR calculation.

Example: Calculating VaR given an expected return

For a $100,000,000 portfolio, the expected 1-week portfolio return and standard deviation are 0.00188 and 0.0125, respectively. **Calculate** the 1-week VaR at 5% significance.

Answer:

$$\begin{aligned} \text{VaR} &= [E(R) - z\sigma] \times \text{portfolio value} \\ &= [0.00188 - 1.65(0.0125)] \times \$100{,}000{,}000 \\ &= -0.018745 \times \$100{,}000{,}000 \\ &= -\$1{,}874{,}500 \end{aligned}$$

The manager can be 95% confident that the maximum 1-week loss will not exceed $1,874,500.

VaR Conversions

VaR, as calculated previously, measured the risk of a loss in asset value over a short time period. Risk managers may, however, be interested in measuring risk over longer time periods, such as a month, quarter, or year. VaR can be converted from a 1-day basis to a longer basis by multiplying the daily VaR by the square root of the number of days (J) in the longer time period (called the **square root rule**). For example, to convert to a weekly VaR, multiply the daily VaR by the square root of 5 (i.e., five business days in a week). We can generalize the conversion method as follows:

$$\text{VaR}(X\%)_{\text{J-days}} = \text{VaR}(X\%)_{\text{1-day}}\sqrt{J}$$

Example: Converting daily VaR to other time bases

Assume that a risk manager has calculated the daily VaR $(10\%)_{\text{dollar basis}}$ of a particular asset to be $12,500. **Calculate** the weekly, monthly, semiannual, and annual VaR for this asset. Assume 250 days per year and 50 weeks per year.

Answer:

The daily dollar VaR is converted to a weekly, monthly, semiannual, and annual dollar VaR measure by multiplying by the square root of 5, 20, 125, and 250, respectively.

$$\text{VaR}(10\%)_{\text{5-days (weekly)}} = \text{VaR}(10\%)_{\text{1-day}}\sqrt{5} = \$12{,}500\sqrt{5} = \$27{,}951$$

$$\text{VaR}(10\%)_{\text{20-days (monthly)}} = \text{VaR}(10\%)_{\text{1-day}}\sqrt{20} = \$12{,}500\sqrt{20} = \$55{,}902$$

$$\text{VaR}(10\%)_{\text{125-days}} = \text{VaR}(10\%)_{\text{1-day}}\sqrt{125} = \$12{,}500\sqrt{125} = \$139{,}754$$

$$\text{VaR}(10\%)_{\text{250-days}} = \text{VaR}(10\%)_{\text{1-day}}\sqrt{250} = \$12{,}500\sqrt{250} = \$197{,}642$$

VaR can also be converted to different confidence levels. For example, a risk manager may want to convert VaR with a 95% confidence level to VaR with a 99% confidence level. This conversion is done by adjusting the current VaR measure by the ratio of the updated confidence level to the current confidence level.

Example: Converting VaR to different confidence levels

Assume that a risk manager has calculated VaR at a 95% confidence level to be $16,500. Now assume the risk manager wants to adjust the confidence level to 99%. **Calculate** the VaR at a 99% confidence level.

Answer:

$$\text{VaR}(1\%) = \text{VaR}(5\%) \times \frac{z_{1\%}}{z_{5\%}}$$

$$\text{VaR}(1\%) = \$16{,}500 \times \frac{2.33}{1.65} = \$23{,}300$$

The VaR Methods

The three main VaR methods can be divided into two groups: linear methods and full valuation methods.

1. **Linear methods** replace portfolio positions with linear exposures on the appropriate risk factor. For example, the linear exposure used for option positions would be delta while the linear exposure for bond positions would be duration. This method is used when calculating VaR with the delta-normal method.
2. **Full valuation methods** fully reprice the portfolio for each scenario encountered over a historical period, or over a great number of hypothetical scenarios developed through historical simulation or Monte Carlo simulation. Computing VaR using full revaluation is more complex than linear methods. However, this approach will generally lead to more accurate estimates of risk in the long run.

Linear Valuation: The Delta-Normal Valuation Method

The **delta-normal approach** begins by valuing the portfolio at an initial point as a relationship to a specific risk factor, *S* (consider only one risk factor exists):

$$V_0 = V(S_0)$$

With this expression, we can describe the relationship between the change in portfolio value and the change in the risk factor as:

$$dV = \Delta_0 \times dS$$

Here, Δ_0 is the sensitivity of the portfolio to changes in the risk factor, *S*. As with any linear relationship, the biggest change in the value of the portfolio will accompany the biggest change in the risk factor. The VaR at a given level of significance, *z*, can be written as:

$$\text{VaR} = |\Delta_0| \times (z\sigma S_0)$$

where:
$z\sigma S_0 = \text{VaR}_S$

Generally speaking, VaR developed by a delta-normal method is more accurate over shorter horizons than longer horizons.

Consider, for example, a fixed income portfolio. The risk factor impacting the value of this portfolio is the change in yield. The VaR of this portfolio would then be calculated as follows:

$$\text{VaR} = \text{modified duration} \times z \times \text{annualized yield volatility} \times \text{portfolio value}$$

Notice here that the volatility measure applied is the volatility of changes in the yield. In future examples, the volatility measured used will be the standard deviation of returns.

Since the delta-normal method is only accurate for linear exposures, non-linear exposures, such as convexity, are not adequately captured with this VaR method. By using a Taylor series expansion, convexity can be accounted for in a fixed income portfolio by using what is known as the **delta-gamma method**. You will see this method in Topic 53. For now, just take note that complexity can be added to the delta-normal method to increase its reliability when measuring non-linear exposures.

Full Valuation: Monte Carlo and Historic Simulation Methods

The **Monte Carlo simulation** approach revalues a portfolio for a large number of risk factor values, randomly selected from a normal distribution. **Historical simulation** revalues a portfolio using actual values for risk factors taken from historical data. These full valuation approaches provide the most accurate measurements because they include all nonlinear relationships and other potential correlations that may not be included in the linear valuation models.

Comparing the Methods

The delta-normal method is appropriate for large portfolios without significant option-like exposures. This method is fast and efficient.

Full-valuation methods, either based on historical data or on Monte Carlo simulations, are more time consuming and costly. However, they may be the only appropriate methods for large portfolios with substantial option-like exposures, a wider range of risk factors, or a longer-term horizon.

Delta-Normal Method

The delta-normal method (a.k.a. the variance-covariance method or the analytical method) for estimating VaR requires the assumption of a **normal distribution**. This is because the method utilizes the expected return and standard deviation of returns. For example, in calculating a daily VaR, we calculate the standard deviation of daily returns in the past and assume it will be applicable to the future. Then, using the asset's expected 1-day return and standard deviation, we estimate the 1-day VaR at the desired level of significance.

The assumption of normality is troublesome because many assets exhibit skewed return distributions (e.g., options), and equity returns frequently exhibit leptokurtosis (fat tails). When a distribution has "fat tails," VaR will tend to underestimate the loss and its associated probability. Also know that delta-normal VaR is calculated using the historical standard deviation, which may not be appropriate if the composition of the portfolio changes, if the estimation period contained unusual events, or if economic conditions have changed.

Example: Delta-normal VaR

The expected 1-day return for a $100,000,000 portfolio is 0.00085 and the historical standard deviation of daily returns is 0.0011. **Calculate** daily value at risk (VaR) at 5% significance.

Answer:

To locate the value for a 5% VaR, we use the Alternative *z*-Table in the appendix to this book. We look through the body of the table until we find the value that we are looking for. In this case, we want 5% in the lower tail, which would leave 45% below the mean that is not in the tail. Searching for 0.45, we find the value 0.4505 (the closest value we will find). Adding the *z*-value in the left hand margin and the *z*-value at the top of the column in which 0.4505 lies, we get 1.6 + 0.05 = 1.65, so the *z*-value coinciding with a 95% VaR is 1.65. (Notice that we ignore the negative sign, which would indicate the value lies below the mean.)

You will also find a Cumulative *z*-Table in the appendix. When using this table, you can look directly for the significance level of the VaR. For example, if you desire a 5% VaR, look for the value in the table which is closest to (1 – significance level) or 1 – 0.05 = 0.9500. You will find 0.9505, which lies at the intersection of 1.6 in the left margin and 0.05 in the column heading.

$$\text{VaR} = \left[\hat{R}_P - (z)(\sigma)\right] V_P$$

$$= [0.00085 - 1.65(0.0011)](\$100{,}000{,}000)$$

$$= -0.000965(\$100{,}000{,}000)$$

$$= -\$96{,}500$$

where:

$\hat{R}_P$ = expected 1-day return on the portfolio

V_P = value of the portfolio

z = z-value corresponding with the desired level of significance

σ = standard deviation of 1-day returns

The interpretation of this VaR is that there is a 5% chance the *minimum* 1-day loss is 0.0965%, or $96,500. (There is 5% probability that the 1-day loss will exceed $96,500.) Alternatively, we could say we are 95% confident the 1-day loss will not exceed $96,500.

If you are given the standard deviation of annual returns and need to calculate a daily VaR, the daily standard deviation can be estimated as the annual standard deviation divided by the square root of the number of (trading) days in a year, and so forth:

$$\sigma_{\text{daily}} \cong \frac{\sigma_{\text{annual}}}{\sqrt{250}}; \sigma_{\text{monthly}} \cong \frac{\sigma_{\text{annual}}}{\sqrt{12}}$$

Delta-normal VaR is often calculated assuming an expected return of zero rather than the portfolio's actual expected return. When this is done, VaR can be adjusted to longer or shorter periods of time quite easily. For example, daily VaR is estimated as annual VaR divided by the square root of 250 (as when adjusting the standard deviation).

Likewise, the annual VaR is estimated as the daily VaR multiplied by the square root of 250. If the true expected return is used, VaR for different length periods must be calculated independently.

Professor's Note: Assuming a zero expected return when estimating VaR is a conservative approach because the calculated VaR will be greater (i.e., farther out in the tail of the distribution) than if the expected return is used.

Since portfolio values are likely to change over long time periods, it is often the case that VaR over a short time period is calculated and then converted to a longer period. The Basel Accord (discussed in the FRM Part II curriculum) recommends the use of a two-week period (10 days).

Professor's Note: For the exam, you will likely be required to make these time conversation calculations since VaR is often calculated over a short time frame.

Advantages of the delta-normal VaR method include the following:

- Easy to implement.
- Calculations can be performed quickly.
- Conducive to analysis because risk factors, correlations, and volatilities are identified.

Disadvantages of the delta-normal method include the following:

- The need to assume a normal distribution.
- The method is unable to properly account for distributions with fat tails, either because of unidentified time variation in risk or unidentified risk factors and/or correlations.
- Nonlinear relationships of option-like positions are not adequately described by the delta-normal method. VaR is misstated because the instability of the option deltas is not captured.

Historical Simulation Method

The historical method for estimating VaR is often referred to as the **historical simulation** method. The easiest way to calculate the 5% daily VaR using the historical method is to accumulate a number of past daily returns, rank the returns from highest to lowest, and identify the lowest 5% of returns. The highest of these lowest 5% of returns is the 1-day, 5% VaR.

Example: Historical VaR

You have accumulated 100 daily returns for your $100,000,000 portfolio. After ranking the returns from highest to lowest, you identify the lowest six returns:

–0.0011, –0.0019, –0.0025, –0.0034, –0.0096, –0.0101

Calculate daily value at risk (VaR) at 5% significance using the historical method.

Answer:

The lowest five returns represent the 5% lower tail of the "distribution" of 100 historical returns. The fifth lowest return (–0.0019) is the 5% daily VaR. We would say there is a 5% chance of a daily loss exceeding 0.19%, or $190,000.

As you will see in Topic 52, the historical simulation method may weight observations and take an average of two returns to obtain the historical VaR. Each observation can be viewed as having a probability distribution with 50% to the left and 50% to the right of a given observation. When considering the previous example, 5% VaR with 100 observations would take the average of the fifth and sixth observations [i.e., (–0.0011 + –0.0019) / 2 = –0.0015]. Therefore, the 5% historical VaR in this case would be $150,000. Either approach (using a given percentile or an average of two) is acceptable for calculating historical VaR, however, using a given percentile, as provided in the previous example, will yield a more conservative estimate since the calculated VaR estimate will be lower.

Professor's Note: On past FRM exams, GARP has calculated historical VaR in a similar fashion to the previous example.

Advantages of the historical simulation method include the following:

- The model is easy to implement when historical data is readily available.
- Calculations are simple and can be performed quickly.
- Horizon is a positive choice based on the intervals of historical data used.
- Full valuation of portfolio is based on actual prices.
- It is not exposed to model risk.
- It includes all correlations as embedded in market price changes.

Disadvantages of the historical simulation method include the following:

- It may not be enough historical data for all assets.
- Only one path of events is used (the actual history), which includes changes in correlations and volatilities that may have occurred only in that historical period.
- Time variation of risk in the past may not represent variation in the future.
- The model may not recognize changes in volatility and correlations from structural changes.
- It is slow to adapt to new volatilities and correlations as old data carries the same weight as more recent data. However, exponentially weighted average (EWMA) models can be used to weigh recent observations more heavily.
- A small number of actual observations may lead to insufficiently defined distribution tails.

Monte Carlo Simulation Method

The Monte Carlo method refers to computer software that generates hundreds, thousands, or even millions of possible outcomes from the distributions of inputs *specified by the user*. For example, a portfolio manager could enter a distribution of possible 1-week returns for each of the hundreds of stocks in a portfolio. On each "run" (the number of runs is specified by the user), the computer selects one weekly return from each stock's distribution of possible returns and calculates a weighted average portfolio return.

The several thousand weighted average portfolio returns will naturally form a distribution, which will approximate the normal distribution. Using the portfolio expected return and the standard deviation, which are part of the Monte Carlo output, VaR is calculated in the same way as with the delta-normal method.

Example: Monte Carlo VaR

A Monte Carlo output specifies the expected 1-week portfolio return and standard deviation as 0.00188 and 0.0125, respectively. **Calculate** the 1-week VaR at 1% significance.

Answer:

$$\text{VAR} = \left[\widehat{R}_P - (z)(\sigma)\right]V_P$$

$$= [0.00188 - 2.33(0.0125)](\$100{,}000{,}000)$$

$$= -0.027245(\$100{,}000{,}000)$$

$$= -\$2{,}724{,}500$$

The manager can be 99% confident that the maximum 1-week loss will not exceed $2,724,500. Alternatively, the manager could say there is a 1% probability that the minimum loss will be $2,724,500 or greater (the portfolio will lose at least $2,724,500).

Advantages of the Monte Carlo method include the following:

- It is the most powerful model.
- It can account for both linear and nonlinear risks.
- It can include time variation in risk and correlations by aging positions over chosen horizons.
- It is extremely flexible and can incorporate additional risk factors easily.
- Nearly unlimited numbers of scenarios can produce well-described distributions.

Disadvantages of the Monte Carlo method include the following:

- There is a lengthy computation time as the number of valuations escalates quickly.
- It is expensive because of the intellectual and technological skills required.
- It is subject to model risk of the stochastic processes chosen.
- It is subject to sampling variation at lower numbers of simulations.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Quantifying Volatility in VaR Models

Topic 52

Exam Focus

Obtaining an accurate estimate of an asset's value that is at risk of loss hinges greatly on the measurement of the asset's volatility (or possible deviation in value over a certain time period). Asset value can be evaluating using a normal distribution; however, deviations from normality will create challenges for the risk manager in measuring both volatility and value at risk (VaR). In this topic, we will discuss issues with volatility estimation and different weighting methods that can be used to determine VaR. The advantages, disadvantages, and underlying assumptions of the various methodologies will also be discussed. For the exam, understand why deviations from normality occur and have a general understanding of the approaches to measuring VaR (parametric and nonparametric).

LO 52.1: Explain how asset return distributions tend to deviate from the normal distribution.

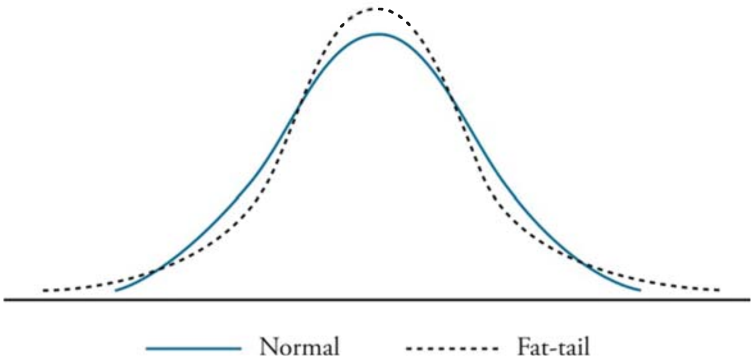
LO 52.2: Explain reasons for fat tails in a return distribution and describe their implications.

LO 52.3: Distinguish between conditional and unconditional distributions.

Three common deviations from normality that are problematic in modeling risk result from asset returns that are fat-tailed, skewed, or unstable.

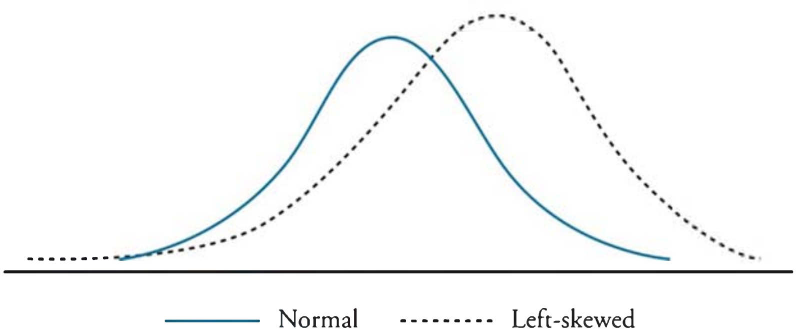
Fat-tailed refers to a distribution with a higher probability of observations occurring in the tails relative to the normal distribution. As illustrated in Figure 1, there is a larger probability of an observation occurring further away from the mean of the distribution. The first two moments (mean and variance) of the distributions are similar for the fat-tailed and normal distribution. However, in addition to the greater mass in the tails, there is also a greater probability mass around the mean for the fat-tailed distribution. Furthermore, there is less probability mass in the intermediate range (around +/– one standard deviation) for the fat-tailed distribution compared to the normal distribution.

Figure 1: Illustration of Fat-Tailed and Normal Distributions



A distribution is **skewed** when the distribution is not symmetrical. A risk manager is more concerned when there is a higher probability of a large negative return than a large positive return. This is referred to as left-skewed and is illustrated in Figure 2.

Figure 2: Left-Skewed and Normal Distributions



In modeling risk, a number of assumptions are necessary. If the parameters of the model are **unstable**, they are not constant but vary over time. For example, if interest rates, inflation, and market premiums are changing over time, this will affect the volatility of the returns going forward.

Deviations From the Normal Distribution

The phenomenon of "fat tails" is most likely the result of the volatility and/or the mean of the distribution changing over time. If the mean and standard deviation are the same for asset returns for any given day, the distribution of returns is referred to as an **unconditional distribution** of asset returns. However, different market or economic conditions may cause the mean and variance of the return distribution to change over time. In such cases, the return distribution is referred to as a **conditional distribution**.

Assume we separate the full data sample into two normally distributed subsets based on market environment with **conditional** means and variances. Pulling a data sample at different points of time from the full sample could generate fat tails in the unconditional distribution even if the conditional distributions are normally distributed with similar means but different volatilities. If markets are efficient and all available information is reflected in stock prices, it is not likely that the first moments or conditional means of the distribution vary enough to make a difference over time.

The second possible explanation for "fat tails" is that the second moment or volatility is time-varying. This explanation is much more likely given observed changes in interest rate volatility (e.g., prior to a much-anticipated Federal Reserve announcement). Increased market uncertainty following significant political or economic events results in increased volatility of return distributions.

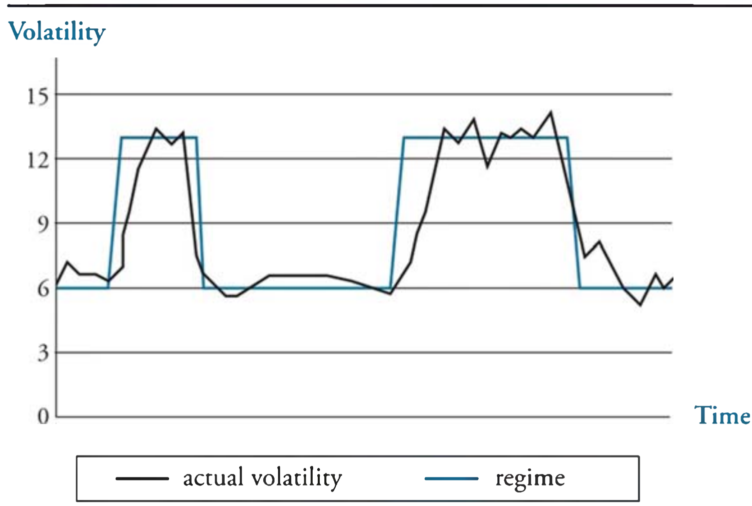
Market Regimes and Conditional Distributions

LO 52.4: Describe the implications of regime switching on quantifying volatility.

A **regime-switching volatility model** assumes different market regimes exist with high or low volatility. The conditional distributions of returns are always normal with a constant mean but either have a high or low volatility. Figure 3 illustrates a hypothetical regime-switching model for interest rate volatility. Note that the true interest rate volatility depicted by the solid line is either 13 basis points per day (bp/day) or 6bp/day. The actual observed returns deviate around the high volatility 13bp/day level and the low volatility 6bp/day. In this example, the unconditional distribution is not normally distributed. However, assuming time-varying volatility, the interest rate distributions are *conditionally normally distributed*.

The probability of large deviations from normality occurring are much less likely under the regime-switching model. For example, the interest rate volatility in Figure 3 ranges from 5.7bp/day to 13.6bp/day with an overall mean of 8.52bp/day. However, the 13.6bp/day has a difference of only 0.6bp/day from the conditional high volatility level compared to a 5.08bp/day difference from the unconditional distribution. This would result in a fat-tailed unconditional distribution. The regime-switching model captures the conditional normality and may resolve the fat-tail problem and other deviations from normality.

Figure 3: Actual Conditional Return Volatility Under Market Regimes



If we assume that volatility varies with time and that asset returns are conditionally normally distributed, then we may be able to tolerate the fat-tail issue. In the next section we demonstrate how to estimate conditional means and variances. However, despite efforts to more accurately model financial data, extreme events do still occur. The model (or distribution) used may not capture these extreme movements. For example, value at risk (VaR) models are typically utilized to model the risk level apparent in asset prices. VaR assumes asset returns follow a normal distribution, but as we have just discussed, asset return distributions tend to exhibit fat tails. As a result, VaR may underestimate the actual loss amount.

However, some tools exist that serve to complement VaR by examining the data in the tail of the distribution. For example, stress testing and scenario analysis can examine extreme events by testing how hypothetical and/or past financial shocks will impact VaR. Also, extreme value theory (EVT) can be applied to examine just the tail of the distribution and some classes of EVT apply a separate distribution to the tail. Despite not being able to accurately capture events in the tail, VaR is still useful for approximating the risk level inherent in financial assets.

Value at Risk

LO 52.5: Explain the various approaches for estimating VaR.

LO 52.6: Compare and contrast different parametric and non-parametric approaches for estimating conditional volatility.

LO 52.7: Calculate conditional volatility using parametric and non-parametric approaches.

A value at risk (VaR) method for estimating risk is typically either a historical-based approach or an implied-volatility-based approach. Under the historical-based approach, the shape of the conditional distribution is estimated based on historical time series data.

Historical-based approaches typically fall into three sub-categories: parametric, nonparametric, and hybrid.

1. The **parametric approach** requires specific assumptions regarding the asset returns distribution. A parametric model typically assumes asset returns are normally or lognormally distributed with time-varying volatility. The most common example of the parametric method in estimating future volatility is based on calculating historical variance or standard deviation using "mean squared deviation." For example, the following equation is used to estimate future variance based on a window of the *K* most recent returns data.[1]

$$\sigma_t^2 = \left(r^2_{t-K,t-K+1} + \cdots + r^2_{t-3,t-2} + r^2_{t-2,t-1} + r^2_{t-1,t}\right)/K$$

1. In order to adjust for one degree of freedom related to the conditional mean, the denominator in the formula is K – 1. In practice, adjusting for the degrees of freedom makes little difference when large sample sizes are used.

If we assume asset returns follow a random walk, the mean return is zero. Alternatively, an analyst may assume a conditional mean different from zero and a volatility for a specific period of time.

Professor's Note: The delta-normal method is an example of a parametric approach.

Example: Estimating a conditional mean

Assuming K = 100 (an estimation window using the most recent 100 asset returns), **estimate** a conditional mean assuming the market is known to decline 15%.

Answer:

The daily conditional mean asset return, μ_t, is estimated to be –15bp/day.

$$\mu_t = -1500\text{bp}/100\text{days} = -15\text{bp/day}$$

2. The **nonparametric approach** is less restrictive in that there are no underlying assumptions of the asset returns distribution. The most common nonparametric approach models volatility using the historical simulation method.

3. As the name suggests, the **hybrid approach** combines techniques of both parametric and nonparametric methods to estimate volatility using historical data.

The **implied-volatility-based approach** uses derivative pricing models such as the Black-Scholes-Merton option pricing model to estimate an implied volatility based on current market data rather than historical data.

Parametric Approaches for VaR

The RiskMetrics® [i.e., exponentially weighted moving average (EWMA) model] and GARCH approaches are both exponential smoothing weighting methods. RiskMetrics® is actually a special case of the GARCH approach. Both exponential smoothing methods are similar to the historical standard deviation approach because all three methods:

- Are parametric.
- Attempt to estimate conditional volatility.
- Use recent historical data.
- Apply a set of weights to past squared returns.

Professor's Note: The RiskMetrics® approach is just an EWMA model that uses a pre-specified decay factor for daily data (0.94) and monthly data (0.97).

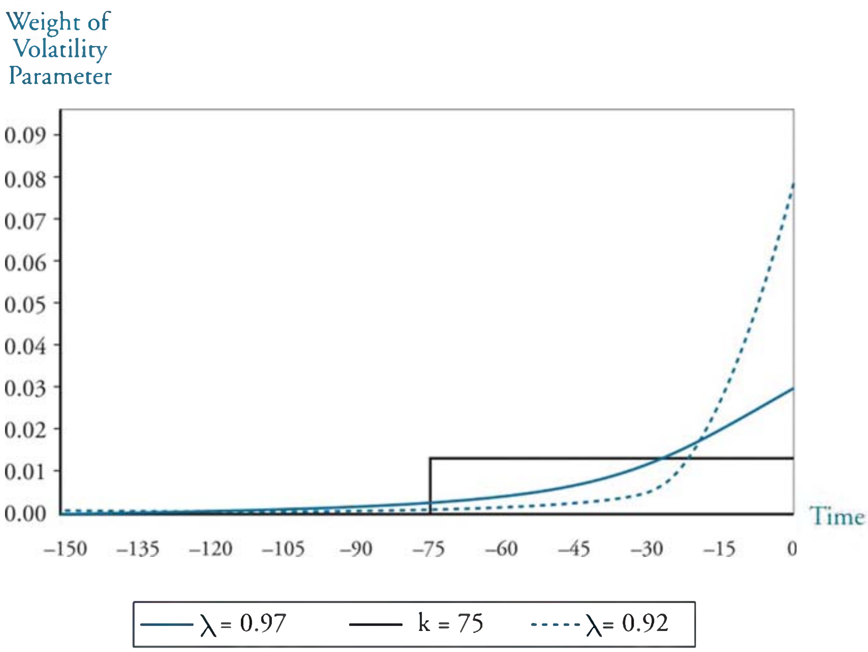
The only major difference between the historical standard deviation approach and the two exponential smoothing approaches is with respect to the weights placed on historical returns that are used to estimate future volatility. The historical standard deviation approach assumes all *K* returns in the window are equally weighted. Conversely, the exponential smoothing methods place a higher weight on more recent data, and the weights decline exponentially to zero as returns become older. The rate at which the weights change, or smoothness, is determined by a parameter λ (known as the decay factor) raised to a power. The parameter λ must fall between 0 and 1 (i.e., $0 < \lambda < 1$); however, most models use parameter estimates between 0.9 and 1 (i.e., $0.9 < \lambda < 1$).

Figure 4 illustrates the weights of the historical volatility for the historical standard deviation approach and RiskMetrics® approach. Using the RiskMetrics® approach, conditional variance is estimated using the following formula:

$$\sigma_t^2 = (1-\lambda)\left(\lambda^0 r^2_{t-1,t} + \lambda^1 r^2_{t-2,t-1} + \lambda^2 r^2_{t-3,t-2} + \cdots + \lambda^N r^2_{t-N-1,t-N}\right)$$

where:
N = the number of observations used to estimate volatility

Figure 4: Comparison of Exponential Smoothing and Historical Standard Deviation



Professor's Note: You may have noticed in Figure 4 that K *(the number of observations used to calculate the historical standard deviation) is 75, but* N *(the number of terms in the RiskMetrics® formula) is more than 75. There is no inconsistency here because the series* $[(1 - \lambda)\lambda^0 + (1 - \lambda)\lambda^1 + \ldots]$ *only sums to one if* N *is infinite. In practice,* N *is chosen so that the first* K *terms (in this example) sum to a number close to one.*

Example: Calculating weights using the RiskMetrics® approach

Using the RiskMetrics® approach, **calculate** the weight for the most current historical return, t = 0, when $\lambda = 0.97$.

Answer:

The weight for the most current historical return, t = 0, when $\lambda = 0.97$ is calculated as follows:

$$(1 - \lambda)\lambda^t = (1 - 0.97)0.97^0 = 0.03$$

Example: Calculating weights using the historical standard deviation approach

Calculate the weight for the most recent return using historical standard deviation approach with K = 75.

Answer:

All historical returns are equally weighted. Therefore, the weights will all be equal to 0.0133 (i.e., 1 / K = 1 / 75 = 0.0133).

Figure 5 summarizes the most recent weights for the volatility parameters using the three approaches used in Figure 4. Parameter λ values of 0.92 and 0.97 are used for the example of the RiskMetrics® approaches in Figure 4.

Figure 5: Summary of RiskMetrics® and Historical Standard Deviation Calculations

	Weight of Volatility Parameter		
	$(1-\lambda)\lambda^t$	1/k	$(1-\lambda)\lambda^t$
t	$\lambda = 0.97$	k = 75	$\lambda = 0.92$
0	0.0300	0.0133	0.0800
–1	0.0291	0.0133	0.0736
–2	0.0282	0.0133	0.0677
–3	0.0274	0.0133	0.0623
–4	0.0266	0.0133	0.0573

Example: Applying a shorter estimation window

How would a shorter estimation window of K = 40 impact forecasts using the historical standard deviation method?

Answer:

Using a shorter estimation window (K = 40) for the historical standard deviation method results in forecasts that are more volatile. This is in part due to the fact that each observation has more weight, and extreme observations therefore have a greater impact on the forecast. However, an advantage of using a smaller *K* for the estimation window is the model adapts more quickly to changes.

Example: Applying a smaller λ parameter

How would a smaller λ parameter in the RiskMetrics® approach impact forecasts?

Answer:

Using a smaller *K* in the historical simulation model is similar to using a smaller λ parameter in the RiskMetrics® approach. It results in a higher weight to recent observations and a smaller sample window. As illustrated by Figure 4, a λ parameter closer to one results in less weight on recent observations and a larger sample window with a slower exponential smoothing decay in information.

GARCH

A more general exponential smoothing model is the GARCH model. This is a time-series model used by analysts to predict time-varying volatility. Volatility is measured with a general GARCH(p,q) model using the following formula:

$$\sigma_t^2 = a + b_1 r_{t-1,t}^2 + b_2 r_{t-2,t-1}^2 + \cdots + b_p r_{t-p,t-p+1}^2 + c_1\sigma_{t-1}^2 + c_2\sigma_{t-2}^2 + \cdots + c_q\sigma_{t-q}^2$$

where:

parameters a, b_1 through b_p, and c_1 through c_q = parameters estimated using historical data with *p* lagged terms on historical returns squared and *q* lagged terms on historical volatility

A GARCH(1,1) model would look like this:

$$\sigma_t^2 = a + b r_{t-1,t}^2 + c\sigma_{t-1}^2$$

Example: GARCH vs. RiskMetrics®

Show how the GARCH(1,1) time-varying process with a = 0 and b + c = 1 is identical to the RiskMetrics® model.

Answer:

Using these assumptions and substituting 1 – c for *b* results in the following special case of the GARCH(1,1) model as follows:

$$\sigma_t^2 = (1-c) r_{t-1,t}^2 + c\sigma_{t-1}^2$$

Substituting λ for *c* in this equation results in the common notation for the RiskMetrics® approach. Therefore, the GARCH model is less restrictive and more general than the RiskMetrics® model. The GARCH model using a larger number of parameters can more accurately model historical data. However, a model with more parameters to estimate also incurs more estimation risk, or noise, that can cause the GARCH model to have less ability to forecast future returns.

Nonparametric vs. Parametric VaR Methods

Three common types of nonparametric methods used to estimate VaR are: (1) historical simulation, (2) multivariate density estimation, and (3) hybrid. These nonparametric methods exhibit the following advantages and disadvantages over parametric approaches.

Advantages of nonparametric methods compared to parametric methods:

- Nonparametric models do not require assumptions regarding the entire distribution of returns to estimate VaR.
- Fat tails, skewness, and other deviations from some assumed distribution are no longer a concern in the estimation process for nonparametric methods.
- Multivariate density estimation (MDE) allows for weights to vary based on how relevant the data is to the current market environment, regardless of the timing of the most relevant data.
- MDE is very flexible in introducing dependence on economic variables (called *state variables* or *conditioning variables*).
- Hybrid approach does not require distribution assumptions because it uses a historical simulation approach with an exponential weighting scheme.

Disadvantages of nonparametric methods compared to parametric methods:

- Data is used more efficiently with parametric methods than nonparametric methods. Therefore, large sample sizes are required to precisely estimate volatility using historical simulation.
- Separating the full sample of data into different market regimes reduces the amount of usable data for historical simulations.
- MDE may lead to data snooping or over-fitting in identifying required assumptions regarding the weighting scheme identification of relevant conditioning variables and the number of observations used to estimate volatility.
- MDE requires a large amount of data that is directly related to the number of conditioning variables used in the model.

Nonparametric Approaches for VaR

Historical Simulation Method

The six lowest returns for an estimation window of 100 days (K = 100) are listed in Figure 6. Under the historical simulation, all returns are weighted equally based on the number of observations in the estimation window (1/K). Thus, in this example, each return has a weight of 1/100, or 0.01.

Example: Calculating VaR using historical simulation

Calculate VaR of the 5th percentile using historical simulation and the data provided in Figure 6.

Figure 6: Historical Simulation Example

Six Lowest Returns	*Historical Simulation Weight*	*HS Cumulative Weight*
–4.70%	0.01	0.0100
–4.10%	0.01	0.0200
–3.70%	0.01	0.0300
–3.60%	0.01	0.0400
–3.40%	0.01	0.0500
–3.20%	0.01	0.0600

Answer:

Calculating VaR of 5% requires identifying the 5th percentile. With 100 observations, the 5th percentile would be the 5th lowest return. However, observations must be thought of as a random event with a probability mass centered where the observation occurs, with 50% of its weight to the left and 50% of its weight to the right. Thus, the 5th percentile is somewhere between the 5th and 6th observation. In our example, the 5th lowest return, –3.40%, represents the 4.5th percentile, and we must interpolate to obtain the 5th percentile at –3.30% [calculated as (–3.4% + –3.20%) / 2].

Professor's Note: As was mentioned in the VaR Methods reading, the calculation of historical VaR may differ depending on the method used. You may use a given percentile return or interpolate to obtain the percentile return as was done in the previous example. On past FRM exams, GARP has just used the percentile in question, so in the previous example, the historical VaR of 5% would be based on –3.4%.

Notice that regardless of how far away in the 100-day estimation window the lowest observations occurred, they will still carry a weight of 0.01. The hybrid approach described next uses exponential weighting similar to the RiskMetrics® approach to adjust the weighting more heavily toward recent returns.

Hybrid Approach

The hybrid approach uses historical simulation to estimate the percentiles of the return and weights that decline exponentially (similar to GARCH or RiskMetrics®). The following three steps are required to implement the hybrid approach.

Step 1: Assign weights for historical realized returns to the most recent K returns using an exponential smoothing process as follows:

$$[(1-\lambda)/(1-\lambda^K)],\ [(1-\lambda)/(1-\lambda^K)]\lambda^1, \ldots,\ [(1-\lambda)/(1-\lambda^K)]\lambda^{K-1}$$

Step 2: Order the returns.

Step 3: Determine the VaR for the portfolio by starting with the lowest return and accumulating the weights until x percentage is reached. Linear interpolation may be necessary to achieve an exact x percentage.

In Step 1, there are several equations in between the second and third terms. These equations change the exponent on the last decay factor term to reflect observations that have occurred t days ago. For example, assume 100 observations and a decay factor of 0.96. For the hybrid weight for an observation that occurred one period ago, you would use the following equation: $[(1-0.96)/(1-0.96^{100})] = 0.0407$. For the hybrid weight of an observation two periods ago, you use this equation: $[(1-0.96)/(1-0.96^{100})] \times 0.96^{(100-99)} = 0.0391$. The hybrid weight five periods ago would equal: $[(1-0.96)/(1-0.96^{100})] \times 0.96^{(100-96)} = 0.0346$.

Example: Calculating weight using the hybrid approach

Suppose an analyst is using a hybrid approach to determine a 5% VaR with the most recent 100 observations (K = 100) and λ = 0.96 using the data in Figure 7. Note that the data in Figure 7 are already ranked as described in Step 2 of the hybrid approach. Therefore, the six lowest returns out of the most recent 100 observations are listed in Figure 7. The weights for each observation are based on the number of observations (K = 100) and the exponential weighting parameter (λ = 0.96) using the formula provided in Step 1.

Figure 7: Hybrid Example Illustrating Six Lowest Returns (where K = 100 and λ = 0.96)

Rank	*Six Lowest Returns*	*Number of Past Periods*	*Hybrid Weight*	*Hybrid Cumulative Weight**
1	–4.70%	2	0.0391	0.0391
2	–4.10%	5	0.0346	0.0736
3	–3.70%	55	0.0045	0.0781
4	–3.60%	25	0.0153	0.0934
5	–3.40%	14	0.0239	0.1173
6	–3.20%	7	0.0318	0.1492

*Cumulative weights are slightly affected by rounding error.

Calculate the hybrid weight assigned to the lowest return, –4.70%.

Answer:

The hybrid weight is calculated as follows:

$$[(1 - \lambda) / (1 - \lambda^K)]\lambda^1 = [(1 - 0.96) / (1 - 0.96^{100})]0.96 = 0.0391$$

Note: Since this observation is only two days old, it has the second highest weight assigned out of the 100 total observations in the estimation window.

Example: Calculating VaR using the hybrid approach

Using the information in Figure 7, **calculate** the initial VaR at the 5th percentile using the hybrid approach.

Answer:

The lowest and second lowest returns have cumulative weights of 3.91% and 7.36%, respectively. Therefore, we must interpolate to obtain the 5% VaR percentile. The point halfway between the two lowest returns is interpolated as –4.40% [(–4.70% + –4.10%) / 2] with a cumulative weight of 5.635% calculated as follows: (7.36% + 3.91%) / 2. Further interpolation is required to find the 5th percentile VaR level somewhere between –4.70% and –4.40%.

For the initial period represented in Figure 7, the 5% VaR using the hybrid approach is calculated as:

$$4.7\% - (4.70\% - 4.40\%)[(0.05 - 0.03910) / (0.05635 - 0.03910)]$$
$$= 4.70\% - 0.3\%(0.63188) = 4.510\%$$

Example: Calculating revised VaR

Assume that over the next 20 days there are no extreme losses. Therefore, the six lowest returns will be the same returns in 20 days, as illustrated in Figure 8. Notice that the weights are less for these observations because they are now further away. **Calculate** the revised VaR at the 5th percentile using the information in Figure 8.

Figure 8: Hybrid Example Illustrating Six Lowest Return After 20 Days (where K = 100 and λ = 0.96)

Rank	*Six Lowest Returns*	*Number of Past Periods*	*Hybrid Weight*	*Hybrid Cumulative Weight**
1	–4.70%	22	0.0173	0.0173
2	–4.10%	25	0.0153	0.0325
3	–3.70%	75	0.0020	0.0345
4	–3.60%	45	0.0068	0.0413
5	–3.40%	34	0.0106	0.0519
6	–3.20%	27	0.0141	0.0659

*Cumulative weights are slightly affected by rounding error.

Answer:

The 5th percentile for calculating VaR is somewhere between –3.6% and –3.4%. The point halfway between these points is interpolated as –3.5% with a cumulative weight of 4.66% [(4.13% + 5.19%) / 2]. The 5% VaR using the hybrid approach is calculated as:

$$3.5\% - (3.5\% - 3.4\%)[(0.05 - 0.0466) / (0.0519 - 0.0466)]$$
$$= 3.5\% - 0.1\%(0.6415) = 3.436\%$$

Multivariate Density Estimation (MDE)

Under the MDE model, conditional volatility for each market state or regime is calculated as follows:

$$\sigma_t^2 = \sum_{i=1}^{K} \omega(x_{t-i}) r_{t-i}^2$$

where:

x_{t-i} = the vector of relevant variables describing the market state or regime at time $t - i$

$\omega(x_{t-i})$ = the weight used on observation $t - i$, as a function of the "distance" of the state x_{t-i} from the current state x_t

The kernel function, $\omega(x_{t-i})$, is used to measure the relative weight in terms of "near" or "distant" from the current state. The MDE model is very flexible in identifying dependence on state variables. Some examples of relevant state variables in an MDE model are interest rate volatility dependent on the level of interest rates or the term structure of interest rates, equity volatility dependent on implied volatility, and exchange rate volatility dependent on interest rate spreads.

Return Aggregation

LO 52.8: Explain the process of return aggregation in the context of volatility forecasting methods.

When a portfolio is comprised of more than one position using the RiskMetrics® or historical standard deviation approaches, a single VaR measurement can be estimated by assuming asset returns are all normally distributed. The covariance matrix of asset returns is used to calculate portfolio volatility and VaR. The delta-normal method requires the calculation of *N* variances and [N × (N – 1)] / 2 covariances for a portfolio of *N* positions. The model is subject to estimation error due to the large number of calculations. In addition, some markets are more highly correlated in a downward market, and in such cases, VaR is underestimated.

The historical simulation approach requires an additional step that aggregates each period's historical returns weighted according to the relative size of each position. The weights are based on the market value of the portfolio positions today, regardless of the actual allocation of positions *K* days ago in the estimation window. A major advantage of this approach compared to the delta-normal approach is that no parameter estimates are required. Therefore, the model is not subject to estimation error related to correlations and the problem of higher correlations in downward markets.

A third approach to calculating VaR estimates the volatility of the vector of aggregated returns and assumes normality based on the strong law of large numbers. The strong law of large numbers states that an average of a very large number of random variables will end up converging to a normal random variable. However, this approach can only be used in a well-diversified portfolio.

Implied Volatility

LO 52.9: Evaluate implied volatility as a predictor of future volatility and its shortcomings.

Estimating future volatility using historical data requires time to adjust to current changes in the market. An alternative method for estimating future volatility is implied volatility. The Black-Scholes-Merton model is used to infer an implied volatility from equity option prices. Using the most liquid at-the-money put and call options, an average implied volatility is extrapolated using the Black-Scholes-Merton model.

A big *advantage* of implied volatility is the forward-looking predictive nature of the model. Forecast models based on historical data require time to adjust to market events. The implied volatility model reacts immediately to changing market conditions.

The implied volatility model does, however, exhibit some *disadvantages*. The biggest disadvantage is that implied volatility is model dependent. A major assumption of the model is that asset returns follow a continuous time lognormal diffusion process. The volatility parameter is assumed to be constant from the present time to the contract maturity date. However, implied volatility varies through time; therefore, the Black-Scholes-Merton model is misspecified. Options are traded on the volatility of the underlying asset with what is known as "vol" terms. In addition, at a given point in time, options with the same underlying assets may be trading at different vols. Empirical results suggest implied volatility is on average greater than realized volatility. In addition to this upward bias in implied volatility, there is the problem that available data is limited to only a few assets and market factors.

Mean Reversion and Long Time Horizons

LO 52.10: Explain long horizon volatility/VaR and the process of mean reversion according to an AR(1) model.

LO 52.11: Calculate conditional volatility with and without mean reversion.

LO 52.12: Describe the impact of mean reversion on long horizon conditional volatility estimation.

To demonstrate mean reversion, consider a time series model with one lagged variable:

$$X_i = a + b \times X_{i-1}$$

This type of regression, with a lag of its own variable, is known as an autoregressive (AR) model. In this case, since there is only one lag, it is referred to as an AR(1) model. The long-run mean of this model is evaluated as [a / (1 – b)]. The key parameter in this long-run mean equation is *b*. Notice that if b = 1, the long-run mean is infinite (i.e., the process is a random walk). If *b*, however, is less than 1, then the process is mean reverting (i.e., the time series will trend toward its long-run mean). In the context of risk management, it is helpful to evaluate the impact of mean revision on variance.

Note that the single-period conditional variance of the rate of change is σ^2 and that the two-period variance is $(1 + b^2)\sigma^2$. If b = 1, the typical variance (i.e., square root volatility) would occur as this represents a random walk. If b < 1, the process is mean reverting. For example, the two-period volatility *without* mean reversion would be equal to:

$$\sqrt{2\sigma^2} = 1.41\sigma$$

With mean reversion (e.g., b = 0.8), the two-period volatility would be less:

$$\sqrt{(1+0.8^2)\sigma^2} = 1.28\sigma$$

Understanding the impact of mean reversion is especially important in the context of arbitrage and other trading strategies. For example, a convergence trade assumes explicitly that the spread between a long and short position is mean reverting. If mean reversion exists, the long horizon risk (and the resulting VaR calculation) is smaller than square root volatility.

Professor's Note: Remember that, like volatility, VaR can be extended to a longer-term basis by multiplying VaR by the square root of the number of days (i.e., the square root rule). For example, to convert daily VaR to weekly VaR, multiply the daily VaR by the square root of 5.

Backtesting VaR Methodologies

Backtesting is the process of comparing losses predicted by the value at risk (VaR) model to those actually experienced over the sample testing period. If a model were completely accurate, we would expect VaR to be exceeded (this is called an *exception*) with the same frequency predicted by the confidence level used in the VaR model. In other words, the probability of observing a loss amount greater than VaR is equal to the significance level (x%). This value is also obtained by calculating one minus the confidence level.

For example, if a VaR of $10 million is calculated at a 95% confidence level, we expect to have exceptions (losses exceeding $10 million) 5% of the time. If exceptions are occurring with greater frequency, we may be underestimating the actual risk. If exceptions are occurring less frequently, we may be overestimating risk.

There are three desirable attributes of VaR estimates that can be evaluated when using a backtesting approach. The first desirable attribute is that the VaR estimate should be *unbiased*. To test this property, we use an indicator variable to record the number of times an exception occurs during a sample period. For each sample return, this indicator variable is recorded as 1 for an exception or 0 for a non-exception. The average of all indicator variables over the sample period should equal x% (i.e., the significance level) for the VaR estimate to be unbiased.

A second desirable attribute is that the VaR estimate is *adaptable*. For example, if a large return increases the size of the tail of the return distribution, the VaR amount should also be increased. Given a large loss amount, VaR must be adjusted so that the probability of the next large loss amount again equals x%. This suggests that the indicator variables, discussed previously, should be independent of each other. It is necessary that the VaR estimate account for new information in the face of increasing volatility.

A third desirable attribute, which is closely related to the first two attributes, is for the VaR estimate to be *robust*. A strong VaR estimate produces only a small deviation between the number of expected exceptions during the sample period and the actual number of exceptions. This attribute is measured by examining the statistical significance of the autocorrelation of extreme events over the backtesting period. A statistically significant autocorrelation would indicate a less reliable VaR measure.

By examining historical return data, we can gain some clarity regarding which VaR method actually produces a more reliable estimate in practice. In general, VaR approaches that are nonparametric (e.g., historical simulation and the hybrid approach) do a better job at producing VaR amounts that mimic actual observations when compared to parametric methods such as an exponential smoothing approach (e.g., GARCH). The likely reason for this performance difference is that nonparametric approaches can more easily account for the presence of fat tails in a return distribution. Note that higher levels of λ (the exponential weighing parameter) in the hybrid approach will perform better than lower levels of λ. Finally, when testing the autocorrelation of tail events, we find that the hybrid approach performs better than exponential smoothing approaches. In other words, the hybrid approach tends to reject the null hypothesis that autocorrelation is equal to zero *fewer times* than exponential smoothing approaches.

Key Concepts

LO 52.1

Three common deviations from normality that are problematic in modeling risk result from asset returns that are fat-tailed, skewed, or unstable. Fat-tailed refers to a distribution with a higher probability of observations occurring in the tails relative to the normal distribution. A distribution is skewed when the distribution is not symmetrical and there is a higher probability of outliers. Parameters of the model that vary over time are said to be unstable.

LO 52.2

The phenomenon of "fat tails" is most likely the result of the volatility and/or the mean of the distribution changing over time.

LO 52.3

If the mean and standard deviation are the same for asset returns for any given day, the distribution of returns is referred to as an unconditional distribution of asset returns. However, different market or economic conditions may cause the mean and variance of the return distribution to change over time. In such cases, the return distribution is referred to as a conditional distribution.

LO 52.4

A regime-switching volatility model assumes different market regimes exist with high or low volatility. The probability of large deviations from normality (such as fat tails) occurring are much less likely under the regime-switching model because it captures the conditional normality.

LO 52.5

Historical-based approaches of measuring VaR typically fall into three sub-categories: parametric, nonparametric, and hybrid.

- The parametric approach typically assumes asset returns are normally or lognormally distributed with time-varying volatility (i.e., historical standard deviation or exponential smoothing).
- The nonparametric approach is less restrictive in that there are no underlying assumptions of the asset returns distribution (i.e., historical simulation).
- The hybrid approach combines techniques of both parametric and nonparametric methods to estimate volatility using historical data.

LO 52.6

A major difference between the historical standard deviation approach and the two exponential smoothing approaches is with respect to the weights placed on historical returns. Exponential smoothing approaches give more weight to recent returns, and the historical standard deviation approach weights all returns equally.

LO 52.7

The RiskMetrics® and GARCH approaches are both exponential smoothing weighting methods. RiskMetrics® is actually a special case of the GARCH approach. Exponential smoothing methods are similar to the historical standard deviation approach because they are parametric, attempt to estimate conditional volatility, use recent historical data, and apply a set of weights to past squared returns.

LO 52.8

When a portfolio is comprised of more than one position using the RiskMetrics® or historical standard deviation approaches, a single VaR measurement can be estimated by assuming asset returns are all normally distributed. The historical simulation approach for calculating VaR for multiple portfolios aggregates each period's historical returns weighted according to the relative size of each position. The weights are based on the market value of the portfolio positions today, regardless of the actual allocation of positions *K* days ago in the estimation window. A third approach to calculating VaR for portfolios with multiple positions estimates the volatility of the vector of aggregated returns and assumes normality based on the strong law of large numbers.

LO 52.9

The implied-volatility-based approach for measuring VaR uses derivative pricing models such as the Black-Scholes-Merton option pricing model to estimate an implied volatility based on current market data rather than historical data.

LO 52.10

With an AR(1) model, long-run mean is computed as: $[a / (1 - b)]$. If b equals 1, the long-run mean is infinite (i.e., the process is a random walk). If b is less than 1, then the process is mean reverting.

LO 52.11

Under the context of mean reversion, the single-period conditional variance of the rate of change is σ^2, and the two-period variance is $(1 + b^2)\sigma^2$. Without mean reversion (i.e., $b = 1$), the two-period volatility is equal to the square root of $2\sigma^2$. With mean reversion (i.e., $b < 1$), the two-period volatility will be less than the volatility from no mean reversion.

LO 52.12

If mean reversion exists, the long horizon risk (and resulting VaR calculation) will be smaller than square root volatility.

Concept Checkers

1. Fat-tailed asset return distributions are most likely the result of time-varying:
   A. volatility for the unconditional distribution.
   B. means for the unconditional distribution.
   C. volatility for the conditional distribution.
   D. means for the conditional distribution.

2. The problem of fat tails when measuring volatility is least likely:
   A. in a regime-switching model.
   B. in an unconditional distribution.
   C. in a historical standard deviation model.
   D. in an exponential smoothing model.

3. Which of the following is not a disadvantage of nonparametric methods compared to parametric methods?
   A. Data is used more efficiently with parametric methods than nonparametric methods.
   B. Identifying market regimes and conditional volatility increases the amount of usable data as well as the estimation error for historical simulations.
   C. MDE may lead to data snooping or over-fitting in identifying required assumptions regarding an appropriate kernal function.
   D. MDE requires a large amount of data that is directly related to the number of conditioning variables used in the model.

4. The lowest six returns for a portfolio are shown in the following table.

Six lowest returns with hybrid weightings

	Six Lowest Returns	*Hybrid Weight*	*Hybrid Cumulative Weight*
1	–4.10%	0.0125	0.0125
2	–3.80%	0.0118	0.0243
3	–3.50%	0.0077	0.0320
4	–3.20%	0.0098	0.0418
5	–3.10%	0.0062	0.0481
6	–2.90%	0.0027	0.0508

What will the 5% VaR be under the hybrid approach? Assume each observation is a random event with 50% to the left and 50% to the right of each observation.
   A. –3.10%.
   B. –3.04%.
   C. –2.96%.
   D. –2.90%.

5. Which of the following statements is an advantage of the implied volatility method in estimating future volatility? The implied volatility:
   A. model reacts immediately to changing market conditions.
   B. model is not model dependent.
   C. is constant through time.
   D. is biased upward and is therefore more conservative.

Concept Checker Answers

1. **A** The most likely explanation for "fat tails" is that the second moment or volatility is time-varying for the unconditional distribution. For example, this explanation is much more likely given observed changes in volatility in interest rates prior to a much anticipated Federal Reserve announcement. Examining a data sample at different points of time from the full sample could generate fat tails in the unconditional distribution, even if the conditional distributions are normally distributed.

2. **A** The regime-switching model captures the conditional normality and may resolve the fat-tailed problem and other deviations from normality. A regime-switching model allows for conditional means and volatility. Thus, the conditional distribution can be normally distributed even if the unconditional distribution is not.

3. **B** The use of market regimes and identifying conditional means and volatility actually reduces—not increases—the amount of data from the full sample. The full sample of data is split into subgroups used to estimate conditional volatility. Therefore, the amount of data available for estimating future volatility is significantly reduced.

4. **C** The fifth and sixth lowest returns have cumulative weights of 4.81% and 5.08%, respectively. The point halfway between these two returns is interpolated as –3.00% with a cumulative weight of 4.945%, calculated as follows: (4.81% + 5.08%) / 2. Further interpolation is required to find the 5th percentile VaR level with a return somewhere between –3.00% and –2.90%. The 5% VaR using the hybrid approach is calculated as:

   $$3.00\% - (3.00\% - 2.90\%)[(0.05 - 0.04945) / (0.0508 - 0.04945)]$$
   $$= 3.00\% - 0.10\%(0.0005 / 0.00135) = 2.96\%$$

   Notice that the answer has to be between –2.90% and –3.00%, so –2.96% is the only possible answer.

5. **A** The only advantage listed is that the implied volatility model reacts immediately to changing market conditions. Forecast models based on historical data require time to adjust to market events. Disadvantages include the following: (1) implied volatility is model dependent; (2) a major assumption of the model is that asset returns follow a continuous time lognormal diffusion process and are assumed to be constant but that implied volatility varies through time; and (3) implied volatility is biased upward.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Putting VaR to Work

Topic 53

Exam Focus

Derivatives and portfolios containing derivatives and other assets create challenges for risk managers in measuring value at risk (VaR). In this topic, risk measurement approaches are discussed for linear and non-linear derivatives. The advantages and disadvantages and underlying assumptions of the various approaches are presented. In addition, Taylor Series approximation is addressed, with examples of applying this theory to VaR approaches. Finally, structured Monte Carlo (SMC), stress testing, and worst case scenario (WSC) analysis are presented as useful methods in extending VaR techniques to more appropriately measure risk for complex derivatives and scenarios.

Linear vs. Non-Linear Derivatives

LO 53.1: Explain and give examples of linear and non-linear derivatives.

A derivative is described as *linear* when the relationship between an underlying factor and the derivative is linear in nature. For example, an equity index futures contract is a linear derivative, while an option on the same index is non-linear. The delta for a linear derivative must be constant for all levels of the underlying factor, but not necessarily equal to one.

For example, the rate on a foreign currency forward contract is defined as:

$$F_{t,T} = S_t(1+R_D)/(1+R_F)$$

Where $F_{t,T}$ is the forward rate at time t for the period T-t, S_t is the spot exchange rate, R_D is the domestic interest rate, and R_F is the foreign interest rate. The value at risk (VaR) of the forward is related to the spot rate, S_t, and the foreign and domestic interest rates. Assuming fixed interest rates for very short time intervals, we can approximate the forward rate, $F_{t,T}$, with the interest rate differential as a constant K that is not a function of time as follows:

$$F_{t,T} = S_t(1 + R_D)/(1 + R_F) \approx KS_t$$

Furthermore, the continuously compounded return on the foreign forward contract, $\Delta f_{t,t+1}$, is approximately equal to the return on the spot rate, $\Delta s_{t,t+1}$. This can be shown mathematically where the ln of the constant K is very close to zero and the approximate relationship is simplified as follows:

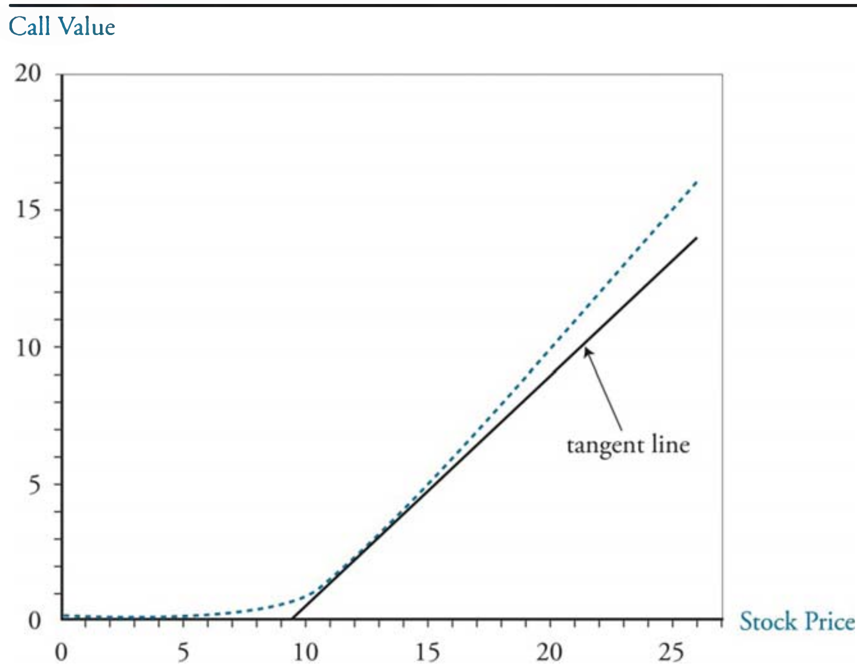
$$\Delta f_{t,t+1} = \ln(F_{t+1,T-1}/F_{t,T}) = \ln(S_{t+1}/S_t) + \ln(\Delta K) \approx \ln(S_{t+1}/S_t)$$

Changes in exchange rates can therefore be approximated by changes in spot rates. The VaR of a spot position is approximately equal to a forward position exchange rate if the only relevant underlying factor is the exchange rate. As this illustrates, many derivatives that are referred to as linear are actually only approximately linear. If we account for the changes in the two interest rates, the actual relationship would be nonlinear. Thus, the notion of linearity or nonlinearity is a function of the definition of the underlying risk factor.

The value of a *nonlinear* derivative is a function of the change in the value of the underlying asset and is dependent on the state of the underlying asset. A call option is a good example of a nonlinear derivative. The value of the call option does not increase (decrease) at a constant rate when the underlying asset increases (decreases) in value.

The change in the value of the call option is dependent in part on how far away the market value of the stock is from the exercise price. Thus, the relationship of the stock to the exercise price, *S/X*, captures the distance the option is from being in-the-money. Figure 1 illustrates how the value of the call option does not change at a constant rate with the change in the value of the underlying asset. The curved line represents the actual change in value of the call option based on the Black-Scholes-Merton model. The tangent line at any point on the curve illustrates how this is not a linear change in value. Furthermore, the slope of the line increases as the stock price increases. The percentage change in the call value given a change in the underlying stock price will be different for different stock price levels.

Figure 1: Call Option Value Given Underlying Stock Price



LO 53.2: Describe and calculate VaR for linear derivatives.

In general, the VaR of a long position in a linear derivative is $VaR_p = \Delta VaR_f$, where VaR_f is the VaR of the underlying factor and the derivative's delta, Δ, is the sensitivity of the derivative's price to changes in the underlying factor. Delta is assumed to be positive because we're modeling a long position. The local delta is defined as the percentage change in the derivative's price for a 1% change in the underlying asset. For small changes in the underlying price of the asset the change in price of the derivative can be extrapolated based on the local delta.

Example: Futures contract VaR

Determine how a risk manager could estimate the VaR of an equity index futures contract. Assume a 1-point increase in the index increases the value of a long position in the contract by $500.

Answer:

This relationship is shown mathematically as: $F_t = \$500S_t$, where F_t is the futures contract and S_t is the underlying index. The VaR of the futures contract is calculated as the amount of the index point movement in the underlying index, S_t, times the multiple, $500 as follows: $VaR(F_t) = \$500VaR(S_t)$.

Taylor Approximation

LO 53.3: Describe the delta-normal approach for calculating VaR for non-linear derivatives.

LO 53.4: Describe the limitations of the delta-normal method.

Suppose we create a table that shows the relationship of the call value to the stock price. The original stock price and call option value are $11.00 and $1.41, respectively. The Black-Scholes-Merton model is used to calculate the call value for different stock prices. Figure 2 summarizes some of the points.

Figure 2: Change in Call Value Given a Change in Stock Price (numbers reflect small rounding error)

Stock Price, *S*	$7.00	$8.00	$9.00	$10.00	$10.89	$11.00
Value of Call, *C*	$0.00	$0.05	$0.23	$0.69	$1.32	$1.41
Percentage Decrease in *S*	–36.36%	–27.27%	–18.18%	–9.09%	–1.00%	
Percentage Decrease in *C*	–100.00%	–96.76%	–83.31%	–51.06%	–6.35%	
Delta ($\Delta C\% / \Delta S\%$)	2.74	3.55	4.58	5.62	6.35	

The **delta** is calculated in Figure 2 by dividing the percentage change in the call value by the percentage change in the stock price (delta = $\Delta C\% / \Delta S\%$). The **local delta** is the slope of the line at any point of the nonlinear relationship for a 1% change in the stock price. The local delta can be used to estimate the change in the value of the call option given a *small* change in the value of the stock price.

Example: Call option VaR

Suppose a 6-month call option with a strike price, *X*, of \$10 is currently trading for \$1.41, when the market price of the underlying stock is \$11. A 1% decrease in the stock price to \$10.89 results in a 6.35% decrease in the call option with a value of \$1.32. If the annual volatility of the stock is σ = 0.1975 and the risk-free rate of return is 5%, **calculate** the one day 5% VaR for this call option.

Answer:

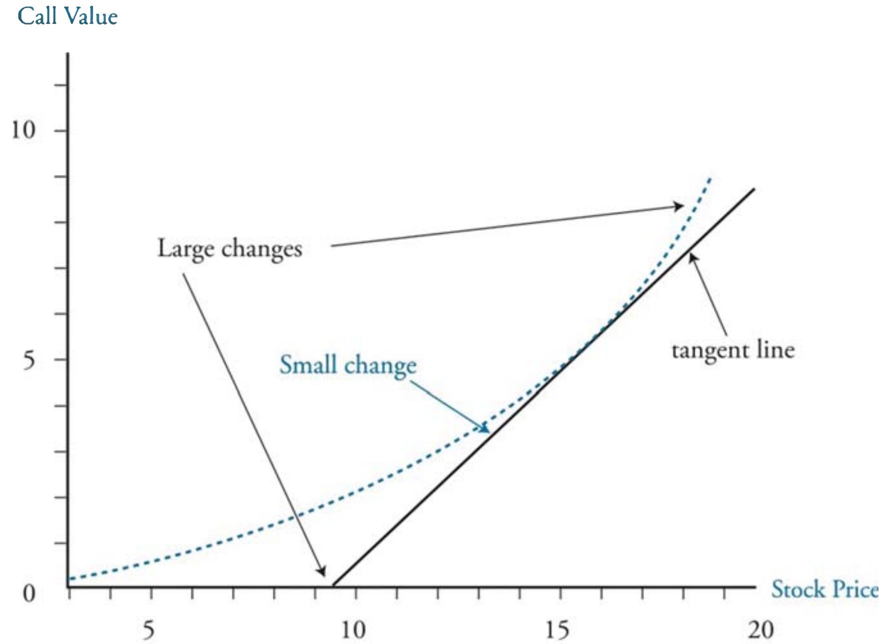
The daily volatility is approximately equal to 1.25% ($0.1975 / \sqrt{250}$). The 5% VaR for the stock price is equivalent to a one standard deviation move, or 1.65 for the normal curve. Assuming a random walk or 0 mean daily return, the 5% VaR of the underlying stock is 0 – 1.25%(1.65) = –2.06%. A 1% change in the stock price results in a 6.35% change in the call option value, therefore, the delta = 0.0635/0.01 = 6.35. For small moves, delta can be used to estimate the change in the derivative given the VaR for the underlying asset as follows: $VaR_{call} = \Delta VaR_{stock}$ = 6.35(2.06%) = 0.1308 or 13.1%. In words, the 5% VaR implies there is a 5% probability that the call option value will decline by 13.1% or more. Note this estimate is only an approximation for small changes in the underlying stock. The precise change can be calculated using the Black-Scholes-Merton model.

Figure 3 illustrates that the slope of the line is only useful in estimating the call value with small changes in the underlying stock value. The gap between the tangency line representing the delta or slope of the line at the tangency point widens the further away the estimate is from the point of tangency. The first derivative of a function tells us the slope of the line at any given point. The second derivative tells us the rate of change. This information is summarized mathematically in the **Taylor Series approximation** of the function f(x) as follows:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

The Taylor Series states that the change in value of any function can be expressed by adjusting the original function value, $f(x_0)$ plus the slope of the line, $f'(x_0)$, times the change in the *x* variable plus the rate of the change measured by the last term above. The last term captures the convexity or curvature. This is still an approximation, but it is much closer than the linear estimation.

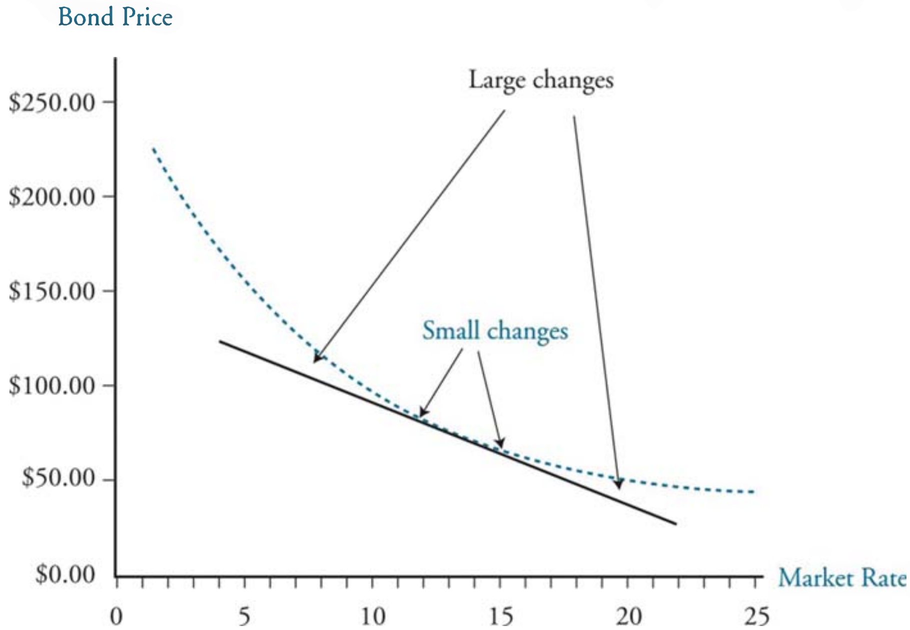
Figure 3: Call Option Example of Measurement Error Resulting from Convexity



Bond Example

As we will discuss in Topic 58, the **price-yield (P-Y) curve** depicts the change in the value of a bond as market rates of interest change. This is another example of a nonlinear relationship. Figure 4 illustrates the P-Y curve for a 20-year treasury bond with no embedded options. The straight line represents the duration of the bond. Duration is a linear estimation of the change in bond price given a change in interest rates and is only good for very small changes. Conversely, for large changes in interest rates, the gap between the P-Y curve and the tangent line represents the estimation error. Measuring the convexity in addition to the duration of the bond gives a much better approximation of the change in bond price for a given change in market rates. The use of duration and convexity to estimate bond prices as interest rates change is similar to the use of the delta and of the gamma approximation of the impact of fluctuations in the underlying factor on the value of an option. Both approximations are based on the Taylor Series that uses first and second derivatives of a known pricing model.

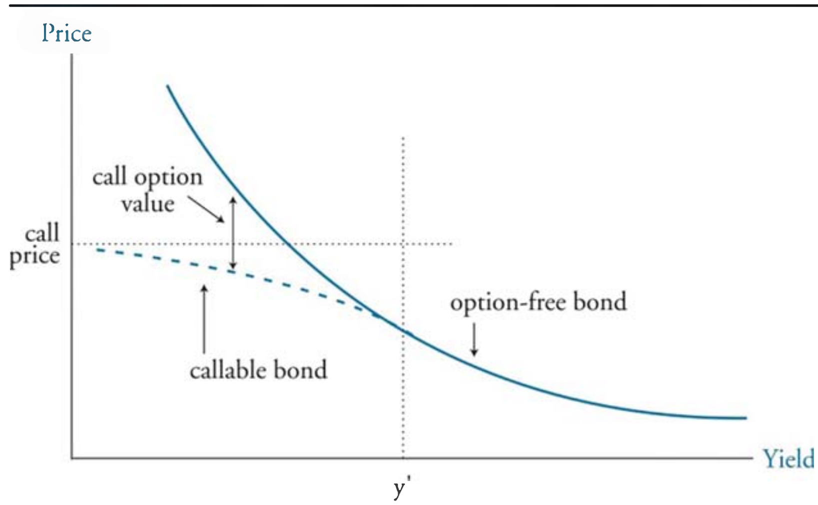
Figure 4: Measurement Error Resulting from Convexity in Bond Pricing



Consider a bond that is callable. The price-yield curve in Figure 5 illustrates that the call feature causes the P-Y curve to become concave as market interest rates approach the level where the bond will be called. Thus, the Taylor approximation is not useful because the callable bond is not a "well-behaved" function. In other words, the embedded call option causes the P-Y curve to deviate from the quadratic function that can be approximated by a polynomial of order two using the Taylor series.

Another example of a security with an embedded option are mortgage-backed securities (MBS). Borrowers will prepay loans early with significant drops in market interest rates. This causes the MBS to act similar to a bond that is called in. Unpredictable changes in duration due to early payoffs of MBS make the securities difficult to price and hedge. A convexity adjustment alone is not sufficient to estimate the change in the underlying security's value based on changes in market rates. The function explaining the relationship between the MBS value and market rates of interest does not behave similar at low and high levels of interest rates.

Figure 5: Price-Yield Curves for Callable and Noncallable Bonds



The Delta-Normal and Full Revaluation Methods

LO 53.5: Explain the full revaluation method for computing VaR.

LO 53.6: Compare delta-normal and full revaluation approaches for computing VaR.

Both the delta-normal and full revaluation methods measure the risk of nonlinear securities. The **full revaluation approach** calculates the VaR of the derivative by valuing the derivative based on the underlying value of the index after the decline corresponding to an *x*% VaR of the index. This approach is accurate, but can be highly computational. The revaluation of portfolios that include more complex derivatives (i.e., mortgage backed securities, or options with embedded features) are not easily calculated due to the large number of possible scenarios.

The **delta-normal approach** calculates the risk using the delta approximation ($VaR_p = \Delta VaR_f$), which is linear or the delta-gamma approximation,

$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$, which adjusts for the curvature of the underlying relationship. This approach simplifies the calculation of more complex securities by approximating the changes based on linear relationships (delta).

The Monte Carlo Approach

LO 53.7: Explain structured Monte Carlo, stress testing, and scenario analysis methods for computing VaR, and identify strengths and weaknesses of each approach.

LO 53.8: Describe the implications of correlation breakdown for scenario analysis.

The **structured Monte Carlo (SMC) approach** simulates thousands of valuation outcomes for the underlying assets based on the assumption of normality. The VaR for the portfolio of derivatives is then calculated from the simulated outcomes. The general equation assumes the underlying asset has normally distributed returns with a mean of μ and a standard deviation of σ. An example of a simulation equation is as follows:

$$s_{t+1,i} = s_t e^{\mu + \sigma \times z}$$

where $s_{t+1,i}$ is the simulated value for a continuously compounded return, based on a random draw, z_i, from a normal distribution with the given first and second moments. Therefore, the draws from the normal distribution are denoted by $z_1, z_2, z_3, \ldots z_N$, and the N scenarios are $\mu + \sigma z_1, \mu + \sigma z_2, \mu + \sigma z_3, \ldots, \mu + \sigma z_N$. The N outcomes are then ordered and the $(1 - x/100) \times$ Nth value is the x% value.

An *advantage* of the SMC approach is that it is able to address multiple risk factors by assuming an underlying distribution and modeling the correlations among the risk factors. For example, a risk manager can simulate 10,000 outcomes and then determine the probability of a specific event occurring. In order to run the simulations, the risk manager just needs to provide parameters for the mean and standard deviation and assume all possible outcomes are normally distributed.

A *disadvantage* of the SMC approach is that in some cases it may not produce an accurate forecast of future volatility and increasing the number of simulations will not improve the forecast.

Example: SMC approach

Suppose a risk manager requires a VaR measurement of a long straddle position. **Demonstrate** how a SMC approach will be implemented to estimate the VaR for a long straddle position.

Answer:

The straddle represents a portfolio of a long call and long put that anticipates a large movement up or down in the underlying stock. The typical VaR measurement would require an estimate of the underlying stock moving more than one standard deviation. However, in a straddle position, the VaR occurs when the stock does not move in price or only moves a small amount. The SMC approach simulates thousands of possible movements in the underlying stock and then uses those outcomes to estimate the VaR for the straddle position.

Correlations During Crisis

The key point here is that in times of crisis, correlations increase (some substantially) and strategies that rely on low correlations fall apart in those times. Certain economic or crisis events can cause diversification benefits to deteriorate in times when the benefits are most needed. A contagion effect occurs with a rise in volatility and correlation causing a different return generating process. Some specific examples of events leading to the breakdown of historical correlation matrices are the Asian crisis, the U.S. stock market crash of October 1987, the events surrounding the failure of Long-Term Capital Management (LTCM), and the recent global credit crisis.

A simulation using the SMC approach is not capable of predicting scenarios during times of crisis if the covariance matrix was estimated during normal times. Unfortunately, increasing the number of simulations does not improve predictability in any way.

For example, the probability of a four or more standard deviation event occurring based on the normal curve is 6.4 out of 100,000 times. However, suppose the number of times the daily return for the equity index is four or more standard deviations based on historical returns is approximately 500 out of 100,000 times. Based on the historical data a four or more standard deviation event is expected to occur once every 0.8 years, not once every 62 years implied by the normal curve.

Stress Testing

During times of crisis, a contagion effect often occurs where volatility and correlations both increase, thus mitigating any diversification benefits. *Stressing* the correlation is a method used to model the contagion effect that could occur in a crisis event.

One approach for stress testing is to *examine historical crisis* events, such as the Asian crisis, October 1987 market crash, etc. After the crisis is identified, the impact on the current portfolio is determined. The *advantage* of this approach is that no assumptions of underlying asset returns or normality are needed. The biggest *disadvantage* of using historical events for stress testing is that it is limited to only evaluating events that have actually occurred.

The **historical simulation approach** does not limit the analysis to specific events. Under this approach, the entire data sample is used to identify "extreme stress" situations for different asset classes. For example, certain historical events may impact the stock market more than the bond market. The objective is to identify the five to ten worst weeks for specific asset classes and then evaluate the impact on today's portfolio. The *advantage* of this approach is that it may identify a crisis event that was previously overlooked for a specific asset class. The focus is on identifying extreme changes in valuation instead of extreme movements in underlying risk factors. The *disadvantage* of the historical simulation approach is that it is still limited to actual historical data.

An alternative approach is to analyze different predetermined *stress scenarios*. For example, a financial institution could evaluate a 200bp increase in short-term rates, an extreme inversion of the yield curve or an increase in volatility for the stock market. As in the previous method, the next step is then to evaluate the effect of the stress scenarios on the current portfolio.

An *advantage* to scenario analysis is that it is not limited to the evaluation of risks that have occurred historically. It can be used to address any possible scenarios. A *disadvantage* of the stress scenario approach is that the risk measure is deceptive for various reasons. For example, a shift in the domestic yield curve could cause estimation errors by overstating the risk for a long and short position and understating the risk for a long-only position. Asset-class-specific risk is another disadvantage of the stress scenario approach. For example, emerging market debt, mortgage-backed securities, and bonds with embedded options all have unique asset class specific features such that interest rate risk only explains a portion of total risk. Addressing asset class risks is even more crucial for financial institutions specializing in certain products or asset classes.

Worst Case Scenario Measure

LO 53.9: Describe worst-case scenario (WCS) analysis and compare WCS to VaR.

The **worst case scenario** (WCS) assumes that an unfavorable event will occur with certainty. The focus is on the distribution of worst possible outcomes given an unfavorable event. An expected loss is then determined from this worst case distribution analysis. Thus, the WCS information extends the VaR analysis by estimating the extent of the loss given an unfavorable event occurs.

In other words, the tail of the original return distribution is more thoroughly examined with another distribution that includes only probable extreme events. For example, within the lowest 5% of returns, another distribution can be formed with just those returns and a 1% WCS return can then be determined. Recall that VaR provides a value of the minimum loss for a given percentage, but says nothing about the severity of the losses in the tail. WCS analysis attempts to complement the VaR measure with analysis of returns in the tail.

Example: WCS approach

Suppose a risk manager simulates the data in Figure 6 using 10,000 random vectors for time horizons, *H*, of 20 and 100 periods. **Demonstrate** how a risk manager would interpret results for the 1% VaR and 1% WCS for a 100 period horizon.

Figure 6: Simulated Worst Case Scenario (WCS) Distribution

Time Horizon = H	*H = 20*	*H = 100*
Expected number of $Z_i < -2.33$	0.40	1.00
Expected number of $Z_i < -1.65$	1.00	4.00
Expected WCS	–1.92	–2.74
WCS 1 percentile	–3.34	–3.85
WCS 5 percentile	–2.69	–3.17

Answer:

Based on the simulation results in Figure 6, the 1% VaR assuming normality corresponds to –2.33 and over the next 100 trading periods a return worse than –2.33 is expected to occur one time. The 1% worst case scenario, denoted in this example by Z_i is –3.85. Thus, over the next 100 trading periods a return worse than –2.33 is expected to occur one time. More importantly, the size of that return is expected to be –2.74, with a 1% probability that the return will be –3.85 or lower.

Key Concepts

LO 53.1

A derivative is described as linear when the relationship between an underlying factor and the derivative's value is linear in nature (e.g., a forward currency contract). A nonlinear derivative's value is a function of the change in the value of the underlying asset and is dependent on the state of the underlying asset (e.g., a call option).

LO 53.2

In general, the VaR of a linear derivative is $VaR_p = \Delta VaR_f$, where the derivative's local delta, Δ, is the sensitivity of the derivative's price to a 1% change in the underlying asset's value.

LO 53.3

The last term of the following Taylor Series approximation adjusts for the curvature of the nonlinear derivative in addition to the slope or delta.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

LO 53.4

More complex derivatives such as mortgage backed securities or bonds with embedded options do not have "well-behaved" quadratic functions. The curvature of the function relating the nonlinear derivative's value to the underlying factor changes for different levels of the underlying factor. Thus, the Taylor Series approximation is not sufficient to capture the shift in curvature.

LO 53.5

The full revaluation approach calculates the VaR of the derivative by valuing the derivative based on the underlying value of the index after the decline corresponding to an *x*% VaR of the index. This approach is accurate, but can be highly computational.

LO 53.6

The delta-normal approach calculates the risk using the delta approximation ($VaR_p = \Delta VaR_f$) which is linear or the delta-gamma approximation, $f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$, which adjusts for the curvature of the underlying relationship.

LO 53.7

The structured Monte Carlo (SMC) approach simulates thousands of possible movements in the underlying asset and then uses those outcomes to estimate the VaR for a portfolio of derivatives.

An advantage of the SMC approach is that it is able to address multiple risk factors by generating correlated scenarios based on a statistical distribution. A disadvantage of the SMC approach is that in some cases it may not produce an accurate forecast of future volatility and increasing the number of simulations will not improve the forecast.

LO 53.8

Crisis events cause diversification benefits to deteriorate due to a contagion effect that occurs when a rise in volatility and correlation result in a different return generating process for the underlying asset. This creates problems when using simulations for scenario analysis due to the fact that a simulation using the SMC approach cannot predict scenarios during times of crisis if the covariance matrix was estimated during normal times.

LO 53.9

The worst case scenario (WCS) extends VaR risk measurement by estimating the extent of the loss given an unfavorable event.

Concept Checkers

1. A call option and a mortgage backed security derivative are good examples of:
   A. a linear and nonlinear derivative, respectively.
   B. a nonlinear and linear derivative, respectively.
   C. linear derivatives.
   D. nonlinear derivatives.

2. Which of the following statements is(are) true?

   I. A linear derivative's delta must be constant for all levels of value for the underlying factor.
   II. A nonlinear derivative's delta must be constant for all levels of value for the underlying factor.

   A. I only.
   B. II only.
   C. Both I and II.
   D. Neither I nor II.

3. Which of the following statements regarding the Taylor Series approximation is(are) true?

   I. The second derivative of the function for the relationship between the derivative and underlying asset estimates the rate of change in the slope.
   II. The Taylor Series approximation can be used to estimate the change in all nonlinear derivative values.

   A. I only.
   B. II only.
   C. Both I and II.
   D. Neither I nor II.

4. Which of the following statements regarding the measurement of risk for non-linear derivatives is(are) true?

   I. A disadvantage of the delta-normal approach is that it is highly computational.
   II. The full revaluation approach is most appropriate for portfolios containing mortgage backed securities or options with embedded features.

   A. I only.
   B. II only.
   C. Both I and II.
   D. Neither I nor II.

5. Which of the following statements is incorrect? A contagion effect:
   A. occurs with a rise in both volatility and correlation.
   B. causes a different return generating process in the underlying asset.
   C. results from a crisis event.
   D. increases diversification benefits.

Concept Checker Answers

1. **D** A *nonlinear* derivative's value is a function of the change in the value of the underlying asset and is dependent on the state of the underlying asset.

2. **A** The delta of a linear derivative must be constant. The delta, or slope, of a nonlinear derivative changes for different levels of the underlying factor.

3. **A** The Taylor Series of order two is represented mathematically as:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$$

The first derivative tells us the delta, or slope of the line. The second derivative tells us the rate of change. The last term including the second derivative captures the convexity or curvature. This approximation is only useful for "well-behaved" quadratic functions of order two.

4. **D** Both the delta-normal and full revaluation methods measure the risk of nonlinear securities. The *full revaluation approach* calculates the VaR of the derivative by valuing the derivative based on the underlying value of the index after the decline corresponding to an *x*% VaR of the index. This approach is accurate, but can be highly computational; therefore, it does not work well for portfolios of more complex derivatives such as mortgage-backed securities, swaptions, or options with embedded features. The *delta-normal approach* calculates the risk using the delta approximation, which is linear or the delta-gamma approximation, which adjusts for the curvature of the underlying relationship. This approach simplifies the calculation of more complex securities by approximating the changes.

5. **D** A *contagion effect* occurs with a rise in volatility and correlation causing a different return generating process. Some specific examples of events leading to the breakdown of historical correlation matrices causing a contagion effect are the Asian crisis and the U.S. stock market crash of October 1987. A contagion effect often occurs where volatility and correlations both increase, thus mitigating any diversification benefits.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Measures of Financial Risk

Topic 54

Exam Focus

The assumption regarding the shape of the underlying return distribution is critical in determining an appropriate risk measure. The mean-variance framework can only be applied under the assumption of an elliptical distribution such as the normal distribution. The value at risk (VaR) measure can calculate risk measures when the return distribution is non-elliptical, but the measurement is unreliable and no estimate of the amount of loss is provided. Expected shortfall is a more robust risk measure that satisfies all the properties of a coherent risk measure with less restrictive assumptions. For the exam, focus your attention on the calculation of VaR, properties of coherent risk measures, and the expected shortfall methodology.

Mean-Variance Framework

LO 54.1: Describe the mean-variance framework and the efficient frontier.

The traditional mean-variance model estimates the amount of financial risk for portfolios in terms of the portfolio's expected return (i.e., mean) and risk (i.e., standard deviation or variance). Under the **mean-variance framework**, it is necessary to assume that return distributions for portfolios are elliptical distributions. The most commonly known elliptical probability distribution function is the normal distribution.

The **normal distribution** is a continuous distribution that illustrates all possible outcomes for random variables. Recall that the standard normal distribution has a mean of zero and a standard deviation of one. If returns are normally distributed, approximately 66.7% of returns will occur within plus or minus one standard deviation of the mean. Approximately 95% of the observations will occur within plus or minus two standard deviations of the mean. Thus, given this type of distribution, returns are more likely to occur closer to the mean return.

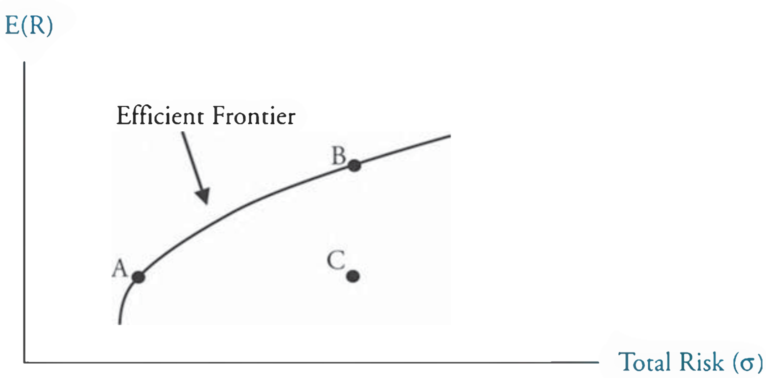
Portfolio managers are concerned with measuring downside risk and therefore are particularly interested in measuring the possibility of outcomes to the left or below the expected mean return. If the return distribution is symmetrical (like the normal distribution), then the standard deviation is an appropriate measure of risk when determining the probability that an undesirable outcome will occur.

If we assume that return distributions for all risky securities are normally distributed, then we can choose portfolios based on the expected returns and standard deviations of all possible combinations of risky securities. Figure 1 illustrates the concept of the **efficient frontier.**

In theory, all investors prefer securities or portfolios that lie on the efficient frontier. Consider portfolios A, B, and C in Figure 1. If you had to choose between portfolios A and C, which one would you prefer and why? Since portfolios A and C have the same expected return, a risk-averse investor would choose the portfolio with the least amount of risk (which would be Portfolio A). Now if you had to choose between portfolios B and C, which one would you choose and why? Because portfolios B and C have the same amount of risk, a risk-averse investor would choose the portfolio with the higher expected return (which would be Portfolio B). We say that Portfolio B dominates Portfolio C with respect to expected return, and that Portfolio A dominates Portfolio C with respect to risk. Likewise, all portfolios on the efficient frontier dominate all other portfolios in the investment universe of risky assets with respect to either risk, return, or both.

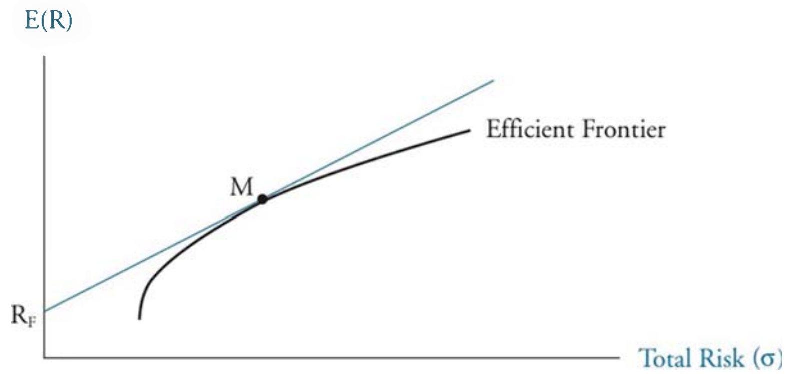
There are an almost unlimited number of combinations of risky assets to the right and below the efficient frontier. However, in the absence of a risk-free security, portfolios to the left and above the efficient frontier are not possible. Therefore, all investors will choose some portfolio on the efficient frontier. If an investor is more risk-averse, she may choose a portfolio on the efficient frontier closer to Portfolio A. If an investor is less risk-averse, she will choose a portfolio on the efficient frontier closer to Portfolio B.

Figure 1: The Efficient Frontier



If we now assume that there is a risk-free security, then the mean-variance framework is extended beyond the efficient frontier. Figure 2 illustrates that the optimal set of portfolios now lie on a straight line that runs from the risk-free security through the **market portfolio**, *M*. All investors will now seek investments by holding some portion of the risk-free security and the market portfolio. To achieve points on the line to the right of the market portfolio, an investor who is very aggressive will borrow money (at the risk-free rate) and invest in more of the market portfolio. More risk-averse investors will hold some combination of the risk-free security and the market portfolio to achieve portfolios on the line segment between the risk-free security and the market portfolio.

Figure 2: The Efficient Frontier with the Risk-Free Security



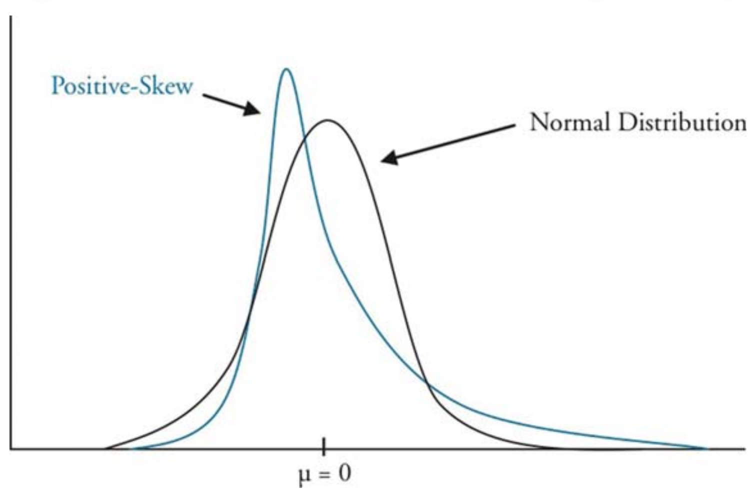
Mean-Variance Framework Limitations

LO 54.2: Explain the limitations of the mean-variance framework with respect to assumptions about the return distributions.

The use of the standard deviation as a risk measurement is not appropriate for non-normal distributions. If the shape of the underlying return density function is not symmetrical, then the standard deviation does not capture the appropriate probability of obtaining undesirable return outcomes.

Figure 3 illustrates two probability distribution functions. One probability distribution function is the normal distribution with a mean of zero. The other probability distribution is positively skewed. This positively skewed distribution has the same mean and standard deviation as the normal distribution. The degree of skewness alters the entire distribution. For the positively skewed distribution, outcomes below the mean are more likely to occur closer to the mean. Clearly normality is an important assumption when using the mean-variance framework. Thus, the mean-variance framework is unreliable when the assumption of normality is not met.

Figure 3: Normal Distribution vs. Positively-Skewed Distribution



Value at Risk

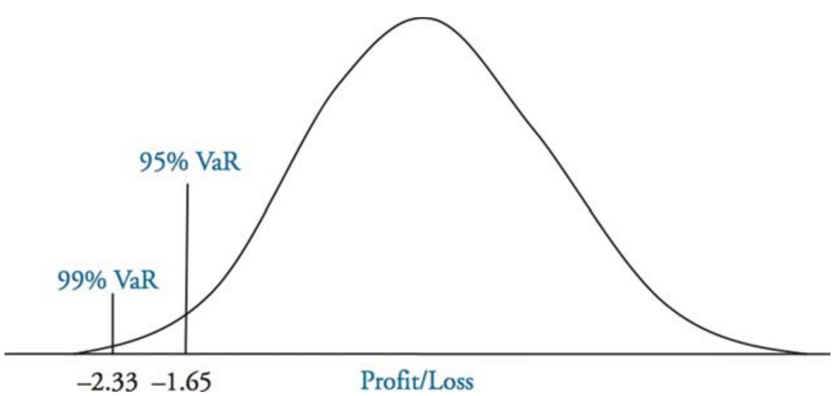
LO 54.3: Define the Value-at-Risk (VaR) measure of risk, describe assumptions about return distributions and holding period, and explain the limitations of VaR.

Value at risk (VaR) is interpreted as the worst possible loss under normal conditions over a specified period. Another way to define VaR is as an estimate of the maximum loss that can occur with a given confidence level. If an analyst says, "for a given month, the VaR is $1 million at a 95% level of confidence," then this translates to mean "under normal conditions, in 95% of the months (19 out of 20 months), we expect the fund to either earn a profit or lose no more than $1 million." Analysts may also use other standard confidence levels (e.g., 90% and 99%). Recall that delta-normal VaR can be computed using the following expression: $[\mu - (z)(\sigma)]$.

A major limitation of the VaR measure for risk is that two arbitrary parameters are used in the calculation—the confidence level and the holding period. The confidence level indicates the likelihood or probability that we will obtain a value greater than or equal to VaR. The holding period can be any pre-determined time period measured in days, weeks, months, or years.

Figure 4 illustrates VaR parameters for a confidence level of 95% and 99%. As you can see, the level of risk is dependent on the degree of confidence chosen. VaR increases when the confidence level increases. In addition, VaR will increase at an increasing rate as the confidence level increases.

Figure 4: VaR Measurements for a Normal Distribution



The second arbitrary parameter is the holding period. VaR will increase with increases in the holding period. The rate at which VaR increases is determined in part by the mean of the distribution. If the return distribution has a mean, μ, equal to 0, then VaR rises with the square root of the holding period (i.e., the square root of time). If the return distribution has a $\mu > 0$, then VaR rises at a lower rate and will eventually decrease. Thus, the mean of the distribution is an important determinant for estimating how VaR changes with changes in the holding period.

VaR estimates are also subject to both model risk and implementation risk. Model risk is the risk of errors resulting from incorrect assumptions used in the model. Implementation risk is the risk of errors resulting from the implementation of the model.

Another major limitation of the VaR measure is that it does not tell the investor the amount or magnitude of the actual loss. VaR only provides the maximum value we can lose for a given confidence level. Two different return distributions may have the same VaR, but very different risk exposures. A practical example of how this can be a serious problem is when a portfolio manager is selling out-of-the-money options. For a majority of the time, the options will have a positive return and, therefore, the expected return is positive. However, in the unfavorable event that the options expire in-the-money, the resulting loss can be very large. Thus, different strategies focusing on lowering VaR can be very misleading since the magnitude of the loss is not calculated.

To summarize, VaR measurements work well with elliptical return distributions, such as the normal distribution. VaR is also able to calculate the risk for non-normal distributions; however, VaR estimates may be unreliable in this case. Limitations in implementing the VaR model for determining risk result from the underlying return distribution, arbitrary confidence level, arbitrary holding period, and the inability to calculate the magnitude of losses. The measure of VaR also violates the coherent risk measure property of subadditivity when the return distribution is not elliptical. This property is further explained in the next LO.

Coherent Risk Measures

LO 54.4: Define the properties of a coherent risk measure and explain the meaning of each property.

In order to properly measure risk, one must first clearly define what is meant by a measure of risk. If we allow *R* to be a set of random events and $\rho(R)$ to be the risk measure for the random events, then **coherent risk measures** should exhibit the following properties:

1. **Monotonicity**: a portfolio with greater future returns will likely have less risk: $R_1 \geq R_2$, then $\rho(R_1) \leq \rho(R_2)$

2. **Subadditivity**: the risk of a portfolio is at most equal to the risk of the assets within the portfolio: $\rho(R_1 + R_2) \leq \rho(R_1) + \rho(R_2)$

3. **Positive homogeneity**: the size of a portfolio, β, will impact the size of its risk: for all $\beta > 0$, $\rho(\beta R) = \beta\rho(R)$

4. **Translation invariance**: the risk of a portfolio is dependent on the assets within the portfolio: for all constants *c*, $\rho(c + R) = \rho(R) - c$

The first, third, and fourth properties are more straightforward properties of well-behaved distributions. Monotonicity infers that if a random future value R_1 is always greater than a random future value R_2, then the risk of the return distribution for R_1 is less than the risk of the return distribution for R_2. Positive homogeneity suggests that the risk of a position is proportional to its size. Positive homogeneity should hold as long as the security is in a liquid market. Translation invariance implies that the addition of a sure amount reduces the risk at the same rate as the cash needed to make the position acceptable.

Subadditivity is the most important property for a coherent risk measure. The property of subadditivity states that a portfolio made up of sub-portfolios will have equal or less risk than the sum of the risks of each individual sub-portfolio. This assumes that when individual risks are combined, there may be some diversification benefits or, in the worst case, no diversification benefits and no greater risk. This implies grouping or adding risks does not increase the overall aggregate risk amount.

Expected Shortfall

LO 54.5: Explain why VaR is not a coherent risk measure.

LO 54.6: Explain and calculate expected shortfall (ES), and compare and contrast VaR and ES.

Value at risk is the minimum percent loss, equal to a pre-specified worst case quantile return (typically the 5th percentile return). **Expected shortfall** (ES) is the expected loss given that the portfolio return already lies below the pre-specified worst case quantile return (i.e., below the 5th percentile return). In other words, expected shortfall is the mean percent loss among the returns falling below the q-quantile. Expected shortfall is also known as **conditional VaR** or **expected tail loss** (ETL).

For example, assume an investor is interested in knowing the 5% VaR (the 5% VaR is equivalent to the 5th percentile return) for a fund. Further, assume the 5th percentile return for the fund equals –20%. Therefore, 5% of the time, the fund earns a return less than –20%. The value at risk is –20%. However, VaR does not provide good information regarding the expected size of the loss if the fund performs in the lower 5% of the possible outcomes. That question is answered by the expected shortfall amount, which is the expected value of all returns falling below the 5th percentile return (i.e., below –20%). Therefore, expected shortfall will equal a larger loss than the VaR. In addition, unlike VaR, ES has the ability to satisfy the property of subadditivity.

The ES method provides an estimate of how large of a loss is expected if an unfavorable event occurs. VaR did not provide any estimate of the magnitude of losses, only the probability that they might occur. The property of subadditivity under the ES framework is also beneficial in eliminating another problem for VaR. When adjusting both the holding period and confidence level at the same time, an ES surface curve showing the interactions of both adjustments is convex. This implies that the ES method is more appropriate than the VaR method in solving portfolio optimization problems.

ES is similar to VaR in that both provide a common consistent risk measure across different positions. ES can be implemented in determining the probability of losses the same way that VaR is implemented as a risk measure, and they both appropriately account for correlations.

However, ES is a more appropriate risk measure than VaR for the following reasons:

- ES satisfies all of the properties of coherent risk measurements including subadditivity. VaR only satisfies these properties for normal distributions.
- The portfolio risk surface for ES is convex because the property of subadditivity is met. Thus, ES is more appropriate for solving portfolio optimization problems than the VaR method.
- ES gives an estimate of the magnitude of a loss for unfavorable events. VaR provides no estimate of how large a loss may be.
- ES has less restrictive assumptions regarding risk/return decision rules.

LO 54.7: Describe spectral risk measures, and explain how VaR and ES are special cases of spectral risk measures.

A more general risk measure than either VaR or ES is known as the **risk spectrum** or risk aversion function. The risk spectrum measures the weighted averages of the return quantiles from the loss distributions. ES is a special case of this risk spectrum measure. When modeling the ES case, the weighting function is set to [1 / (1 – confidence level)] for tail losses. All other quantiles will have a weight of zero.

VaR is also a special case of spectral risk measure models. The weighting function with VaR assigns a probability of one to the event that the *p*-value equals the level of significance (i.e., $p = \alpha$), and a probability of zero to all other events where $p \neq \alpha$. Thus, the ES measure places equal weights on tail losses while VaR places no weight on tail losses.

In order for a risk measure to be coherent, it must give higher losses at least the same weight as lower losses. In the ES case, all losses are given the same weight. This suggests that investors are risk-neutral with respect to losses. This is contradictory to the common notion that investors are risk-averse. In the VaR case, only the loss associated with a *p*-value equal to α is given any weight. Greater losses are given no weight at all. This implies that investors are risk-seekers. Thus, the ES and VaR measures are inadequate in that the weighting function is not consistent with risk aversion.

Scenario Analysis

LO 54.8: Describe how the results of scenario analysis can be interpreted as coherent risk measures.

The results of scenario analysis can be interpreted as coherent risk measures by first assigning probabilities to a set of loss outcomes. These losses can be thought of as tail drawings of the relevant distribution function. The expected shortfall for the distribution can then be computed by finding the arithmetic average of the losses. Therefore, the outcomes of scenario analysis must be coherent risk measurements, because ES is a coherent risk measurement.

Scenario analysis can also be applied in situations where there are numerous distribution functions involved. It can be shown that the ES, the highest ES from a set of comparable expected shortfalls based on different distribution functions, and the highest expected shortfall from a set of highest losses are all coherent risk measures. For example, assume you are considering a set of *n* loss outcomes out of a family of distribution functions. The

ES is obtained from each distribution function. If there is a set of *m* comparable expected shortfalls, that each have a different corresponding loss distribution function, then the maximum of these expected shortfalls is a coherent risk measure. Thus, in cases where n = 1, the ES is the same as the probable maximum loss because there is only one tail loss in each scenario. If *m* equals one, then the highest expected loss from a single scenario analysis is a coherent measure. In cases where *m* is greater than one, the highest expected of *m* worst case outcomes is a coherent risk measure.

Key Concepts

LO 54.1

The traditional mean-variance model estimates the amount of financial risk for portfolios in terms of the portfolio's expected return (mean) and risk (standard deviation or variance). A necessary assumption for this model is that return distributions for the portfolios are elliptical distributions.

The efficient frontier is the set of portfolios that dominate all other portfolios in the investment universe of risky assets with respect to risk and return. When a risk-free security is introduced, the optimal set of portfolios consists of a line from the risk-free security that is tangent to the efficient frontier at the market portfolio.

LO 54.2

The mean-variance framework is unreliable when the underlying return distribution is not normal or elliptical. The standard deviation is not an accurate measure of risk and does not capture the probability of obtaining undesirable return outcomes when the underlying return density function is not symmetrical.

LO 54.3

Value at risk (VaR) is a risk measurement that determines the probability of an occurrence in the left-hand tail of a return distribution at a given confidence level. VaR is defined as: $[\mu - (z)(\sigma)]$. The underlying return distribution, arbitrary choice of confidence levels and holding periods, and the inability to calculate the magnitude of losses result in limitations in implementing the VaR model when determining risk.

LO 54.4

The properties of a coherent risk measure are:

- Monotonicity: $Y \geq X \Rightarrow \rho(Y) \leq \rho(X)$
- Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$
- Positive homogeneity: $\rho(hx) = h\rho(X)$ for $h > 0$
- Translation invariance: $\rho(X + n) = \rho(X) - n$

LO 54.5

Subadditivity, the most important property for a coherent risk measure, states that a portfolio made up of sub-portfolios will have equal or less risk than the sum of the risks of each individual sub-portfolio. VaR violates the property of subadditivity.

LO 54.6

Expected shortfall is a more accurate risk measure than VaR for the following reasons:

- ES satisfies all the properties of coherent risk measurements including subadditivity.
- The portfolio risk surface for ES is convex since the property of subadditivity is met. Thus, ES is more appropriate for solving portfolio optimization problems than the VaR method.
- ES gives an estimate of the magnitude of a loss for unfavorable events. VaR provides no estimate of how large a loss may be.
- ES has less restrictive assumptions regarding risk/return decision rules.

LO 54.7

ES is a special case of the risk spectrum measure where the weighting function is set to 1 / (1 – confidence level) for tail losses that all have an equal weight, and all other quantiles have a weight of zero. The VaR is a special case where only a single quantile is measured, and the weighting function is set to one when *p*-value equals the level of significance, and all other quantiles have a weight of zero.

LO 54.8

The outcomes of scenario analysis are coherent risk measurements, because expected shortfall is a coherent risk measurement. The ES for the distribution can be computed by finding the arithmetic average of the losses for various scenario loss outcomes.

Concept Checkers

1. The mean-variance framework is inappropriate for measuring risk when the underlying return distribution:
   A. is normal.
   B. is elliptical.
   C. has a kurtosis equal to three.
   D. is positively skewed.

2. Assume an investor is very risk-averse and is creating a portfolio based on the mean-variance model and the risk-free asset. The investor will most likely choose an investment on the:
   A. left-hand side of the efficient frontier.
   B. right-hand side of the efficient frontier.
   C. line segment connecting the risk-free rate to the market portfolio.
   D. line segment extending to the right of the market portfolio.

3. $\rho(X + Y) \leq \rho(X) + \rho(Y)$ is the mathematical equation for which property of a coherent risk measure?
   A. Monotonicity.
   B. Subadditivity.
   C. Positive homogeneity.
   D. Translation invariance.

4. Which of the following is not a reason that expected shortfall (ES) is a more appropriate risk measure than value at risk (VaR)?
   A. For normal distributions, only ES satisfies all the properties of coherent risk measurements.
   B. For non-elliptical distributions, the portfolio risk surface formed by holding period and confidence level is more convex for ES.
   C. ES gives an estimate of the magnitude of a loss.
   D. ES has less restrictive assumptions regarding risk/return decision rules than VaR.

5. If the weighting function in the general risk spectrum measure is set to 1 / (1 – confidence level) for all tail losses, then the risk spectrum is a special case of:
   A. value at risk.
   B. mean-variance.
   C. expected shortfall.
   D. scenario analysis.

Concept Checker Answers

1. **D** The mean-variance framework is only appropriate when the underlying distribution is elliptical. The normal distribution is a special case of elliptical distributions where skewness is equal to zero and kurtosis is equal to three. If there is any skewness, the distribution function will not be symmetrical, and standard deviation will not be an appropriate risk measure.

2. **C** Under the mean-variance framework, when a risk-free security is included in the analysis, the optimal set of portfolios lies on a straight line that runs from the risk-free security to the market portfolio. All investors will hold some portion of the risk-free security and the market portfolio. More risk-averse investors will hold some combination of the risk-free security and the market portfolio to achieve portfolios on the line segment between the risk-free security and the market portfolio.

3. **B** The property of subadditivity states that a portfolio made up of sub-portfolios will have equal or less risk than the sum of the risks of each individual sub-portfolio.

4. **A** VaR and ES both satisfy all the properties of coherent risk measures for normal distributions. However, only ES satisfies all the properties of coherent risk measures when the assumption of normality is not met.

5. **C** Expected shortfall is a special case of the risk spectrum measure that is found by setting the weighting function to 1 / (1 – confidence level) for tail losses that all have an equal weight.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Binomial Trees

Topic 55

Exam Focus

This topic introduces the binomial model for valuing options on stock and serves as an introduction to the Black-Scholes-Merton model you'll encounter in the next topic. For the exam, you should be able to calculate the value of a European or American option using a one-step or two-step binomial model. This will require you to know the formulas for the sizes of upward and downward movements as well as the risk-neutral probabilities in both up and down states.

A One-Step Binomial Model

LO 55.1: Calculate the value of an American and a European call or put option using a one-step and two-step binomial model.

A **one-step binomial model** is best described within a two-state world where the price of a stock will either go up once or down once, and the change will occur one step ahead at the end of the holding period.

The Replicating Portfolio

The replicating portfolio is the key to understanding how to value options. In general, the replicating portfolio is a concept that holds that the outlay for a bankruptcy-free stock position should be the same as the outlay for a long call position with the same payoff.

To see how this works, let's first define some terms. Then we'll work through a calculation:

- P = the stock's current price.
- X = the call option's exercise price.
- t = the time to option expiration.
- i = the risk-free interest rate.
- S_U = the stock value in "up" state.
- S_D = the stock value in "down" state.
- c = the value of the call option today.

Example: Creating a replicating portfolio

Calculate the value of the call option where:

P = \$100
X = \$125
t = 1 year
i = 8%
S_U = \$200
S_D = \$50

The one-period binomial trees for the stock and the call option are shown in Figure 1.

Figure 1: One-Period Binomial Trees

Stock:

P = \$100 → S_U = \$200
P = \$100 → S_D = \$50

Call Option:

c = ?? → c_U = max (\$0, \$200 – \$125) = \$75
c = ?? → c_D = max (\$0, \$50 – \$125) = \$0

Answer:

The process used to establish a replicating portfolio can be broken down into four steps:

Step 1: Construct the bankruptcy-free portfolio. As shown in Figure 1, the stock's minimum value one year from today is \$50, which means we should borrow the present value of \$50: \$50/(1.08) = \$46.30. With the borrowed money and \$53.70 of our own money, purchase one share of stock for \$100. This stock-plus-borrowing combination is the bankruptcy-free portfolio, since:

- If the stock goes up to \$200, you will be able to repay your \$50 loan. Your net return (excluding out-of-pocket costs) will be \$200 – \$50 = \$150.
- If the stock goes down to \$50, you will still be able to repay your \$50 loan. Your net return will be \$50 – \$50 = \$0.

Current net cash outlay = \$53.70. Stock value net of loan = \$0 or \$150.

Step 2: Replicate the future returns. To replicate the stock-plus-borrowing transaction (i.e., the bankruptcy-free portfolio), we want our net cash returns from our option position (excluding out-of-pocket expenses) to equal \$0 or \$150 at the end of the holding period. Since the profit diagram for one call option indicated a net return of \$0 or \$75 at the end of the holding period, purchasing *two* call options will provide the same end-of-period return as the bankruptcy-free portfolio.

Step 3: Align the dollar cost of the option and the portfolio. If the dollar outlay for the bankruptcy-free portfolio is $53.70, the dollar cost of the two option contracts should also be $53.70, since they have the same risk and return. *Arbitrage forces this condition.*

Step 4: Value the option. Since arbitrage forces the outlay of the bankruptcy-free portfolio to be the same as the outlay for two options, each option must cost half of the outlay for the bankruptcy-free portfolio. If two contracts cost $53.70, then c = $53.70/2 = $26.85. *Note*: To solve for the call price, all we needed were six data items: P, X, i, t, S_U and S_D.

Using the Hedge Ratio to Develop the Replicating Portfolio

The value of the option can also be solved by creating a **perfect hedge**.

- **Hedging** is the elimination of price variation through the short sale of an asset exhibiting the same price volatility as the asset to be hedged. A perfect hedge creates a riskless position.
- The **hedge ratio** indicates the number of asset units needed to completely eliminate the price volatility of one call option.

In the previous example we could have created a perfect hedge had we sold one share of stock short at $100 and purchased two call options. No matter which way the stock price moves, the hedged portfolio will be worth $50:

- If the stock price *falls* to $50, the two options will have zero value, so the net asset position is the gain on the short sale: $100 – $50 = $50.
- If the stock price *rises* to $200, the two calls will have a combined value of $150, leaving a net asset value of $50 after considering a $100 loss ($100 – $200) from the short position in the stock.

Since the terminal value of this strategy (short one share and long two calls) always nets $50, the present value of the strategy is $50/1.08 = $46.30. Therefore, the value of one call option must be $26.85 ($100 – 2c = $46.30, c = $26.85).

The hedge ratio tells us how many units of the stock are to be shorted per long call option to make the hedge work. In the single-period model, the hedge ratio may be calculated as follows:

$$\text{HR} = \frac{c_U - c_D}{S_U - S_D} = \frac{\$75 - \$0}{\$200 - \$50} = 0.5$$

A hedge ratio of 0.5 says that one option contract is needed for each half-share of stock. The reciprocal of the hedge ratio is equivalent to the number of option contracts to buy per share of stock that was sold short.

Synthetic Call Replication

A combination of the hedge ratio, the stock price, and the present value of the borrowings can be used to price the call option:

call price = hedge ratio × [stock price – PV(borrowing)]

Using the data from the previous example, the call price is:

call price = 0.5 × ($100 – $46.30) = $26.85

The hedge ratio is also known as the option **delta**, which is a key measure of option sensitivity, as you will see in Topic 57 when we discuss the Greek letters for option pricing.

Risk-Neutral Valuation

The one-step binomial model can also be expressed in terms of probabilities and call prices. The sizes of the upward and downward movements are defined as functions of the volatility and the length of the "steps" in the binomial model:

$$U = \text{size of the up-move factor} = e^{\sigma\sqrt{t}}$$

$$D = \text{size of the down-move factor} = e^{-\sigma\sqrt{t}} = \frac{1}{e^{\sigma\sqrt{t}}} = \frac{1}{U}$$

where:
σ = annual volatility of the underlying asset's returns
t = the length of the step in the binomial model

The risk-neutral probabilities of upward and downward movements are then calculated as follows:

$$\pi_u = \text{probability of an up move} = \frac{e^{rt} - D}{U - D}$$

$$\pi_d = \text{probability of a down move} = 1 - \pi_u$$

where:
r = continuously compounded annual risk-free rate

Professor's Note: These two probabilities are not the actual probability of an up or down move. They are risk-neutral probabilities that would exist if investors were risk-neutral.

We can calculate the value of an option on the stock by:

- Calculating the payoff of the option at maturity in both the up-move and down-move states.
- Calculating the expected value of the option in one year as the probability-weighted average of the payoffs in each state.
- Discounting the expected value back to today at the risk-free rate.

Example: Risk neutral approach to option valuation

The current price of Downhill Ski Equipment, Inc., is \$20. The annual standard deviation is 14%. The continuously compounded risk-free rate is 4% per year. Assume Downhill pays no dividends. **Compute** the value of a 1-year European call option with a strike price of \$20 using a one-period binomial model.

Answer:

The up-move and down-move factors are:

$$U = e^{0.14 \times \sqrt{1}} = 1.15$$

$$D = \frac{1}{1.15} = 0.87$$

The risk-neutral probabilities of an up move and down move are:

$$\pi_u = \frac{e^{0.04 \times 1} - D}{U - D} = \frac{1.0408 - 0.87}{1.15 - 0.87} = 0.61$$

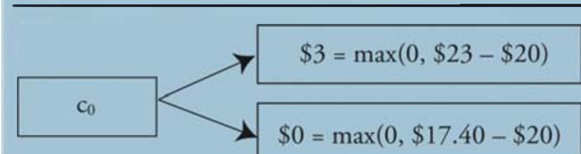
$$\pi_d = 1 - 0.61 = 0.39$$

The binomial tree for the stock is shown in Figure 2:

Figure 2: Binomial Tree—Stock

The binomial tree for the option is shown in Figure 3:

Figure 3: Binomial Tree—Option



Notice that the call option is in-the-money in the "up" state, so its value is \$3. It is out-of-the-money in the "down" state, so its value is zero.

The expected value of the option in one year is:

$$c_U \times \pi_U + c_D \times \pi_D \text{ or } (\$3 \times 0.61) + (\$0 \times 0.39) = \$1.83$$

The present value of the option's expected value is:

$$c_0 = \frac{\$1.83}{e^{0.04 \times 1}} = \frac{\$1.83}{1.0408} = \$1.76$$

Example: Put option valuation using put-call parity

The current price of Downhill Ski Equipment, Inc., is \$20, the risk-free rate is 4% per year, and the price of a 1-year call option with a strike price of \$20 is \$1.76. **Compute** the value of a 1-year European put option on Downhill Ski Equipment with a strike price of \$20.

Answer:

$$\text{put} = \text{call} - \text{stock} + Xe^{-rT}$$

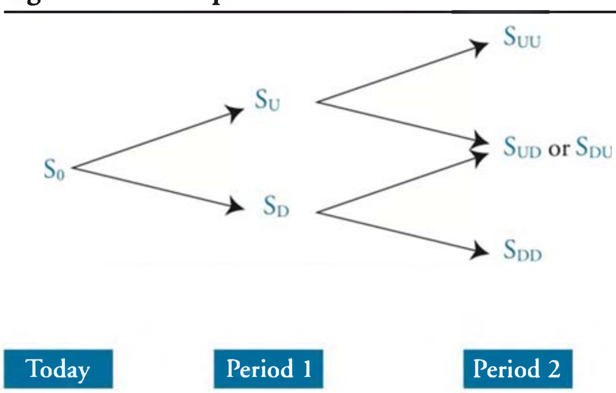
Using the information provided, we have:

$$\text{put} = \$1.76 - \$20 + [\$20 \times e^{-(0.04)\times(1)}] = \$0.98$$

Two-Step Binomial Model

In the two-period and multi-period models, the *tree* is expanded to provide for a greater number of potential outcomes. The stock price tree for the two-period model is shown in Figure 4.

Figure 4: Two-Step Binomial Model Stock Price Tree



Example: Option valuation with a two-step binomial model

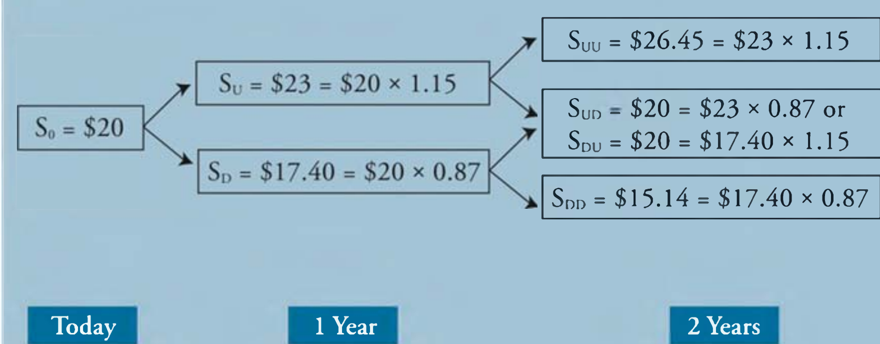
Let's continue with the Downhill Ski Equipment example. The current price of Downhill Ski Equipment, Inc., is \$20. The annual standard deviation is 14%. The risk-free rate is 4% per year.

Assume Downhill pays no dividends. Using information from the previous example, **compute** the values of a 2-year European call and a 2-year European put option with strike prices of \$20.

Answer:

First **compute** the theoretical value of the stock in each period using the up and down stock price movements from the preceding examples, as shown in Figure 5:

Figure 5: Theoretical Stock Value



From this, the values of the call option in each of the possible outcomes can be determined. Notice that the only time that the option is in-the-money is when two upward price movements lead to an ending price of \$26.45 and a call value of \$6.45. The expected value of the option at the end of year 2 is the value of the option in each state multiplied by the probability of that state occurring.

$$\begin{aligned}\text{expected call value in 2 years} &= (0.61 \times 0.61 \times \$6.45) + (0.61 \times 0.39 \times \$0)\\ &+ (0.39 \times 0.61 \times \$0) + (0.39 \times 0.39 \times \$0)\\ &= (0.3721 \times \$6.45) = \$2.40\end{aligned}$$

The value of the option today is the expected value in two years discounted at the risk-free rate of 4%:

$$\text{call option value} = \frac{\$2.40}{e^{(0.04)\times(2)}} = \$2.21$$

$$\text{put} = \text{call} - \text{stock} + Xe^{-rT} = \$2.21 - \$20 + \$20e^{-(0.04)(2)} = \$0.67$$

Assessing Volatility

LO 55.2: Describe how volatility is captured in the binomial model.

Notice from the previous examples that a high standard deviation will result in a large difference between the stock price in an up state, S_U, and the stock price in a down state, S_D. If the standard deviation were zero, the binomial tree would collapse into a straight line and S_U would equal S_D. Obviously, the higher the standard deviation, the greater the dispersion between stock prices in up and down states. Therefore volatility, as measured here by standard deviation, can be captured by evaluating stock prices at each time period considered in the tree.

Modifying the Binomial Model

LO 55.4: Explain how the binomial model can be altered to price options on: stocks with dividends, stock indices, currencies, and futures.

The binomial option pricing model can be altered to value a stock that pays a continuous **dividend yield**, q. Since the total return in a risk-neutral setting is the risk-free rate, r, and dividends provide a positive yield, capital gains must be equal to r – q. The risk-neutral probabilities of upward and downward movements incorporate a dividend yield as follows:

$$\pi_u = \frac{e^{(r-q)t} - D}{U - D}$$

$$\pi_d = 1 - \pi_u$$

The equations for the size of the up-move and down-move factors will be the same. Options on stock indices are valued in a similar fashion to stocks with dividends, because it is assumed that stocks underlying the index pay a dividend yield equal to q.

For options on currencies, it is assumed that a foreign currency asset provides a return equal to the foreign risk-free rate of interest, r_{FC}. As a result, the upward probability in the binomial model is altered by replacing e^{rt} with $e^{(r_{DC}-r_{FC})t}$ such that:

$$\pi_u = \frac{e^{(r_{DC}-r_{FC})t} - D}{U - D}$$

The binomial model can also incorporate the unique characteristics of options on futures. Since futures contracts are costless to enter into, they are considered, in a risk-neutral setting, to be zero growth instruments. To account for this characteristic, e^{rt} is simply replaced with a 1 so that:

$$\pi_u = \frac{1 - D}{U - D}$$

American Options

Valuing American options with a binomial model requires the consideration of the ability of the holder to exercise early. In the case of a two-step model, that means determining whether early exercise is optimal at the end of the first period. If the payoff from early exercise (the intrinsic value of the option) is greater than the option's value (the present value of the expected payoff at the end of the second period), then it is optimal to exercise early.

Example: American put option valuation

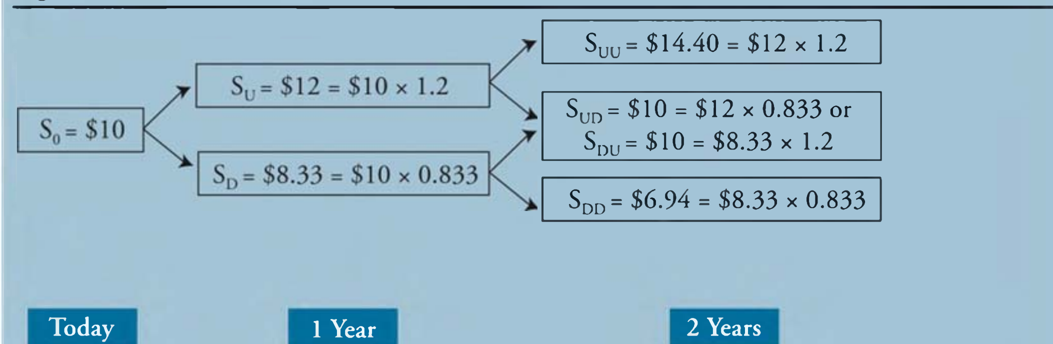
The current price of Uphill Mountaineering is $10. The up-move factor is 1.20, and the down-move factor is 0.833. The probability of an up move is 0.51, and the probability of a down move is 0.49. The risk-free rate is 2%. **Compute** the value of a 2-year American put option with strike price of $12.

Professor's Note: The calculation of the risk-neutral probabilities depends on the length of the time step. So, for a 2-year option with two time steps, the change in t is 1 year. For example, the probability of an up move in the information above is calculated as: $(e^{0.02 \times 1} - 0.833) / (1.2 - 0.833) = 0.51$.

Answer:

The stock price tree is shown in Figure 6.

Figure 6: Stock Price Tree



The $12 put option is in-the-money when the stock price finishes at $10 or at $6.94; the option is worth $2.00 ($12 – $10) or $5.06 ($12 – $6.94). It is out-of-the-money at $14.40. The year 1 value of the expected payoff on the option in year 2, given that the year 1 move is an up move, is:

$$\frac{(\$0.00 \times 0.51) + (\$2.00 \times 0.49)}{e^{(0.02)(1)}} = \$0.96$$

The payoff from early exercise at the year 1 up node is:

max ($12 – $12, 0), since the option is at-the-money

Early exercise is not optimal in this case because the option is worth more unexercised ($0.96), than if exercised ($0).

At the down node at the end of year 1, the value of the expected option payoff in year 2 is:

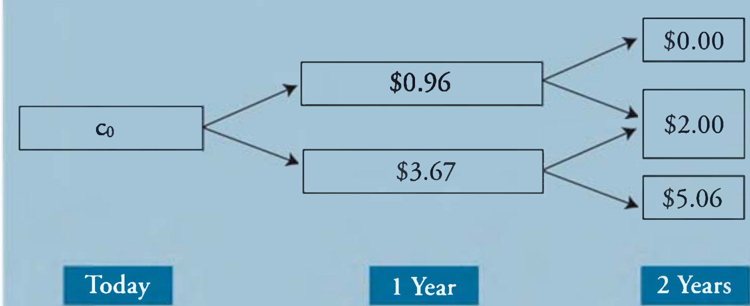
$$\frac{(\$2.00 \times 0.51) + (\$5.06 \times 0.49)}{e^{(0.02)(1)}} = \$3.43$$

The payoff from early exercise at the down node at the end of the first year is:

max ($12 – $8.33) = $3.67

In this case, early exercise is optimal because the option is worth more if exercised ($3.67) than if not exercised ($3.43). The option tree is shown in Figure 7.

Figure 7: Option Tree



The value of the option today is calculated as:

$$\frac{(\$0.96 \times 0.51) + (\$3.67 \times 0.49)}{e^{(0.02)(1)}} = \$2.24$$

Note that $3.67 appears in the bottom node of year 1 since the early exercise value ($3.67) exceeds the unexercised value ($3.43).

Professor's Note: When evaluating American options, you need to assess early exercise at each node in the tree. This includes the initial node (node 0). If the option price today (calculated via the binomial model) is less than the value of early exercise today, then the option should be exercised early. In the previous example, if the value of the option today was worth less than $2, the option would be exercised today since the put option is currently equal to $2.

Increasing the Number of Time Periods

LO 55.3: Describe how the value calculated using a binomial model converges as time periods are added.

If we shorten the length of the intervals in a binomial model, there are more intervals over the same time period, more branches to consider, and more possible ending values. If we continue to shrink the length of intervals in the model until they are what mathematicians call "arbitrarily small," we approach continuous time as the limiting case of the binomial model. The model for option valuation in the next topic (the Black-Scholes-Merton model) is a continuous time model. The binomial model "converges" to this continuous time model as we make the time periods arbitrarily small.

Key Concepts

LO 55.1

The value of a European option can be calculated using a binomial tree, as the probability-weighted expected value of the option at maturity discounted at the risk-free rate.

Given the volatility of the underlying stock and the length of the steps in the binomial tree, the size of the up- and down-move factors are calculated as:

$$U = \text{size of the up-move factor} = e^{\sigma\sqrt{t}}$$

$$D = \text{size of the down-move factor} = \frac{1}{U}$$

The risk-neutral probabilities of up and down moves are calculated as:

$$\pi_u = \text{probability of an up move} = \frac{e^{rt} - D}{U - D}$$

$$\pi_d = \text{probability of a down move} = 1 - \pi_u$$

where:
r = annual continuously compounded risk-free rate

The value of the comparable European put option can be calculated using put-call parity, which is put = call – stock + Xe^{-rT}.

The value of an American option reflects the early exercise features. An American option will be exercised at the end of the first period if the intrinsic value is greater than the discounted value of the expected option payoff at the end of the second period.

LO 55.2

The higher the standard deviation, the greater the dispersion between stock prices in up and down states. Therefore volatility, as measured here by standard deviation, can be captured by evaluating stock prices at each time period considered in the tree.

LO 55.3

As the period covered by a binomial model is divided into arbitrarily small, discrete time periods, the model results converge to those of the continuous-time model.

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LO 55.4

The binomial option pricing model can be altered to value a stock that pays a continuous dividend yield, q.

$$\pi_u = \frac{e^{(r-q)t} - D}{U - D}$$

$$\pi_d = 1 - \pi_u$$

Options on stock indices are valued in a similar fashion to stocks with dividends.

For options on currencies, upward probability in the binomial model is altered by replacing e^{rt} with $e^{(r_{DC}-r_{FC})t}$ such that:

$$\pi_u = \frac{e^{(r_{DC}-r_{FC})t} - D}{U - D}$$

The binomial model can also incorporate the unique characteristics of options on futures.

Concept Checkers

1. The stock price is currently $80. The stock price annual up-move factor is 1.15. The risk-free rate is 3.9%. The value of a 2-year European call option with an exercise price of $62 using a two-step binomial model is closest to:
   A. $0.00.
   B. $18.00.
   C. $23.07.
   D. $24.92.

2. The stock price is currently $80. The stock price will move up by 15% each year. The risk-free rate is 3.9%. The value of a 2-year European put option with an exercise price of $62 using a two-step binomial model is closest to:
   A. $0.42.
   B. $16.89.
   C. $18.65.
   D. $21.05.

3. JTE Corporation is a nondividend-paying stock that is currently priced at $49. An analyst has determined that the annual standard deviation of returns on JTE stock is 8% and that the annual risk-free interest rate on a continuously compounded basis is 5.5%. Calculate the value of a 6-month American call option on JTE stock with a strike price of $50 using a two-period binomial model.
   A. $0.32.
   B. $0.65.
   C. $1.31.
   D. $2.97.

4. A 1-year American put option with an exercise price of $50 will be worth either $8.00 at maturity with a probability of 0.45 or $0 with a probability of 0.55. The current stock price is $45. The risk-free rate is 3%. The optimal strategy is to:
   A. exercise the option because the payoff from exercise exceeds the present value of the expected future payoff.
   B. not exercise the option because the payoff from exercise is less than the discounted present value of the future payoff.
   C. exercise the option because it is currently in-the-money.
   D. not exercise the option because it is currently out-of-the-money.

5. Suppose a 1-year European call option exists on XYZ stock. The current continuously compounded risk-free rate is 3%, and XYZ pays a continuous dividend yield of 2%. Assume an annual standard deviation of 3%. The risk-neutral probability of an up-move for the XYZ call option is:
   A. 0.67.
   B. 0.97.
   C. 1.00.
   D. 1.03.

Concept Checker Answers

1. C

$80

$80 × (1.15) = $92

$80 × (0.87) = $70

$80 × (1.15)² = $106

$80 × (1.15)(0.87) = $80

$80 × (0.87)² = $61

$23.07

$32.56

$10.56

max($0, $106 – $62) = $44

max($0, $80 – $62) = $18

max($0, $61 – $62) = $0

$$U = 1.15$$

$$D = \frac{1}{1.15} = 0.8696$$

$$\pi_U = \frac{\left(e^{0.039}\right) - (0.87)}{1.15 - 0.87} = 0.61$$

$$\pi_D = 1 - 0.61 = 0.39$$

$$\pi_{UU} = 0.61^2 = 0.372$$

$$\pi_{UD} = \pi_{DU} = 0.61 \times 0.39 = 0.238$$

$$\pi_{DD} = 0.39^2 = 0.152$$

c_{UU} = $44
c_{UD} = $18
c_{DU} = $18
c_{DD} = $0

$$c_t = \frac{(0.372 \times \$44) + (0.238 \times \$18) + (0.238 \times \$18) + (0.152 \times \$0)}{e^{(0.039) \times 2}} = \$23.07$$

Professor's Note: You may have a slightly different result due to rounding. Focus on the mechanics of the calculation.

2. A put = call – stock + (exercise price × e^{-rT})

= $23.07 – $80 + [$62 × $e^{-(0.039)(2)}$] = $0.42

3. **C** The up-move factor is $U = e^{\sigma\sqrt{t}} = e^{0.08\sqrt{0.25}} = 1.041$.

The down-move factor is $D = \frac{1}{U} = \frac{1}{1.041} = 0.961$.

The probability of an up move in JTE stock $= \frac{e^{rt} - D}{U - D} = \frac{e^{0.055(0.25)} - 0.961}{1.041 - 0.961} = 0.66$.

The probability of a down move in JTE = 1 – 0.66 = 0.34.

Stock Tree

Option Tree

The $50 call option is in-the-money when the stock price finishes at $53.09 and the call has a value of $3.09.

The present value of the expected payoff in the up node at the end of three months is:

$$\frac{(\$3.09 \times 0.66) + (\$0 \times 0.34)}{e^{0.055 \times 0.25}} = \$2.01$$

Since this is an American option, we need to see if it is optimal to exercise the option early. The payoff from early exercise in the up node of the first 3-month period is max ($51 – 50, 0) = $1.00. Since $1.00 < $2.01, it is not optimal to exercise the option early.

The value of the option today is calculated as:

$$\frac{(\$2.01 \times 0.66) + (\$0 \times 0.34)}{e^{0.055 \times 0.25}} = \$1.31$$

4. **A** The payoff from exercising the option is the exercise price minus the current stock price: $50 – $45 = $5. The discounted value of the expected future payoff is:

$$\frac{(0 \times 0.55) + (8 \times 0.45)}{e^{(0.03) \times 1}} = \$3.49$$

It is optimal to exercise the option early because it is worth more exercised ($5.00) than if not exercised ($3.49).

5. **A** First calculate the size of the up- and down-move factors:

$$U = e^{\sigma\sqrt{t}} = e^{0.03\sqrt{1}} = 1.03$$

$$D = \frac{1}{U} = \frac{1}{1.03} = 0.97$$

The risk-neutral probability of an up move is calculated as follows:

$$\pi_u = \frac{e^{(r-q)t} - D}{U - D} = \frac{e^{(0.03-0.02)} - 0.97}{1.03 - 0.97} = \frac{0.04}{0.06} = 0.67$$

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

The Black-Scholes-Merton Model

Topic 56

Exam Focus

The Black-Scholes-Merton (BSM) option pricing model (often referred to as the Black-Scholes model) is based on the assumption that stock prices are lognormally distributed. In this topic, we examine the calculation of call and put options using the BSM option pricing model. Also, we discuss how volatility can be estimated using a combination of the BSM model and current option prices. For the exam, know how to calculate the value of a call and put option using the BSM model and be able to incorporate dividends into the model if necessary. Put-call parity can be applied to calculate call or put values since the BSM model requires the use of European options.

LO 56.1: Explain the lognormal property of stock prices, the distribution of rates of return, and the calculation of expected return.

LO 56.2: Compute the realized return and historical volatility of a stock.

Lognormal Stock Prices

The model used to develop the **Black-Scholes-Merton** (BSM) **model** assumes stock prices are lognormally distributed:

$$\ln S_T \sim N\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

where:
S_T = stock price at time T
S_0 = stock price at time 0
μ = expected return on stock per year
σ = volatility of the stock price per year
$N[m, s]$ = normal distribution with mean = m and standard deviation = s

Since ln S_T is normally distributed, S_T has a lognormal distribution.

Example: Calculating mean and standard deviation

Assume a stock has an initial price S_0 = $25, an expected annual return of 12%, and an annual volatility of 20%. **Calculate** the mean and standard deviation of the distribution of the stock price in three months.

Answer:

The probability distribution of the stock price, S_T, in three months would be:

$$\ln S_T \sim N\left[\ln 25 + \left[\left(0.12 - \frac{0.2^2}{2}\right) \times 0.25\right], 0.2 \times \sqrt{0.25}\right]$$

$$\ln S_T \sim N(3.244,\ 0.10)$$

Since ln S_T is normally distributed, 95% of the values will fall within 1.96 standard deviations of the mean. Therefore, ln S_T will lie between 3.244±(1.96 × 0.1). Stated another way:

$$e^{3.244-1.96\times0.1} < S_T < e^{3.244+1.96\times0.1}$$

$$21.073 < S_T < 31.187$$

Dividing the mean and standard deviation by *T* results in the continuously compounded annual return of a stock price. Specifically, the continuously compounded annual returns are *normally distributed* with a mean of:

$$\left(\mu - \frac{\sigma^2}{2}\right)$$

and a standard deviation of:

$$\frac{\sigma}{\sqrt{T}}$$

Professor's Note: Notice that the BSM model assumes stock prices are lognormally distributed, but stock returns are normally distributed. Also, notice in the standard deviation formula that volatility will be lower for longer periods of time.

Example: Return distribution

Assume a stock has an expected annual return of 12% and an annual volatility of 20%. **Calculate** the mean and standard deviation of the probability distribution for the continuously compounded average rate of return over a 4-year period.

Answer:

$$\text{mean} = 0.12 - \frac{0.2^2}{2} = 0.10$$

$$\text{standard deviation} = \frac{0.2}{\sqrt{4}} = 0.10$$

Expected Value

Using the properties of a *lognormal distribution*, we can show that the expected value of S_T, $E(S_T)$, is:

$$E(S_T) = S_0 e^{\mu T}$$

where:
μ = expected rate of return

Example: Expected stock price

Assume a stock is currently priced at $25 with an expected annual return of 20%. **Calculate** the expected value of the stock in six months.

Answer:

$$E(S_T) = \$25 \times e^{0.2 \times 0.5} = \$27.63$$

The difference between the expected annual return on a stock, μ, and the mean return, $\left[\mu - \frac{\sigma^2}{2}\right]$, is closely related to the difference between the arithmetic return and the geometric return. The mean return will always be slightly less than the expected return, just as the geometric return will always be slightly less than the arithmetic return.

When computing the **realized return** for a portfolio, we want to chain-link the returns just like in the calculation of a geometric mean. Using a geometric return produces a more accurate representation of portfolio return.

Example: Realized return

Consider a portfolio that has the following asset returns: 5%, –4%, 9%, 6%. **Calculate** the return realized by this portfolio.

Answer:

$$\text{realized portfolio return} = (1.05 \times 0.96 \times 1.09 \times 1.06)^{1/4} - 1 = 3.9\%$$

Estimating Historical Volatility

As we saw in the value at risk (VaR) topics, the volatility for short periods of time can be scaled to longer periods in time. For example, if the weekly standard deviation is 5%, and we want the annual standard deviation, we simply scale it by the square root of the number of periods in a year, or $\sqrt{52}$. So the annual standard deviation in this case is 36.06%. Conversely, if we knew that the annual standard deviation was 36.06%, then the weekly standard deviation can be found using this formula: 36.06% / $\sqrt{52}$, which is 5%.

The volatility estimation process is a matter of collecting daily price data and then computing the standard deviation of the series of corresponding continuously compounded returns. Continuously compounded returns can be calculated as: $\ln(S_i / S_{i-1})$. The annualized volatility is simply the estimated volatility multiplied by the square root of the number of trading days in a year. Typically, 90 to 180 trading days of data is sufficient for this estimation technique, but a common rule of thumb is to use data covering a period equal to the length of the projection period. In other words, to estimate the volatility for the next year, we should use a year's worth of historical data.

Professor's Note: We will examine the calculation for historical volatility later in this topic.

Black-Scholes-Merton Model Assumptions

LO 56.3: Describe the assumptions underlying the Black-Scholes-Merton option pricing model.

The Black-Scholes-Merton model values options in continuous time and is derived from the same no-arbitrage assumption used to value options with the binomial model. In the binomial model, the hedge portfolio is riskless over the next period, and the no-arbitrage option price is the one that ensures that the hedge portfolio will yield the risk-free rate. To derive the BSM model, an "instantaneous" riskless portfolio (one that is riskless over the next instant) is used to solve for the option price based on the same logic.

In addition to the no-arbitrage condition, the assumptions underlying the BSM model are the following:

- **The price of the underlying asset follows a lognormal distribution.** A variable that is lognormally distributed is one where the logs of the values (in this case, the continuous returns) are normally distributed. It has a minimum of zero and conforms to prices better than the normal distribution (which would produce negative prices).
- **The (continuous) risk-free rate is constant and known.**
- **The volatility of the underlying asset is constant and known.** Option values depend on the volatility of the price of the underlying asset or interest rate.
- **Markets are "frictionless."** There are no taxes, no transactions costs, and no restrictions on short sales or the use of short-sale proceeds.
- **The underlying asset has no cash flow,** such as dividends or coupon payments.
- **The options valued are European options,** which can only be exercised at maturity. The model does not correctly price American options.

Black-Scholes-Merton Option Pricing Model

LO 56.4: Compute the value of a European option using the Black-Scholes-Merton model on a non-dividend-paying stock.

The formulas for the BSM model are:

$$c_0 = \left[S_0 \times N(d_1)\right] - \left[X \times e^{-R_f^c \times T} \times N(d_2)\right]$$

$$p_0 = \left\{X \times e^{-R_f^c \times T} \times \left[1 - N(d_2)\right]\right\} - \left\{S_0 \times \left[1 - N(d_1)\right]\right\}$$

where:

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left[R_f^c + \left(0.5 \times \sigma^2\right)\right] \times T}{\sigma \times \sqrt{T}}$$

$$d_2 = d_1 - \left(\sigma \times \sqrt{T}\right)$$

T = time to maturity (as % of a 365-day year)
S_0 = asset price
X = exercise price
R_f^c = continuously compounded risk-free rate
σ = volatility of continuously compounded returns on the stock
$N(\bullet)$ = cumulative normal probability

We've given you the formulas for both call and put values. However, remember that if you're given one of those prices, you can always use **put-call parity** (with continuously compounded interest rates) to calculate the other one:

$$c_0 = p_0 + S_0 - \left(X \times e^{-R_f^c \times T}\right)$$

or

$$p_0 = c_0 - S_0 + \left(X \times e^{-R_f^c \times T}\right)$$

$N(d_1)$ and $N(d_2)$ are found in a table of probability values (i.e., the *z*-table), so any question about the value of an option will provide those values. The rest is a straightforward calculation.

Example: Using the Black-Scholes-Merton model to value a European call option

Suppose that the stock of Vola, Inc., is trading at $50, and there is a call option on Vola available with an exercise price of $45 that expires in three months. The continuously compounded risk-free rate is 5%, and the annualized standard deviation of returns is 12%. Using the Black-Scholes-Merton model, **calculate** the value of the call option.

Answer:

First we must compute d_1 and d_2 as follows:

$$d_1 = \frac{\ln\left(\frac{50}{45}\right) + \left[0.05 + \left(0.5 \times 0.12^2\right)\right] \times 0.25}{0.12 \times \sqrt{0.25}} = 1.99$$

$$d_2 = 1.99 - \left(0.12 \times \sqrt{0.25}\right) = 1.93$$

Now look up these values in the normal probability tables in Figure 1.

Figure 1: Partial Cumulative Normal Distribution Table*

	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.8	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	**0.9732**	0.9738	0.9744	0.9750	0.9756	0.9761	**0.9767**
2.0	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817

* Note: This table is incomplete. To view an example of a complete cumulative normal table, see the table included at the back of this book.

From the table, we determine that $N(d_1)$ is 0.9767 and $N(d_2)$ is 0.9732. Now that we have everything we need to apply the main call option formula, the value of the call is:

$$c_0 = (\$50 \times 0.9767) - \left(\$45 \times e^{-(0.05 \times 0.25)} \times 0.9732\right) = \$48.84 - \$43.25 = \$5.59$$

To price the corresponding put option using the data in our example, we simply solve put-call parity for the put option price.

Example: Calculating put option value

Calculate the value of a Vola 3-month put option with an exercise price of $45, given the information in the previous example.

Answer:

We can use put-call parity to find the value of the comparable put:

$$p_0 = \$5.59 - \$50.00 + [\$45.00 \times e^{-(0.05 \times 0.25)}] = \$0.03$$

We can also use the BSM put formula:

$$p_0 = \left[\$45 \times e^{-0.05 \times 0.25} \times (1 - 0.9732)\right] - \left[\$50 \times (1 - 0.9767)\right] = \$0.03$$

Professor's Note: You should know how to look up $N(d_1)$ and $N(d_2)$ in the normal probability table given d_1 and d_2. It's possible, however unlikely, that you will have to calculate d_1 and d_2 without the formulas. To value the put option, memorize put-call parity and use it to solve for the put value given the call value.

Black-Scholes-Merton Model With Dividends

LO 56.8: Compute the value of a European option using the Black-Scholes-Merton model on a dividend-paying stock.

European Options

Just as we subtracted the present value of expected cash flows from the asset price when valuing forwards and futures, we can subtract it from the asset price in the BSM model. Since the BSM model is in continuous time, in practice $S_0 \times e^{-qT}$ is substituted for S_0 in the BSM formula, where q is equal to the continuously compounded rate of dividend payment. Over time, the asset price is discounted by a greater amount to account for the greater amount of cash flows. Cash flows will increase put values and decrease call values.

Example: Valuing a call option on a stock with a continuous dividend yield

Let's revisit Vola, Inc., and this time we'll assume the stock pays a continuous dividend yield of 2%. Here's the basic information again. Suppose the stock of Vola is trading at \$50, and there is a call option available with an exercise price of \$45 that expires in three months. The continuously compounded risk-free rate is 5%, and the annualized standard deviation of returns is 12%. Using the Black-Scholes-Merton model, **calculate** the value of the call option.

Answer:

The adjusted price of the stock is:

$$e^{-0.02\times 0.25}\times \$50.00 = \$49.75$$

Recalculate d_1 and d_2 using the adjusted price:

$$d_1 = \frac{\ln\left(\frac{49.75}{45}\right)+\left\{0.05+\left[0.5\times(0.12)^2\right]\right\}\times 0.25}{0.12\times\sqrt{0.25}} = 1.91$$

$$d_2 = 1.91-\left(0.12\times\sqrt{0.25}\right) = 1.85$$

Now look up these values in the normal probability tables in Figure 2.

Figure 2: Partial Cumulative Normal Distribution Table*

	0.00	0.01	0.02	0.03	0.04	0.05	0.06
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	**0.9678**	0.9686
1.9	0.9713	**0.9719**	0.9726	0.9732	0.9738	0.9744	0.9750
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803

* Note: This table is incomplete. To view an example of a complete cumulative normal table, see the table included at the back of this book.

From the table, we can determine that $N(d_1)$ is 0.9719 and $N(d_2)$ is 0.9678. The adjusted price is \$49.75. Now that we have everything we need to apply the BSM model, the value of the call is:

$$c_0 = (\$49.75\times 0.9719)-\left(\$45\times e^{-(0.05\times 0.25)}\times 0.9678\right) = \$48.35-\$43.01 = \$5.34$$

The value of the Vola call with no dividend yield was \$5.59 from our earlier example. The 2% dividend yield reduced the call value by \$0.25, from \$5.59 to \$5.34.

On the exam, it may be the case that you are provided with the dollar amount of the dividend rather than the dividend yield. The process for computing option value is similar, but instead of discounting the stock price with a continuously compounded dividend rate, you would compute the present value of the dividend(s) and then subtract that amount from the stock price. The following example demonstrates this technique.

Example: Pricing options on a dividend paying stock

Assume we have a non-dividend paying stock with a current price of $100 and volatility of 20%. If the risk-free rate is 7%, the price of a 6-month at-the-money call option, according to the Black-Scholes-Merton model, will be $7.43, and the corresponding put option price will be $3.99. Now assume that the same stock instead pays a $1 dividend in two months and a $1 dividend in five months. **Compute** the value of a 6-month call option on the dividend paying stock.

Answer:

The present value of the first dividend is $1e^{-0.07(0.1667)} = 0.9884$, and the present value of the second dividend is $1e^{-0.07(0.4167)} = 0.9713$. The stock price then becomes:

$$S_0 = 100 - 0.9884 - 0.9713 = \$98.04$$

We now know the following: $S_0 = \$98.04$; $X = \$100$; $\sigma = 20\%$; $r = 7\%$; $T = 0.5$

d_1 and d_2 are computed as follows:

$$d_1 = \frac{\ln\left(\frac{98.04}{100}\right) + \left(0.07 + \frac{0.2^2}{2}\right) \times 0.5}{0.2\sqrt{0.5}} = 0.1783$$

$$d_2 = 0.1783 - 0.2\sqrt{0.5} = 0.03688$$

From the cumulative standard normal tables, we find:

$$N(d_1) = 0.5708 \text{ and } N(d_2) = 0.5147$$

Substituting back into the call option price formula yields:

$$c = 98.04 \times 0.5708 - 100e^{-0.07 \times 0.5} \times 0.5147 = \$6.26$$

Using put-call parity, the corresponding put option price is:

$$p = \$6.26 + 100e^{-0.07 \times 0.5} - 98.04 = \$4.78$$

Since the dividend reduces the value of the stock, the call value decreased, and the put value increased compared to the non-dividend paying stock.

LO 56.7: Explain how dividends affect the decision to exercise early for American call and put options.

American Options

Recall that when no dividends are paid, there is no difference between European and American call options. This is because the unexercised value of a call option, $S_0 - Xe^{-rT}$, was always more valuable than the exercised value of the option, $S_0 - X$. When a stock pays a dividend, *D*, at time *n*, the exercise decision becomes more complicated.

At the last dividend date before expiration, t_n, the exercised value of the option is:

$$S(t_n) - X$$

If the call option is unexercised and the dividend is paid, its unexercised value is:

$$S(t_n) - D_n - Xe^{-r(T-t_n)}$$

An investor will only exercise when:

$$S(t_n) - X > S(t_n) - D_n - Xe^{-r(T-t_n)}$$

or

$$D_n > X\left(1 - e^{-r(T-t_n)}\right)$$

So the closer the option is to expiration and the larger the dividend, the more optimal early exercise will become. The previous result can be generalized to show that early exercise is not optimal if:

$$D_i \leq X\left(1 - e^{-r(t_{i+1}-t_n)}\right) \text{for } i < n$$

A popular approximation for pricing American call options on dividend paying stocks is **Black's approximation.** Black suggests using the procedure for European options on *T* and t_n and then taking the larger of the two as the price of the American call option.

Consider the situation provided in the previous example. However, instead of evaluating a European option, assume the call option is an American option. We know that the call option value at maturity, T = 6 months, with dividend payments at two months and five months was $6.26. Suppose an investor instead opted to exercise the option immediately before the second dividend payment. Here, exercise may be optimal, if the option is deep in-the-money, because the second \$1 dividend, D_2, is greater than 0.5816.

$$1 > \$100\left(1 - e^{-0.07\times\left(\frac{6}{12}-\frac{5}{12}\right)}\right) = 0.5816$$

In this case, we can apply Black's approximation by computing the call option value assuming early exercise before the second dividend payment. When only considering the first dividend's present value of 0.9884, the stock price becomes \$99.0116. The call option value, according to the Black-Scholes-Merton model, is now \$6.05. Since Black's approximation values the American option as the greater of the two values (\$6.26 > \$6.05), we would value this option at \$6.26.

For American put options, early exercise becomes less likely with larger dividends. The value of the put option increases as the dividend is paid. Early exercise is, therefore, not optimal as long as:

$$D_n \geq X\left(1 - e^{-r(T-t_n)}\right)$$

Valuation of Warrants

LO 56.5: Compute the value of a warrant and identify the complications involving the valuation of warrants.

Warrants are attachments to a bond issue that give the holder the right to purchase shares of a security at a stated price. After purchasing the bond, warrants can be exercised separately or stripped from the bond and sold to other investors. Hence, warrants can be valued as a separate call option on the firm's shares.

One distinction is necessary, though. With call options, the shares are already outstanding, and the exercise of a call option triggers the trading of shares among investors at the strike price of the call options. When an investor exercises warrants, the investor purchases shares directly from the firm. The distinction is that the value of all outstanding shares can be affected by the exercise of warrants, as the amount paid for the shares will (in all likelihood) be less than their pro rata market value, so the value of equity per share will fall with exercise (i.e., dilution can occur).

Assuming there is no benefit to the company from issuing warrants, the value of each warrant is computed as:

$$\frac{N}{N+M} \times \text{value of regular call option}$$

where:
N = number of shares outstanding
M = number of new warrants issued

Thus, the total cost of issuing warrants is *M* times the value of each warrant. With no perceived market benefit, the company's stock price will decline by: M / (N + M) × value of regular call option.

For example, suppose a company with 1 million shares outstanding worth $50 each is contemplating issuing 500,000 warrants. Each warrant would grant the holder the right to purchase one share with a strike price of $65 in two years. Assuming the value of a corresponding two-year European call option is worth $6.00, the value of each warrant would be computed as:

$$\frac{1{,}000{,}000}{1{,}000{,}000+500{,}000}\times 6.00=\$4.00$$

Thus, the total cost of the warrant issue would be 500,000 × $4 = $2 million. In the event that there is no perceived benefit in the marketplace from issuing the warrants, we would expect the initial stock price of $50 to decline by $2 to $48/share:

$$\frac{500{,}000}{1{,}000{,}000+500{,}000}\times 6.00=\$2.00$$

Volatility Estimation

LO 56.6: Define implied volatilities and describe how to compute implied volatilities from market prices of options using the Black-Scholes-Merton model.

Notice in call and put equations that volatility is unobservable. Historical data can serve as a basis for what volatility might be going forward, but it is not always representative of the current market. Consequently, practitioners will use the BSM option pricing model along with market prices for options and solve for volatility. The result is what is known as **implied volatility**. Before we discuss implied volatility further, let's first examine the calculation of historical volatility.

The steps in computing **historical volatility** for use as an input in the BSM continuous-time options pricing model are:

- Convert a time series of *N* prices to returns:

$$R_i=\frac{P_i-P_{i-1}}{P_{i-1}},\ i=1 \text{ to } N$$

- Convert the returns to continuously compounded returns:

$$R_i^c=\ln(1+R_i),\ i=1 \text{ to } N$$

- Calculate the variance and standard deviation of the continuously compounded returns:

$$\sigma^2 = \frac{\sum_{i=1}^{N}\left(R_i^c - \overline{R_i^c}\right)^2}{N-1}$$

$$\sigma = \sqrt{\sigma^2}$$

Recall from Book 2 that continuously compounded returns can be calculated using a set of price data. We introduced the equation for continuously compounded returns as:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

Arriving at the continuously compounded return value is no different than taking the holding period return and then taking the natural log of (1 + holding period return) as illustrated above. For example, if we assume that a stock price is currently valued at \$50 and was \$47 yesterday, the continuously compounded return can be computed as either:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) = \ln\left(\frac{50}{47}\right) = 6.19\%$$

or

$$R_i = \frac{P_i - P_{i-1}}{P_{i-1}} = \frac{50-47}{47} = 6.38\%$$

$$R_i^c = \ln(1+0.0638) = 6.19\%$$

Implied volatility is the value for standard deviation of continuously compounded rates of return that is "implied" by the market price of the option. Of the five inputs into the BSM model, four are observable: (1) stock price, (2) exercise price, (3) risk-free rate, and (4) time to maturity. If we use these four inputs in the formula and set the BSM formula equal to market price, we can solve for the volatility that satisfies the equality.

Volatility enters into the equation in a complex way, and there is no closed-form solution for the volatility that will satisfy the equation. Rather, it must be found by iteration (trial and error). If a value for volatility makes the value of a call calculated from the BSM model lower than the market price, it needs to be increased (and vice versa) until the model value equals market price (remember, option value and volatility are positively related).

Key Concepts

LO 56.1

The Black-Scholes-Merton model suggests that stock prices are lognormal over longer periods of time, but suggests that stock returns are normally distributed.

LO 56.2

The realized return for a portfolio is computed using a geometric return.

LO 56.3

Assumptions underlying the BSM model:

- The price of the underlying asset follows a lognormal distribution.
- The (continuous) risk-free rate is constant and known.
- The volatility of the underlying asset is constant and known.
- Markets are "frictionless."
- The underlying asset generates no cash flows.
- The options are European.

LO 56.4

The formulas for the BSM model are:

$$c_0 = [S_0 \times N(d_1)] - [Xe^{-R_F^c \times T} \times N(d_2)]$$
$$p_0 = \{Xe^{-R_F^c \times T} \times [1 - N(d_2)]\} - \{S_0 \times [1 - N(d_1)]\}$$

Cash flows on the underlying asset decrease call prices and increase put prices.

LO 56.5

Warrants are attachments to a bond issue that give the holder the right to purchase shares of a security at a stated price. Warrants can be valued as a separate call option on the firm's shares.

LO 56.6

Historical volatility is the standard deviation of a past series of continuously compounded returns for the underlying asset. Implied volatility is the volatility that, when used in the Black-Scholes-Merton formula, produces the current market price of the option.

LO 56.7

Dividends complicate the early exercise decision for American-style options because a dividend payment effectively decreases the price of the stock.

LO 56.8

To adjust the BSM model for assets with a continuously compounded rate of dividend payment equal to *q*, S_0e^{-qT} is substituted for S_0 in the formula.

Concept Checkers

1. A European put option has the following characteristics: S_0 = \$50; X = \$45; r = 5%; T = 1 year; and σ = 25%. Which of the following is closest to the value of the put?
   A. \$1.88.
   B. \$3.28.
   C. \$9.07.
   D. \$10.39.

2. A European call option has the following characteristics: S_0 = \$50; X = \$45; r = 5%; T = 1 year; and σ = 25%. Which of the following is closest to the value of the call?
   A. \$1.88.
   B. \$3.28.
   C. \$9.06.
   D. \$10.39.

3. A security sells for \$40. A 3-month call with a strike of \$42 has a premium of \$2.49. The risk-free rate is 3%. What is the value of the put according to put-call parity?
   A. \$1.89.
   B. \$3.45.
   C. \$4.18.
   D. \$6.03.

4. Which of the following is not an assumption underlying the BSM options pricing model?
   A. The underlying asset does not generate cash flows.
   B. Continuously compounded returns are lognormally distributed.
   C. The option can only be exercised at maturity.
   D. The risk-free rate is constant.

5. Stock ABC trades for \$60 and has 1-year call and put options written on it with an exercise price of \$60. The annual standard deviation estimate is 10%, and the continuously compounded risk-free rate is 5%. The value of both the call and put using the BSM option pricing model are closest to:

	Call	Put
A.	\$6.21	\$1.16
B.	\$4.09	\$3.28
C.	\$4.09	\$1.16
D.	\$6.21	\$3.28

Concept Checker Answers

1. A S_0 = \$50; X = \$45; r = 5%; T = 1 year; and σ = 25%.

$$d_1 = \frac{\ln\left(\frac{50}{45}\right) + \left(0.05 + \frac{0.0625}{2}\right)1}{0.25(1)} = \frac{0.18661}{0.25} = 0.74644$$

$d_2 = 0.74644 - 0.25 = 0.49644$

from the cumulative normal table:

$N(-d_1) = 0.2266$

$N(-d_2) = 0.3085$*

$p = 45e^{-0.05(1)}(0.3085) - 50(0.2266) = 1.88$

(*note rounding)

2. C S_0 = \$50; X = \$45; r = 5%; T = 1 year; and σ = 25%.

$$d_1 = \frac{\ln\left(\frac{50}{45}\right) + \left(0.05 + \frac{0.0625}{2}\right)1}{0.25(1)} = \frac{0.18661}{0.25} = 0.74644$$

$d_2 = 0.74644 - 0.25 = 0.49644$

from the cumulative normal table:

$N(d_1) = 0.7731$

$N(d_2) = 0.6915$*

$c = 50(0.7731) - 45e^{-0.05}(0.6915) = 9.055$

(*note rounding)

3. C $p = c + Xe^{-rT} - S = 2.49 + 42\,e^{-0.03 \times 0.25} - 40 = \$4.18$

4. B No arbitrage is possible, and:
   - Asset price (not returns) follows a lognormal distribution.
   - The (continuous) risk-free rate is constant.
   - The volatility of the underlying asset is constant.
   - Markets are "frictionless."
   - The asset has no cash flows.
   - The options are European (i.e., they can only be exercised at maturity).

5. C First, let's compute d_1 and d_2 as follows:

$$d_1 = \frac{\ln\left(\frac{60}{60}\right) + \left[0.05 + \left(0.5 \times 0.10^2\right)\right] \times 1.0}{0.1 \times \sqrt{1.0}} = 0.55$$

$$d_2 = 0.55 - \left(0.1 \times \sqrt{1.0}\right) = 0.45$$

Now look up these values in the normal table at the back of this book. These values are $N(d_1) = 0.7088$ and $N(d_2) = 0.6736$. Hence, the value of the call is:

$$c_0 = \$60(0.7088) - \left[\$60 \times e^{-(0.05 \times 1.0)} \times (0.6736)\right] = \$42.53 - \$38.44 = \$4.09$$

According to put/call parity, the put's value is:

$$p_0 = c_0 - S_0 + \left(X \times e^{-R_c^f \times T}\right) = \$4.09 - \$60.00 + \left[\$60.00 \times e^{-(0.05 \times 1.0)}\right] = \$1.16$$

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Greek Letters

Topic 57

Exam Focus

The level of risk associated with an option position is dependent in large part on the following factors: relationship between the value of a position involving options and the value of the underlying assets; time until expiration; asset value volatility; and the risk-free rate. Measures that capture the effects of these factors are referred to as "the Greeks" due to their names: delta; theta; gamma; vega; and rho. Thus, a large part of this topic covers the evaluation of option Greeks. Once option participants are aware of their Greek exposures, they can more effectively hedge their positions to mitigate risk. This topic also introduces the common hedging concepts of delta-neutral portfolios and portfolio insurance.

Naked and Covered Call Options

LO 57.1: Describe and assess the risks associated with naked and covered option positions.

A **naked position** occurs when one party sells a call option without owning the underlying asset. A **covered position** occurs when the party selling a call option owns the underlying asset.

Suppose a firm can sell 10,000 call options on a stock that is currently trading at \$20. The strike price of the option is \$23, and the option premium is \$4. A naked position would generate \$40,000 in revenue, and as long as the stock price is below \$23 at expiration, the firm can retain the income without cost. However, the initial income will be reduced by \$10,000 for every dollar above \$23 that the stock reaches at expiration. For example, if the stock is at \$30 per share when the option expires, the naked position results in a negative payoff of \$70,000 and a net loss of \$30,000. The potential loss from a naked written position is unlimited, assuming the stock's price can rise without bound. The maximum potential gain is capped at the level of the premium received. If the stock price at expiration is \$23 or less, the writer makes a profit equal to the premium of \$40,000.

With a covered call, the firm owns 10,000 shares of the underlying stock, so if the stock price rises above the \$23 strike price and the option is exercised, the firm will sell shares that it already owns. This minimizes the "cost" of the short options by locking in the revenue from the option sale. However, if the stock falls to \$10 per share, the long stock position decreases in value by \$100,000, which is substantially larger than the premium received from the option sale.

A Stop-Loss Strategy

LO 57.2: Explain how naked and covered option positions generate a stop loss trading strategy.

Stop-loss strategies with call options are designed to limit the losses associated with short option positions (i.e., those taken by call writers). The strategy requires purchasing the underlying asset for a naked call position when the asset rises above the option's strike price. The asset is then sold as soon as it goes below the strike price. The objective here is to hold a naked position when the option is out-of-the-money and a covered position when the option is in-the-money.

The main drawbacks to this simplistic approach are transaction costs and price uncertainty. The costs of buying and selling the asset can become high as the frequency of stock price fluctuations about the exercise price increases. In addition, there is great uncertainty as to whether the asset will be above (or below) the strike price at expiration.

Delta Hedging

LO 57.3: Describe delta hedging for an option, forward, and futures contracts.

LO 57.4: Compute the delta of an option.

The **delta** of an option, Δ, is the ratio of the change in price of the call option, *c*, to the change in price of the underlying asset, *s*, for small changes in *s*. Mathematically:

$$\text{delta} = \Delta = \frac{\partial c}{\partial s}$$

where:
∂c = change in the call option price
∂s = change in the stock price

As illustrated in Figure 1, delta is the slope of the call option pricing function at the current stock price. As shown in Figure 2, call option deltas range from zero to positive one, while put option deltas range from negative one to zero.

Figure 1: Delta of a Call Option

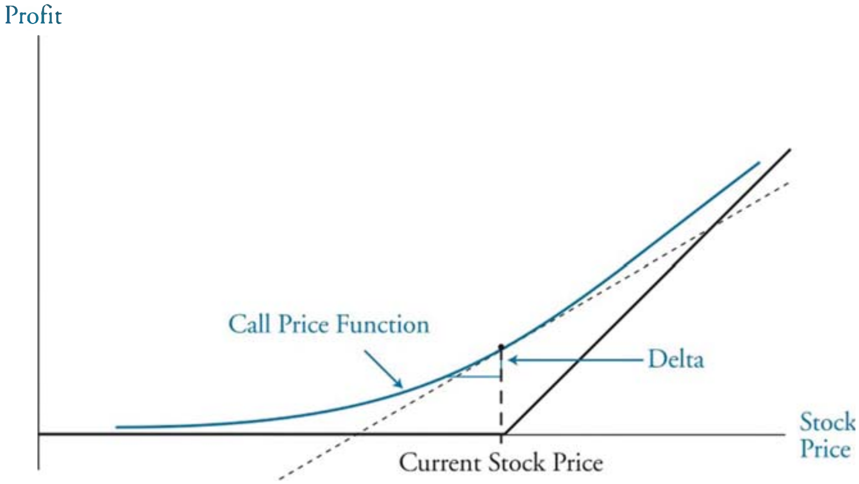
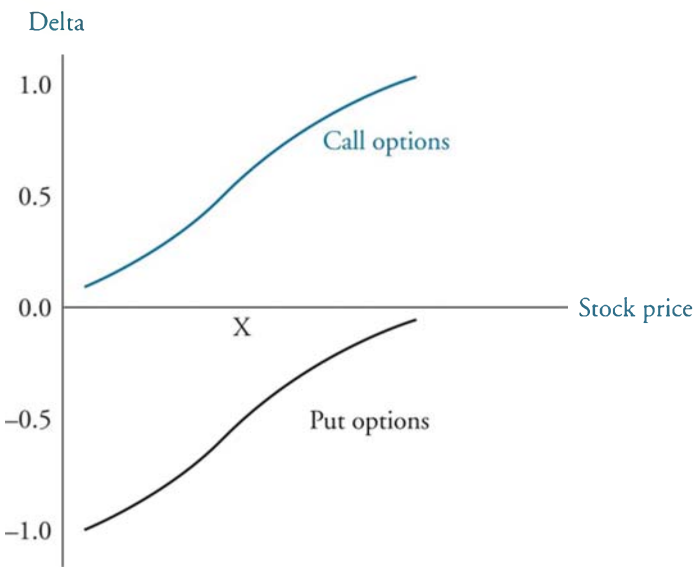


Figure 2: Call and Put Option Deltas



Option Delta

A call delta equal to 0.6 means that the price of a call option on a stock will change by approximately \$0.60 for a \$1.00 change in the value of the stock. To completely hedge a long stock or short call position, an investor must purchase the number of shares of stock equal to delta times the number of options sold. Another term for being completely hedged is **delta-neutral**. For example, if an investor is short 1,000 call options, he will need to be long 600 (0.6 × 1,000) shares of the underlying. When the value of the underlying asset increases by \$1.00, the underlying position increases by \$600, while the value of his option position decreases by \$600. When the value of the underlying asset decreases by \$1.00, there is an offsetting increase in value in the option position.

Delta can also be calculated as the $N(d_1)$ in the Black-Scholes-Merton option pricing model. Recall from the previous topic that d_1 is equal to:

$$d_1 = \frac{\ln(S_0 / X) + (R_F + \sigma^2 / 2) \times T}{\sigma \times \sqrt{T}}$$

Example: Computing delta

Suppose that stock XYZ is trading at \$50, and there is a call option that trades on XYZ with an exercise price of \$45 which expires in three months. The risk-free rate is 5% and the standard deviation of returns is 12% annualized. **Determine** the value of the call option's delta.

Answer:

$$d_1 = \frac{\ln(50 / 45) + (0.05 + 0.12^2 / 2) \times 0.25}{0.12 \times \sqrt{0.25}} = 1.99$$

Next, look up this value in the normal probability tables, which can be found in appendix at the end of this book. From the normal probability tables N(1.99), and, in turn, delta is 0.9767. This means that when the stock price changes by \$1, the option price will change by 0.9767.

Forward Delta

The delta of a forward position is equal to one, implying a one-to-one relationship between the value of the forward contract and its underlying asset. A forward contract position can easily be hedged with an offsetting underlying asset position with the same number of securities.

Professor's Note: When the underlying asset pays a dividend, q, the delta of an option or forward must be adjusted. If a dividend yield exists, the delta for a call option equals $e^{-qT} \times N(d_1)$, the delta of a put option equals $e^{-qT} \times [N(d_1) - 1]$, and the delta of a forward contract equals e^{-qT}.

Futures Delta

Unlike forward contracts, the delta of a futures position is not ordinarily one because of the spot-futures parity relationship. For example, the delta of a futures position is e^{rT} on a stock or stock index that pays no dividends, where *r* is the risk-free rate and *T* is the time to maturity. Assets that pay a dividend yield, *q*, would generate a delta equal to $e^{(r-q)T}$. An investor would hedge short futures positions by going long the amount of the deliverable asset.

Dynamic Aspects of Delta Hedging

LO 57.5: Describe the dynamic aspects of delta hedging and distinguish between dynamic hedging and hedge-and-forget strategy.

As we saw in Figure 1, the delta of an option is a function of the underlying stock price. That means when the stock price changes, so does the delta. When the delta changes, the portfolio will no longer be hedged (i.e., the number of options and underlying stocks will no longer be in balance), and the investor will need to either purchase or sell the underlying asset. This rebalancing must be done on a continual basis to maintain the delta-neutral hedged position.

The goal of a **delta-neutral portfolio** (or delta-neutral hedge) is to combine a position in an asset with a position in an option *so that the value of the portfolio does not change with changes in the value of the asset*. In referring to a stock position, a delta-neutral portfolio can be made up of a risk-free combination of a long stock position and a short call position where the number of calls to short is given by $1/\Delta_c$.

$$\text{number of options needed to delta hedge} = \frac{\text{number of shares hedged}}{\text{delta of call option}}$$

Example: Delta-neutral portfolio—Part 1

An investor owns 60,000 shares of ABC stock that is currently selling for \$50. A call option on ABC with a strike price of \$50 is selling at \$4 and has a delta of 0.60. **Determine** the number of call options necessary to create a delta-neutral hedge.

Answer:

In order to determine the number of call options necessary to hedge against instantaneous movements in ABC's stock price, calculate:

$$\text{number of options needed to delta hedge} = \frac{60{,}000}{0.6} = 100{,}000 \text{ options}$$

$$= 1{,}000 \text{ call option contracts}$$

Because he is long the stock, he needs to short the options.

Example: Delta-neutral portfolio—Part 2

Calculate the effect on portfolio value of a $1.00 increase in the price of ABC stock.

Answer:

Assuming the price of ABC stock increased instantly by $1.00, then the value of the call option position would decrease by $0.60 because the investor is *short* (or has sold) the call option contracts. Therefore, the net impact of the price change would be zero as illustrated here:

total value of increase in stock position = (60,000) × ($1) = $60,000

total value of decrease in option position = (100,000) × (–$0.60)= –$60,000

total change in portfolio value = $60,000 – $60,000 = $0

Recall that when short a call (or other asset), as the price of the underlying rises, the position loses value, and when the price of the underlying declines, the value of the position increases.

Maintaining the Hedge

A key consideration in delta-neutral hedging is that the *delta-neutral position only holds for very small changes in the value of the underlying stock*. Hence, the delta-neutral portfolio must be frequently (continuously) *rebalanced to maintain the hedge*. As the underlying stock price changes, so does the delta of the call option. The delta of the option is an approximation of a nonlinear function: the change in value of the option that corresponds with a change in the value of the underlying asset. As the delta changes, the number of calls that need to be sold to maintain a risk-free position also changes. Hence, continuously maintaining a delta-neutral position can be very costly in terms of transaction costs associated with either closing out options or selling additional contracts.

Adjusting the hedge on a frequently basis is known as **dynamic hedging**. If, instead, the hedge is initially set-up but never adjusted, it is referred to as **static hedging**. This type of hedge is also known as a *hedge-and-forget strategy*.

Example: Delta-neutral portfolio—Part 3

Continuing with the previous example, assume now that the price of the underlying stock has moved to $51, and consequently, the delta of the call option with a strike price of $50 has increased from 0.60 to 0.62. How would the investor's portfolio of stock and options have to be adjusted to maintain the delta-neutral position?

Answer:

In order to determine the number of call options necessary to maintain the hedge against instantaneous movements in ABC's stock price, recalculate the number of short call options needed:

$$\text{number of options needed} = \frac{\text{number of shares hedged}}{\text{delta of call option}} = \frac{60{,}000}{0.62} = 96{,}774$$

She will need 96,774 call options, or approximately 968 option contracts. In other words, 32 option contracts would need to be purchased in order to maintain the delta-neutral position. If the hedge were not modified, then another price change would result in a greater movement in the value of the options than in the underlying stock. With the rebalanced hedge, the change in value of her stock position will again be offset by the change in value of her short position. Assume the price of ABC stock increased (decreased) instantly by $1.00, then the value of the short call option position would decrease (increase) by $0.62. Therefore, the net impact of the price change would be zero:

increase in stock position = (60,000) × ($1) = $60,000

decrease in short position = (96,774) × (–$0.62)= –$60,000

Other Portfolio Hedging Approaches

It's also possible to develop a delta-neutral hedge by buying put options in sufficient numbers so that the current gain or loss on the underlying asset is offset by the current gain or loss on the puts. Hence, similar to the discussion of delta-neutral portfolios using call options, a delta-neutral position can be created by *purchasing* the correct number of put options so that:

$$\Delta \text{ value of puts} = -\ \Delta \text{ value of long stock position}$$

When using puts in constructing a delta-neutral portfolio, *purchase* [1 / (call delta – 1)] put options to protect a share of stock held long. When using calls you would *sell* (1 / call delta) call options for each long share of stock. Rebalancing is just as important with puts as it is with calls.

Example: Delta-neutral portfolio—Part 4

Using our earlier example, assume the investor owns 60,000 shares of ABC stock that is currently selling for \$50. A call option on ABC with a strike price of \$50 is selling at \$4 and has a delta of 0.60. **Determine** the number of put options necessary to create a delta-neutral hedge.

Answer:

First, compute the delta of the put option. The investor knows that the delta of a call option is 0.60. The delta of the put option is then equal to (call delta – 1), or (0.60 – 1) = –0.40. In order to determine the number of put options necessary to hedge against instantaneous movements in ABC's stock price, calculate:

$$\text{number of options needed to delta hedge} = \frac{-60{,}000}{-0.4} = 150{,}000 \text{ options}$$

$$= 1{,}500 \text{ put option contracts}$$

Because he is long the stock, he needs to purchase the put options.

LO 57.6: Define the delta of a portfolio.

The delta of a portfolio of options on a single underlying asset can be calculated as the weighted average delta of each option position in the portfolio:

$$\text{portfolio delta} = \Delta_p = \sum_{i=1}^{n} w_i \Delta_i$$

where:
w_i = the portfolio weight of each option position
Δ_i = the delta of each option position

Therefore, portfolio delta represents the expected change of the overall option portfolio value given a small change in the price of the underlying asset.

Theta, Gamma, Vega, and Rho

LO 57.7: Define and describe theta, gamma, vega, and rho for option positions.

LO 57.8: Explain how to implement and maintain a delta-neutral and a gamma-neutral position.

LO 57.9: Describe the relationship between delta, theta, gamma, and vega.

Theta

Theta, Θ, measures the option's sensitivity to a decrease in time to expiration. Theta is also termed the "time decay" of an option. Theta varies as a function of both time and the price of the underlying asset. Figure 3 illustrates theta as a function of stock price and days until expiration.

Theta for a call option is calculated using the following equation:

$$\Theta = \frac{\partial c}{\partial t}$$

where:
∂c = change in the call price
∂t = change in time

For European call options on non-dividend-paying stocks, theta can be calculated using the Black-Scholes-Merton formula as follows:

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rXe^{-rT}N(d_2)$$

$$\Theta(\text{put}) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rXe^{-rT}N(-d_2)$$

where:

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\left(x^2/2\right)}$$

Note that theta in the above equations is measured in years. It can be converted to a daily basis by dividing by 365. To find the theta for each trading day, you would divide by 252.

Example: Computing theta

Suppose that stock XYZ is trading at $50, and there is a call option that trades on XYZ with an exercise price of $45 which expires in three months. The risk-free rate is 5%, and the standard deviation of returns is 12% annualized. **Determine** the value of the call option's theta per trading day. Assume d_1 is 1.99 and d_2 is 1.93. From the normal probability tables, $N(d_1)$ is 0.9767 and $N(d_2)$ is 0.9732.

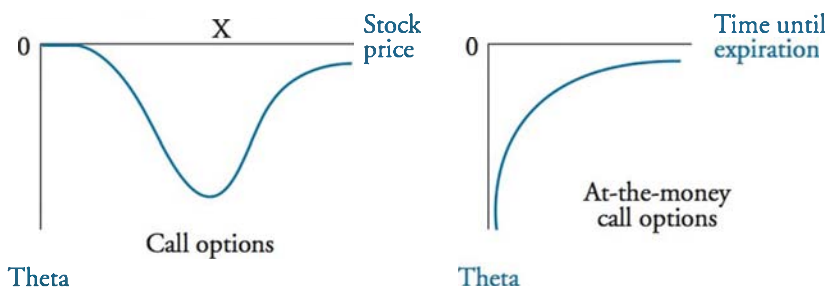
Answer:

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\left(1.99^2/2\right)} = 0.055$$

$$\Theta(\text{call}) = -\frac{50 \times 0.055 \times 0.12}{2\sqrt{0.25}} - 0.05 \times 45e^{-0.05\times0.25} \times 0.9732 = -0.33 - 2.16 = -2.49$$

Theta per trading day is: –2.49 / 252 = –0.00988

Figure 3: Theta as a Function of Stock Price and Time to Expiration



The specific characteristics of theta are as follows:

- Theta affects the value of put and call options in a similar way (e.g., as time passes, most call and put options decrease in value, all else equal).
- Theta varies with changes in stock prices and as time passes.
- Theta is most pronounced when the option is at-the-money, especially nearer to expiration. The left side of Figure 3 illustrates this relationship.
- Theta values are usually negative, which means the value of the option decreases as it gets closer to expiration.
- Theta usually increases in absolute value as expiration approaches. The right side of Figure 3 illustrates this relationship.
- It is possible for a European put option that is in-the-money to have a positive theta value.

Gamma

Gamma, Γ, represents the expected change in the delta of an option. It measures the curvature of the option price function not captured by delta (see Figure 1). The specific mathematical relationship for gamma is:

$$\Gamma = \frac{\partial^2 c}{\partial s^2}$$

where:
$\partial^2 c$ and ∂s^2 = the second partial derivatives of the call and stock prices, respectively

The calculation of gamma for European call or put options on non-dividend-paying stocks can also be found using the following formula, where $N'(x)$ is calculated in the same fashion as it is for theta.

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$

Example: Computing gamma

Suppose that stock XYZ is trading at \$50, and there is a call option that trades on XYZ with an exercise price of \$45 which expires in three months. The risk-free rate is 5%, and the standard deviation of returns is 12% annualized. **Determine** the value of the call option's gamma. Assume d_1 is 1.99 and $N(d_1)$ is 0.9767.

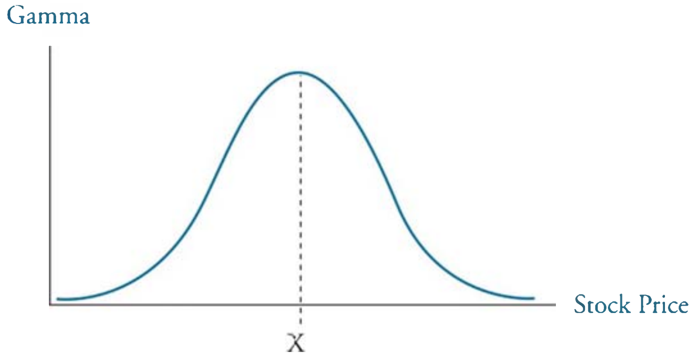
Answer:

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}} = \frac{0.055}{50 \times 0.12 \times \sqrt{0.25}} = \frac{0.055}{3} = 0.0183$$

Gamma measures the rate of change in the option's delta, so for a \$1 change in the price of the stock, the delta will change by 0.0183.

Figure 4 illustrates the relationship between gamma and the stock price for a stock option. As indicated in Figure 4, gamma is largest when an option is at-the-money (at stock price = X). When an option is deep in-the-money or out-of-the-money, changes in stock price have little effect on gamma.

Figure 4: Gamma vs. Stock Price



When gamma is large, delta will be changing rapidly. On the other hand, when gamma is small, delta will be changing slowly. Since gamma represents the curvature component of the call-price function not accounted for by delta, it can be used to minimize the *hedging error* associated with a linear relationship (delta) to represent the curvature of the call-price function.

Delta-neutral positions can hedge the portfolio against small changes in stock price, while gamma can help hedge against relatively large changes in stock price. Therefore, it is not only desirable to create a delta-neutral position but also to create one that is **gamma-neutral**. In that way, neither small nor large stock price changes adversely affect the portfolio's value.

Since underlying assets and forward instruments generate linear payoffs, they have zero gamma and, hence, cannot be employed to create gamma-neutral positions. Gamma-neutral positions have to be created using instruments that are not linearly related to the underlying instrument, such as options. The specific relationship that determines the number of options that must be added to an existing portfolio to generate a gamma-neutral position is $-(\Gamma_p / \Gamma_T)$, where Γ_P is the gamma of the existing portfolio position, and Γ_T is the gamma of a traded option that can be added. Let's take a look at an example.

Example: Creating a gamma-neutral position

Suppose an existing short option position is delta-neutral but has a gamma of –6,000. Here, gamma is negative because we are short the options. Also, assume that there exists a traded option with a delta of 0.6 and a gamma of 1.25. Create a gamma-neutral position.

Answer:

To gamma-hedge, we must buy 4,800 options (6,000 / 1.25). Now the position is gamma-neutral, but the added options have changed the delta position of the portfolio from 0 to 2,880 = 4,800 × 0.6. This means that 2,880 shares of the underlying position will have to be sold to maintain not only a gamma-neutral position, but also the original delta-neutral position.

Relationship Among Delta, Theta, and Gamma

Stock option prices are affected by delta, theta, and gamma as indicated in the following relationship:

$$r\Pi = \Theta + rS\Delta + 0.5\sigma^2 S^2 \Gamma$$

where:
r = the risk-neutral risk-free rate of interest
Π = the price of the option
Θ = the option theta
S = the price of the underlying stock
Δ = the option delta
σ^2 = the variance of the underlying stock
Γ = the option gamma

This equation shows that the change in the value of an option position is directly affected by its sensitivities to the Greeks.

For a delta-neutral portfolio, $\Delta = 0$, so the preceding equation reduces to:

$$r\Pi = \Theta + 0.5\sigma^2 S^2 \Gamma$$

The left side of the equation is the dollar risk-free return on the option (risk-free rate times option value). Assuming the risk-free rate is small, this demonstrates that for large positive values of theta, gamma tends to be large and negative, and vice versa, which explains the common practice of using theta as a proxy for gamma.

Vega

Professor's Note: Vega is not actually a letter of the Greek alphabet, but we still call vega one of the "Greeks" in option pricing.

Vega measures the sensitivity of the option's price to changes in the volatility of the underlying stock. For example, a vega of 8 indicates that for a 1% increase in volatility, the option's price will increase by 0.08. For a given maturity, exercise price, and risk-free rate, the vega of a call is equal to the vega of a put.

Vega for a call option is calculated using the following equation:

$$\text{vega} = \frac{\partial c}{\partial \sigma}$$

where:
∂c = change in the call price
$\partial \sigma$ = change in volatility

Vega for European calls and puts on non-dividend-paying stocks is calculated as:

$$\text{vega} = S_0 N'(d_1)\sqrt{T}$$

Example: Computing vega

Suppose that stock XYZ is trading at $50, and there is a call option that trades on XYZ with an exercise price of $45 which expires in three months. The risk-free rate is 5%, and the standard deviation of returns is 12% annualized. **Determine** the value of the call option's vega. Assume d_1 is 1.99 and $N(d_1)$ is 0.9767.

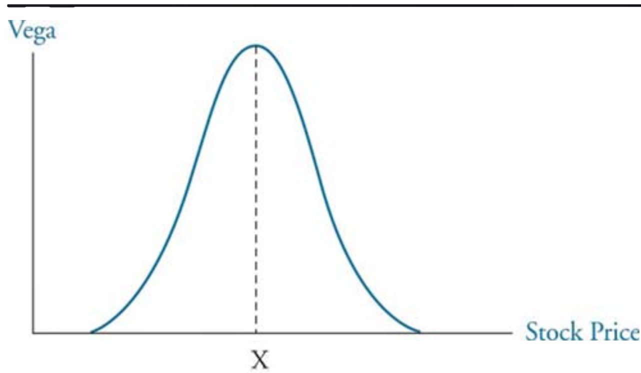
Answer:

$$\text{vega} = S_0 N'(d_1)\sqrt{T} = 50 \times 0.055 \times \sqrt{0.25} = 1.375$$

The interpretation for this value is that for a 1% increase in the volatility of the option (in this example, 12% to 13%), the value of the option will increase by approximately 0.01 × 1.375 = 0.01375.

Options are most sensitive to changes in volatility when they are at-the-money. Deep out-of-the-money or deep in-the-money options have little sensitivity to changes in volatility (i.e., vega is close to zero). The diagram in Figure 5 illustrates this behavior.

Figure 5: Vega of a Stock Option



Rho

Rho, ρ, measures an option's sensitivity to changes in the risk-free rate. Keep in mind, however, that equity options are not as sensitive to changes in interest rates as they are to changes in the other variables (e.g., volatility and stock price). Large changes in rates have only small effects on equity option prices. Rho is a much more important risk factor for fixed-income derivatives.

Rho for a call option is calculated using the following equation:

$$\text{rho} = \frac{\partial c}{\partial r}$$

where:
∂c = change in the call price
∂r = change in interest rate

In-the-money calls and puts are more sensitive to changes in rates than out-of-the-money options. Increases in rates cause larger *increases* for in-the-money call prices (versus out-of-the-money calls) and larger *decreases* for in-the-money puts (versus out-of-the-money puts).

For European calls on a non-dividend-paying stock, rho is measured as:

$$\text{rho(call)} = XTe^{-rT}N(d_2)$$

For European puts, rho is:

$$\text{rho(put)} = -XTe^{-rT}N(-d_2)$$

Example: Computing rho

Suppose that stock XYZ is trading at $50, and there is a call option that trades on XYZ with an exercise price of $45 which expires in three months. The risk-free rate is 5%, and the standard deviation of returns is 12% annualized. **Determine** the value of the call option's rho. Assume d_2 is 1.93 and $N(d_2)$ is 0.9732.

Answer:

$$\text{rho(call)} = 45 \times 0.25 \times e^{-0.05 \times 0.25} \times 0.9732 = 10.81$$

Similar to the interpretation of vega, a 1% increase in the risk-free rate (from 5% to 6%) will increase the value of the call option by approximately 0.01 × 10.81 = 0.1081.

Hedging in Practice

LO 57.10: Describe how hedging activities take place in practice, and describe how scenario analysis can be used to formulate expected gains and losses with option positions.

One of the main problems facing options traders is the expense associated with trying to maintain positions that are neutral to the Greeks. Although delta-neutral positions can be created, it is not as easy to find securities at reasonable prices that can mitigate the negative effects associated with gamma and vega.

To make things somewhat more manageable, large financial institutions usually adjust to a delta-neutral position and then monitor exposure to the other Greeks. Two offsetting situations assist in this monitoring activity. First, institutions that have sold options to their clients are exposed to negative gamma and vega, which tend to become more negative as time passes. In contrast, when the options are initially sold at-the-money, the level of sensitivity to gamma and vega is highest, but as time passes, the options tend to go either in-the-money or out-of-the-money. The farther in- or out-of-the-money an option becomes, the less the impact of gamma and vega on the delta-neutral position.

Scenario analysis involves calculating expected portfolio gains or losses over desired periods using different inputs for underlying asset price and volatility. In this way, traders can assess the impact of changing various factors individually, or simultaneously, on their overall position.

Portfolio Insurance

LO 57.11: Describe how portfolio insurance can be created through option instruments and stock index futures.

Portfolio insurance is the combination of (1) an underlying instrument and (2) either cash or a derivative that generates a floor value for the portfolio in the event that market values decline, while still allowing for upside potential in the event that market values rise.

The simplest way to create portfolio insurance is to buy put options on an underlying portfolio. In this case, any loss on the portfolio may be offset with gains on the long put position.

Simply buying puts on the underlying portfolio may not be feasible because the put options needed to generate the desired level of portfolio insurance may not be available. As an alternative to buying the puts, a synthetic put position can be created with index futures contracts. This is accomplished by selling index futures contracts in an amount equivalent to the proportion of the portfolio dictated by the delta of the required put option. The main reasons traders may prefer synthetically creating the portfolio insurance position with index futures include substantially lower trading costs and relatively higher levels of liquidity.

Key Concepts

LO 57.1

A naked call option is written without owning the underlying asset, whereas a covered call is a short call option where the writer owns the underlying asset. Neither of these positions is a hedged position.

LO 57.2

Stop-loss trading strategies are designed to minimize losses in the event the price of the underlying exceeds the strike price of a short call-option position.

LO 57.3

To completely hedge a long stock or short call position, an investor must purchase the number of shares of stock equal to delta times the number of options sold. Another term for being completely hedged is delta-neutral.

A forward/futures contract position can easily be hedged with an offsetting underlying asset position with the same number of securities.

LO 57.4

The delta of an option, Δ, is the ratio of the change in price of the call option, c, to the change in price of the underlying asset, s, for small changes in s.

LO 57.5

Delta-neutral hedges are sophisticated hedging methods that minimize changes in a portfolio's position due to changes in the underlying security.

Delta-neutral hedges are only appropriate for small changes in the underlying asset and need to be rebalanced when large changes in the asset's value occur.

LO 57.6

The delta of a portfolio is a weighted average of the deltas of each portfolio position.

LO 57.7

Theta, also referred to as the time decay of an option, measures the sensitivity of an option's price to decreases in time to expiration.

Gamma measures the sensitivity of an option's price to changes in the option's delta.

Vega measures the sensitivity of an option's price to changes in the underlying asset's volatility.

Rho measures the sensitivity of an option's price to changes in the level of interest rates.

LO 57.8

Gamma is used to correct the hedging error associated with delta-neutral positions by providing added protection against large movements in the underlying asset's price.

Gamma-neutral positions are created by matching the gamma of the portfolio with an offsetting option position.

LO 57.9

Theta, delta, and gamma directly affect the rate of return of an option portfolio.

Stock option prices are affected by delta, theta, and gamma as indicated in the following relationship:

$$r\Pi = \Theta + rS\Delta + 0.5\sigma^2 S^2 \Gamma$$

where:
r = the risk-neutral risk-free rate of interest
Π = the price of the option
Θ = the option theta
S = the price of the underlying stock
Δ = the option delta
σ^2 = the variance of the underlying stock
Γ = the option gamma

LO 57.10

Hedging usually entails actively managing a delta-neutral position while monitoring the other option Greek sensitivities.

LO 57.11

Portfolio insurance is the combination of (1) an underlying instrument and (2) either cash or a derivative that generates a floor value of the portfolio in the event that market valuations decline, while allowing for upside potential in the event that market valuations rise.

Concept Checkers

1. Which of the following choices will effectively hedge a short call option position that exhibits a delta of 0.5?
   A. Sell two shares of the underlying for each option sold.
   B. Buy two shares of the underlying for each option sold.
   C. Sell the number of shares of the underlying equal to one-half the options sold.
   D. Buy the number of shares of the underlying equal to one-half the options sold.

2. A delta-neutral position exhibits a gamma of –3,200. An existing option with a delta equal to 0.5 exhibits a gamma of 1.5. Which of the following will generate a gamma-neutral position for the existing portfolio?
   A. Buy 4,800 of the available options.
   B. Sell 4,800 of the available options.
   C. Buy 2,133 of the available options.
   D. Sell 2,133 of the available options.

3. Which of the following actions would have to be taken to restore a delta-neutral hedge to the gamma-neutral position created in Question 2?
   A. Buy 1,067 shares of the underlying stock.
   B. Sell 1,067 shares of the underlying stock.
   C. Buy 4,266 shares of the underlying stock.
   D. Sell 4,266 shares of the underlying stock.

4. Portfolio insurance payoffs would not involve which of the following?
   A. Selling call options in the proportion 1/delta.
   B. Buying put options one-to-one relative to the underlying.
   C. Buying and selling the underlying in the proportion of delta of a put.
   D. Buying and selling futures in the proportion of delta of a put.

5. Which of the following statements about the "Greeks" is true?
   A. Rho for fixed income options is small.
   B. Call option deltas range from –1 to +1.
   C. A vega of 10 suggests that for a 1% increase in volatility, the option price will increase by 0.10.
   D. Theta is the most negative for out-of-the-money options.

Concept Checker Answers

1. **D** In order to hedge a short call option position, a manager would have to buy enough of the underlying to equal the delta times the number of options sold. In this case, delta = 0.5, so for every two options sold, the manager would have to buy a share of the underlying security.

2. **C** To create a gamma-neutral position, a manager must add the appropriate number of options that equals the existing portfolio gamma position. In this case, the existing gamma position is –3,200, and an available option exhibits a gamma of 1.5, which translates into buying approximately 2,133 options (3,200 / 1.5).

3. **B** The gamma-neutral hedge requires the purchase of 2,133 options, which will then increase the delta of the portfolio to 1,067 (2,133 × 0.5). Therefore, this would require selling approximately 1,067 shares to maintain a delta-neutral position.

4. **A** Portfolio insurance can be created by all of the statements except selling call options in the proportion 1/delta. This action generates a delta-neutral hedge, not portfolio insurance.

5. **C** Theta is the most negative for at-the-money options. Call option deltas range from 0 to 1. A vega of 10 suggests that for a 1% increase in volatility, the option price will increase by 0.10. Rho for equity options is small.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Prices, Discount Factors, and Arbitrage

Topic 58

Exam Focus

This topic provides an overview of the fundamentals of bond valuation. The value of a bond is simply the present value of its cash flows discounted at the appropriate periodic required return. Discount factors are used for pricing coupon bonds and for determining whether bonds are trading cheap or rich. If a mispricing exists among securities, a riskless arbitrage profit can be made from the violation of the law of one price.

Fundamentals of Bond Valuation

The general procedure for valuing fixed-income securities (or any security) is to take the present values of all the expected cash flows and add them up to get the value of the security.

There are three steps in the bond valuation process:

Step 1: **Estimate the cash flows** over the life of the security. For a bond, there are two types of cash flows: (1) the coupon payments and (2) the return of principal.

Step 2: **Determine the appropriate discount rate** based on the risk of (uncertainty about) the receipt of the estimated cash flows.

Step 3: **Calculate the present value of the estimated cash flows** by multiplying the bond's expected cash flows by the appropriate discount factors.

For a Treasury bond, the appropriate rate used to value the promised cash flows is the risk-free rate. This may be a single rate, used to discount all of the cash flows, or a series of discount rates that correspond to the times until each cash flow arrives.

For non-Treasury securities, we must add a risk premium to the risk-free (Treasury) rate to determine the appropriate discount rate. This risk premium is the added yield to compensate for greater risk (credit risk, liquidity risk, call risk, prepayment risk, and so on). When using a single discount rate to value bonds, the risk premium is added to the risk-free rate to get the appropriate discount rate for all of the expected cash flows.

Calculating the Value of a Coupon Bond

Valuation with a single yield (discount rate). For an option-free coupon bond, the coupon payments can be valued as an annuity. In order to take into account the payment of the par value at maturity, we will enter this final payment as the future value. This is the basic difference between valuing a coupon bond and valuing an annuity.

For simplicity, consider a security that will pay $100 per year for ten years and make a single $1,000 payment at maturity (in ten years). If the appropriate discount rate is 8% for all the cash flows, the value is:

$$\frac{100}{1.08}+\frac{100}{1.08^2}+\frac{100}{1.08^3}+\frac{100}{1.08^4}+\ldots+\frac{100}{1.08^{10}}+\frac{1{,}000}{1.08^{10}}$$

$$= \$1{,}134.20 = \text{present value of expected cash flows}$$

This is simply the sum of the present values of the future cash flows, $100 per year for ten years and $1,000 (the principal repayment) to be received at the end of the tenth year, at the same time as the final coupon payment.

The calculator solution is:

N = 10; PMT = 100; FV = 1,000; I/Y = 8; CPT → PV = –$1,134.20

where:
N = number of years
PMT = the *annual* coupon payment
I/Y = the *annual* discount rate
FV = the par value or selling price at the end of an assumed holding period

Professor's Note: Take note of a couple of points here. The discount rate is entered as a whole number in percent, 8, not 0.08. The ten coupon payments of $100 each are taken care of in the N = 10 entry, the principal repayment is in the FV = 1,000 entry. Lastly, note that the PV is negative; it will be the opposite sign to the sign of PMT and FV. The calculator is just "thinking" that if you receive the payments and future value (you own the bond), you must pay the present value of the bond today (you must buy the bond). That's why the PV amount is negative; it is a cash outflow to a bond buyer. Just make sure that you give the payments and future value the same sign, and then you can ignore the sign on the answer (PV).

Valuation with a single yield and semiannual cash flows. Let's calculate the value of the same bond with semiannual payments.

Rather than $100 per year, the security will pay $50 every six months. Adjust the discount rate of 8% per year to 4% per six months. The par value remains $1,000.

The calculator solution is:

N = 20; PMT = 50; FV = 1,000; I/Y = 4; CPT PV = –1,135.90

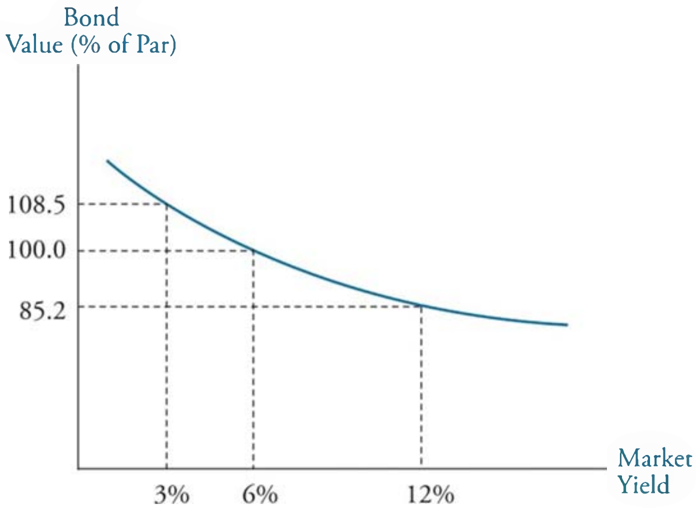
where:
N = number of semiannual periods
PMT = the *semiannual* coupon payment
I/Y = the *semiannual* discount rate
FV = the par value

Price-Yield Curve

Bond values and bond yields are inversely related. An *increase* in the discount rate will *decrease* the present value of a bond's expected cash flows; a decrease in the discount rate will increase the present value of a bond's expected cash flows. The change in bond value in response to a change in the discount rate can be calculated as the difference between the present values of the cash flows at the two different discount rates.

If you plot a bond's yield to its corresponding value, you'll get a graph like the one shown in Figure 1. Here we see that higher prices are associated with lower yields. This graph is called the *price-yield curve*. Note that it is not a straight line but is curved. For option-free bonds, the price-yield curve is convex toward the origin, meaning it looks like half of a smile.

Figure 1: The Price-Yield Profile



Bond Price Quotations

Bonds are quoted on a percentage basis relative to a par value (100). Bonds priced at par are quoted at 100; bonds sold at a discount are priced at less than 100; and bonds sold at a premium are priced at greater than 100. U.S. Treasury notes and bonds use a "32nds" convention. A bond quoted as 97-6 (or 97:06 or 97.6) is interpreted as $97\frac{6}{32}\%$ of par value. If the par amount is $1 million, the price of the bond is 97.1875% of the par amount, or $971,875.00. Corporate and municipal bonds are quoted in eighths (e.g., $102\frac{1}{8}$ indicates the price of the bond is 102.125% of par). In either case, convert a price quotation into a dollar price simply by adding the decimal equivalent of the fraction to the base number of the quote and using that as a percentage of the par amount to compute the price.

A "+" in the quote indicates a half tick. For example, if the price is quoted as 101-12+, then the bond would sell at $101 + \frac{12.5}{32}$.

Discount Factors

LO 58.1: Define discount factor and use a discount function to compute present and future values.

Discount factors are used to determine present values. The discount function is expressed as *d(t)*, where *t* denotes time in years.

Example: Calculating bond value using discount factors

Suppose that the discount factor for the first 180-day coupon period is as follows:

$$d(0.5) = 0.92432$$

Calculate the price of a bond that pays $108 six months from today.

Answer:

Since $1 to be received in six months is worth $0.92432 today, $108 received in six months is worth 0.92432 × $108 = $99.83 today.

Professor's Note: The future value of $1 invested for time t is 1/d(t).

Bonds are securities that promise a future stream of cash flows, so a series of Treasury bond (T-bond) prices can be used to generate the discount function.

Example: Calculating discount factors given bond prices

Figure 2 shows selected T-bond prices for semiannual coupon $100 face value bonds.

Figure 2: Selected Treasury Bond Prices

Prices are from 5/14/06, with t + 1 settlement			
Bond	*Coupon*	*Maturity*	*Price*
1	4.25%	11/15/06	101-16
2	7.25%	5/15/07	105-31+
3	2.00%	11/15/07	101-07
4	12.00%	5/15/08	120-30
5	5.75%	11/15/08	110-13+

Generate the discount factors for the dates indicated.

Answer:

Bond 1:

When this bond matures on 11/15/06, it makes its last interest payment of $2.125 = \left(\frac{0.0425}{2} \times \$100\right)$ plus the principal repayment of 100. The present value of the 102.125 is given as the price of 101-16, or 101.50.

$$\text{price(PV)} = \text{CF}(0.5) \times d(0.5)$$

$$101.5 = 102.125\ d(0.5)$$

Solving for the discount function yields:

$$d(0.5) = 0.9939$$

Moving farther out on the curve, the function becomes slightly more complex, as each point of the curve must be included. For example, to solve for Bond 2, we must include $d(0.5)$ as well.

Bond 2:

The coupon payment at time 0.5 is 3.625 = 7.25 / 2. The final cash flow at time t = 1 is 103.625 = 100 + 3.625. Those two cash flows discounted back to 5/14/06 using the discount function should equal the price of the bond:

$$105\text{-}31+ = 105.9844 = 3.625\ d(0.5) + 103.625\ d(1)$$

Since it's already known that d(0.5) = 0.9939, substitute that value into the equation and solve for $d(1)$:

$$105.9844 = (3.625 \times 0.9939) + [103.625\ d(1)]$$

$$d(1) = 0.9880$$

Using the same methodology for Bonds 3, 4, and 5:

Bond 3:

$$101.2188 = 1.0\ d(0.5) + 1.0\ d(1) + 101\ d(1.5)$$

Thus:

$$d(1.5) = 0.9825$$

Bond 4 and Bond 5:

$$d(2.0) = 0.9731$$

$$d(2.5) = 0.9633$$

Figure 3: Discount Factors

Time to Maturity	*Discount Factor*
0.5	0.9939
1.0	0.9880
1.5	0.9825
2.0	0.9731
2.5	0.9633

Determining Value Using Discount Functions

LO 58.2: Define the "law of one price," explain it using an arbitrage argument, and describe how it can be applied to bond pricing.

LO 58.5: Identify arbitrage opportunities for fixed income securities with certain cash flows.

The discount functions previously mentioned can be used to estimate the value of a bond. Since investors do not care about the origin of a cash flow, all else equal, a cash flow from one bond is just as good as a cash flow from another bond. This phenomenon is commonly referred to as the **law of one price**. If investors are able to exploit a mispricing because of the law of one price, it is referred to as an arbitrage opportunity.

Example: Identifying arbitrage opportunities

Suppose you observe the annual coupon bonds shown in Figure 4.

Figure 4: Observed Bond Yields and Prices

Maturity	*YTM*	*Coupon (annual payments)*	*Price (% of par)*
1 year	4%	0%	96.154
2 years	8%	0%	85.734
2 years	8%	8%	100.000

The 2-year spot rate is 8.167%. Is there an arbitrage opportunity? If so, **describe** the trades necessary to exploit the arbitrage opportunity.

Answer:

The answer is yes, an arbitrage profit may be realized because the YTM on the 2-year zero coupon is too low (8% versus 8.167%), which means the bond is trading *rich* (the bond price is too high). To exploit this violation of the law of one price, buy the 2-year, 8% coupon bond, strip the coupons, and short sell them separately. The discount factors are derived from the prices of the zero-coupon bonds.

Figure 5: Discount Factors

Time to Maturity	*Discount Factor*
1.0	0.96154
2.0	0.85734

To demonstrate the process of exploiting the arbitrage opportunity here, consider the following 3-step process (the dollar amounts given are arbitrary):

Step 1: Buy $1 million of the 2-year, 8% coupon bonds because they are trading *cheap*.
Step 2: Short sell $80,000 of the 1-year, zero-coupon bonds at 96.154.
Step 3: Short sell $1.08 million of the 2-year, zero-coupon bonds at 85.734.

Figure 6: Cash Flow Diagram

Time = 0		*1 year*		*2 years*	
–1,000,000.00	(cost of 2-year, 8% coupon bonds)	+80,000	(coupon, interest)	+1,080,000	(coupon, interest)
+76,923.20*	(proceeds 1-year, 0% bonds)	–80,000	(maturity)		
+925,927.20**	(proceeds 2-year, 0% bonds)			–1,080,000	(maturity)
+2,850.40	Net	0		0	

*76,923.20 = 0.96154 × 80,000

**925,927.20 = 0.85734 × 1,080,000

The result is receiving *positive income today* in return for *no future obligation*, which is an *arbitrage opportunity*. The selling of the 2-year STRIPS would force the price down to 85.469 (the price at which the YTM = 8.167%), at which point the arbitrage opportunity would disappear.

Treasury Coupon Bonds and Treasury STRIPS

LO 58.3: Identify the components of a U.S. Treasury coupon bond, and compare and contrast the structure to Treasury STRIPS, including the difference between P-STRIPS and C-STRIPS.

Zero-coupon bonds issued by the Treasury are called **STRIPS** (separate trading of registered interest and principal securities). STRIPS are created by request when a coupon bond is presented to the Treasury. The bond is "stripped" into two components: principal and coupon (**P-STRIPS** and **C-STRIPS**, respectively).

The Treasury can also retire a STRIP by gathering the parts up to **reconstitute**, or remake, the coupon bond. C-STRIPS can be put with any bond to reconstitute, but P-STRIPS are identified with specific bonds—the original bond that it was stripped from. What this means is that the value of a P-STRIP comes from the underlying bond. If the underlying was cheap, the P-STRIP will be cheap. If the underlying was rich, the P-STRIP will also be rich.

STRIPS are of interest to investors because:

- Zero-coupon bonds can be easily used to create any type of cash flow stream and thus match asset cash flows with liability cash flows (e.g., to provide for college expenses, house-purchase down payment, or other liability funding). This mitigates reinvestment risk. (The concept of reinvestment risk will be discussed in later topics.)
- Zero-coupon bonds are more sensitive to interest rate changes than are coupon bonds. This could be an issue for asset-liability management or hedging purposes.

STRIPS do have some disadvantages, which include the following:

- They can be illiquid.
- Shorter-term C-STRIPS tend to trade rich.
- Longer-term C-STRIPS tend to trade cheap.
- P-STRIPS typically trade at fair value.
- Large institutions can potentially profit from STRIP mispricings relative to the underlying bonds. They can do this by either buying Treasuries and stripping them or reconstituting STRIPS. Because of the cost involved with stripping/reconstituting, investors generally pay a premium for zero-coupon bonds.

Constructing a Replicating Portfolio

LO 58.4: Construct a replicating portfolio using multiple fixed income securities to match the cash flows of a given fixed income security.

Suppose a 2-year fixed income security exists with \$100 face value and a 10% coupon rate. The coupons are paid on a semiannual basis, and the security's YTM is assumed to be 4.5%. The present value of this bond, Bond 1, and its cash flows are calculated as follows:

$$PV_{B1} = \frac{5}{1.0225^1} + \frac{5}{1.0225^2} + \frac{5}{1.0225^3} + \frac{105}{1.0225^4} = \$110.41$$

or:

$$N = 2 \times 2 = 4;\ I/Y = 4.5/2 = 2.25\%;\ FV = 100;\ PMT = 10/2 = 5;\ CPT \Rightarrow PV = \$110.41$$

If this bond is determined to be trading "cheap," then a trader can conduct an arbitrage trade by purchasing the undervalued bond and shorting a replicating portfolio that mimics the bond's cash flows. To demonstrate the creation of a replicating portfolio, assume the following four fixed income securities exist in addition to the bond we are trying to replicate.

Bond	*Coupon*	*PV*	*FV*	*Time Horizon*
2	7%	\$101.22	\$100	6 months (0.5 years)
3	12%	\$107.25	\$100	12 months (1 year)
4	5%	\$100.72	\$100	18 months (1.5 years)
5	6%	\$102.84	\$100	24 months (2 years)

To create a replicating portfolio using multiple fixed-income securities, we must determine the face amounts of each bond to purchase, F_i, which match Bond 1 cash flows in each semiannual period.

$$\text{Bond 1 } CF_t = F_2 \times \frac{7\%}{2} + F_3 \times \frac{12\%}{2} + F_4 \times \frac{5\%}{2} + F_5 \times \frac{6\%}{2}$$

When doing this calculation by hand, it is easiest to start from the end—with the bond that matches Bond 1's time horizon. In this case, that security is Bond 5. Since the other bonds do not make payments in 24 months, they are not considered in this first step (i.e., their face amounts are multiplied by zero).

$$\$105 = F_2 \times 0 + F_3 \times 0 + F_4 \times 0 + F_5 \times \left(100 + \frac{6}{2}\right)\%$$

Solving this equation for F_5 yields the face amount percentage we need to purchase of Bond 5 (F_5 = 101.94). Since the coupon rate on Bond 5 is lower than that of Bond 1, it makes sense that we'll need to purchase more of Bond 5 (101.94%) than the \$100 face value of Bond 1. We can now use the value of F_5 to solve for F_4.

$$\$5 = F_2 \times 0 + F_3 \times 0 + F_4 \times \left(100 + \frac{5}{2}\right)\% + 101.94 \times \frac{6\%}{2}$$

The remaining unknowns (F_2 and F_3) are solved in a similar fashion. The replicating portfolio can now be purchased (or sold for the arbitrage trade) using the below face amount percentages. Notice, in the last two rows of the following table, how the total cash flows from these four bonds exactly matches the cash flows from Bond 1.

The cash flows from the replicating portfolio are computed by multiplying each bond's initial cash flows by face amount percentage. For example, regarding Bond 5, the 2-year cash flow is computed as $103 × 1.0194 = $105, and the 1-year cash flow is computed as $3 × 1.0194 = $3.0582.

	Coupon	*Face Amount*	*CF (t = 0.5)*	*CF (t = 1)*	*CF (t = 1.5)*	*CF (t = 2)*
Bond 2	7%	1.73%	1.7906			
Bond 3	12%	1.79%	0.1074	1.8974		
Bond 4	5%	1.89%	0.0473	0.0473	1.9373	
Bond 5	6%	101.94%	3.0582	3.0582	3.0582	105
Total CFs			5	5	5	105
Bond 1 CFs			5	5	5	105

Computing Price Between Coupon Dates

LO 58.6: Differentiate between "clean" and "dirty" bond pricing and explain the implications of accrued interest with respect to bond pricing.

LO 58.7: Describe the common day-count conventions used in bond pricing.

All of our computations to this point have assumed that the number of remaining periods in the bond's life is an integer. In other words, we have assumed that the bond's valuation took place on a coupon date. Frequently, bonds are not purchased on a coupon date, and we must deal with fractional periods in the valuation process. We must account for three items in this situation: accrued interest, fractional period compounding, and the day-count convention of the bond.

Accrued Interest

When a bond is purchased, the buyer must pay the owner for any interest earned up through the settlement date. This is called accrued interest (AI) and is computed as:

$$AI = c\left(\frac{\text{number of days from last coupon to the settlement date}}{\text{number of days in coupon period}}\right)$$

where:

c = coupon payment

Example: Computing accrued interest

A $1,000 par value U.S. corporate bond pays a semiannual 10% coupon. Assume the last coupon was paid 90 days ago and there are 30 days in each month. **Compute** the accrued interest.

Answer:

Accrued interest is computed as follows:

$$AI = \$50\left(\frac{90}{180}\right) = \$25$$

Day-Count Convention

Several day-count conventions are used in practice in the bond markets:

- Actual/actual (in period).
- Actual/365.
- Actual/365 (366 in leap year).
- Actual/360.
- 30/360.
- 30E/360 (E is for Europe).

The day count used will depend on the type of security. For example, U.S. government bonds pay coupons semiannually and have an actual/actual day count. U.S. corporate and municipal bonds pay semiannual interest with a 30/360 day count. U.S. government agencies pay annually, semiannually, and quarterly coupons (depending on the type of bond) with a 30/360 day count.

Clean and Dirty Bond Pricing

We need to modify the pricing formula to incorporate the appropriate day count convention. Specifically, the bond pricing equation becomes:

$$P = \frac{C}{(1+y)^{w}} + \frac{C}{(1+y)^{1+w}} + \frac{C}{(1+y)^{2+w}} + \ldots + \frac{C}{(1+y)^{n-1+w}} + \frac{M}{(1+y)^{n-1+w}}$$

where:
P = price
C = semiannual coupon
y = periodic required yield
n = number of periods remaining
M = par value of the bond
w = the number of days until the next coupon payment divided by the number of days in the coupon period.

When expressing *w* in the preceding equation, the number of days to use for the coupon period is determined by the appropriate day count convention. For example, the denominator is 180 for semiannual bonds that use the 30/360 convention. This equation computes the *dirty price* of the bond, since it includes the discounted value of the first full coupon payment even though the accrued interest belongs to the seller of the bond.

Example: Computing the dirty price of a bond

Suppose the bond from the previous example (10% coupon bond with $1,000 par value) is a U.S. corporate bond that pays coupons semiannually on January 1 and July 1. Assume that it is now April 1, 2015, and the bond matures on July 1, 2025. **Compute** the dirty price of this bond if the required annual yield is 8%.

Answer:

This is a U.S. corporate bond so it uses the 30/360 day count convention. Under this convention the number of days between the settlement date (April 1, 2015) and the next coupon payment (July 1, 2015) is 90 days (= 3 months at 30 days per month). That means $w = 90/180 = 0.5$. Since $n = 21$, $n - 1 + w = 20.5$.

Applying the new equation:

$$P = 50/(1 + 0.04)^{0.5} + 50/(1+0.04)^{1.5} + ... + 50/(1+0.04)^{20.5} + 1{,}000/(1+0.04)^{20.5} = \$1{,}162.87$$

We can use the time-value calculator to solve this long equation directly in two steps. First, we compute the value of the bond immediately after the January 2015 coupon payment, when it was a 21-period bond:

$$N = 21;\ PMT = 50;\ I/Y = 4;\ FV = 1{,}000;\ CPT \rightarrow PV = 1{,}140.29$$

Then, 90 days later, on April 1, 2015, the dirty price of the bond is:

$$\$1{,}140.29 \times (1.04)^{0.5} = \$1{,}162.87$$

The dirty price is the price that the seller of the bond must be paid to give up ownership. It includes the present value of the bond plus the accrued interest. The clean price is the dirty price less accrued interest:

clean price = dirty price – accrued interest

Example: Computing the clean price of a bond

Compute the clean price of the U.S. corporate bond in the previous two examples.

Answer:

clean price = \$1,162.87 – \$25.00 = \$1,137.87

Note that the dirty price includes the discounted value of the next coupon so that the method of calculating accrued interest does not matter. As long as the clean price is calculated as: dirty price – accrued interest, the sum of the clean price and accrued interest will equal the dirty price.

Professor's Note: The dirty price of the bond is sometimes referred to as the full price or invoice price. The clean price of the bond is sometimes referred to as the flat price or quoted price.

Key Concepts

LO 58.1

The cash flows of a coupon bond consist of periodic coupon payments and a par value payment at maturity.

The price of a bond consists of an ordinary annuity portion (the coupons) and a lump sum single cash flow (the par amount).

Bond prices can be generated from discount functions. Prices are calculated by summing the product of each cash flow and its applicable discount rate.

LO 58.2

Since investors do not care about the origin of a cash flow, all else equal, a cash flow from one bond is just as good as a cash flow from another bond. This phenomenon is commonly referred to as the law of one price.

LO 58.3

Treasury STRIPS can be used to create specific fixed-income cash flow streams. P-STRIPS typically trade at fair value, while longer-term C-STRIPS tend to trade cheap, and shorter-term C-STRIPS tend to trade rich.

LO 58.4

To create a replicating portfolio using multiple fixed-income securities, we must determine the face amounts of each fixed-income security to purchase, which match the cash flows from the bond we are trying to replicate.

LO 58.5

An arbitrage profit can be made if violations of the law of one price exist. By short selling a "rich" security and using the proceeds to purchase a similar "cheap" security, an investor can make a riskless profit with no investment.

LO 58.6

Valuation of bonds between coupon payment dates requires the calculation of accrued interest and a modification to the bond pricing formula. Values derived between coupon dates will include accrued interest. This is also known as the dirty price. Subtracting the accrued interest from the dirty price gives the clean price of the bond.

LO 58.7

Accrued interest calculations vary across classes of bonds because of differing day count conventions. The most common day count conventions are actual/actual and 30E/360.

Concept Checkers

Use the following information for Questions 1 and 2.

Maturity	*Coupon*	*Price*
6 months	5.5%	101.3423
1 year	14.0%	102.1013
2 years	8.5%	99.8740

1. Which of the following is the closest to the discount factor for *d*(0.5)?
   A. 0.8923.
   B. 0.9304.
   C. 0.9525.
   D. 0.9863.

2. Which of the following is the closest to the discount factor for *d*(1.0)?
   A. 0.8897.
   B. 0.9394.
   C. 0.9525.
   D. 0.9746.

3. A Treasury bond is quoted at a price of 106-17+. The price of the bond as a percent of par is closest to:
   A. 106.1700%.
   B. 106.1750%.
   C. 106.5313%.
   D. 106.5469%.

4. P-STRIPS typically:
   A. trade rich.
   B. trade cheap.
   C. trade at fair value.
   D. Not enough information.

5. Which of the following statements about STRIPS is correct? STRIPS:
   I. have less interest rate sensitivity than coupon bonds.
   II. tend to be highly liquid.

   A. I only.
   B. II only.
   C. Both I and II.
   D. Neither I nor II.

Concept Checker Answers

1. **D** $101.3423 = 102.75\ d(0.5)$

   $d(0.5) = 0.9863$

2. **A** $102.1013 = 7\ d(0.5) + 107\ d(1)$

   $102.1013 = 7(0.9863) + 107\ d(1)$

   $95.1972 = 107\ d(1)$

   $d(1) = 0.8897$

3. **D** The price of the bond is $106\ ^{17.5}/_{32}\%$ of par, or 106.5469%.

4. **C** P-STRIPS usually trade at fair value. This means that the cheapness or richness of the underlying bond passes on to the P-STRIP.

5. **D** STRIPS can be relatively illiquid and have more interest rate sensitivity than coupon bonds. Because of the cost to strip/reconstitute, only large institutional investors can potentially profit from doing so. STRIPS are often used with hedging strategies for asset-liability management such as matching maturity dates with a liability stream.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Spot, Forward, and Par Rates

Topic 59

Exam Focus

Any bond can be partitioned into a series of periodic cash flows. If we compute the present value of each cash flow, viewed as STRIPS, and add them up, we arrive at the value of the bond. In other words, a bond is really a package of STRIPS. Using this framework enables us to relate the yield curve directly to the spot curve. The spot curve may then be manipulated to compute a forward curve that represents interest rates between future periods implied by the current spot curve. In either case, STRIPS or discount factors can be used to calculate prices.

Annual Compounding vs. Semiannual Compounding

LO 59.1: Calculate and interpret the impact of different compounding frequencies on a bond's value.

Most financial institutions pay and charge interest over much shorter periods than annually. For instance, if an account pays interest every six months, we say interest is compounded semiannually. Every three months represents quarterly compounding, and every month is monthly compounding.

Use the following formula to find the future value of a bond using different compounding methods:

$$FV_n = PV_0 \times \left[1 + \frac{r}{m}\right]^{m \times n}$$

where:
r = annual rate
m = number of compounding periods per year
n = number of years

Assume \$100 was invested for four years earning 10% compounded semiannually. After four years the future value would be:

$$FV_n = 100 \times \left[1 + \frac{0.10}{2}\right]^{2\times 4} = \$147.75$$

Assuming annual compounding the future value would be:

$$FV_n = 100 \times \left[1 + \frac{0.10}{1}\right]^{1\times4} = \$146.41$$

The additional \$1.34 (= 147.75 – 146.41) is the extra interest earned from the compounding effect of interest on interest. Although the differences do not seem profound, the effects of compounding are magnified with larger values, greater number of compounding periods per year, and/or higher nominal interest rates.

Holding Period Return

We can rearrange the previous future value of a bond calculation and solve for the rate of return, *r* (i.e., the holding period return). The rate of return on a bond is as follows:

$$r = m\left[\left(\frac{FV_n}{PV_0}\right)^{\frac{1}{m\times n}} - 1\right]$$

Assume \$100 was initially invested and grew to \$147.75 after four years. Using semiannual compounding yields the following rate of return:

$$r = 2\left[\left(\frac{\$147.75}{\$100}\right)^{\frac{1}{2\times4}} - 1\right] = 10\%$$

Professor's Note: Recall from the Time Value of Money reading in Book 2 that this type of problem can be easily solved with a financial calculator. N = 4 × 2; FV = 147.75; PV = –100; CPT I/Y = 5% × 2 = 10%.

Deriving Discount Factors from Swap Rates

LO 59.2: Calculate discount factors given interest rate swap rates.

In the previous topic, we generated discount factors given a series of bond prices. In a similar fashion, we can also derive discount factors given a series of interest rate swap rates. Recall from Book 3 that in an interest rate swap, payments are exchanged based on a notional amount. Although this notional amount is never technically exchanged between counterparties in an interest rate swap, it is used to determine the size of both the fixed and floating leg payments.

If we were to hypothetically exchange the notional amount at swap maturity, it would be easy to see similarities between the fixed leg of a swap and a fixed coupon paying bond, with the fixed leg payments acting like semiannual, fixed coupon payments and the notional amount acting like the bond principal payment at maturity (i.e., its terminal value). Similarly, if the notional amount was exchanged at swap maturity, the floating leg of the swap would resemble a floating rate bond, with semiannual, floating coupon payments and a bond principal payment at maturity.

In an interest rate swap, the fixed receiver (floating payer) "buys" the fixed leg, and the fixed payer (floating receiver) "sells" the fixed leg. Thus, we use fixed swap rates to derive discount factors. For this calculation, swap rates represent bond coupon payments and the swap notional amount represents the bond's par value.

Example: Computing discount rates from swap rates

Given the following swap rates, **compute** the discount factors for maturities ranging from six months to two years assuming a notional swap amount of $100.

Figure 1: Swap Rates

Maturity (Years)	*Swap Rates*
0.5	0.65%
1.0	0.80%
1.5	1.02%
2.0	1.16%

Answer:

The six-month discount rate is computed as:

$$\left(100+\frac{0.65}{2}\right)d(0.5)=100$$

$$d(0.5) = 0.9968$$

The 1-year discount rate, given the six-month discount rate, is then computed as:

$$\left(\frac{0.8}{2}\right)d(0.5)+\left(100+\frac{0.8}{2}\right)d(1.0)=100$$

$$\left(\frac{0.8}{2}\right)(0.9968)+\left(100+\frac{0.8}{2}\right)d(1.0)=100$$

$$d(1.0) = 0.9920$$

Figure 2 shows all discount factors for maturities ranging from six months to two years.

Figure 2: Discount Factors

Maturity (Years)	*Discount Factor*
0.5	0.9968
1.0	0.9920
1.5	0.9848
2.0	0.9771

The Spot Rate Curve

LO 59.3: Compute spot rates given discount factors.

A *t*-period **spot rate**, denoted as *z(t)*, is the yield to maturity on a zero-coupon bond that matures in *t*-years (assuming semiannual compounding). The **spot rate curve** is the graph of the relationship between spot rates and maturity. The spot rate curve can be derived from either a series of STRIPS prices, or the comparable discount factors.

Example: Computing a spot rate

The price of a 6-month $100 par value STRIP is 99.2556. **Calculate** the 6-month annualized spot rate.

Answer:

You can use the Texas Instruments BAII Plus® to solve this problem. Here are the keystrokes:

$$N = 1;\ PV = -99.2556;\ PMT = 0;\ FV = 100;\ CPT \rightarrow I/Y = 0.75\%$$

$$z(0.5) = 0.75\% \times 2 = 1.50\%$$

Recall from the previous topic that the *t*-period *discount factor* is the present value today of $1 to be received at the end of period *t*. For semiannual coupon bonds, the *t*-year discount factor is related to the *t*-year spot rate as follows:

$$z(t) = 2\left[\left(\frac{1}{d(t)}\right)^{1/2t} - 1\right]$$

Notice that the 6-month discount factor (0.992556) is just the price of the 6-month STRIP (99.2556) expressed in decimal form. This means that either spot rates or discount factors can be used to price coupon bonds.

Example: Computing spot rates from STRIP prices

Given the STRIPS prices in Figure 3, **compute** the discount factors and spot rates for maturities ranging from six months to three years, and **graph** the spot rate curve.

Figure 3: STRIPS Prices and Discount Factors

Maturity (Years)	*STRIPS Price*	*Discount Factor*
0.5	99.2556	0.992556
1.0	97.8842	0.978842
1.5	96.2990	0.962990
2.0	94.3299	0.943299
2.5	92.1205	0.921205
3.0	89.7961	0.897961

Answer:

Consider the calculations for the 2.5-year maturity. In this case:

$$N = 5;\ PV = -92.1205;\ PMT = 0;\ FV = 100;\ CPT \rightarrow I/Y = 1.655\%$$

$$z(2.5) = 1.655\% \times 2 = 3.31\% \ \text{ or } \ 2\left[\left(\frac{100}{92.1205}\right)^{1/5} - 1\right] = 3.31\%$$

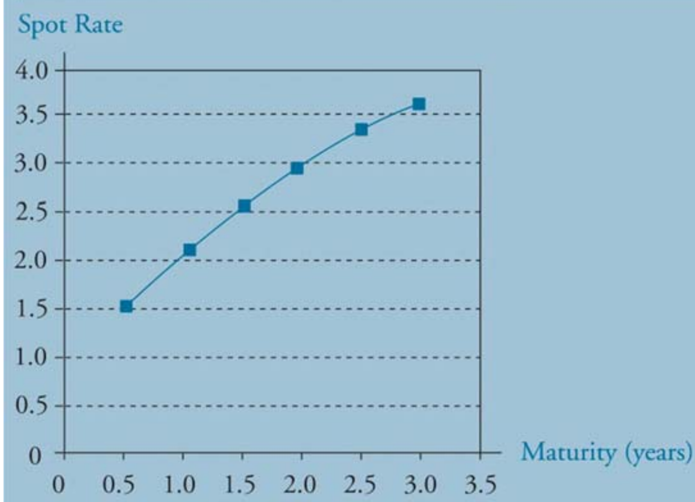
The spot rates for each maturity are shown in Figure 4.

Figure 4: Spot Rates

Maturity (Years)	*Spot Rate*
0.5	1.50%
1.0	2.15%
1.5	2.53%
2.0	2.94%
2.5	3.31%
3.0	3.62%

The resulting spot rate curve is shown in Figure 5.

Figure 5: Spot Rate Curve



Forward Rates

LO 59.4: Interpret the forward rate, and compute forward rates given spot rates.

Forward rates are interest rates that span future periods. Given the spot rates as in Figures 4 and 5, it is possible to compute forward rates implied by that spot curve. The spot rates in Figures 4 and 5 are the appropriate rates that an investor should expect to realize for various periods for a risk-free investment starting today. Should the investor be concerned whether the investment is composed of a single instrument or a series of shorter investments rolled over consecutively? No, because if the risk is the same, the realized return must be the same, regardless of how the investment is packaged. This concept is at the core of forward rate analysis.

For example, suppose an investor is faced with the following two strategies, based on the spot curve in Figures 4 and 5:

1. Buy a 1-year STRIP yielding 2.15%.

2. Buy a 6-month (0.5-year) STRIP yielding 1.50% and then roll that into another 6-month STRIP in six months at the 6-month forward rate.

The investor will be indifferent about which investment to use if both offer the same return at the end of one year. The spot curve can be used to compute what the forward rate must be for an investor to be indifferent between the two strategies. This process is called **bootstrapping**.

Example: Computing a forward rate

Compute the 6-month forward rate in six months, given the following spot rates:

$z(0.5) = 1.50\%$

$z(1.0) = 2.15\%$

Answer:

In order for strategies 1 and 2 to realize the same return, the 6-month forward rate, f(1.0), on an investment that matures in one year must solve the following equation:

$$\left(1+\frac{0.0215}{2}\right)^2 = \left(1+\frac{0.0150}{2}\right)^1 \times \left(1+\frac{f(1.0)}{2}\right)^1$$

$$\Rightarrow f(1.0) = 0.028 = 2.80\%$$

Example: Computing a forward rate

Compute the 6-month forward rate in one year, given the following spot rates:

$z(1.0) = 2.15\%$

$z(1.5) = 2.53\%$

Answer:

The 6-month forward rate, f(1.5), on an investment that matures in 1.5 years must solve the following equation:

$$\left(1+\frac{0.0253}{2}\right)^3 = \left(1+\frac{0.0215}{2}\right)^2 \times \left(1+\frac{f(1.5)}{2}\right)^1$$

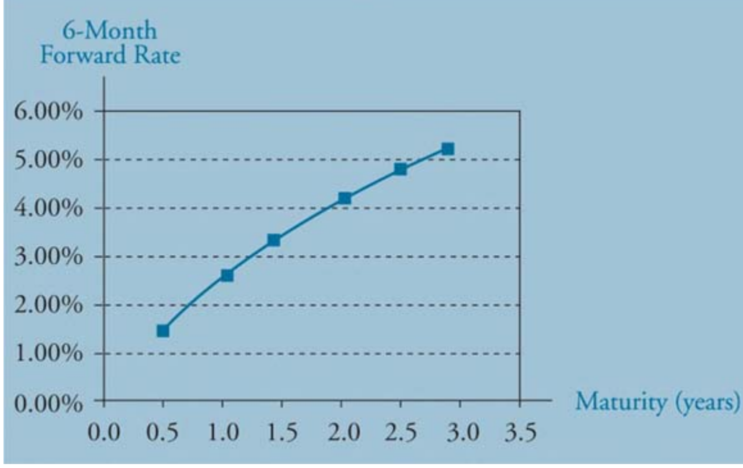
$$\Rightarrow f(1.5) = 3.29\%$$

The remaining 6-month forward rates are shown in Figure 6, and the forward rate curve is shown in Figure 7.

Figure 6: Spot Rates and Forward Rates

Maturity (Years)	*Spot Rate*	*6-Month Forward Rate*
0.5	1.50%	1.50%
1.0	2.15%	2.80%
1.5	2.53%	3.29%
2.0	2.94%	4.18%
2.5	3.31%	4.80%
3.0	3.62%	5.18%

Figure 7: Forward Rate Curve



Par Rates

LO 59.5: Define par rate and describe the equation for the par rate of a bond.

The **par rate** at maturity is the rate at which the present value of a bond equals its par value. By assuming a 2-year bond pays semiannual coupons and has a par value of $100, the 2-year par rate can be computed by incorporating bond discount factors from each semiannual period as follows:

$$\frac{\text{par rate}}{2}\left[d(0.5)+d(1.0)+d(1.5)+d(2.0)\right]+100\times d(2.0)=100$$

For example, by plugging in discount factors from Figure 2 (in LO 59.2), we can compute the 2-year par rate as:

$$\frac{\text{par rate}}{2}(0.9968+0.9920+0.9848+0.9771)+100\times 0.9771=100$$

$$\frac{\text{par rate}}{2}(3.9507)+97.71=100$$

$$\text{par rate}=1.16\%$$

From the example in LO 59.2, notice how the par rate of 1.16% is exactly equal to the year 2 swap rate of 1.16%. This equality occurs because swap rates are, in fact, par rates. Therefore, because we used swap rates to represent bond coupon payments when deriving discount factors, we can also say that par rates represent bond coupon payments when a bond's price is equal to its par value.

We can generalize the par rate equation above to compute the par rate for any maturity, C_T, assuming a par value of $1 as follows:

$$\frac{C_T}{2} \times \sum_{t=1}^{2T} d\left(\frac{t}{2}\right) + d(T) = 1$$

The sum of the discount factors: $\sum_{t=1}^{2T} d\left(\frac{t}{2}\right)$ is known as the **annuity factor**, A_T. Therefore, the notation can be simplified as follows:

$$\frac{C_T}{2} \times A_T + d(T) = 1$$

Note that similar to spot rate and forward rate curves, we can also construct a par rate curve (i.e., a swap rate curve). When the spot rate and forward rate curves are flat, the par rate curve will also be flat. In addition, note that bonds or swaps will cease to trade at par when interest rate and discount factors change since these changes will impact present value calculations.

Pricing a Bond Using Spot, Forward, and Par Rates

LO 59.6: Interpret the relationship between spot, forward and par rates.

Any bond can be split into a series of periodic cash flows. Each cash flow in isolation can be considered a STRIP. If the present value of each cash flow is computed and summed, the resulting number should be equivalent to the bond's price. The appropriate discount rate for each cash flow is the spot rate. Discount factors can also be used because spot rates can be derived from discount factors.

Because spot rates and the implied forward rates are so closely related, it makes no difference which one is used to compute present values. A spot rate or a sequence of forward rates can be used to compute the present value. For example, a 1-year spot rate can be used to discount a cash flow taking place in one year, or a 6-month spot rate and the 6-month implied forward rate six months from now can be used. Both approaches will give the same present value, since they both span the same period.

Example: Calculating the price of a bond

Suppose a 1-year Treasury bond (T-bond) pays a 4% coupon semiannually. **Compute** its price using the discount factors, spot rates, forward rates, and par rates from Figure 8.

Figure 8: Discount Factors, Spot Rates, Forward Rates, and Par Rates

Maturity (Years)	*Discount Factor*	*Spot Rate*	*6-Month Forward Rate*	*Par Rates*
0.5	0.992556	1.50%	1.50%	1.5000%
1.0	0.978842	2.15%	2.80%	2.1465%
1.5	0.962990	2.53%	3.29%	2.5225%
2.0	0.943299	2.94%	4.18%	2.9245%
2.5	0.921205	3.31%	4.80%	3.2839%
3.0	0.897961	3.62%	5.18%	3.5823%

Answer:

Using discount factors:

$$\text{bond price} = (\$2 \times 0.992556) + (\$102 \times 0.978842) = \$101.83$$

Using annuity and discount factors:

$$\text{bond price} = [\$2 \times (0.992556 + 0.978842)] + (\$100 \times 0.978842) = \$101.83$$

Using spot rates:

$$\text{bond price} = \frac{\$2}{\left(1+\frac{0.0150}{2}\right)^1} + \frac{\$102}{\left(1+\frac{0.0215}{2}\right)^2} = \$101.83$$

Using forward rates:

$$\text{bond price} = \frac{\$2}{\left(1+\frac{0.0150}{2}\right)^1} + \frac{\$102}{\left(1+\frac{0.0150}{2}\right)^1 \times \left(1+\frac{0.0280}{2}\right)^1} = \$101.83$$

Using par rates:

$$\text{bond price} = \$100 + \left[\left(\$2 - \frac{2.1465}{2}\right) \times (0.992556 + 0.978842)\right] = \$101.83$$

An interesting observation from Figure 8 is that each spot rate is approximately equal to the average of the forward rates of equal or lower term. For example, the spot rate in Year 3 is approximately:

$$3.62\% \approx \frac{1.50\% + 2.80\% + 3.29\% + 4.18\% + 4.80\% + 5.18\%}{6}$$

Exact spot rates are a complex average of forward rates, but a simple average of forward rates will provide a good approximation. This means that as spot rates increase over time, forward rates are greater than corresponding spot rates. However, as spot rates increase or decrease with term, forward rates will also fluctuate above or below spot rates.

Another observation from Figure 8 is that par rates are near, but slightly below, corresponding spot rates. This relationship occurs because the spot rate curve is not flat. Given an upward-sloping spot curve, par rates will be below corresponding spot rates, and given a downward-sloping spot curve, par rates will be above corresponding spot rates.

Effect of Maturity on Bond Prices and Returns

LO 59.7: Assess the impact of maturity on the price of a bond and the returns generated by bonds.

To analyze the effect of maturity on bond prices, we must compare coupon rates to corresponding forward rates over an arbitrary time period. In general, bond prices will tend to increase with maturity when coupon rates are above the relevant forward rates. The opposite holds when coupon rates are below the relevant forward rates (i.e., bond prices will tend to decrease with maturity in this scenario).

To analyze the effect of maturity on bond returns, assume two investors would like to invest over a 3-year time horizon. One investor invests in 6-month STRIPS and rolls them over for 3 years (i.e., when the first 6-month contract expires, he will invest in the next 6-month contract and so on for 3 years). The other investor just invests in a 3-year bond.

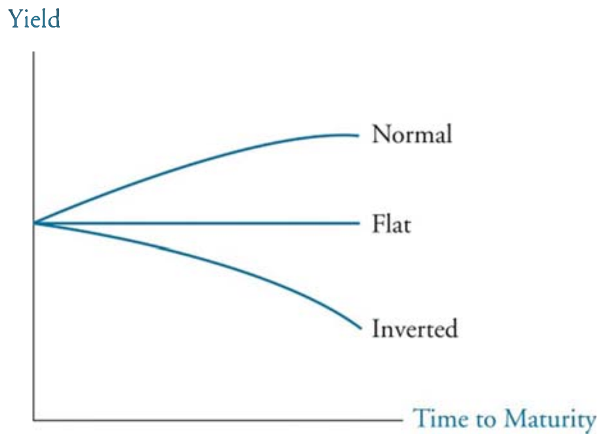
When short-term rates are above the forward rates utilized by bond prices, the investors who rolls over shorter-term investments will tend to outperform investors who invest in longer-term investments. The opposite holds when short-term rates are below the forward rates (i.e., the investor in long-term investments will outperform). If some short-term rates are lower than forward rates and some are higher, then a more detailed analysis will be required to determine which investor outperformed.

Yield Curve Shapes

LO 59.8: Define the "flattening" and "steepening" of rate curves and describe a trade to reflect expectations that a curve will flatten or steepen.

Historically, the yield curve has taken on three fundamental shapes, as shown in Figure 9.

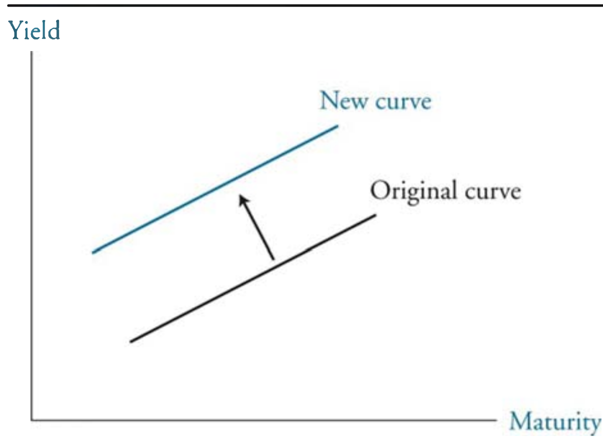
Figure 9: Yield Curve Shapes



A *normal* yield curve is one in which long-term rates are greater than short-term rates, so the curve has a *positive slope*. A *flat* yield curve represents the situation where the yield on all maturities is essentially the same. An *inverted* yield curve reflects the condition where long-term rates are less than short-term rates, giving the yield curve a *negative slope*.

When the yield curve undergoes a **parallel shift**, the yields on all maturities change in the same direction and by the same amount. As indicated in Figure 10, the slope of the yield curve remains unchanged following a parallel shift.

Figure 10: Parallel Yield Curve Shift

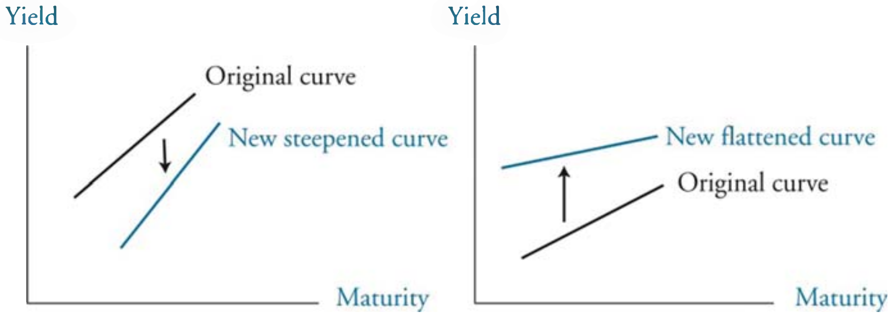


When the yield curve undergoes a **nonparallel shift**, the yields for the various maturities change by differing amounts. The slope of the yield curve after a nonparallel shift is not the same as it was prior to the shift. Nonparallel shifts fall into two general categories: twists and butterfly shifts.

Yield curve twists refer to yield curve changes when the slope becomes either flatter or steeper. With an upward-sloping yield curve, a *flattening* of the yield curve means that the spread between short- and long-term rates has narrowed. Conversely, a *steepening* of the yield curve occurs when spreads widen.

As shown in Figure 11, the most common shifts tend to be either a downward shift and a steepened curve or an upward shift and a flattened curve.

Figure 11: Nonparallel Yield Curve Shifts—Twists

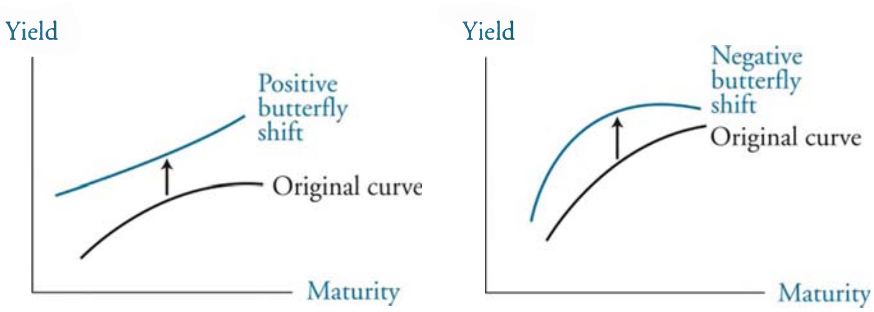


There are two ways the yield curve would be perceived as flattening or steepening. The curve will flatten when long-term rates fall by more than short-term rates or when short-term rates rise by more than long-term rates. The curve will steepen when long-term rates rise by more than short-term rates or when short-term rates fall by more than long-term rates.

Given an upward-sloping yield curve, if a trader expects the curve to steepen, he is anticipating that the spread between short- and long-term rates will widen. Therefore, he would sell short a long-term rate and buy a short-term rate, because he expects bond prices in the long-term to fall (as rates increase, bond prices fall). If this trader instead expected the curve to flatten, he is anticipating that the spread between short- and long-term rates will narrow. Therefore, he would buy a long-term rate and sell short a short term rate, because he expects bond prices in the long-term to rise.

Yield curve butterfly shifts refer to changes in the degree of curvature. A positive butterfly means that the yield curve has become less curved. For example, if rates increase, the short and long maturity yields increase by more than the intermediate maturity yields, as shown in Figure 12. A negative butterfly means that there is more curvature to the yield curve. For example, if rates increase, intermediate term yields increase by more than the long and short maturity yields, as shown in Figure 12.

Figure 12: Nonparallel Yield Curve Shifts—Butterfly Shifts



Key Concepts

LO 59.1

Annual compounding means paying interest once a year, while semiannual compounding means paying interest once every six months.

LO 59.2

Similar to a series of bond prices, discount factors can also be derived from a series of interest rate swap rates. To make this calculation, swap rates are treated as bond coupon payments and the swap notional amount represents the bond's par value.

LO 59.3

A *t*-period spot rate is the yield to maturity on a zero-coupon bond that matures in *t*-years. The spot rate curve is the graph of the relationship between spot rates and maturity. The spot rate curve can be derived from either a series of STRIPS prices, or the comparable discount factors.

LO 59.4

Forward rates are interest rates corresponding to a future period implied by the spot curve. Bootstrapping is the process of computing forward rates from spot rates.

LO 59.5

The par rate at maturity is the rate at which the present value of a bond equals its par value. Par rates are the same as swap rates and can be accessed via the swap rate curve.

LO 59.6

A spot rate is approximately equal to the average of the forward rates of equal or lower term. As spot rates increase over time, forward rates are greater than corresponding spot rates.

Given an upward-sloping spot rate curve, par rates are near, but slightly below, corresponding spot rates. This relationship occurs because the spot rate curve is not flat.

LO 59.7

In general, bond prices will increase with maturity when coupon rates are above relevant forward rates. A bond's return will depend on the duration of the investment and the relationship between spot and forward rates.

LO 59.8

When the yield curve undergoes a parallel shift, the yield on all maturities change in the same direction and by the same amount. The slope of the yield curve remains unchanged following a parallel shift.

When the yield curve undergoes a nonparallel shift, the yields for the various maturities do not necessarily change in the same direction or by the same amount. The slope of the yield curve after a nonparallel shift is not the same as it was prior to the shift.

- Twists refer to yield curve changes when the slope becomes either flatter or more steep. A flattening of the yield curve means that the spread between short- and long-term rates has narrowed.
- Butterfly shifts refer to changes in curvature of the yield curve. A positive butterfly means that the yield curve has become less curved. A negative butterfly means that there is more curvature to the yield curve.

Concept Checkers

Use the following data to answer Questions 1 and 2.

Maturity (Years)	*STRIPS Price*	*Spot Rate*	*Forward Rate*
0.5	98.7654	2.50%	2.50%
1.0	97.0662	3.00%	3.50%
1.5	95.2652	3.26%	3.78%
2.0	93.2775	?.??%	?.??%

1. The 6-month forward rate in 1.5 years (ending in year 2.0) is closest to:
   A. 4.04%.
   B. 4.11%.
   C. 4.26%.
   D. 4.57%.

2. The value of a 1.5-year, 6% semiannual coupon, $100 par value bond is closest to:
   A. $102.19.
   B. $103.42.
   C. $104.00.
   D. $105.66.

3. The 4-year spot rate is 8.36% and the 3-year spot rate is 8.75%. What is the 1-year forward rate three years from today (assuming these are annual rates)?
   A. 0.39%.
   B. 7.20%.
   C. 8.56%.
   D. 9.93%.

4. Given the interest rates, which of the following is closest to the price of a 4-year bond that has a par value of $1,000 and makes 10% coupon payments annually?

   - Current 1-year spot rate = 5.5%.
   - 1-year forward rate one year from today = 7.63%.
   - 1-year forward rate two years from today = 12.18%.
   - 1-year forward rate three years from today = 15.50%.

   A. $844.55.
   B. $995.89.
   C. $1,009.16.
   D. $1,085.62.

5. Given the following bonds and forward rates:

Maturity	*YTM*	*Coupon*	*Price*
1 year	4.5%	0%	95.694
2 years	7%	0%	87.344
3 years	9%	0%	77.218

- 1-year forward rate one year from today = 9.56%.
- 1-year forward rate two years from today = 10.77%.
- 2-year forward rate one year from today = 11.32%.

Which of the following statements about the forward rates, based on the bond prices, is true?

A. The 1-year forward rate one year from today is too low.
B. The 2-year forward rate one year from today is too high.
C. The 1-year forward rate two years from today is too low.
D. The forward rates and bond prices provide no opportunities for arbitrage.

Concept Checker Answers

1. **C** First compute the 2-year spot rate:

   N = 4; PV = –93.2775; PMT = 0; FV = 100; CPT I/Y = 1.755%

   $z(0.5) = 1.755\% \times 2 = 3.51\%$

   Next compute the forward rate in 1.5 years ending in year 2.

$$\left(1+\frac{0.0351}{2}\right)^4=\left(1+\frac{0.0326}{2}\right)^3\times\left(1+\frac{f(2.0)}{2}\right)^1$$

$$\Rightarrow f(2.0)=4.26\%$$

2. **C** $\text{bond price}=\dfrac{\$3}{\left(1+\dfrac{0.0250}{2}\right)^1}+\dfrac{\$3}{\left(1+\dfrac{0.0300}{2}\right)^2}+\dfrac{\$103}{\left(1+\dfrac{0.0326}{2}\right)^3}=\$104.00$

3. **B** $\dfrac{(1.0836)^4}{(1.0875)^3}-1=7.20\%$

4. **C** The easiest way to find the bond value is to first calculate the appropriate spot rates to discount each cash flow.

   $S_1 = 5.5\%$

   $S_2 = [(1.055)(1.0763)]^{1/2} - 1 = 6.56\%$

   $S_3 = [(1.055)(1.0763)(1.1218)]^{1/3} - 1 = 8.39\%$

   $S_4 = [(1.055)(1.0763)(1.1218)(1.155)]^{1/4} - 1 = 10.13\%$

   Then use the spot rates to discount each cash flow and take the sum of the discounted cash flows to find the value of the bond.

$$\text{bond price}=\frac{\$100}{1.055}+\frac{\$100}{1.0656^2}+\frac{\$100}{1.0839^3}+\frac{\$1,100}{1.1013^4}=\$1,009.16$$

   Note that you could also do this in one step using the forward rates, but breaking the problem into two steps makes the math easier to do on your calculator.

5. **C** Given the bond spot rates on the zero-coupon bonds, the appropriate forward rates should be:

   - 1-year forward rate one year from today = $[(1 + 0.07)^2 / (1 + 0.045)] - 1 = 0.0956$, or 9.56%
   - 1-year forward rate two years from today = $[(1 + 0.09)^3 / (1 + 0.07)^2] - 1 = 0.1311$, or 13.11%
   - 2-year forward rate one year from today = $[(1 + 0.09)^3 / (1 + 0.045)] = 1.2393$. $1.2393^{0.5} - 1 = 0.1132 = 11.32\%$

   The 1-year forward rate two years from today is too low.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Returns, Spreads, and Yields

Topic 60

Exam Focus

Bonds with coupons that are greater than market rates are said to trade at a premium, while bonds with coupon rates less than market rates are said to be trading at a discount. For coupon bonds, yield to maturity (YTM) is not a good measure of actual returns to maturity. When a bondholder receives coupon payments, the investor runs the risk that these cash flows will be reinvested at a rate of return that is lower than the original promised yield on the bond. This is known as reinvestment risk. For the exam, know how to calculate YTM given different compounding frequencies.

Realized Return

LO 60.1: Distinguish between gross and net realized returns, and calculate the realized return for a bond over a holding period including reinvestments.

The **gross realized return** for a bond is its end-of-period total value minus its beginning-of-period value divided by its beginning-of-period value. The end-of-period total value will include both ending bond price and any coupons paid during the period. If we denote current bond price at time t as BV_t, coupons received during time period t as C_t, and initial bond price as BV_{t-1}, then the realized return for a bond from time period $t-1$ to t is computed as follows:

$$R_{t-1,t} = \frac{BV_t + C_t - BV_{t-1}}{BV_{t-1}}$$

Example: Calculating gross realized return

What is the gross realized return for a bond that is currently selling for \$112 if it was purchased exactly six-months ago for \$105 and paid a \$2 coupon today?

Answer:

Substituting the appropriate values into the realized return equation, we get:

$$R_{t-1,t} = \frac{\$112 + \$2 - \$105}{\$105}$$

$$R_{t-1,t} = 8.57\%$$

The **net realized return** for a bond is its gross realized return minus per period financing costs. Cost of financing would arise from borrowing cash to purchase the bond. Even though borrowing cash to pay for the entire price of the bond would technically reduce the initial cash outlay to zero, convention is to use the initial bond price as the beginning-of-period value.

Example: Calculating net realized return

What is the net realized return for a bond that is currently selling for $112 and paid a $2 coupon today if its purchase price of $105 was entirely financed at an annual rate of 0.6% exactly six-months ago?

Answer:

Substituting the appropriate values into the realized return equation and then subtracting per period financing costs, we get:

$$R_{t-1,t} = \frac{\$112 + \$2 - \$105}{\$105} - \frac{0.6\%}{2}$$

$$R_{t-1,t} = 8.57\% - 0.3\% = 8.27\%$$

In order to compute the realized return for a bond over multiple periods, we must keep track of the rates at which coupons received are reinvested. When a bondholder receives coupon payments, the investor runs the risk that these cash flows will be reinvested at a rate that is lower than the expected rate. For example, if interest rates go down across the board, the reinvestment rate will also be lower. This is known as **reinvestment risk**.

Example: Calculating realized return with reinvested coupons

What is the realized return for a bond that is currently selling for $112 if it was purchased exactly one year ago for $105, paid a $2 coupon today, and paid a $2 coupon six months ago? Assume the coupon received six months ago was reinvested at an annual rate of 1%.

Answer:

$$R_{t-1,t} = \frac{\$112 + \$2 + \left[\$2 \times \left(1 + \frac{1\%}{2}\right)\right] - \$105}{\$105}$$

$$R_{t-1,t} = \frac{\$112 + \$2 + 2.01 - \$105}{\$105} = 10.49\%$$

Bond Spread

LO 60.2: Define and interpret the spread of a bond, and explain how a spread is derived from a bond price and a term structure of rates.

The market price of a bond may differ from the computed price of a bond using spot rates or forward rates. Any difference between bond market price and bond price according to the term structure of interest rates is known as the **spread** of a bond. A bond's spread is a relative measure of value which helps investors identify whether investments are trading cheap or rich relative to the yield curve (i.e., the term structure of rates).

Recall the calculation of bond price using forward rates from the previous topic. Assume a 2-year bond pays annual coupon payments, *C*, and a principal payment, *P*, at the end of year two. The bond's price will be computed by discounting all cash flows by corresponding 1-year forward rates as follows:

$$\text{bond price} = \frac{C}{[1+f(1.0)]} + \frac{C+P}{[1+f(1.0)]\times[1+f(2.0)]}$$

If the market price of this bond trades at a premium or discount to this computed price, we can find the spread of the bond by adding a spread, *s*, to the forward rates as follows:

$$\text{market bond price} = \frac{C}{[1+f(1.0)+s]} + \frac{C+P}{[1+f(1.0)+s]\times[1+f(2.0)+s]}$$

By deriving this spread, we can identify how much the bond is trading cheap or rich in terms of the bond's return. For example, rather than saying the market price of the bond is trading 10 cents cheap, relative to the price determined by the term structure of rates, we can say that that the bond is trading 4.9 basis points cheap. Note that this spread could be the result of either bond-specific factors or sector-specific factors.

Yield to Maturity

LO 60.3: Define, interpret, and apply a bond's yield-to-maturity (YTM) to bond pricing.

LO 60.4: Compute a bond's YTM given a bond structure and price.

The **yield to maturity**, or YTM, of a fixed-income security is equivalent to its internal rate of return. The YTM is the discount rate that equates the present value of all cash flows associated with the instrument to its price.

For a security that pays a series of known annual cash flows, the computation of yield uses the following relationship:

$$P = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + ... + \frac{C_N}{(1+y)^N}$$

where:
P = the price of the security
C_k = the annual cash flow in year k
N = term to maturity in years
y = the annual yield or YTM on the security

Example: Yield to maturity

Suppose a fixed-income instrument offers annual payments in the amount of $100 for ten years. The current value for this instrument is $700. **Compute** the YTM on this security.

Answer:

The YTM is the *y* that solves the following equation:

$$\$700 = \frac{\$100}{(1+y)^1} + \frac{\$100}{(1+y)^2} + \frac{\$100}{(1+y)^3} + ... + \frac{\$100}{(1+y)^{10}}$$

We can solve for YTM using a financial calculator:

N = 10; PMT = 100; PV = –700; CPT ⇒ I/Y = 7.07%

If cash flows occur more frequently than annually, the previous equation can be rewritten as:

$$P = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + ... + \frac{C_n}{(1+y)^n}$$

where:
n = N × m = the number of periods (years multiplied by payments per year)
C_k = the periodic cash flow in time period k
y = the periodic yield or periodic interest rate

Example: Periodic yield and YTM

Suppose now that the security in the previous example pays the $100 semiannually for five years. **Compute** the periodic yield and the YTM on this security.

Answer:

The periodic yield is the *y* that solves the following equation:

$$\$700 = \frac{\$100}{(1+y)^1} + \frac{\$100}{(1+y)^2} + \frac{\$100}{(1+y)^3} + \ldots + \frac{\$100}{(1+y)^{10}}$$

Using a financial calculator:

N = 10; PMT = 100; PV = –700; CPT ⇒ I/Y = 7.07%

Why is this the same value as in the previous example? Remember that this yield corresponds to a 6-month period. To compute the annual YTM, we must multiply the periodic yield by the number of periods per year, m = 2. This produces a YTM of 14.14%.

The yield to maturity can be viewed as the **realized return** on the bond assuming all cash flows are reinvested at the YTM.

Example: Realized return

Suppose a bond pays $50 every six months for five years and a final payment of $1,000 at maturity in five years. If the price is $900, **calculate** the realized return on the security. Assume all cash flows are reinvested at the YTM.

Answer:

The semiannual rate is the *y* that solves the following equation:

$$\$900 = \frac{\$50}{(1+y)^1} + \frac{\$50}{(1+y)^2} + \frac{\$50}{(1+y)^3} + \ldots + \frac{\$50 + \$1{,}000}{(1+y)^{10}}$$

Using a financial calculator, we arrive at a semiannual discount rate of 6.3835% and a YTM of 12.77%:

N = 10; PMT = 50; PV = –900; FV = 1,000; CPT ⇒ I/Y = 6.3835;
YTM = 6.3835 × 2 = 12.77%

The yield to maturity calculated above (2 × the semiannual discount rate) is referred to as a **bond equivalent yield** (BEY), and we will also refer to it as a semiannual YTM or semiannual-pay YTM. If you are given yields that are identified as BEY, you will know that you must divide by two to get the semiannual discount rate. With bonds that make annual coupon payments, we can calculate an **annual-pay yield to maturity**, which is simply the internal rate of return for the expected annual cash flows.

For zero-coupon Treasury bonds, the convention is to quote the yields as BEYs (semiannual-pay YTMs).

Example: Calculating YTM for zero-coupon bonds

A 5-year Treasury STRIP is priced at $768. **Calculate** the semiannual-pay YTM and annual-pay YTM.

Answer:

The direct calculation method, based on the geometric mean, is:

$$\text{semiannual-pay YTM or BEY} = \left[\left(\frac{1{,}000}{768}\right)^{\frac{1}{10}} - 1\right] \times 2 = 5.35\%$$

$$\text{annual-pay YTM} = \left(\frac{1{,}000}{768}\right)^{\frac{1}{5}} - 1 = 5.42\%$$

The Limitations of Traditional Yield Measures

Reinvestment risk is a major threat to the bond's computed YTM, as it is assumed in such calculations that the coupon cash flows can be reinvested at a rate of return that's equal to the computed yield (i.e., if the computed yield is 8%, it is assumed the investor will be able to reinvest all coupons at 8%).

Reinvestment risk applies not only to coupons but also to the repayment of principal. Thus, it is present with bonds that can be prematurely retired, as well as with amortizing bonds where both principal and interest are received periodically over the life of the bond. Reinvestment risk becomes more of a problem with *longer term bonds* and with bonds that carry *larger coupons*. Reinvestment risk, therefore, is high for long-maturity, high-coupon bonds and is low for short-maturity, low-coupon bonds.

The realized yield on a bond is the actual compound return that was earned on the initial investment. It is usually computed at the end of the investment horizon. For a bond to have a **realized yield** equal to its YTM, all cash flows prior to maturity must be reinvested at the YTM, and the bond must be held until maturity. If the "average" reinvestment rate is below the YTM, the realized yield will be below the YTM. For this reason, it is often stated that: *The yield to maturity assumes cash flows will be reinvested at the YTM and assumes that the bond will be held until maturity.*

LO 60.5: Calculate the price of an annuity and a perpetuity.

Calculating the Price of an Annuity

We can easily calculate the price of cash flows (annuities) if given the YTM and cash flows.

Example: Present value of an annuity

Suppose a fixed-income instrument offers annual payments in the amount of $100 for 10 years. The YTM for this instrument is 10%. **Compute** the price (PV) of this security.

Answer:

The price is the PV that solves the following equation:

$$PV = \frac{\$100}{(1+0.10)^1} + \frac{\$100}{(1+0.10)^2} + \frac{\$100}{(1+0.10)^3} + \ldots + \frac{\$100}{(1+0.10)^{10}}$$

Using a financial calculator the price equals $614.46:

N = 10; PMT = 100; I/Y = 10; CPT ⇒ PV = $614.46

Calculating the Price of a Perpetuity

The perpetuity formula is straightforward and does not require an iterative process:

$$\text{PV of a perpetuity} = \frac{C}{y}$$

where:
C = the cash flow that will occur every period into perpetuity
y = yield to maturity

Example: Price of perpetuity

Suppose we have a security paying $1,000 annually into perpetuity. The interest rate is 10%. **Calculate** the price of the perpetuity.

Answer:

We don't need a financial calculator to do this calculation. The price of the perpetuity is simply $10,000:

$$PV = \frac{\$1{,}000}{0.10} = \$10{,}000$$

Spot Rates and YTM

LO 60.6: Explain the relationship between spot rates and YTM.

In the previous topic, we discussed the calculation of spot rates and examined how to value a bond given a spot rate curve. Pricing a bond using YTM is similar to using spot rates in that YTM is a blend of the given spot rates. Consider the following example.

Example: Spot rates and YTM

A bond with a $100 par value pays a 5% coupon annually for 4 years. The spot rates corresponding to the payment dates are as follows:

Year 1: 4.0%
Year 2: 4.5%
Year 3: 5.0%
Year 4: 5.5%

Assume the price of the bond is $98.47. **Show** the calculation of the price of the bond using spot rates and **determine** the YTM for the bond.

Answer:

The formula for the price of the bond using the spot rates is as follows:

$$P = \frac{5}{(1.04)} + \frac{5}{(1.045)^2} + \frac{5}{(1.05)^3} + \frac{105}{(1.055)^4}$$

$$\$98.47 = 4.81 + 4.58 + 4.32 + 84.76$$

Now compute the YTM:

$$\$98.47 = \frac{5}{(1+\text{YTM})} + \frac{5}{(1+\text{YTM})^2} + \frac{5}{(1+\text{YTM})^3} + \frac{105}{(1+\text{YTM})^4}$$

$$\text{FV} = \$100;\ \text{PV} = -\$98.47;\ \text{PMT} = 5;\ \text{N} = 4;\ \text{CPT} \rightarrow \text{I / Y} = 5.44\%$$

$$\text{YTM} = 5.44\%$$

We see from this example that the YTM is closest to the 4-year spot rate. This is because the largest cash flow occurs at year 4 as the bond matures. If the spot curve is upward sloping, as in this example, the YTM will be less than the 4-year spot (i.e., the last spot rate). If the spot curve is flat, the YTM will be equal to the 4-year spot, and if the spot curve is downward sloping, the YTM will be greater than the 4-year spot.

The Relationship Between YTM, Coupon Rate, and Price

LO 60.7: Define the coupon effect and explain the relationship between coupon rate, YTM, and bond prices.

A bond's price reflects its relative value in the market based on several factors. Assume that a firm issues a bond at par, meaning that the market rate for the bond is precisely that of the coupon rate. Immediately after this bond is issued and before the market has time to adjust, the bond will trade at *par*. After the bond begins trading in the market, the same cannot be said. The price of the bond will reflect market conditions.

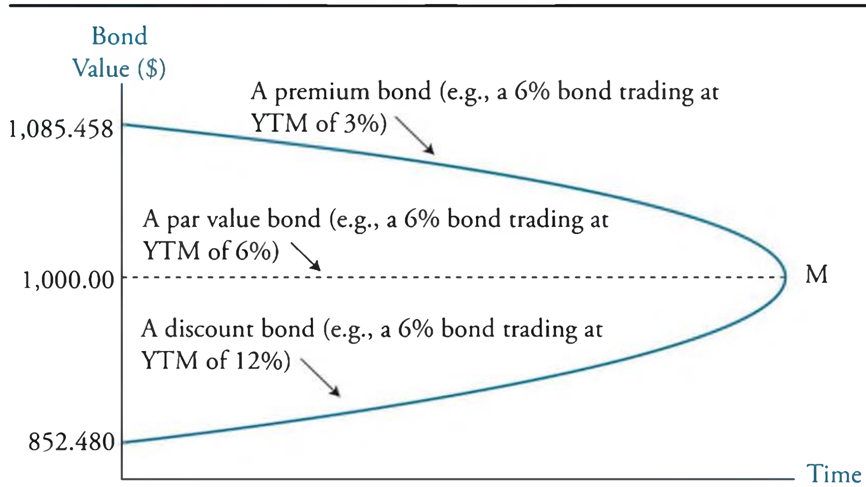
For example, suppose that after the bond was issued, market interest rates declined substantially. Investors in the bond would receive coupon rates substantially higher than what the market currently offers. Because of this, the price of the bond would adjust upward. This bond is a **premium bond**. If interest rates were to increase substantially after the bond was issued, investors would have to be compensated for the fact that the coupon rate of the bond is substantially lower than those offered currently in the market. The price would adjust downward as a consequence. The bond would be referred to as a **discount bond**.

Therefore:

- If coupon rate > YTM, the bond will sell for more than par value, or at a **premium**.
- If coupon rate < YTM, the bond will sell for less than par value, or at a **discount**.
- If coupon rate = YTM, the bond will sell for **par value**.

Over time, the price of premium bonds will gradually fall until they trade at par value at maturity. Similarly, the price of discount bonds will gradually rise to par value as maturity gets closer. This converging effect is known as **pull to par**. The change in bond value associated with the passage of time for three 6% coupon-paying bonds with YTMs of 3%, 6%, and 12% is presented graphically in Figure 1.

Figure 1: Premium, Par, and Discount Bonds



Coupon Effect

If two bonds are identical in all respects except their coupon, *the bond with the smaller coupon will be more sensitive to interest rate changes.* That is, for any given change in yield, the smaller-coupon bond will experience a bigger percentage change in price than the larger-coupon bond. All else being equal:

- The lower the coupon rate, the greater the interest-rate risk.
- The higher the coupon rate, the lower the interest-rate risk.

Figure 2 summarizes the relationship between bond price sensitivity and coupon size. The bonds have equal maturities but different coupons. Assume semiannual coupons for both bonds.

Figure 2: Bond Price Reactions to Changes in Yield

	Price Change From Par ($1,000)	
Change in Interest Rates	*20-year, 8%*	*20-year, 12%*
–2%	+231.15	+171.59
–1%	+106.77	+80.23
0%	0	0
+1%	–92.01	–70.73
+2%	–171.59	–133.32

Coupon effect: For the same change in interest rates, the 20-year, 8% bond experiences a greater change in price than the 20-year, 12% bond. This suggests that bonds with similar maturities, but different coupon rates, can have different yield to maturities.

Return Decomposition

LO 60.8: Explain the decomposition of P&L for a bond into separate factors including carry roll-down, rate change, and spread change effects.

Return decomposition for a bond breaks down bond profitability or loss (P&L) into component parts. This decomposition of P&L helps bond investors understand how their investments are making or losing money. A bond's profitability or loss is generated through price appreciation and explicit cash flows (i.e., cash-carry), such as coupons and financing costs. Bond total price appreciation can be broken down into three component parts for price effect analysis: carry-roll-down, rate changes, and spread change. Note that the sum of all component parts will equal total price appreciation.

The **total price appreciation** for a bond is equal to its price at t minus its price at t–1. Given the passage of time, total price appreciation results from moving along the original term structure, R , from t–1 to t and accounts for changes in bond spread, s, from t–1 to t. Mathematically, it can be represented as follows:

$$\text{total price appreciation} = BV_t(R_t, s_t) - BV_{t-1}(R_{t-1}, s_{t-1})$$

The **carry-roll-down component** accounts for price changes due to interest rate movements from the original term structure to an expected term structure, R′ , as the bond matures. Different carry-roll-down scenarios for this expected term structure will be discussed in the next LO. This component does not account for spread changes.

$$\text{carry-roll-down} = BV_t(R'_t, s_{t-1}) - BV_{t-1}(R_{t-1}, s_{t-1})$$

The **rate changes component** accounts for price changes due to interest rate movements from an expected term structure to the original term structure that exists at time *t*. Similar to carry-roll-down, this component does not account for spread changes.

$$\text{rate changes} = BV_t(R_t, s_{t-1}) - BV_t(R'_t, s_{t-1})$$

The **spread change component** accounts for price changes due to changes in the bond's spread from *t*–1 to *t*. Expected changes in the spread are frequently the subject of investments for traders who are betting that a security is trading either cheap or rich.

$$\text{spread change} = BV_t(R_t, s_t) - BV_t(R_t, s_{t-1})$$

Carry-Roll-Down Scenarios

LO 60.9: Identify the most common assumptions in carry roll-down scenarios, including realized forwards, unchanged term structure, and unchanged yields.

As mentioned, the carry-roll-down component of total price appreciation considers movement to an expected term structure. Traders make investment return calculations based on their expectations, and many traders will consider scenarios where rates do not change. Given this expectation, term structure choices for no change scenarios include: realized forwards, unchanged term structure, and unchanged yields.

The **realized forward scenario** assumes that forward rates are equal to expected future spot rates, and over the investment horizon, these forward rates will be realized. This implies, for example, that an investor could earn the same return by investing in a 5-year bond or by investing in a 3-year bond and then a 2-year bond after the 3-year bond expires. This means that the one-period gross realized return will equal the prevailing one-period rate.

For example, suppose the 1-year spot rate is 5% and the 2-year spot rate is 7%. Under the realized forward scenario, the 1-year forward rate in one year must be 9%, because investing for two years at 7% will generate approximately the same annual return as investing for the first year at 5% and the second year at 9%. In other words, the 2-year rate of 7% is the average of the expected future 1-year rates of 5% and 9%. The 9% rate is called the implied forward rate.

If the implied forward rates over the investment horizon are realized, then an investor will earn the same return by either rolling over short-term investments or investing in a long-term bond and reinvesting its coupons at short-term rates. However, if realized forward rates are greater than implied forward rates, the strategy of rolling over one-period bonds will generate a higher return. Conversely, if realized forward rates are less than implied forward rates, the long-term bond strategy will be more profitable. Thus, the chosen investment strategy will depend on how investor expectations of rates compare to implied forward rates.

The **unchanged term structure scenario** simply assumes that the term structure will remain unchanged over the investment horizon. This means that the gross realized return will depend greatly on the relationship between the bond's coupon rate and the last forward rate before the bond matures. This scenario implies that there is a risk premium built into forward rates. For example, if the term structure is upward-sloping and remains unchanged, the term structure shape must reflect an investor risk premium that increases over the investment horizon.

As the name suggests, the **unchanged yields scenario** assumes that bond yields remain unchanged over the investment horizon. This means that the one-period gross realized return will equal a bond's yield (i.e., its yield to maturity). Thus, this scenario assumes that bond coupon payments are reinvested at the YTM. As stated earlier, there are limitations to this reinvestment assumption since the term structure is unlikely to be flat and remain unchanged.

Key Concepts

LO 60.1

The gross realized return for a bond is the difference between end-of-period total value (including end-of-period price and coupons) and starting period value divided by starting period value:

$$R_{t-1,t} = \frac{BV_t + C_t - BV_{t-1}}{BV_{t-1}}$$

The net realized return for a bond is its gross realized return minus per period financing costs.

LO 60.2

The difference between bond market price and bond model price (according to the term structure) is known as the spread. A bond's spread helps investors identify whether fixed-income investments are trading cheap or rich.

LO 60.3

Yield is an internal rate of return found by equating the present value of the cash flows to the current price of the security. An iterative process is used for the actual computation of yield. On a financial calculator, it can be found by inputting all other variables and solving for YTM.

LO 60.4

For a security that pays a series of known annual cash flows, the computation of yield uses the following relationship:

$$P = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \ldots + \frac{C_N}{(1+y)^N}$$

LO 60.5

The present value, or price, of a perpetuity can be found by dividing the coupon payment by the YTM.

LO 60.6

When pricing a bond, YTM or spot rates can be used. The YTM will be a blend of the spot rates for the bond.

LO 60.7

If two bonds are identical in all respects except their coupon, the bond with the smaller coupon will be more sensitive to interest rate changes.

When the bond is trading at par, the coupon rate is equal to the YTM. When the bond is trading below par, the coupon rate is less than the YTM, and is said to trade at a discount. When a bond is trading above par, the coupon rate is greater than the YTM, and the bond then trades at a premium.

LO 60.8

A bond's P&L is generated through price appreciation and explicit cash flows. Total price appreciation can be broken down into three component parts for price effect analysis: carry-roll-down, rate changes, and spread change.

LO 60.9

For an expected term structure, no change scenarios include: realized forwards, unchanged term structure, and unchanged yields. Realized forwards assume that forward rates are equal to expected future spot rates. Unchanged term structure assumes that the term structure will remain unchanged. Unchanged yields assume that bond yields remain unchanged.

Concept Checkers

1. An annuity pays \$10 every year for 100 years and currently costs \$100. The YTM is closest to:
   A. 5%.
   B. 7%.
   C. 9%.
   D. 10%.

2. A \$1,000 par bond carries a 7.75% semiannual coupon rate. Prevailing market rates are 8.25%. The price of the bond is:
   A. less than \$1,000.
   B. greater than \$1,000.
   C. \$1,000.
   D. Not enough information to determine.

3. A \$1,000 par bond carries a coupon rate of 10%, pays coupons semiannually, and has 13 years remaining to maturity. Market rates are currently 9.25%. The price of the bond is closest to:
   A. \$586.60.
   B. \$1,036.03.
   C. \$1,055.41.
   D. \$1,056.05.

4. Reinvestment risk would not occur if:
   A. interest rates shifted over the time period the bond is held.
   B. the bonds were callable.
   C. bonds are issued at par.
   D. only zero-coupon bonds are purchased.

5. An investment pays \$50 annually into perpetuity and yields 6%. Which of the following is closest to the price?
   A. \$120.
   B. \$300.
   C. \$530.
   D. \$830.

Concept Checker Answers

1. **D** N = 100; PMT = 10; PV = –100; CPT ⇒ I/Y = 10%

2. **A** Since the coupon rate is less than the market interest rate, the bond is a discount bond and trades less than par.

3. **D** N = 26; PMT = 50; I/Y = 4.625; FV = 1,000; CPT ⇒ PV = $1,056.05

4. **D** Callable bonds have reinvestment risk because the principal can be prematurely retired. The higher the coupon, the higher the reinvestment risk, holding all else constant. A bond being issued at par has nothing to do with reinvestment risk.

5. **D** PV = C/I = $50 / 0.06 = $833.33

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

One-Factor Risk Metrics and Hedges

Topic 61

Exam Focus

This topic looks at ways to measure and hedge risk for fixed income securities. The three main concepts covered are DV01, duration, and convexity. DV01 is an acronym for the dollar value of a basis point, which measures how much the price of a bond changes from a one basis point change in yield. Duration measures the percentage change in a bond's value for a specific unit's change in rates. Utilizing both DV01 and duration can measure price volatility, but they do not capture the curvature in the relationship between bond yield and price. In order to capture the curvature effects of the price-yield relationship, we use convexity to complement these measures. For the exam, be able to compare, contrast, and calculate DV01, duration, and convexity.

Interest Rate Factors

LO 61.1: Describe an interest rate factor and identify common examples of interest rate factors.

Measures of interest rate sensitivity allow investors to evaluate bond price changes as a result of interest rate changes. Being able to properly measure price sensitivity can be useful in the following situations:

1. Hedgers must understand how the bond being hedged as well as the hedging instrument used will respond to interest rate changes.
2. Investors need to determine the optimal investment to make in the event that expected changes in rates do in fact occur.
3. Portfolio managers would like to know the portfolio level of volatility for expected changes in rates.
4. Asset/liability managers need to match the interest rate sensitivity of their assets with the interest rate sensitivity of their liabilities.

In order to estimate bond price changes, we need to have some idea as to how interest rates will change going forward. Price changes are based on **interest rate factors**, which are random variables that influence individual interest rates along the yield curve. For this topic, we will be evaluating price sensitivity based on parallel shifts in the yield curve. This is a one-factor approach (i.e., single-factor approach) which assumes that a change in one rate (e.g., 20-year rate) will impact all other rates along the curve in a similar fashion.

Dollar Value of a Basis Point

LO 61.2: Define and compute the DV01 of a fixed income security given a change in yield and the resulting change in price.

The DV01 is the "dollar value of an 01," meaning the change in a fixed income security's value for every one basis point change in interest rates. The "01" refers to one basis point (i.e., 0.0001). DV01 is the absolute change in bond price for every basis point change in yield, which is essentially a basis point's price value. Therefore, an equivalent term for DV01 is PVBP, the price value of a basis point.

DV01 is computed using the following formula:

$$\text{DV01} = -\frac{\Delta \text{BV}}{10{,}000 \times \Delta y}$$

where:
ΔBV = change in bond value
Δy = change in yield

Example: Computing DV01

Suppose the yield on a zero-coupon bond declines from 3.00% to 2.99%, and the price of the zero increases from \$17.62 to \$17.71. **Compute** the DV01.

Answer:

$$\text{DV01} = -\frac{\Delta \text{BV}}{10{,}000 \times \Delta y}$$

$$\text{DV01} = -\frac{\$17.71 - \$17.62}{10{,}000 \times (-0.0001)} = \frac{\$0.09}{1} = \$0.09$$

The DV01 formula is preceded by a negative sign, so when rates decline and prices increase, DV01 will be positive.

DV01 Application to Hedging

LO 61.3: Calculate the face amount of bonds required to hedge an option position given the DV01 of each.

Sensitivity measures like DV01 are commonly used to compute **hedge ratios**. Hedge ratios provide the relative sensitivity between the position to be hedged and the instrument used to hedge the position. For example, if the hedge ratio is 1, that means that the hedging instrument and the position have the same interest rate sensitivity.

The goal of a hedge is to produce a combined position (the initial position combined with the hedge position) that will not change in value for a small change in yield. This is expressed as:

dollar price change of position = dollar price change of hedging instrument

$$HR = \frac{\text{DV01 (per \$100 of initial position)}}{\text{DV01 (per \$100 of hedging instrument)}}$$

Example: Computing the hedge ratio

Suppose a 30-year semiannual coupon bond has a DV01 of 0.17195624, and a 15-year semiannual coupon bond will be used as the hedging instrument. The 15-year bond has a DV01 of 0.10458173. **Compute** the hedge ratio.

Answer:

$$HR = \frac{0.17195624}{0.10458173} = 1.644$$

For every $1 par value of the 30-year bond, short $1.644 of par of the 15-year bond.

Professor's Note: On the exam, if you are given a yield beta, be sure to use it. The yield beta is the relationship between the yield of the initial position and the implied yield of the hedging instrument. In the above example, if the yield beta is anything other than 1, you would multiple the hedge ratio by the yield beta.

Example: Computing the amount of bonds needed to hedge

An investor takes a long position in an option worth $100 million. The option has a DV01 of 0.141. The investor wishes to hedge this option position with a 15-year zero-coupon bond which increases in price from $56.40 to $56.58 when yields drop by one basis point. **Calculate** the face amount of the bond required to hedge this option position.

Answer:

First compute the DV01 of the bond position:

$$\text{DV01}^{B} = -\frac{\$56.58 - \$56.40}{10{,}000 \times -0.0001} = 0.18$$

To determine the face amount of the bond required to hedge this option exposure, we use the following approach:

$$\text{face value} = \text{option position} \times \frac{\text{DV01}^{0}}{\text{DV01}^{B}}$$

$$\text{face value} = 100\text{M} \times \frac{0.141}{0.18} = \$78.33\text{M}$$

So in order to hedge this $100 million option position, the investor must short $78.33 million in face value of the bond.

Duration

LO 61.4: Define, compute, and interpret the effective duration of a fixed income security given a change in yield and the resulting change in price.

Duration is the most widely used measure of bond price volatility. A bond's price volatility is a function of its coupon, maturity, and initial yield. Duration captures the impact of all three of these variables in a single measure. Just as important, a bond's duration and its price volatility are directly related (i.e., the longer the duration, the more price volatility there is in a bond). Of course, such a characteristic greatly facilitates the comparative evaluation of alternative bond investments. For this LO, we will explain three duration measures: Macaulay, modified, and effective.

Macaulay duration is an estimate of a bond's interest rate sensitivity based on the time, in years, until promised cash flows will arrive. Since a 5-year zero-coupon bond has only one cash flow five years from today, its Macaulay duration is five. The change in value in response to a 1% change in yield for a 5-year zero-coupon bond is approximately 5%. A 5-year coupon bond has some cash flows that arrive earlier than five years from today (the coupons), so its Macaulay duration is less than five [the higher the coupon, the less the price sensitivity (duration) of a bond].

Macaulay duration is the earliest measure of duration, and because it was based on the time, duration is often stated as years. Since Macaulay duration is based on the expected cash flows for an option-free bond, it is not an appropriate estimate of the price sensitivity of bonds with embedded options.

Modified duration is derived from Macaulay duration and offers a slight improvement over Macaulay duration in that it takes the current YTM into account:

$$\text{modified duration} = \frac{\text{Macaulay duration}}{(1 + \text{periodic market yield})}$$

Modified duration can also be computed as follows, given the initial bond value and how value changes for a given change in yield:

$$\text{modified duration} = \frac{1}{\text{BV}} \frac{\Delta \text{BV}}{\Delta y}$$

Like Macaulay duration, and for the same reasons, modified duration is not an appropriate measure of interest rate sensitivity for bonds with embedded options. For callable and putable bonds, we instead use the formula for **effective duration**. Note that for option-free bonds, effective duration (based on small yield changes) and modified duration will be very similar. Effective duration is computed as follows:

$$\text{effective duration} = \frac{\text{BV}_{-\Delta y} - \text{BV}_{+\Delta y}}{2 \times \text{BV}_0 \times \Delta y}$$

where:
$\text{BV}_{-\Delta y}$ = estimated price if yield decreases by a given amount, Δy
$\text{BV}_{+\Delta y}$ = estimated price if yield increases by a given amount, Δy
BV_0 = initial observed bond price
Δy = change in required yield, in decimal form

Example: Computing duration

Suppose there is a 15-year, option-free noncallable bond with an annual coupon of 7% trading at par. **Compute** and **interpret** the bond's duration for a 50 basis point increase and decrease in yield.

Answer:

If interest rates rise by 50 basis points (0.50%), the estimated price of the bond falls to 95.586%.

N = 15; PMT = 7.00; FV = 100; I/Y = 7.50%; CPT ⇒ PV = –95.586

If interest rates fall by 50 basis points, the estimated price of the bond is 104.701%. Therefore, the duration of the bond is:

$$\text{duration} = \frac{104.701 - 95.586}{2(100)(0.005)} = 9.115$$

So, for a 100 basis point (1%) change in required yield, the expected price change is 9.115%. In other words, if the yield on this bond goes up by 1%, the price should fall by about 9.115%. If yield drops by 1%, the price of the bond should rise by approximately 9.115%.

DV01 vs. Duration

LO 61.5: Compare and contrast DV01 and effective duration as measures of price sensitivity.

While DV01 measures the change in *dollar value* of a security for every basis point change in rates, duration measures the *percentage* change in a security's value for a unit change in rates.

As it turns out, we can use duration to calculate the DV01 as follows:

$$\text{DV01} = \text{duration} \times 0.0001 \times \text{bond value}$$

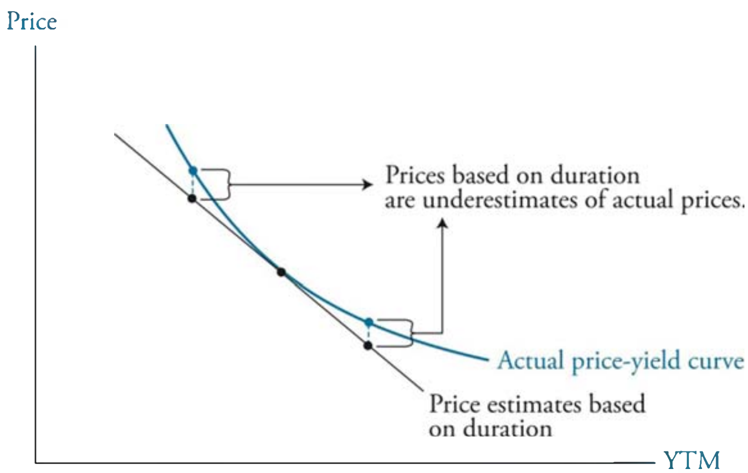
Duration is more convenient than DV01 in an investing context, in that a high duration number can easily alert an investor of a large percentage change in value. However, when analyzing trading or hedging situations, percentage changes are not that useful because dollar amounts of the two sides of the transaction are different. In this case, DV01 would be more useful.

Convexity

LO 61.6: Define, compute, and interpret the convexity of a fixed income security given a change in yield and the resulting change in price.

Duration is a good approximation of price changes for relatively small changes in interest rates. Like DV01, duration is a linear estimate since it assumes that the price change will be the same regardless of whether interest rates go up or down. As rate changes grow larger, the curvature of the bond price-yield relationship becomes more important, meaning that a linear estimate of price changes will contain errors. Figure 1 illustrates why convexity is important and why estimates of price changes based solely on duration are inaccurate.

Figure 1: Duration-Based Price Estimates vs. Actual Bond Prices



Convexity is a measure of the curvature in the relationship between bond yield and price. An understanding of convexity can illustrate how a bond's duration changes as interest rates change. The formula for convexity is as follows:

$$\text{convexity} = \frac{1}{BV}\frac{d^2BV}{dy^2}$$

The second term in this equation is the second derivative of the price-yield function. The first derivative measures how price changes with yields (i.e., duration), while the second derivative measures how the first derivative changes with yields (i.e., convexity). Generally, the higher the convexity number, the higher the price volatility.

While a precise calculation of convexity involves the use of calculus, an approximate measure of convexity can be generated as follows:

$$\text{convexity} = \frac{BV_{-\Delta y} + BV_{+\Delta y} - 2 \times BV_0}{BV_0 \times \Delta y^2}$$

Example: Computing convexity

Suppose there is a 15-year option-free noncallable bond with an annual coupon of 7% trading at par. If interest rates rise by 50 basis points (0.50%), the estimated price of the bond is 95.586%. If interest rates fall by 50 basis points, the estimated price of the bond is 104.701%. **Calculate** the convexity of this bond.

Answer:

$$\text{convexity} = \frac{104.701 + 95.586 - 2(100)}{(100)(0.005)^2} = 114.8$$

Unlike duration, a convexity of 114.8 cannot be conveniently converted into some measure of potential price volatility. Indeed, the convexity value means nothing in isolation, although a higher number does mean more price volatility than a lower number. This value can become very useful, however, when it is used to measure a bond's *convexity effect*, because it can be combined with a bond's duration to provide a more accurate estimate of potential price change.

Price Change Using Both Duration and Convexity

Now, by combining duration and convexity, a far more accurate estimate of the percentage change in the price of a bond can be obtained, especially for large swings in yield. That is, the amount of convexity embedded in a bond can be accounted for by adding the convexity effect to duration effect as follows:

$$\text{percentage price change} \approx \text{duration effect} + \text{convexity effect}$$
$$= \left[-\text{duration} \times \Delta y \times 100\right] + \left[\left(\tfrac{1}{2}\right) \times \text{convexity} \times (\Delta y)^2 \times 100\right]$$

Example: Estimating price changes with the duration/convexity approach

Using the duration/convexity approach, estimate the effect of a 150 basis point increase and decrease on a 15-year, 7%, option-free bond currently trading at par. The bond has a duration of 9.115 and a convexity of 114.8.

Answer:

Using the duration/convexity approach:

$$\Delta BV\_\% \approx [-9.115 \times -0.015 \times 100] + \left[\left(\tfrac{1}{2}\right) \times 114.8 \times (-0.015)^2 \times 100\right]$$

$$= 13.6725\% + 1.2915\% = 14.9640\%$$

$$\Delta BV_{+}\% \approx [-9.115 \times 0.015 \times 100] + \left[\left(\tfrac{1}{2}\right) \times 114.8 \times (0.015)^2 \times 100\right]$$

$$= -13.6725\% + 1.2915\% = -12.3810\%$$

Portfolio Duration and Convexity

LO 61.7: Explain the process of calculating the effective duration and convexity of a portfolio of fixed income securities.

The duration of a portfolio of individual securities equals the weighted sum of the individual durations. Each security's weight is its value taken as a percentage of the overall portfolio value.

$$\text{duration of portfolio} = \sum_{j=1}^{K} w_j \times D_j$$

where:
D_j = duration of bond j
w_j = market value of bond j divided by market value of portfolio
K = number of bonds in portfolio

Like portfolio duration, portfolio convexity is calculated as the value-weighted average of the individual bond convexities within a portfolio.

Example: Computing portfolio duration

Assume there are three bonds in a portfolio, with portfolio weightings and individual durations shown as follows:

Coupon	*Maturity (years)*	*YTM*	*Price (% of par)*	*Weights*	*Duration*
1%	5	0.75%	102.916	20%	4.02
2%	15	1.25%	109.579	35%	9.63
3%	30	2.125%	118.297	45%	13.75

Calculate the portfolio duration.

Answer:

$$\text{duration of the portfolio} = (0.20 \times 4.02) + (0.35 \times 9.63) + (0.45 \times 13.75) = 10.36$$

A significant problem with using portfolio duration as a measure of interest rate exposure is its implication that all the yields for every bond in the portfolio are perfectly correlated. This is a severely limiting assumption and should be of particular concern in global portfolios because it is unlikely that yields across national borders are perfectly correlated.

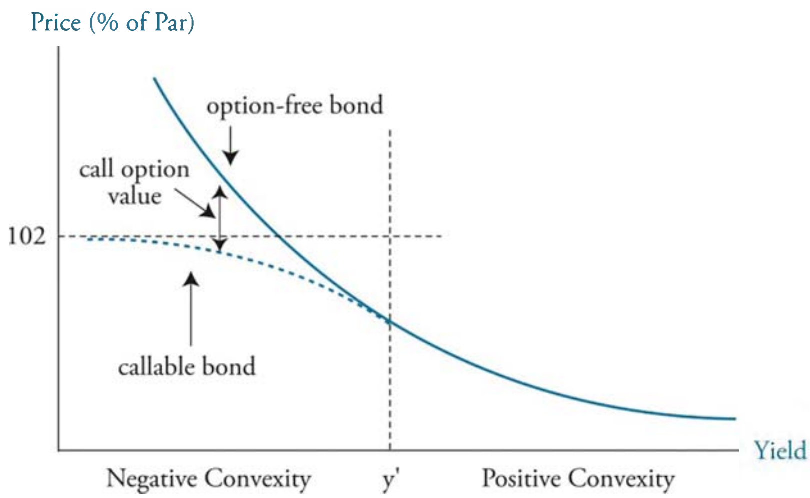
Negative Convexity

LO 61.8: Explain the impact of negative convexity on the hedging of fixed income securities.

With **callable debt**, the upside price appreciation in response to decreasing yields is limited (sometimes called price compression). Consider the case of a bond that is currently callable at 102. The fact that the issuer can call the bond at any time for $1,020 per $1,000 of face value puts an effective upper limit on the value of the bond. As Figure 2 illustrates, as yields fall and the price approaches $1,020, the price-yield curve rises more slowly than that of an identical but noncallable bond. When the price begins to *rise at a decreasing rate* in response to further decreases in yield, the price-yield curve "bends over" to the left and exhibits **negative convexity**.

Thus, in Figure 2, so long as yields remain *below level* y', callable bonds will exhibit *negative convexity*; however, at yields *above level* y', those same callable bonds will exhibit *positive convexity*. In other words, at higher yields the value of the call options becomes very small, so that a callable bond will act very much like a noncallable bond. It is only at lower yields that the callable bond will exhibit negative convexity.

Figure 2: Price-Yield Function of a Callable vs. an Option-Free Bond



In terms of price sensitivity to interest rate changes, the slope of the price-yield curve at any particular yield tells the story. Note that as yields fall, the slope of the price-yield curve for the callable bond decreases, becoming almost zero (flat) at very low yields. This tells us how a call feature affects price sensitivity to changes in yield. At higher yields, the interest rate risk of a callable bond is very close or identical to that of a similar option-free bond. At lower yields, the price volatility of the callable bond will be much lower than that of an identical, but noncallable, bond.

Convexity is an exposure to volatility, so as long as interest rates move, bond returns will increase when convexity is positive. Conversely, when convexity is negative, movement in either direction reduces returns. In other words, if an investor wishes to be "long volatility," a security exhibiting positive convexity should be chosen, and if short volatility is desired, a security exhibiting negative convexity should be chosen.

Constructing a Barbell Portfolio

LO 61.9: Construct a barbell portfolio to match the cost and duration of a given bullet investment, and explain the advantages and disadvantages of bullet versus barbell portfolios.

A **barbell strategy** is typically used when an investment manager uses bonds with short and long maturities, thus forgoing any intermediate-term bonds. A **bullet strategy** is used when an investment manager buy bonds concentrated in the intermediate maturity range.

The advantages and disadvantages of a barbell versus a bullet portfolio are dependent on the investment manager's view on interest rates. If the manager believes that rates will be especially volatile, the barbell portfolio would be preferred over the bullet portfolio.

Professor's Note: Since duration is linearly related to maturity, it is possible for a bullet and a barbell strategy to have the same duration. However, since convexity increases with the square of maturity, these two strategies will have different convexities.

Assume that a portfolio manager is considering buying a \$100 million U.S. Treasury security with coupons paying 1⅝s due in 10 years, at a cost of \$99,042,300. The manager is initially comfortable with the pricing of the bond at its present yield of 1.72%, but in looking at two other Treasury bonds on either side of the selected bond, one with a shorter maturity and one with a longer maturity, the manager wishes to consider alternatives.

Consider the three bonds in the following table, with maturities of 5 years, 10 years, and 30 years:

Coupon	*Maturity*	*Price*	*Yield*	*Duration*	*Convexity*
¾	5 years	100.0175	0.74%	4.12	21.9
1⅝	10 years	99.0423	1.72%	7.65	59.8
2¾	30 years	97.4621	2.88%	14.93	310.5

Instead of buying the "bullet" investment of 10-year 1⅝s, the manager is considering a "barbell" strategy, whereby he would buy the shorter maturity and the longer maturity bonds. The barbell portfolio can be constructed to have the same cost and duration as the individual bullet investment as follows:

V^5 = value in barbell portfolio of the 5-year bonds

V^{30} = value in barbell portfolio of the 30-year bonds

The barbell will have the same cost when: $V^5 + V^{30} = \$99,042,300$

The duration of the barbell equals the duration of the bullet when:

$$\frac{V^5}{99,042,300} \times 4.12 + \frac{V^{30}}{99,042,300} \times 14.93 = 7.65$$

With this equation, we can compute the proportion of each bond to purchase:

$P \times 4.12 + (1 - P) \times 14.93 = 7.65$

$4.12P + 14.93 - 14.93P = 7.65$

$-10.81P = -7.28$

$P = 0.6735$

Thus, V^5 = 67.35% of the portfolio and V^{30} = 32.65% of the portfolio.

The combined convexity of the barbell portfolio can be computed as follows:

$67.35\% \times 21.9 + 32.65\% \times 310.5 = 116.1$

The barbell's convexity of 116.1 is greater than the bullet's convexity of 59.8. Thus, for the same amount of duration risk, the barbell portfolio has greater convexity.

Looking at the weighted yield of the barbell portfolio we have:

$$67.35\% \times 0.74\% + 32.65\% \times 2.88\% = 1.439\%$$

This is compared to the bullet's yield of 1.72%. Thus, the barbell portfolio will not do as well as the bullet portfolio, assuming yields stay the same. If the manager projects that rates will be especially volatile, the barbell portfolio may be preferred.

Key Concepts

LO 61.1

Interest rate factors are random variables that influence individual interest rates along the yield curve.

LO 61.2

DV01 is the absolute change in bond price for every basis point change in yield, which is essentially a basis point's price value.

The DV01 formula is:

$$DV01 = -\frac{\Delta BV}{10{,}000 \times \Delta y}$$

where:
ΔBV = change in bond value
Δy = change in yield

LO 61.3

The hedge ratio when hedging a bond with another bond is calculated as:

$$HR = \frac{\text{DV01 (initial position)}}{\text{DV01 (hedging instrument)}}$$

LO 61.4

Duration measures the percentage change in a security's value for a particular unit's change in rates. The formula for effective duration is:

$$\text{duration} = \frac{BV_{-\Delta y} - BV_{+\Delta y}}{2 \times BV_0 \times \Delta y}$$

LO 61.5

DV01 works better for hedgers, while duration is more convenient for traditional investors.

LO 61.6

Convexity is a measure of the degree of curvature in the price-yield relationship:

$$\text{convexity} = \frac{BV_{-\Delta y} + BV_{+\Delta y} - 2 \times BV_0}{BV_0 \times \Delta y^2}$$

LO 61.7

$$\text{duration of portfolio} = \sum_{j=1}^{K} w_j \times D_j$$

where:
D_j = duration of bond j
w_j = market value of bond j divided by market value of portfolio
K = number of bonds in portfolio

Convexity for the entire portfolio is simply the value-weighted average of each individual security's convexity within the portfolio.

LO 61.8

Convexity is an exposure to volatility so as long as interest rates move; returns will increase when convexity is positive. If an investor wishes to be "long volatility," a positively convex security should be chosen.

LO 61.9

A barbell strategy is typically used when an investment manager uses bonds with short and long maturities, thus forgoing any intermediate-term bonds. A bullet strategy is used when an investment manager buy bonds concentrated in the intermediate maturity range.

Concept Checkers

Use the following information to answer Questions 1 and 2.

An investor has a short position valued at $100 in a 10-year, 5% coupon, T-bond with a YTM of 7%. Assume discounting occurs on a semiannual basis.

1. Which of the following is closest to the dollar value of a basis point (DV01)?
   A. 0.065.
   B. 0.056.
   C. 0.047.
   D. 0.033.

2. Using a 20-year T-bond with a DV01 of 0.085 to hedge the interest rate risk in the 10-year bond mentioned above, which of the following actions should the investor take?
   A. Buy $130.75 of the hedging instrument.
   B. Sell $130.75 of the hedging instrument.
   C. Buy $76.50 of the hedging instrument.
   D. Sell $76.50 of the hedging instrument.

3. The duration of a portfolio can be computed as the sum of the value-weighted durations of the bonds in the portfolio. Which of the following is the most limiting assumption of this methodology?
   A. All the bonds in the portfolio must change by the same yield.
   B. The yields on all the bonds in the portfolio must be perfectly correlated.
   C. All the bonds in the portfolio must be in the same risk class or along the same yield curve.
   D. The portfolio must be equally weighted.

4. Estimate the percentage price change in bond price from a 25 basis point increase in yield on a bond with a duration of 7 and a convexity of 243.
   A. 1.67% decrease.
   B. 1.67% increase.
   C. 1.75% increase.
   D. 1.75% decrease.

5. An investor is estimating the interest rate risk of a 14% semiannual pay coupon bond with 6 years to maturity. The bond is currently trading at par. The effective duration and convexity of the bond for a 25 basis point increase and decrease in yield are closest to:

	Duration	Convexity
A.	3.970	23.20
B.	3.740	23.20
C.	3.970	20.80
D.	3.740	20.80

Concept Checker Answers

1. **A** For a 7% bond, N = 10 × 2 = 20; I/Y = 7/2 = 3.5%; PMT = 5/2 = 2.5; FV = 100; CPT → PV = –85.788

   For a 7.01% bond, N = 20; I/Y = 7.01/2 = 3.505%; PMT = 2.5; FV = 100; CPT → PV = –85.723

   For a 6.99% bond, N = 20; I/Y = 6.99/2 = 3.495%; PMT = 2.5; FV = 100; CPT → PV = –85.852

   $DV01{+}\Delta y = |\,85.788 - 85.723\,| = 0.065$

   $DV01{-}\Delta y = |\,85.788 - 85.852\,| = 0.064$

2. **C** The hedge ratio is 0.065 / 0.085 = 0.765. Since the investor has a short position in the bond, this means the investor needs to buy \$0.765 of par value of the hedging instrument for every \$1 of par value for the 10-year bond.

3. **B** A significant problem with using portfolio duration is that it assumes all yields for every bond in the portfolio are perfectly correlated. However, it is unlikely that yields across national borders are perfectly correlated.

4. **A** $\Delta BV_+\% \approx [-7\times 0.0025\times 100]+\left|\left(\tfrac{1}{2}\right)\times 243\times (0.0025)^2\times 100\right| = -1.67\%$

5. **C** N = 12; PMT = 7; FV = 100; I/Y = 13.75/2 = 6.875%; CPT → PV = 100.999

   N = 12; PMT = 7; FV = 100; I/Y = 14.25/2 = 7.125%; CPT → PV = 99.014

   $\Delta y = 0.0025$

   $$\text{Duration} = \frac{100.999 - 99.014}{2(100)0.0025} = 3.970$$

   $$\text{Convexity} = \frac{100.999 + 99.014 - 200}{(100)(0.0025)^2} = 20.8$$

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Multi-Factor Risk Metrics and Hedges

Topic 62

Exam Focus

This topic divides the term structure of interest rates into several regions, and makes assumptions regarding how rates change for each region. Key rate analysis measures a portfolio's exposure to changes in a few key rates—for instance, 2-year, 5-year, 10-year, and 30-year rates. The key rate method is straightforward and assumes that rates change in the region of the key rate chosen. The forward-bucket method is similar to the key rate approach, but instead uses information from a greater array of rates, specifically those built into the forward rate curve. For the exam, understand how to apply key rate shift analysis and be able to calculate key rate '01, key rate duration, and the face amount of hedging positions given a specific key rate exposure profile.

Weaknesses of Single-Factor Approaches

LO 62.1: Describe and assess the major weakness attributable to single-factor approaches when hedging portfolios or implementing asset liability techniques.

A single-factor approach to measuring and hedging risk in fixed income markets is quite limiting because it assumes that within the term structure of interest rates (typically referred to as the yield curve), all rate changes are driven by a single factor (i.e., the term structure shifts in a parallel fashion).

It is more realistic to instead recognize that rates in different regions of the term structure are not always correlated. The risk that rates along the term structure move differently (i.e., nonparallel shifts) is called **yield curve risk**. The single-factor approach does not protect against yield curve risk; however, it is easy to compute and understand for hedging or asset-liability management, as only one security is needed to hedge the risk of a large portfolio.

The simplifying assumption that rates of all terms move up or down by the same amount based on one factor, such as the 10-year par rate (i.e., the 10-year swap rate), is restrictive; thus, practitioners apply multi-factor approaches. These multi-factor approaches, such as key rate and bucket approaches, assume that rate changes are a function of two or even more factors.

Key Rate Exposures

LO 62.2: Define key rate exposures and know the characteristics of key rate exposure factors including partial '01s and forward-bucket '01s.

Key rate exposures help describe how the risk of a bond portfolio is distributed along the term structure, and they assist in setting up a proper hedge for a bond portfolio. Key rate exposures are utilized for measuring and hedging risk in bond portfolios using rates from the most liquid bonds available, which are generally government bonds that have been issued recently and are selling at or near par.

Similar to key rate exposures, **partial '01s** are utilized for measuring and hedging risk in swap portfolios (or a portfolio with a combination of bonds and swaps). These partial '01s are derived from the most liquid money market and swap instruments for which a swap curve is usually constructed. **Forward-bucket '01s** are also used in swap and combination bond/swap contexts, but instead of measuring risk based on other securities, they measure risk based on changes in the shape of the yield curve. Thus, forward-bucket '01s enable us to understand a portfolio's yield curve risk. Partial '01s and forward-bucket '01s are similar to key rate approaches, but use more rates, which divide the term structure into many more regions.

Key Rate Shift Technique

LO 62.3: Describe key-rate shift analysis.

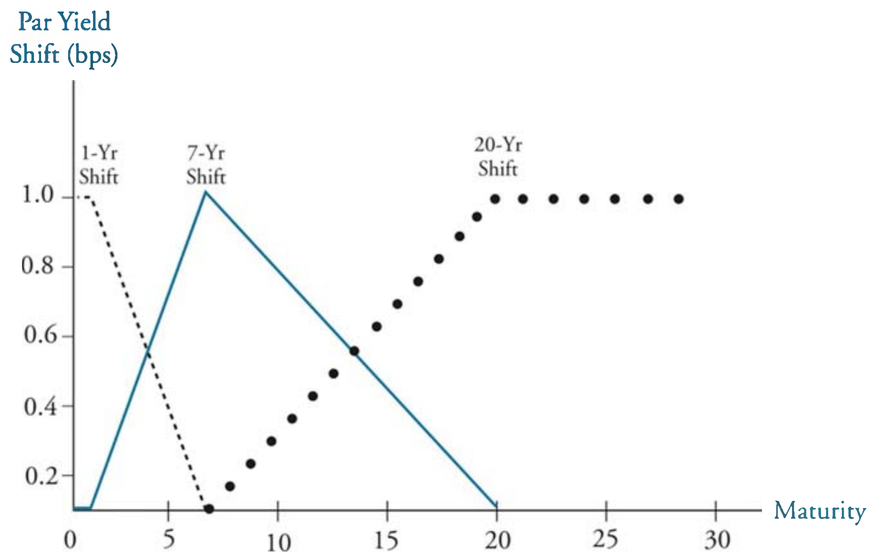
Key rate shift analysis makes the simplifying assumption that all rates can be determined as a function of a few "key rates." To cover risk across the entire term structure, a small number of key rates are used, pertaining only to the most liquid government securities.

The most common key rates used for the U.S. Treasury and related markets are par yield bonds—2-, 5-,10- and 30-year par yields. If one of these key rates shifts by one basis point, it is called a **key rate shift**. Note that par yields are also referred to as par rates.

The key rate shift technique is an approach to nonparallel shifts in the yield curve, which allows for changes in all rates to be determined by changes from selected key rates.

For example, assume that there are three key rates: 1-year, 7-year, and 20-year par yields. The key rate technique indicates that changes in each key rate will affect rates from the term of the previous key rate to the term of the subsequent key rate. In this case, the 1-year key rate will affect all rates from 0 to 7 years; the 7-year key rate affects all rates from 1 year to 20 years; and the 20-year key rate affects all rates from 7 years to the end of the curve. If one assumes a simplistic one basis point effect, the impact of each key rate will be one basis point at each key rate and then a linear decline to the subsequent key rate. This key rate shift behavior is illustrated in Figure 1.

Figure 1: Key Rate Shifts



This is obviously a somewhat limiting and simplistic approach. However, the key rate approach is appealing because (1) key rates are affected by a combination of rates closest to them; (2) key rates are mostly affected by the closest key rate; (3) key rate effects are smooth (do not jump across maturity); and (4) a parallel shift across the yield curve results.

Key Rate '01 and Key Rate Duration

LO 62.4: Define, calculate, and interpret key rate '01 and key rate duration.

The following example demonstrates the calculation of key rate '01 and key rate duration, using a 30-year zero-coupon bond. Zero-coupon securities are also referred to as STRIPS (separate trading of registered interest and principal securities). Investors of zero-coupon bonds receive payment from STRIPS at maturity.

Figure 2: Key Rate '01s and Durations of a C-STRIP

	(1) Value	*(2) Key Rate '01*	*(3) Key Rate Duration*
Initial value	25.11584		
2-year shift	25.11681		
5-year shift	25.11984	–0.0040	–1.59
10-year shift	25.13984		
30-year shift	25.01254		

Column (1) in Figure 2 provides the initial price of a C-STRIP and its present value after application of key rate one basis point shifts.

Column (2) in Figure 2 includes the key rate '01s (i.e., the key rate DV01s). A **key rate '01** is the effect of a dollar change of a one basis point shift around each key rate on the value of the security. For example, the key rate '01 with respect to the 5-year shift is calculated as follows:

$$-\frac{1}{10{,}000}\frac{25.11984-25.11584}{0.01\%}=-0.0040$$

This implies that the C-STRIP increases in price by 0.0040 per $100 face value for a positive one basis point 5-year shift. Like DV01, the key rate '01 is negative when value, after a given shift, increases relative to the initial value. The key rate '01 would be positive if value, after a given shift, declines relative to the initial value.

Continuing with the same 5-year shift, its **key rate duration** is calculated as follows:

$$-\frac{1}{25.11584}\frac{25.11984-25.11584}{0.01\%}=-1.59$$

Completing Column (3) in Figure 2 and summing all key rate durations would give us the effective duration of the 30-year C-STRIP. Note that key rate duration can also be computed using the corresponding key rate '01 and initial value as follows:

$$\frac{-0.0040}{25.11584}\times 10{,}000=-1.59$$

Hedging Applications

LO 62.5: Describe the key rate exposure technique in multi-factor hedging applications; summarize its advantages and disadvantages.

For every basis point shift in a key rate, the corresponding key rate '01 provides the *dollar change* in the value of the bond. Similarly, key rate duration provides the approximate *percentage change* in the value of the bond. Key rate duration works off 100 basis point changes, so it is the percentage of price movement for every 100 basis point change in rates.

The important information to be collected from these calculations is the bond's price sensitivity to shifts in each key rate. Key rate shifts allow for better hedging of a bond position, and when summed across all key rates, assume a parallel shift across all maturities in the maturity spectrum.

As mentioned, key rate exposure analysis is a useful tool for measuring bond price sensitivity; however, it makes very strong assumptions about how the term structure behaves. It assumes that the rate of a given term is affected only by the key rates that surround it. In reality, shifts are not always perfectly linear.

LO 62.6: Calculate the key rate exposures for a given security, and compute the appropriate hedging positions given a specific key rate exposure profile.

Suppose a 30-year semiannual-paying noncallable bond pays a $4,500 semiannual coupon in a flat rate environment of 5% across all maturities. If we assume a one basis point shift in the key rates used (2-, 5-, 10- and 30-year key rates), the subsequent key rate effects on the security are as shown in Figure 3.

Figure 3: Key-Rate Exposure of 30-Year Semiannual Pay Non-Callable Bond

	(1) Value	*(2) Key Rate '01*	*(3) Key Rate Duration*
Initial value	139,088.95		
2-year shift	139,083.96	4.99	0.36
5-year shift	139,074.21	14.74	1.06
10-year shift	139,015.04	73.91	5.31
30-year shift	139,024.25	64.70	4.65
Total		158.34	11.38

To illustrate hedging based on key rates, suppose that four other securities (shown below) exist in addition to the noncallable bond just discussed and that each of these new hedging securities have the following key rate exposures:

- A 2-year security only has a 2-year key rate exposure of 0.015 per $100 face value.
- A 5-year security has exposures over the 2-year and 5-year key rate of 0.0025 and 0.035, respectively, per $100 face value.
- A 10-year security has exposures over the 2-year, 5-year, and 10-year key rates of 0.003, 0.015, and 0.1, respectively, per $100 face value.
- A 30-year security only has exposure to the 30-year key rate of 0.15 per $100 face value.

It is assumed in this example that the 2-year bond and the 30-year bond are trading at par, so their only exposure is to the key rate corresponding to the maturity date. Using the key rate exposures from Figure 3 generates the following set of equations to establish the hedge:

2-year key-rate exposure: $$\frac{0.015}{100}\times F_2 + \frac{0.0025}{100}\times F_5 + \frac{0.003}{100}\times F_{10} = 4.99$$

5-year key-rate exposure: $$\frac{0.035}{100}\times F_5 + \frac{0.015}{100}\times F_{10} = 14.74$$

10-year key-rate exposure: $$\frac{0.1}{100}\times F_{10} = 73.91$$

30-year key-rate exposure: $$\frac{0.15}{100}\times F_{30} = 64.70$$

By simultaneously solving for F_2, F_5, F_{10}, and F_{30}, these equations indicate that the investor needs to short the 2-year security in the face amount of $16,745, short the 5-year in the face amount of $10,439, short the 10-year in the face amount of $73,910, and short the 30-year in the face amount of $43,133. Combining these short positions with the initial bond position will immunize the portfolio from changes in rates close to the key rates selected.

As is the case with most duration-based hedging techniques, the assumption of interest rate movements drives the effectiveness of immunized strategies. There are two factors at work when using key rates in an immunization-type setting. If interest rates change more dramatically than indicated, the immunized position will not perform as expected. This nonperformance will be exacerbated given larger changes in interest rates. More importantly, however, is the assumption of how interest rates will change around and between key rates. If the assumed rate shifts do not change in accordance with the assumed path indicated by the key rate technique, the effectiveness of the immunized position will be decreased. Losses or gains will accrue, which will directly affect the immunization strategy.

Simply stated, using the key rates in an immunized setting will only be an approximation of the effectiveness of immunization. This is a direct result of the dependence of the technique on the ultimate size and movement of rates in and around the key rates chosen. The only way immunization will work perfectly in a real-world setting is if all sources of interest rate changes are perfectly matched.

Partial '01s and Forward-Bucket '01s

LO 62.7: Relate key rates, partial '01s and forward-bucket '01s, and calculate the forward bucket '01 for a shift in rates in one or more buckets.

Key rate shifts utilize just a few key rates, and express position exposures in terms of hedging securities. For example, if we assume the key rates are 2-year, 5-year, 10-year, and 30-year par yields, each exposure is measured and hedged separately, and all four securities are needed to hedge the fixed income position.

With more complex portfolios that contain swaps, partial '01s and forward-bucket '01s are often used instead of key rates. These approaches are similar to the key rate approach, but instead divide the term structure into more parts. Risk along the yield curve is thus measured more frequently, in fact daily.

For example, swap market participants fit a par rate curve (i.e., swap rate curve) daily or even more frequently, using a group of observable par rates and short-term money market/futures rates. A partial '01 will measure the change in the value of the portfolio from a one basis point decrease in the fitted rate and subsequent refitting of the curve. In other words, with partial '01s, yield curve shifts are able to be fitted more precisely because we are constantly fitting securities.

The forward-bucket '01 approach is a more direct and mechanical approach for looking at exposures. Forward-bucket '01s are computed by shifting the forward rate over several regions of the term structure, one region at a time, after the term structure is divided into various buckets. For example, under this approach, we can analyze bond price changes after shifting the forward rate over the 2- to 5-year term bucket. Figure 4 illustrates the computation of forward-bucket '01s of a 5-year swap given a 0–2 year bucket and a 2–5 year bucket.

Figure 4: Forward-Bucket Exposures

		Forward Rate %			
Term	*Cash Flow*	*Current*	*0–2 Year Shift*	*2–5 Year Shift*	*Shift All*
0.5	1.06	1.012	1.022	1.012	1.022
1.0	1.06	1.248	1.258	1.248	1.258
1.5	1.06	1.412	1.422	1.412	1.422
2.0	1.06	1.652	1.662	1.652	1.662
2.5	1.06	1.945	1.945	1.955	1.955
3.0	1.06	2.288	2.288	2.298	2.298
3.5	1.06	2.614	2.614	2.624	2.624
4.0	1.06	2.846	2.846	2.856	2.856
4.5	1.06	3.121	3.121	3.131	3.131
5.0	101.06	3.321	3.321	3.331	3.331
PV		**99.9955**	**99.9760**	**99.9679**	**99.9483**
Forward-Bucket ‘01			**0.0196**	**0.0276**	**0.0472**

This example demonstrates how semiannual rates shift using a forward-bucket ‘01 approach. The PV line is the present value of the cash flows under the initial forward-rate curve, as well as under each of the “shifted” curve scenarios. To compute the forward-bucket ‘01 for each shift, take the difference between the shifted and the initial present values, and change the sign. For example, for the 0–2 year shift, the forward-bucket ‘01 is: (99.9760 – 99.9955), or 0.0196.

Hedging Across Forward-Bucket Exposures

LO 62.8: Construct an appropriate hedge for a position across its entire range of forward bucket exposures.

Suppose a counterparty enters into a euro 5×10 payer swaption with a strike of 4.044% on May 28, 2010. This payer swaption gives the buyer the right to pay a fixed rate of 4.044% on a 10-year euro swap in five years. The underlying is a 10-year swap for settlement on May 31, 2015. Figure 5 gives the forward-bucket ‘01s of this swaption for four different buckets, along with other swaps for hedging purposes.

Since the overall forward-bucket ‘01 of the payer swaption is negative (–0.0380), as rates rise, the value of the option to pay a fixed rate of 4.044% in exchange for a floating rate worth par also rises.

Figure 5: Forward-Bucket Exposures

Security	*Rate*	*0–2*	*2–5*	*5–10*	*10–15*	*All*
5×10 payer swaption	4.044%	0.0010	0.0016	–0.0218	–0.0188	–0.0380
5-year swap	2.120%	0.0196	0.0276	0.0000	0.0000	0.0472
10-year swap	2.943%	0.0194	0.0269	0.0394	0.0000	0.0857
15-year swap	3.290%	0.0194	0.0265	0.0383	0.0323	0.1164
5×10 swap	4.044%	0.0000	0.0000	0.0449	0.0366	0.0815

Continuing with this example, Figure 6 shows forward-bucket exposures of three different ways to hedge this payer swaption (as of May 28, 2010) using the securities presented in Figure 5. As you can see, the third hedge is the best option since this hedge best neutralizes risk in each of the buckets (the lowest net position indicates when risk is best neutralized).

Figure 6: Hedging with Forward-Bucket Exposures

Security / Portfolio	*0–2*	*2–5*	*5–10*	*10–15*	*All*
5×10 payer swaption	0.001	0.0016	–0.0218	–0.0188	–0.0380
Hedge #1:					
Long 44.34% of 10-year swaps	0.0086	0.0119	0.0175		0.038
Net position	0.0096	0.0135	–0.0043	–0.0188	0.000
Hedge #2:					
Long 46.66% of 5×10 swaps			0.0209	0.0171	0.038
Net position	0.001	0.0016	–0.0009	–0.0017	0.000
Hedge #3:					
Long 57.55% of 15-year swaps	0.0112	0.0153	0.022	0.0186	0.067
Short 61.55% of 5-year swaps	–0.012	–0.017			–0.029
Net position	0.0002	–0.0001	0.0002	–0.0002	0.000

Estimating Portfolio Volatility

LO 62.9: Apply key rate and multi-factor analysis to estimating portfolio volatility.

Key rates and bucket analysis allow a manager to use more than a single factor to manage interest rate risk effects on a portfolio. These multi-factor approaches work well not only in estimating changes in the level of the portfolio, but also in the estimation of portfolio volatility because it incorporates correlation effects between various interest rate assumptions.

Suppose one has information related to the volatility effects of two key rates. In this case, a manager can use traditional portfolio volatility relationships not only to incorporate the volatility impacts of each individual key rate, but also to incorporate the correlation between each key rate. The bucket technique works in a similar fashion, but because it is based on estimating forward rate effects, the number of inputs and correlation pairs that must be incorporated is greater.

Key Concepts

LO 62.1

The single-factor approach to hedging risk of fixed-income portfolios is limiting because it assumes that all future rate changes are driven by a single factor.

LO 62.2

Key rate exposures hedge risk by using rates from a small number of available liquid bonds. Partial '01s are used with swaps, and use a greater number of securities. Forward-bucket '01s are also used with swaps and use predefined regions to determine changes due to shifts in forward rates. Forward-bucket '01s enable us to understand a portfolio's yield curve risk.

LO 62.3

The key rate shift technique is a multi-factor approach to nonparallel shifts in the yield curve that allows for changes in all rates to be determined by changes in key rates. Choices have to be made regarding which key rates shift and how the key rate movements relate to prior or subsequent maturity key rates.

LO 62.4

Key rate '01s are calculated as follows:

$$DV01^k = -\frac{1}{10{,}000}\frac{\Delta BV}{\Delta y^k}$$

Key rate durations are calculated as follows:

$$D^k = -\frac{1}{BV}\frac{\Delta BV}{\Delta y^k}$$

LO 62.5

For every basis point shift in a key rate, the corresponding key rate '01 provides the dollar change in the value of the bond. Similarly, key rate duration provides the approximate percentage change in the value of the bond.

LO 62.6

Hedging positions can be created in response to shifts in key rates by equating individual key rate exposures adjacent to key rate shifts to the overall key rate exposure for that particular key rate change. The resulting positions indicate either long or short positions in securities to protect against interest rate changes surrounding key rate shifts.

LO 62.7

A partial '01 is the change in the value of the portfolio from a one basis point decrease in the fitted rate and subsequent refitting of the curve. Forward-bucket '01s are computed by shifting the forward rate over several regions of the term structure, one region at a time, after the term structure is divided into various buckets.

LO 62.8

In order to set up a proper hedge for a swap position across an entire range of forward-bucket exposures, the hedger will determine the various forward-bucket exposures for several different swaps and select the hedge that contains the lowest forward-bucket exposures in net position.

LO 62.9

Multifactor approaches to hedging, such as key rate and bucket shift approaches, can be used to estimate portfolio volatility effects because they incorporate correlations across a variety of interest rate effects.

Concept Checkers

1. The main problem associated with using single-factor approaches to hedge interest rate risk is:
   A. no method can hedge interest rate risk.
   B. single-factor models assume mean-reversion between one short-term and one long-term rate.
   C. single-factor models assume effects across the entire curve are dictated by one rate.
   D. single-factor models assume risk-free securities have credit exposure.

2. Using key rates of 2-year, 5-year, 7-year, and 20-year exposures assumes all of the following except that the:
   A. 2-year rate will affect the 5-year rate.
   B. 7-year rate will affect the 20-year rate.
   C. 5-year rate will affect the 7-year rate.
   D. 2-year rate will affect the 20-year rate.

Use the following information to answer Questions 3 and 4.

The following table provides the initial price of a C-STRIP and its present value after application of a one basis point shift in four key rates.

	Value
Initial value	25.11584
2-year shift	25.11681
5-year shift	25.11984
10-year shift	25.13984
30-year shift	25.01254

3. What is the key rate '01 for a 30-year shift?
   A. –0.058.
   B. 0.024
   C. 0.103.
   D. 0.158.

4. What is the key-rate duration for a 30-year shift?
   A. –4.57.
   B. 15.80.
   C. 38.60.
   D. 41.13.

5. Assume you own a security with a 2-year key rate exposure of $4.78, and you would like to hedge your position with a security that has a corresponding 2-year key rate exposure of 0.67 per $100 of face value. What amount of face value would be used to hedge the 2-year exposure?
   A. $478.
   B. $239.
   C. $713.
   D. $670.

Concept Checker Answers

1. **C** Single-factor models assume that any change in any rate across the maturity spectrum can indicate changes across any other portion of the curve.

2. **D** Key rate exposures assume that key rates chosen adjacent to the rate of interest are affected, not across other key rates.

3. **C** Key rate '01 with respect to the 30-year shift is calculated as follows:

$$-\frac{1}{10,000}\frac{25.01254-25.11584}{0.01\%}=0.103$$

This implies that the C-STRIP decreases in price by 0.103 per 100 face amount for a positive one basis point 30-year shift.

4. **D** Key-rate duration for the 30-year shift is calculated as follows:

$$-\frac{1}{25.11584}\frac{25.01254-25.11584}{0.01\%}=41.13$$

5. **C** $\frac{0.67}{100}\times F=\$4.78$

$F=\$713.43$

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Country Risk: Determinants, Measures and Implications

Topic 63

Exam Focus

Sovereign risks vary across countries. Factors such as the country's political risk, legal risk, position in the economic growth cycle, and economic diversity affect the investor's overall risk. Rating agencies rate sovereign risks, but tend to be slow to change sovereign ratings and are prone to regional biases, herding behavior, and in some cases upwardly biased ratings. In the last three decades, countries have relied less on bank debt and have issued more bonds. This has given rise to analyzing country risk by using sovereign default risk spreads. For the exam, be able to compare and contrast the advantages of sovereign debt ratings and sovereign default risk spreads. Understand the issues that can impede the value of ratings, such as herding behavior, timeliness, and rating biases. Also, be able to identify sources of sovereign risk and describe the consequences that result from sovereign defaults. Finally, be able to describe sovereign defaults across time, including the main time periods that saw the majority of defaults.

Sources of Country Risk

LO 63.1: Identify sources of country risk

Investors are increasingly exposed to **country risk**, both directly and indirectly. First, innovations in financial markets have made investing in non-domestic companies easier for investors. Individuals now have a range of investment options including mutual funds that invest in foreign markets, exchange-traded funds (ETFs), and certificates issued by domestic banks representing shares of foreign stock that the bank holds in trust but that are traded on domestic exchanges [called American depositary receipts (ADRs) in the United States and global depositary receipts (GDRs) in European markets]. Second, companies are increasingly global and many firms rely on growth in foreign markets to drive profits and returns to shareholders. Risk exposures vary across countries and there are different sources of risk that contribute to these variations.

Key sources of country risk include: (1) where the country is in the economic growth life cycle, (2) political risks, (3) the legal systems of countries, including both the structure and the efficiency of legal systems, and (4) the disproportionate reliance of a country on one commodity or service.

Country Risk Exposure

LO 63.2: Explain how a country's position in the economic growth life cycle, political risk, legal risk, and economic structure affect its risk exposure.

The key sources of country risk affect a country's risk exposure in the following ways:

Economic Growth Life Cycle

More mature markets and companies within those markets are less risky than those firms and countries in the early stages of growth. Young companies rely on a stable macro environment and often have limited resources to withstand setbacks relative to mature companies. Early growth and emerging market countries are more vulnerable and are thus exposed to higher levels of risk than mature countries. Early growth countries generally have higher growth rates in recoveries and larger declines in gross domestic product (GDP) growth in downturns than their more mature counterparts. Equity markets are impacted as well. For example, in 2008, U.S. and European equity markets dropped approximately 25% to 30% while emerging markets fell 50% or more. This is evidence that there is greater risk in early growth global investing even in countries that have sound legal systems and are well governed.

Political Risk

Political risk is broad and includes everything from whether a country is a democracy or a dictatorship to the smoothness with which a country transfers political power (e.g., military coups vs. democratic elections). There are at least four components of political risk, including:

- *Continuous versus discontinuous risk.* While it may be surprising to those living in a democracy, some investors prefer the stability of investing in companies in countries with autocratic leadership (i.e., one leader controls decision making). The notion is that government policies are locked in and predictable compared to a democracy where an election can change government policies fairly significantly. Risks in democracies are continuous but generally low. In contrast, risks in dictatorships are discontinuous. Policies change much less frequently, but changes are often severe and difficult to protect against (i.e., discontinuous risk). Studies are mixed regarding which system, authoritarian or democratic, results in higher economic growth.
- *Corruption.* There are costs associated with government corruption. Corruption costs can be likened to an implicit tax, directly reducing company profits and returns and indirectly reducing investor returns. Because the "tax" is not explicit and may also result in legal sanctions against the firm operating in the corrupt system (e.g., if a firm is caught bribing an official), it increases risk. Using survey data, Transparency International ranks countries based on the level of corruption that is present. According to the 2014 survey, Denmark is recognized as the least corrupt country while North Korea is perceived as the most corrupt in the survey (see https://www.transparency.org/cpi2014/results for survey results and an interactive map that lets the user compare countries and regions of the world visually).

- *Physical violence*. There are economic costs (e.g., insurance and security costs) and physical costs (e.g., possible physical harm to employees or investors) associated with countries in conflict. The Institute for Peace and Economics publishes a Global Peace Index (see http:/www.visionofhumanity.org/#/page/indexes/global-peace-index for the map).
- *Nationalization and expropriation risk*. Firms that perform well may see their profits expropriated via arbitrary taxation by governments. A firm may be nationalized, in which case the owners will receive much less than the true value of the company. Risks are greater in countries where nationalization and/or expropriation are possible.

Legal Risk

The protection of property rights (i.e., the structure of the legal system) and the speed with which disputes are settled (i.e., the efficiency of the legal system) affect risk. If disputes cannot be settled in a timely fashion, it is in essence the same as a system that does not protect property rights at all. Several non-government organizations have created an international property rights index which can be found at: http://internationalpropertyrightsindex.org/. North America and Western Europe are ranked the highest, while Latin America and Africa are ranked the lowest in terms of protection of property rights. Not surprisingly, many of the countries that are ranked the least corrupt by Transparency International are also ranked high on the property rights index.

Economic Structure

A disproportionate reliance on a single commodity or service in an economy increases a country's risk exposure. For example, the economies of some countries rely almost exclusively on oil production. Other countries have the bulk of their economic output tied to banking or insurance while still others depend on tourism. A downturn in the demand (and/or price) for the good or service on which the country is dependent can devastate the economy, increasing risks for businesses and investors. It is not only the affected industry but also all of the businesses in the country/region that can be affected by a downturn. The United National Conference on Trade and Development (UNCTAD) measures the degree of dependence on commodities in emerging markets (see http://unctad.org/en/PublicationsLibrary/suc2014d7_en.pdf). The study indicates that Africa and Latin America are especially dependent on commodity exports, resulting in economies that are highly sensitive to changes in commodity prices, which increases country risk.

Professor's Note: This phenomenon can happen in regions, countries, and even neighborhoods. For example, when much of the trading was shut down in the futures pits in Chicago in July 2015, there were many reports that described the economic impact on the "region," restaurants, retailers, barbershops, and shoeshine stands, all facing imminent demise as a result of the death of the key economic engine in the neighborhood.

Investors must assess all of the sources of country risk not only in isolation but also in conjunction with each other, when analyzing non-domestic investment opportunities.

Evaluating Country Risk

LO 63.3: Evaluate composite measures of risk that incorporate all types of country risk and explain limitations of the risk services.

There are numerous services that attempt to evaluate country risk in its entirety, including:

- **Political Risk Services (PRS)**. This for-profit firm (i.e., the risk assessments are available to paying members of the service) evaluates more than 100 countries on the key areas of country risk. Political, economic, and financial risk dimensions are evaluated using 22 variables to measure risks. The firm provides a composite score as well as a score on each of the three dimensions. In the 2015 update, Syria was ranked lowest (i.e., the riskiest country) and Switzerland was ranked highest (i.e., the least risky country) in terms of aggregated country risks.
- ***Euromoney.*** The magazine surveys 400 economists who assess country risk factors and rank countries from 0 to 100 with higher numbers indicating lower risk. See http://www.euromoneycountryrisk.com/ for risk assessments.
- ***The Economist.*** Currency risk, sovereign debt risk, and banking risks are assessed by the magazine to develop country risk scores. In this risk measure, low scores indicate lower risk and high scores indicate higher risk.
- **The World Bank**. The World Bank compiles risk measures from several sources on 215 countries. Risk measures assess six areas, including the level of corruption, government effectiveness, political stability, rule of law, voice and accountability, and regulatory quality. Positive numbers imply less risk and negative numbers imply more risk. See http://info.worldbank.org/governance/wgi/index.aspx#home for risk assessments.

For investors and businesses to use country risk ratings, they must be meaningful to their specific concerns. There are many limitations associated with these risk services that may diminish their value to businesses and investors. First, there is no standardization across the information providers. The World Bank's ratings are scaled around zero. Higher scores indicate lower risk in the *Euromoney* and PRS rankings while lower scores indicate lower risk in *The Economist's* rankings. This makes it difficult to compare country risk assessments across providers. Second, the methodologies used to generate scores are often developed by non-businesses and may have more relevance to economists and policymakers than to businesses and investors. Third, the scores are better used as rankings than as true scores. For example, a country with a ranking of 70 in *The Economist* rankings can be interpreted as riskier than a country with a ranking of 35. However, one cannot conclude, using the rankings, that a country with a score of 70 is twice as risky as a country with a score of 35.

Sovereign Defaults

LO 63.4: Compare instances of sovereign default in both foreign currency debt and local currency debt, and explain common causes of sovereign defaults.

Sovereign default risk refers to the risk that holders of government-issued debt fail to receive the full amount of promised interest and principal payments during the specified time period. Sovereign default risk can be used as a proxy for country risk. Sovereign default categories include foreign currency defaults and local currency defaults.

Foreign Currency Defaults

Throughout history, some governments have relied on debt borrowed from other countries or banks in those countries. The debt, denominated in the foreign currency, is called **foreign currency debt**. Countries often default on foreign currency debt. Countries may be without the foreign currency to meet the obligation and are unable to print money in a foreign currency to repay the debt. A large proportion of sovereign defaults are foreign currency defaults. Between 2000 and 2014 there have been at least 12 foreign currency defaults. The largest sovereign default during the period was Argentina, defaulting on more than $82 billion. The debt was subsequently restructured.

Over the last 200 years there have been many instances of default. The defaults primarily occurred in seven distinct time periods: 1824–1834, 1867–1882, 1890–1900, 1911–1921, 1931–1940, 1976–1989, and 1998–2003. In Europe, the greatest number of defaults occurred from 1931 to 1940. Nine countries defaulted, including Hungary in 1931, Austria, Bulgaria, and Germany in 1932, Romania and Serbia-Yugoslavia in 1933, Poland in 1936, and Italy and Turkey in 1940. The same period saw a rash of foreign currency defaults in Latin America. Fourteen countries defaulted, primarily in 1931 and 1932. The early 1980s was another key period for Latin American defaults with 16 countries defaulting, 15 of which occurred between 1980 and 1983. Many Latin American countries also defaulted in the 1826–1832 period (14 countries defaulted) and the 1867–1880 period (13 countries defaulted). There have also been defaults in Asia and Africa over the last 50 years.

In a study on sovereign defaults between 1975 and 2004 conducted by Standard and Poor's, the firm notes:

1. Countries were more likely to default on funds borrowed from banks than on sovereign bond issues.

2. Latin America accounts for a large proportion of sovereign defaults in the last 50 years (measured in dollar value terms).

Except for the 1990s, in each of the last five decades Latin America has accounted for at least 60% of the foreign currency defaults. In fact, Latin America has been at the center of sovereign defaults for the last 200 years. To put it in historical perspective, in the 19th century Latin America attracted capital from France, Britain, and Spain because of abundant natural resources. Latin American countries did not have significant domestic savings and thus borrowed heavily in gold and in foreign currency during this period. Maturities were long, usually greater than 20 years. Military coups and conflicts were the primary triggers of default on this debt. Between 1820 and 1914, 58 of the 77 sovereign defaults worldwide occurred in Latin America. During the 115-year period, Uruguay spent 12% of the time in default (the shortest period) and Honduras, on the high end, spent 79% of the time in default. Between 1820 and 1940, Latin American countries collectively spent 38% of the time in default.

Local Currency Defaults

Many of the countries that defaulted on foreign currency debt over the last several decades were simultaneously defaulting on local (i.e., domestic) country debt. An S&P study of local currency defaults since 1975 indicates that 23 issuers have defaulted. Defaulting countries range from Russia in 1998–1999 to Argentina in 2002–2004. The largest local currency default in dollar terms occurred in 1990 when Brazil defaulted on $62 billion of debt. Russia defaulted on $39 billion of ruble debt in 1998–1999.

A study by Moody's finds that countries are increasingly defaulting on both foreign and local currency debts concurrently as shown in Figure 1.

Figure 1: Changes in Concurrent Default in Foreign and Local Currency Debt

Default Type	*1960–1996*	*1997–2007*
Foreign Currency Only	57%	29%
Local Currency Only	38%	29%
Both Foreign and Local	5%	42%

You can see that the instance of concurrent defaults (i.e., local currency and foreign currency simultaneous defaults) has increased from 5% in the 1960 to 1996 period to 42% in the 1997 to 2007 period. Foreign currency only and local currency only defaults fell during the same period.

It is difficult to explain why countries default on local currency debt. It would seem that countries would simply print more money to meet their obligations. However, there are three reasons that help explain local currency defaults.

1. **The Gold Standard.** Prior to 1971, some countries followed the gold standard. This means the country was required to have gold reserves to back currency. The gold standard thus limited the amount of currency a country could print, reducing its flexibility in terms of printing currency to repay debt.

2. **Shared Currency.** The euro is an example of a shared currency. The advantage of a shared currency is convenience for businesses, tourists, and so on. It eliminates the costs of converting currencies and increases transparency. However, a shared currency limits the abilities of individual countries to print money. For example, during the recent Greek debt crisis, as a member of the European Union (EU), Greece was not able to print currency to pay off debt.

3. **Currency Debasement.** There are costs associated with printing money. Printing money may devalue and debase the currency. It also leads to higher inflation, sometimes exponentially higher inflation. There are costs associated with default and costs associated with printing money. Some countries decide that the costs associated with default are the lesser of the two evils.

Consequences of Sovereign Default

LO 63.5: Describe the consequences of sovereign default.

Historically, defaults were often followed by military actions. For example, when Egypt defaulted in the 1880s, Britain used military force to take over the government. In the 20th century, countries that defaulted suffered a loss of reputation, making it more difficult and more expensive to borrow in the future. Countries also experienced turmoil in stock and bond markets, faced a decrease in real output, and had to deal with political instability as a result of default. An examination of research on sovereign defaults leads to the following conclusions:

- **GDP growth.** Gross domestic product (GDP) growth falls between 0.5% and 2.0% following a sovereign default. However, the decline is short-lived. The majority of the decline is experienced in the first year following the default.
- **Sovereign ratings and borrowing costs.** One study finds that ratings of countries that have defaulted at least once since 1970 are one to two grades lower than the ratings of similar countries that have not defaulted. Also, borrowing costs are 0.5%–1.0% higher. The effects lessen over time.
- **Trade retaliation.** Sovereign default can cause trade retaliation. Export businesses are most affected. One study finds an average 8% drop in bilateral trade following a default. The study also finds the effects can last up to 15 years.
- **Fragile banking systems.** One study finds, based on 149 countries between 1975 and 2000, that there is a 14% probability of a banking crisis following a sovereign default, which is 11% higher than for non-defaulting countries.
- **Political change.** Sharp currency devaluations often follow defaults. While studies have not examined political change following defaults per se, they have studied political change following currency devaluations. Countries that default on debt are more likely to see a change in the president or prime minister (a 45% increase in the probability of a change) and the top finance minister or head of the central bank (a 64% increase in the probability of a change).

Factors Influencing Sovereign Default Risk

LO 63.6: Describe factors that influence the level of sovereign default risk; explain and assess how rating agencies measure sovereign default risks.

Individuals, companies, and governments default for many of the same reasons. Primarily, they each borrow more than they can afford in good times and find themselves unable to repay the borrowed funds during downturns. Several factors influence a country's sovereign default risk. They are as follows:

1. **The country's level of indebtedness.** The level of indebtedness is the most fundamental factor used to determine the risk of default. One must consider not only the country's debts to foreign banks and investors, but also the amount the country owes its own citizens (e.g., for social safety nets such as welfare and universal health care). For comparison purposes, debt is typically scaled to a country's GDP. Figure 2 lists the debts as a percent of GDP for several of the top 20 indebted countries, based on data from the CIA's *The World Factbook*[1]. Note that some of the highest debt countries (as a percent of GDP), such as Japan and France, are considered highly creditworthy by financial markets and rating agencies. This means that the level of indebtedness is not the only factor determining the risk of default. Other countries on the list, such as Zimbabwe, have high default risk. Portugal, Spain, and Greece are also on the list. These countries had high credit ratings prior to the 2007–2009 financial crisis, but have struggled to repay debt since the crisis.

1. *The World Factbook*, 2015, Central Intelligence Agency.

Figure 2: Government Debt as a Percent of GDP

Country	*Government Debt as a Percent of GDP*
Japan	227.70%
Zimbabwe	181.00%
Greece	174.50%
Italy	134.10%
Portugal	131.00%
Spain	97.60%
France	95.50%
Canada	92.60%

Debt in the United States has steadily increased as a percent of GDP since the financial crisis and is 72.6% according to *The World Factbook*. However, according to other sources, this number is greater than 100% as of 2014.

2. **Pension funds and social services.** Countries with greater pension commitments and health care commitments (e.g., Medicare in the United States) have higher default risk. Also, as these commitments increase as the population ages, countries with older populations face greater risks.

3. **Tax receipts.** The greater the tax receipts, the more able a country is to make debt payments. A larger tax base should increase a country's revenues (i.e., tax receipts) and therefore lower default risk.

4. **Stability of tax receipts.** Governments must pay debt obligations in both good and bad economic times. This means the revenue stream must be stable to meet these fixed obligations. Countries with more diversified economies are more likely to have stable tax receipts. Countries like Jamaica, which depend on tourism, and Peru, which depend on silver and copper production, have more sovereign default risk than the governments of larger, more diversified economies such as India. Also, sales and value added tax systems are generally more stable than income tax systems.

5. **Political risk.** Autocracies may be more likely to default than democracies because, as noted above, defaults put pressure on, and may cause a change in, the leadership of the country. There may be less pressure on the leaders of dictatorships if the country defaults. Also, the more independent the central bank, the more difficult it may be for a country to print money.

6. **Backing from other countries/entities.** Rating agencies and other market participants reassessed and decreased the estimated default risk of Spain, Greece, and Portugal when those countries joined the EU. It is assumed that stronger countries like Germany will help weaker countries and not allow them to default. However, there is no guarantee. It is an implicit, not explicit, backing.

In summary, sovereign default risk is multi-faceted and must be analyzed from many perspectives. The country's level of indebtedness, obligations to its citizens for things like pensions and medical care, and its tax systems are all relevant to assessing the risk of default.

In addition, the trustworthiness of the government and the nature of the economy must also be considered when evaluating sovereign default risk.

Rating Agencies and Default Risk

Rating agencies have distinct advantages when it comes to assessing sovereign default risk. First, the companies have been measuring default risk in corporations for more than a century. One would expect that at least some of the skills would transfer to sovereign risk evaluation. Additionally, investors are accustomed to the ratings used for companies, so understanding the risk of a AAA rated country relative to a C rated country is somewhat intuitive.

Moody's was rating nearly 50 governments by 1929. However, interest in government bonds abated following the Great Depression and World War II. As such, the interest in bond ratings also abated. Interest in government bonds picked up slowly in the 1970s and has continued to grow. By 1994 there were 49 rated countries, rated from AAA (the average rating from the 1970s to 1984, 15 countries) to A rated (the average rating for countries added between 1985 and 1989, 19 countries) to BBB rated (the average rating for countries added between 1990 and 1994, 15 countries).

The market for sovereign bonds has increased dramatically since 1994. Moody's, S&P, and Fitch currently rate more than 100 countries each. Moody's and S&P have improved the ratings, currently providing two ratings for each country, a local currency rating for domestic currency bonds and a foreign currency rating for borrowings in a foreign currency. Figure 3 provides some examples of Moody's sovereign bond ratings.

Figure 3: Sample of Moody's Sovereign Bond Ratings

Country	*Foreign Currency Rating*	*Foreign Currency Outlook*	*Local Currency Rating*	*Local Currency Outlook*
Argentina	Caa1	NEG (Negative)	Caa1	NEG (Negative)
Brazil	Baa2	NEG (Negative)	Baa2	NEG (Negative)
Honduras	B3	POS (Positive)	B3	POS (Positive)
Paraguay	Ba1	STA (Stable)	Ba1	STA (Stable)
Venezuela	Caa3	STA (Stable)	Caa3	STA (Stable)

The outlook column indicates the view of a potential ratings change. NEG indicates that there is a possibility of a downgrade, STA indicates a stable rating, and POS indicates the possibility of a ratings upgrade.

Generally, the local currency rating is at least as high as the foreign currency rating, because, as noted, countries can print money in local currency to repay debt. It is however possible for the local currency rating to be lower, as was the case with India in March 2010. In that instance, the local currency rating was Ba2 and the foreign currency rating was Baa3. Ratings are usually the same across agencies but may diverge.

Sovereign ratings may change across time, but change much less frequently than corporate bond ratings. Historically, there has been little change in ratings from year to year. For example, using Fitch's bond ratings data (from Fitch's ratings transitions tables) between 1995 and 2008, the calculated probability of a AAA rated government keeping the AAA rating was 99.42%. A BBB rated sovereign had an approximately 88% probability of remaining the same, an 8% chance of an upgrade, and a 4% change of a downgrade. This leads to one of the major criticisms of sovereign ratings, that they do not change quickly enough to alert investors of looming risks. More recent data indicates that ratings are changing more frequently, reflecting the true sovereign risks in a more timely fashion. For example, data based on 2010–2012 ratings, shows a BBB rated country has only a 57% probability of maintaining the same rating (compared to the 88% probability based on 1995–2008 data, noted above).

How Rating Agencies Measure Risk

The three main rating agencies use similar processes to determine sovereign ratings. The goal is to determine the creditworthiness of the country. The agencies focus on the default risk faced by banks and private bondholders, not official creditors such as the World Bank and the International Monetary Fund (IMF). S&P ratings focus on the probability of default while Moody's ratings focus on both the probability of default and the severity of the default (i.e., the rating attempts to capture the expected recovery rate in the event of default). Default is defined as the failure to pay principal or interest payments on the due date (i.e., outright default) or rescheduling or restructuring the debt (i.e., restructuring default).

The ratings process includes:

- **Evaluating factors that may contribute to default.** These factors are related to the economic, political, and institutional detail of the country with respect to its ability to repay debt. Regarding the sovereign ratings methodology profile, S&P uses variables classified based on political risk, economic structure, economic growth prospects, fiscal flexibility, general government debt burden, offshore and contingent liabilities, monetary flexibility, external liquidity, and the country's external debt burden. Fitch and Moody's use similar frameworks for analyzing the risk of default.
- **Ratings recommendation.** An analyst prepares a draft report, recommending the rating. A committee, composed of 5 to 10 people, debates each category (i.e., political risk, economic structure, and so on) and then votes on a score. The committee decides the final rating, following closing arguments.
- **Foreign currency versus local currency ratings.** As noted, the local currency rating is generally equal to or better than the foreign currency rating. The difference between the two is largely based on the country's monetary policy independence. Countries that have given up monetary policy independence (e.g., EU countries) will see their foreign and local currency ratings converge. Countries with floating rate exchange regimes that fund borrowing through sound domestic markets will have the greatest difference between the two ratings. Agencies use two approaches to arrive at the foreign versus local currency ratings:
    1. *Notch-up approach*. The foreign currency rating is the key indicator of sovereign default risk and the local currency rating is "notched up" based on the domestic debt market and other domestic factors.

    2. *Notch-down approach*. The local currency rating is the key indicator of sovereign credit risk and the foreign currency rating is "notched down" based on foreign exchange issues and constraints.

- Ratings review process. Ratings are reviewed on a periodic basis. In addition, news can trigger a ratings review. For example, an economic disaster in one country may trigger a review not only in that country, but also in neighboring countries that may be subject to a contagion effect.

Do Ratings Measure Risk?

Rating agencies argue that despite some mistakes, there is a high correlation between the sovereign risk ratings and sovereign default. Upon examination of S&P's cumulative default rates from 1975 to 2012, it appears that the rating agencies do indeed deliver on the promise. For example, AAA rated countries have a zero percent probability of default across the specified time horizon, while a CCC rated country sees the probability increase from approximately 40% to 66% across the time horizon.

However, rating agencies have been criticized on a number of counts. These include:

- **Ratings are biased upward**. Some argue that rating agencies are too optimistic when it comes to rating sovereigns and corporations. The conflict of interest that is often cited as a problem in corporate ratings (i.e., the company pays the rating agency for the rating) is not a problem with sovereign ratings. The country does not pay the rating agencies. However, there are some other conflicts that may result in an upward bias that are discussed below.
- **Herd behavior**. When one agency upgrades or downgrades a country, the other agencies tend to follow suit. This lack of independence reduces the benefit of having three rating agencies.
- **Not timely enough**. Investors need rating agencies to update ratings in a timely fashion. Some market participants feel that the agencies take too long to change ratings, leaving investors unprotected in the event of a crisis.
- **Overreaction leads to a vicious cycle**. Some argue that the agencies overreact to crises, lowering ratings too much in response to a crisis, creating a feedback effect that worsens the crisis.
- **Ratings failures**. One study examined multiple ratings changes by S&P's and Moody's, when the changes occur during the course of a single year. These multiple changes imply that the agencies were incorrect in their initial assessments of the country. The study offered several possible explanations for the ratings failures:
  - *Bad information*. Agencies rely on information from governments. There is significant variability in the amount and quality of the data that agencies receive across countries. If governments hide the truth and reveal only positive information, ratings will be incorrect. This may also help explain the upward bias discussed above.
  - *Overburdened analysts*. Rating sovereigns generates limited revenues for agencies. Analysts are spread thin and often rate four or five countries. This leads analysts to rely on common information available in the market rather than doing their own research. This may also contribute to the herd behavior discussed above.

- *Conflicts of interest resulting from revenue challenges.* Rating agencies generally do not charge users for sovereign ratings. This means revenues must come either from the issuers or from other businesses that arise from the sovereign ratings business such as ratings evaluation services, risk management services, and market indices. Rating agencies generate significant revenues from sub-sovereigns (e.g., states, provinces, counties, and cities). If some agency revenues arise from sovereigns or sub-sovereigns, there is an incentive for the agency to refrain from issuing harsh judgments against a country. A sovereign ratings downgrade is usually followed by a series of sub-sovereign downgrades. Sub-sovereigns will therefore fight the sovereign downgrade, contributing to the upward bias of sovereign ratings.
- *Other conflicts of interest.* As indicated above, there are off-shoot businesses from the core sovereign ratings business. These businesses may generate enough revenue to influence ratings. Typically, rating agency employees do not go work for the sovereigns they rate, thus this issue does not generally pose a conflict of interest.

Professor's Note: Along with the criticism that rating agencies do not react quickly enough to changes in country risk profiles, is the criticism that agencies show "regional bias." That is, the rating agencies underrate entire regions, most notably Africa and Latin America. The criticism is answered with the claim that the past is a good predictor of the future and Latin America has a lot of defaults in its past, thus deserving of lower ratings. In other cases, it is argued that sovereign ratings are too high (i.e., upward bias). Issues of conflicts of interest and poor information may contribute to the upward bias of ratings. In general, sovereign risks are complex and rating agencies do not generate significant revenues from this business and may understaff as a result. These issues all contribute to criticisms levied against rating agencies as it relates to measuring sovereign default risks.

The Sovereign Default Spread

LO 63.7: Describe the advantages and disadvantages of using the sovereign default spread as a predictor of defaults.

There has been significant growth in the sovereign bond market, beginning in the 1980s. More countries are deliberately avoiding bank debt and issuing bonds instead. As a result, there is another measure of sovereign default risk that may be used, the **sovereign default spread**. This measure is generated by the market and is continuously updated as sovereign bonds are traded. It is in essence a comparison of the sovereign bond yield compared to a riskless investment in the same currency and maturity. For example, a 10-year, dollar-denominated Brazilian bond was yielding 4.5% in July 2015. At the same time, the 10-year U.S. Treasury bond was yielding 2.47%. Because the U.S. Treasury bond is assumed to be risk free, the 2.03% difference between the two bond yields reflects the market's assessment of the default risk of the Brazilian bond. The 2.03% difference is the sovereign default spread.

Figure 4 shows the comparison of default spreads and Moody's ratings for a sample of dollar-denominated bonds in July 2015.

Figure 4: Default Spreads on Dollar-Denominated Latin American Bonds

Country	*Interest Rate on Dollar-Denominated 10-year Sovereign Bond*	*Interest Rate on 10-year U.S. Treasury Bond*	*Sovereign Default Spread*	*Moody's Sovereign Bond Rating*
Brazil	4.50%	2.47%	2.03%	Baa2
Colombia	4.05%	2.47%	1.58%	Baa3
Peru	3.93%	2.47%	1.46%	Baa2
Mexico	3.92%	2.47%	1.45%	Baa1

This figure yields some interesting information regarding the use of default spreads versus bond ratings. The most obvious comparison is Brazil and Peru. Moody's rated the bonds of the two countries the same, Baa2. However, the market demands a higher rate of return, and thus is seeing more risk, in the Brazilian bonds compared to the Peruvian bonds. The difference in default risk spreads between the two is 57 basis points (i.e., 2.03% – 1.46%). There is a strong, but not perfect, correlation between sovereign risk ratings and default risk spreads. However, there are some distinct advantages of sovereign default spreads relative to sovereign bond ratings.

Advantages of Default Risk Spreads

1. **Changes occur in real time.** Market-based spreads are more dynamic than ratings. As bonds trade and bond yields rise and fall, default risk spreads change, revealing information about the market's perception of risk. For example, in late 2005 the risk spreads of Brazil and Venezuela were close, 3.18% and 3.09%, respectively. Spreads started to diverge between 2006 and 2009. By December 2010, the default risk spread had widened to 10.26% on the Venezuelan bonds and narrowed to 1.32% on the Brazilian bonds. It is clear that the market's perception of the relative risk of the two countries was changing, as evidenced by the diverging spreads.

2. **Granularity.** Default risk spreads are more granular. Instead of A versus B or good versus bad, there are much finer comparisons to be made based on a 2.03% yield spread on Brazilian bonds (rated Baa2) compared to a 1.46% spread on Peruvian bonds (rated Baa2) compared to a 1.58% spread on Colombian bonds (rated Baa3).

3. **Adjust quickly to new information.** Similar to the advantage that changes occur in real time, yield spreads adjust more quickly to new information regarding the sovereign relative to bond ratings. This means investors are signaled earlier of impending threats and can adjust portfolios accordingly.

Disadvantages of Default Risk Spreads

1. **Need for a risk-free security.** In order to calculate a default risk spread, there must be a risk-free security in the currency in which the bonds are issued.

2. **Cannot compare local currency bonds.** Local currency bonds do not have a risk-free security with which to compare. You cannot compare local currency bonds with each other because differences in yields may reflect differences in expected inflation across

countries. Also, even with dollar-denominated bonds, it is the assumption that U.S. Treasury bonds are default risk free that makes calculating a yield spread meaningful.

3. **Greater volatility.** Default risk spreads are volatile and changes in spreads may be affected by variables that are unrelated to the default risk of the sovereign. For example, investor demand for the bonds and changes in liquidity can affect spreads but often have nothing to do with default risk.

Studies that have examined default risk spreads and ratings generally conclude:

- *Default risk spreads are "leading indicators."* The spreads widen before a ratings downgrade and narrow before a ratings upgrade.
- *Default risk spreads are positively correlated with ratings and with default.* In other words, low rated sovereign bonds are more likely to trade at higher yields (and yield spreads) and are more likely to default.
- *A ratings change provides information to the market, despite the longer lag time relative to default risk spreads.* Rating agencies use market information to make ratings changes. The market reacts to ratings and rating changes when pricing bonds. This means that both ratings and default risk spreads are useful to market participants in evaluating and understanding sovereign default risk.

Key Concepts

LO 63.1

Key sources of country risk include where the country is in the economic growth life cycle, political risks, the legal systems of countries, including both the structure and the efficiency of legal systems, and the disproportionate reliance of a country on one commodity or service.

LO 63.2

Regarding economic growth life cycle, more mature markets and companies within those markets are less risky than those firms and countries in the early stages of growth.

Regarding political risk, there are at least four components of political risk, including the level of corruption in the country, the occurrences of physical violence due to wars or civil unrest, the possibility of nationalization and expropriations, and the continuity and severity of risks versus discontinuous risks.

Regarding legal risks, the protection of property rights and the speed with which disputes are settled affect default risk.

Regarding economic structure, a disproportionate reliance on a single commodity or service in an economy increases a country's risk exposure.

LO 63.3

Companies such as Political Risk Services (PRS) and organizations such as *The Economist* and *Euromoney* evaluate more than 100 countries on key areas of country risk. Some are critical of these composite risk measures because they are not readily comparable with each other due to a lack of standardization across the information providers. Also, the methodologies used to generate scores are often developed by non-business entities and may have more relevance to economists and policymakers than to businesses and investors. Finally, the scores are better used as rankings than as a way to interpret the relative risk of countries.

LO 63.4

There are many causes of sovereign defaults. It is easier to understand foreign currency defaults than local currency defaults. Countries are often without the foreign currency to meet the debt obligation and are unable to print money to repay the debt. This makes up a large proportion of sovereign defaults.

Many of the countries that defaulted on foreign currency debt over the last several decades were simultaneously defaulting on local country debt. Three reasons may explain local currency defaults: (1) the use of the gold standard prior to 1971 made it more difficult for some countries to print money, (2) shared currencies, such as the euro, make it impossible for countries to control their own monetary policy, and (3) some counties must conclude that the costs of currency debasement and potentially higher inflation are greater than the costs of default.

LO 63.5

Historically, defaults were often followed by military actions. Research suggests the following additional consequences of sovereign defaults:

- GDP growth falls between 0.5% and 2.0% following a sovereign default.
- Borrowing costs are 0.5% to 1.0% higher following default.
- Sovereign default can cause trade retaliation.
- One study finds, based on 149 countries between 1975 and 2000, that there is a 14% probability of a banking crisis following a sovereign default, which is 11% higher than for non-defaulting countries.
- Sharp currency devaluations often follow defaults.

LO 63.6

Several factors determine a country's sovereign default risk. The country's level of indebtedness, obligations such as pension commitments and social service commitments, the country's level of and stability of tax receipts, political risks, and backing from other countries or entities all impact a country's likelihood of defaulting on sovereign debt.

Rating agencies consider several factors when evaluating default risk. These factors are related to the economic, political, and institutional characteristics of a country with respect to its ability to repay debt. The ratings process includes an analyst preparing a draft report and recommending a rating. A committee votes on a score and decides the final rating. Ratings are reviewed periodically and may also be reviewed following a news event that could affect the likelihood of default.

Rating agencies have been criticized on a number of counts, including the fact that ratings are biased upward, there is herd behavior among the major rating agencies (i.e., S&P, Moody's, and Fitch), sovereign rating changes are too slow to change, rating agencies often overreact to news about a country, and ratings are simply wrong in some cases.

LO 63.7

Advantages of default risk spreads relative to sovereign bond ratings are that changes occur in real time, risk premiums adjust to new information more quickly, and there is more granularity in default risk spreads than in risk ratings.

Disadvantages of default risk spreads include the fact that a default risk-free instrument is required with which to compare the sovereign yield, spreads are more volatile and may react to factors that have little to do with default risk (e.g., changes in liquidity and investor demand), and local currency bonds cannot be compared with each other because differences may reflect differences in expected inflation rather than differences in default risk.

Concept Checkers

1. Ravi Chowdhury, a portfolio manager for a hospital foundation, is considering the inclusion of sovereign bonds in the fixed income portion of the foundation's portfolio. Chowdhury, much to the surprise of his colleagues, plans to purchase the bonds of a country that has long been under authoritarian rule. He cites "lower political risk" when asked about his investment decision. Which of the following statements is most likely what Chowdhury means by his assertion of lower risk?
   A. Authoritarian regimes are more likely to control corruption in government agencies.
   B. Government policies that may affect debt repayment are often more stable under an authoritarian regime.
   C. Relative to a democracy, risks are greater on a day-to-day basis, but the effects are less detrimental overall.
   D. In most authoritarian countries, property rights are protected and property disputes are settled quickly.

2. In an attempt to understand country risk, Mary Ann Small, an analyst at Global Funds, examines multiple sources of information to determine the truest measure of risk. She considers sovereign risk ratings, default risk spreads, and composite measures of risk. Which of the following sources relies on surveys of several hundred economists to measures sovereign risk?
   A. Political Risk Services.
   B. *The Economist*.
   C. Standard and Poor.
   D. *Euromoney*.

3. Which of the following statements regarding foreign currency defaults is true?
   A. African countries are responsible for the greatest number of defaults in the last 50 years.
   B. Prior to the 20th century, no country had ever defaulted on funds borrowed in a foreign currency.
   C. Latin America accounted for more than 60% of foreign currency defaults in the 1990s.
   D. Countries are more likely to default on funds borrowed from foreign banks than on sovereign bond issues.

4. Following a sovereign default:
   A. borrowing costs rise 1.0% to 2.5% and the country is more likely to suffer a banking crisis.
   B. countries are more likely to suffer banking crises and GDP growth generally falls 2.0% to 3.5%, but the drop is usually short-lived.
   C. ratings fall one to two notches and GDP growth falls between 0.5% to 2.0%.
   D. trade retaliations are likely to occur but are short-lived, lasting only six months to one year.

5. Which of the following statements contributes to the upward bias of sovereign risk ratings?
   A. Historical evidence on sovereign defaults for regions such as Latin America leads rating agencies to overestimate those countries' abilities to meet their debt obligations.
   B. A lack of independence between the rating agencies contributes to the upward bias of ratings.
   C. Rating agencies' reliance on information provided by governments who may hide the truth and reveal only information that puts the country in a positive light.
   D. Analysts are spread thin and rely on common information rather than their own research.

Concept Checker Answers

1. **B** Some investors prefer the stability of investing in countries with autocratic governments because government policies are locked in and generally more predictable compared to democratic countries where an election can significantly change government policies. Risks in a democracy are continuous, but usually low. In contrast, risks in a dictatorship are discontinuous. Policies change much less frequently, but changes are often severe and difficult to protect against. Chowdhury is willing to accept the bigger, discontinuous risk as a tradeoff for the more frequent, but less damaging, continuous risk.

2. **D** Numerous services attempt to evaluate country risk in its entirety. They include Political Risk Services (PRS), *The Economist*, *Euromoney*, and the World Bank. *Euromoney* surveys 400 economists who assess country risk factors and rank countries from 0 to 100, with higher numbers indicating lower risk.

3. **D** Historically, countries have been more likely to default on foreign bank debt than on sovereign bonds. Latin America is responsible for the greatest number of foreign currency defaults over the last five decades with more than 60% of defaults in each decade with the exception of the 1990s. Over the last 200 years there are many instances of default. The defaults primarily occur in seven distinct time periods: 1824–1834, 1867–1882, 1890–1900, 1911–1921, 1931–1940, 1976–1989, and 1998–2003. Thus, countries did borrow and default in the 19th century.

4. **C** An examination of research on sovereign defaults leads to the following conclusions: (1) gross domestic product (GDP) growth falls between 0.5% and 2.0% following a sovereign default, and the decline is short-lived, (2) ratings of countries that have defaulted at least once since 1970 are one to two grades lower than the ratings of similar countries that have not defaulted and borrowing costs are 0.5% to 1.0% higher with the effects lessening over time, (3) sovereign default can cause trade retaliation, lasting up to 15 years, with export businesses most sharply affected, (4) a banking crisis is more likely following a default, and (5) sharp currency devaluations and changes in a country's leadership often follow defaults.

5. **C** Some argue that the rating agencies are too optimistic when it comes to rating sovereigns and corporations. The conflict of interest that is often cited as a problem in corporate ratings (i.e., that the company pays the rating agency for the rating) is not a problem with sovereign ratings. The country does not pay the rating agencies. However, there are other conflicts that may influence the bias. There is a regional bias against Latin America and Africa, that some argue biases those ratings downward. Herd behavior does exist but does not, per se, contribute to an upward bias. Analysts do rely on common information but this too should not contribute to the upward bias of ratings. Agencies do rely on information provided by governments which may be overly optimistic and contribute to upwardly biased ratings.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

External and Internal Ratings

Topic 64

Exam Focus

Credit ratings can apply to whole companies or individual investments. Agencies determine external ratings with both qualitative and quantitative methods, and the historical relationship between ratings and subsequent defaults is quite strong. Banks generate their own internal ratings, and they may use an at-the-point approach or a through-the-cycle approach. For the exam, have a general understanding on how external and internal credit ratings are established.

External Credit Ratings

LO 64.1: Describe external rating scales, the rating process, and the link between ratings and default.

External rating scales are designed to convey information about either a specific instrument, called an **issue-specific credit rating**, or information about the entity that issued the instrument, which is called an **issuer credit rating**, or both. The ratings are typically one-dimensional, and the **ratings scale** goes from the highest rating to the lowest in uniform increments. The highest grade on the scale represents the lowest amount of risk (maximum safety), and each move down the scale represents a reduction in safety, or an increase in risk. An example is Moody's ratings, which starts at Aaa and moves down to Aa, A, Baa, Ba, B, Caa, Ca, and C. Each successive move represents an increase in the expected loss caused by a default on the issue. Although the scale has many levels from Aaa to C, when applied to bonds, there is an important division that occurs between Baa and Ba. Ratings Baa and above are designated **investment grade**, and ratings Ba and below represent **non-investment grade**. Most other external ratings agencies have a similar set boundary between investment and non-investment grade bonds.

Professor's Note: Recall that Standard and Poor's uses a slightly different scale than Moody's. S&P uses the following ratings: AAA, AA, A, BBB, BB, B, CCC, CC, C, and D. The switch from investment grade to non-investment grade (i.e., junk) occurs between BBB and BB. A rating of D is considered a default rating.

The rating process requires the existence of an adequate amount of information and a defined analytical framework that can be applied globally. The **ratings process** usually consists of the following steps:

1. Conducting qualitative analysis (e.g., competition and quality of management).

2. Conducting quantitative analysis, which would include financial ratio analysis.

3. Meeting with the firm's management.

4. Meeting of the committee in the rating agency assigned to rating the firm.

5. Notifying the rated firm of the assigned rating.

6. Opportunity for the firm to appeal or offer new information.

7. Disseminating the rating to the public via the news media.

After the initial rating, the ratings agency monitors the firm and adjusts the rating as needed.

LO 64.6: Describe a ratings transition matrix and explain its uses.

Researchers have composed tables (known as a transition matrices) that show the frequency of default, as a percent, over given time horizons for bonds that began the time horizon with a given rating. These tables use historical data to report that for bonds that began a 5-year period with an Aa rating, for example, a certain percent defaulted during the five years. These tables demonstrate that the higher the credit rating, the lower the default frequency. An example of a transition matrix can be seen in Figure 1.

Figure 1: Transition Matrix from Moody's

Rating From:	*Rating To:*							
	Aaa	*Aa*	*A*	*Baa*	*Ba*	*B*	*Caa-C*	*Default*
Aaa	91.75%	7.26%	0.79%	0.17%	0.02%	0.00%	0.00%	0.00%
Aa	1.32%	90.71%	6.92%	0.75%	0.19%	0.04%	0.01%	0.06%
A	0.08%	3.02%	90.24%	5.67%	0.76%	0.12%	0.03%	0.08%
Baa	0.05%	0.33%	5.05%	87.50%	5.72%	0.86%	0.18%	0.31%
Ba	0.01%	0.09%	0.59%	6.70%	82.58%	7.83%	0.72%	1.48%
B	0.00%	0.07%	0.20%	0.80%	7.29%	80.62%	6.23%	4.78%
Caa-C	0.00%	0.03%	0.06%	0.23%	1.07%	7.69%	75.24%	15.69%

Example: Ratings Migration

Given the following one-year transition matrix, what is the probability that a B rated firm will default over a two-year period?

Rating From	*Rating To*			
	A	B	C	Default
A	90%	5%	5%	0%
B	5%	85%	5%	5%
C	0%	5%	80%	15%

Answer:

At the end of year 1, there is a 5% chance of default and an 85% chance that the firm will maintain a B rating.

In year 2, there is a 5% chance of default if the firm was rated B after 1 year (85% × 5% = 4.25%). There is a 0% chance of default if the firm was rated A after 1 year (5% × 0% = 0%). Also, there is a 15% chance of default if the firm was rated C after 1 year (5% × 15% = 0.75%).

The probability of default is 5% from year 1 plus 5% chance of default from year 2 (i.e., 4.25% + 0% + 0.75%) for a total probability of default over a two-year period of 10%.

LO 64.2: Describe the impact of time horizon, economic cycle, industry, and geography on external ratings.

Ratings agencies determine the external rating of a firm or bond using current information with the goal of indicating the probability of future events such as default and/or loss. The probability of default given any rating at the beginning of a cycle *increases with the horizon*. The increase in the default rates, or cumulative default rate, is much more dramatic for non-investment grade bonds. In addition to the condition of the firm, forecasted events in the horizon will affect the probabilities. The most notable events are the *economic and industrial cycles*. Since the rating should apply to a long horizon, in many cases, ratings agencies try to give *a rating that incorporates the effect of an average cycle*. This practice leads to the ratings being relatively stable over an economic or industrial cycle. Unfortunately, this averaging practice may lead to an over- or underestimate during periods when the economic conditions deviate too far from an average cycle. Also, the default rate of lower-grade bonds is correlated with the economic cycle, while the default rate of high-grade bonds is fairly stable.

Ratings agencies apply their ratings to different types of firms around the world, and the ratings may be interpreted differently given a specific industry and geographic location. Evidence shows that for a *given rating category, default rates can vary from industry to industry* (e.g., a higher percentage of banks with a given rating will default when compared with firms in other industries with the same rating). However, *geographic location does not seem to cause a similar variation of default* for a given rating class.

One firm may receive different ratings from different agencies. The degree to which ratings discrepancies may exist for a given firm can vary by industry and geographic location. The more ambiguous the data is, the more the ratings tend to vary, and the degree of the data's ambiguity can also vary by industry and geographic location. In addition, the ratings delivered by more specialized and regional agencies tend to be less homogeneous than those delivered by major players like S&P and Moody's.

LO 64.3: Explain the potential impact of ratings changes on bond and stock prices.

The evidence supporting the impact of ratings changes on bonds is not surprising:

- *A rating downgrade* is likely to make the *bond price decrease* (stronger evidence).
- *A rating upgrade* is likely to make the *bond price increase* (weaker evidence).

As indicated, the relationship is asymmetric in that the underperformance of recently downgraded bonds is more statistically significant than the over-performance of recently upgraded bonds. Measuring the exact effect can be difficult because a rating change can occur along with other changes in the firm (e.g., a restructuring and changes in the economy such as fluctuating interest rates). Also, some theorize that ratings changes tend to lag the inflow of information, and the market anticipates the change, so the impact on the price is minimized.

For stocks, the change in bond ratings has an even more asymmetric effect on the stock prices than it does on bond prices:

- *A rating downgrade* is likely to lead to a *stock price decrease* (moderate evidence).
- *A rating upgrade* is somewhat likely to lead to a *stock price increase* (evidence is mixed).

The relationship between a change in ratings and the stock price can be complex, and the effect is usually related either to the reason for or the direction of the rating change, or both. A downgrade from a fall in earnings will generally decrease stock prices, but a downgrade from an increase in leverage may leave the price of stocks the same or even increase them. Since firms tend to release good news more readily than bad news, downgrades may be more of a surprise, so downgrades affect stock prices more than upgrades when the firm reveals the good news associated with the upgrade prior to its occurrence.

Evolution of Internal Credit Ratings

LO 64.4: Compare external and internal ratings approaches.

The core business of a bank is to lend money; however, in order to continually make solid lending decisions, it is increasingly important for banks to create their own internal credit ratings. Today, a bank's process for developing internal credit ratings is largely based on techniques developed and applied by external credit rating agencies. Since external ratings models have been thoroughly tested and validated, it makes sense for banks to apply these techniques when assessing the creditworthiness of their own borrowers.

Previous internal credit ratings approaches were very simplistic, as they often just identified a company as being either a good or a bad borrower. This process lacked the ability to assign unique interest rates based on individual probability of default (PD) and loss given default (LGD). As a result, average interest rates were assigned to good borrowers based on average PDs and average recovery rates.

Two key factors have contributed to the increase in sophistication of internal credit ratings: the growing use of external credit rating agency language in the financial markets and the encouragement of Basel II rules to refine the approach for calculating credit risk capital requirements. Internal credit ratings models continue to improve, but key issues still exist regarding objectivity, data quality, time horizon, and consistency with external ratings.

Internal Credit Ratings

LO 64.5: Explain and compare the through-the-cycle and at-the-point internal ratings approaches.

Banks have increasingly been formulating their own internal ratings systems, which can vary from bank to bank. A given bank may have more than one system, such as an **at-the-point approach**, to score a company. This approach's goal is to predict the credit quality over a relatively short horizon of a few months or, more generally, a year. Banks use this approach and employ quantitative models (e.g., logit models) to determine the credit score. Other at-the-point models include those that are based on arbitrage between debt and equity markets, and structural models.

A bank may also use a **through-the-cycle approach**, which focuses on a longer time horizon and includes the effects of forecasted cycles. The approach uses more qualitative assessments. Given the stability of the ratings over an economic cycle, when using through-the-cycle approaches, high-rated firms may be underrated during growth periods and overrated during the decline of a cycle.

Some evidence suggests that ratings based on at-the-point methodologies tend to vary more over an economic cycle than ratings based on through-the-cycle methodologies. Generally though, the through-the-cycle and at-the-point approaches are not comparable. The users of the ratings should select the type of rating that suits their goal and horizon. However, some researchers have formulated models that derive longer-term, through-the-cycle ratings from a sufficient history of short-term, at-the-point probability of defaults.

Because of their short-term focus, the use of at-the-point approaches may be *procyclical* (i.e., they tend to amplify the business cycle). The reason for this is as follows:

economic downturn → rating downgrades → decrease in loans and economic activity
economic upturn → rating upgrades → increase in loans and economic activity

Furthermore, the changes in ratings and lending policies can *lag* the economic cycle, so just when the economy hits a trough and is about to start expanding, the banks may downgrade firms and restrict the credit they need to participate in the expansion.

LO 64.7: Describe the process for and issues with building, calibrating and backtesting an internal rating system.

One method for building an internal rating system is to create ratings that resemble those set by ratings agencies. A bank can accomplish this task by assigning weights to financial ratios and risk factors that have been deemed most important by the rating agency analyst. Internal rating templates can then be constructed to properly score a company (e.g., 0-100). This score is based on the pre-determined weights of important financial ratios and risk factors that contribute to the determination of a company's creditworthiness. To ensure the weights used are an accurate representation of reality, a comparison of a sample of internal ratings and external ratings is appropriate.

Internal rating systems are established to determine the credit risk of a bank's loans. In addition, they are also used for managing the bank's loan portfolio by assisting with the calculation of economic capital required. In order to accomplish these two objectives, internal ratings systems should properly reflect information from cumulative default probability tables.

However, before banks can link default probabilities to internal ratings, it is necessary to backtest the current internal rating system. Sufficient historical data of 11–18 years is appropriate to properly validate these ratings.[1] If a bank's transition matrix is found to be unstable, then different matrices will need to be constructed. Once a robust rating system is found, the link between the internal ratings and default rates can be established.

LO 64.8: Identify and describe the biases that may affect a rating system.

An internal rating system may be biased by several factors. The following list identifies the main factors[2]:

- *Time horizon bias*: mixing ratings from different approaches to score a company (i.e., at-the-point and through-the-cycle approaches).
- *Homogeneity bias*: inability to maintain consistent ratings methods.
- *Principal/agent bias*: moral hazard could result if bank employees do not act in the interest of management.
- *Information bias*: ratings assigned based on insufficient information.
- *Criteria bias*: allocation of ratings is based on unstable criteria.
- *Scale bias*: ratings may be unstable over time.
- *Backtesting bias*: incorrectly linking rating system to default rates.
- *Distribution bias*: using an incorrect distribution to model probability of default.

1 Carey and Hrycay. 2001. Parametrizing credit risk models with ratings data, *Journal of Banking and Finance*, 25, 197-270.

2 Servigny and Renault, *Measuring and Managing Credit Risk*. Chapter 2, Appendix 2C. New York: McGraw-Hill, 2004.

Key Concepts

LO 64.1

External rating scales are designed to convey information about either a specific instrument, called an issue-specific credit rating, or information about the entity that issued the instrument, which is called an issuer credit rating, or both.

The usual steps in the external ratings process include qualitative and quantitative analysis, a meeting with the firm's management, a meeting of the committee in the rating agency assigned to rating the firm, notification of the firm being rated of the assigned rating, an opportunity for the firm to appeal the rating, and an announcement of the rating to the public.

LO 64.2

The probability of default given any rating at the beginning of a cycle increases with the horizon.

Although external ratings have had a fairly good record in indicating relative rates of default, they are designed to be relatively stable over the business cycle (i.e., using an average cycle approach), which can produce errors in severe cycles.

Interpreting external ratings may vary based on the industry but not necessarily on the geographic location of the firm. Ratings delivered by more specialized and regional agencies tend to be less homogeneous than those delivered by major players like S&P and Moody's.

LO 64.3

Generally for bonds, a ratings downgrade is likely to make the price decrease, and an upgrade is likely to make the price increase. For stocks, a ratings downgrade is likely to lead to a stock price decrease, and an upgrade is somewhat likely to lead to a price increase.

LO 64.4

Since external credit ratings models have been thoroughly tested and validated, it makes sense for banks to apply these techniques when developing internal credit ratings to assess the creditworthiness of their own borrowers.

LO 64.5

The internal at-the-point ratings approach to score a company is usually short term, uses quantitative models like logit models, and produces scores that tend to vary over the economic cycle.

The internal through-the-cycle ratings approach to score a company has a longer horizon, uses more qualitative information, and tends to be more stable through the economic cycle.

Internal ratings can have a procyclical effect on the economy since banks often change ratings with a lag with respect to the change in the economy. Thus, after the economic trough has been reached, it is possible that a bank may downgrade a company poised for recovery with the use of additional credit from the bank.

LO 64.6

Transition matrices show the frequency of default, as a percentage, over given time horizons for bonds that began the time horizon with a given rating. These tables demonstrate that the higher the credit rating, the lower the default frequency.

LO 64.7

In order to build an internal rating system, banks should create ratings that resemble those set by ratings agencies. However, before banks can link default probabilities to internal ratings, it is necessary to backtest the current internal rating system.

LO 64.8

An internal rating system may be biased by several factors, including time horizon bias, information bias, criteria bias, and backtesting bias.

Concept Checkers

1. Which of the following is not part of the external ratings process? A(n):
   A. qualitative assessment.
   B. meeting with the representatives of the firm.
   C. determination of a fair market price of the bond or company.
   D. opportunity for the company being rated to appeal the rating.

2. External credit ratings scales indicate:
   A. the probability of default or the probability of loss.
   B. the probability of default but not the probability of loss.
   C. the probability of loss but not the probability of default.
   D. neither the probability of loss nor the probability default.

3. The longer the time horizon, the higher the incidence of default for a given rating. This effect:
   A. is equal for low-rated and high-rated bonds.
   B. is stronger for low-rated bonds than for high-rated bonds.
   C. is stronger for high-rated bonds than for low-rated bonds.
   D. has not been studied enough to be documented.

4. Given the effort by ratings agencies to incorporate the effect of an average cycle in external ratings, the ratings tend to:
   A. underestimate the probability of default in an economic expansion.
   B. overestimate the probability of default in an economic recession.
   C. underestimate the probability of default in an economic recession.
   D. be unbiased in all phases of the business cycle.

5. With respect to the effect on the price of a bond, the effect of a bond upgrade will:
   A. be positive and stronger than the downward effect of a bond downgrade.
   B. be positive and weaker than the downward effect of a bond downgrade.
   C. have about the same negative effect, in absolute value terms, as a bond downgrade.
   D. be negative and about equal to that of a bond downgrade.

Concept Checker Answers

1. **C** The ratings process does not directly determine prices. The other steps do occur, along with a quantitative assessment, a meeting of the committee in the rating agency assigned to the firm, and the release of the rating to the public.

2. **A** External credit ratings scales indicate either the probability of default, the probability of loss, or both.

3. **B** There is a very strong increase of defaults over time for low-rated bonds. High-rated bonds tend to have much more stable rates of default over time.

4. **C** Because the ratings agencies give ratings that tend to reflect an average business cycle and are generally stable through the cycle, a firm's probability of defaulting during a severe downturn may be underestimated based on the given rating.

5. **B** A bond's upgrade will have a positive effect on the bond's price, but the negative effect of a bond downgrade is generally stronger.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Capital Structure in Banks

Topic 65

Exam Focus

This topic discusses a bottom-up approach to calculating economic capital for credit risk and issues related to that approach. Since a bank holds many assets, we need to examine the expected and unexpected loss in a portfolio setting. The portfolio expected loss is the sum of the individual expected losses; however, portfolio unexpected loss is significantly less than the sum of individual unexpected losses due to diversification effects. We will derive an expression for unexpected loss equal to a fraction of the exposure amount. As you will see, default and credit migration increase the unexpected loss of a risky asset (i.e., a loan). For the exam, be familiar with the calculations of expected loss, unexpected loss, and the risk contribution of each asset in a portfolio. Also, know that economic capital is used to cover unexpected losses.

Credit Risk Factors

LO 65.2: Identify and describe important factors used to calculate economic capital for credit risk: probability of default, exposure, and loss rate.

The **probability of default (PD)**, also referred to as expected default frequency (EDF), is the likelihood that a borrower will default; however, this measure is not necessarily the creditor's greatest concern. A borrower may briefly default and then quickly correct the situation by making a payment or paying interest charges or penalties for missed payments. Creditors must rely on other measures of risk in addition to PD.

The **exposure amount (EA)**, also referred to as exposure at default (EAD), is the loss exposure stated as a dollar amount (e.g., the loan balance outstanding). EA can also be stated as a percentage of the nominal amount of the loan or the maximum amount available on a credit line.

The **loss rate (LR)**, also referred to as loss given default (LGD), represents the likely percentage loss if the borrower defaults. The severity of a default is equally as important to the creditor as the likelihood that the default would occur in the first place. If the default is brief and the creditor suffers no loss as a result, it is less of a concern than if the default is permanent and the creditor suffers significant losses. Both PD and LR are expressed as percentages. Note that, by definition, LR = 1 – recovery rate (RR). Therefore, the factors that affect the loss rate will also impact the recovery rate.

Expected Loss

LO 65.3: Define and calculate expected loss (EL).

Expected loss (EL) is defined as the anticipated deterioration in the value of a risky asset that the bank has taken onto its balance sheet. EL is calculated as the product of EA, PD, and LR:

$$EL = EA \times PD \times LR$$

This expected loss equation describes the average behavior of a risky asset. Over time, the value of the asset will fluctuate above and below its average level. At maturity, in most cases the asset will not have defaulted; however, a fraction of the time default will occur bringing a significant decrease in value. The EL measure does not capture the variation in the risky asset's value. This variation is referred to as *unexpected loss*.

The unanticipated loss on the risky asset can arise from the incidence of default or credit migration. Default is a positive probability event for even the safest of borrowers. Banks can estimate the likelihood of default using historical data, the method employed by rating agencies. On the other hand, default can be estimated using models based on the "option" view of the firm such as the Merton model (discussed in the FRM Part II curriculum). This approach views the firm as holding a call option with a strike price equal to the value of the outstanding debt. If the value of the firm is less than the value of its debt obligations, the firm will default.

Credit migration denotes the possible deterioration in creditworthiness of the borrower. While a shift in migration may not result in immediate default, the probability of such an event increases. It is also possible for the reverse to occur, that is, the credit quality of the obligor improves over time.

Unexpected Loss

LO 65.4: Define and calculate unexpected loss (UL).

LO 65.5: Estimate the variance of default probability assuming a binomial distribution.

As mentioned, **unexpected loss** (UL) represents the variation in expected loss. The observation that the unexpected loss represents the variability of potential losses can be modeled using the typical definition of standard deviation. If UL_H denotes the unexpected loss at the horizon for asset value V_H, then:

$$UL_H \equiv \sqrt{var(V_H)}$$

In the following equation, the subscript H will be dropped but be aware that we are focused on the horizon date, H. After some algebra, we derive the following expression:

$$UL = EA \times \sqrt{PD \times \sigma^2_{LR} + LR^2 \times \sigma^2_{PD}}$$

Since we assume a two-state model, the variance of PD is simply the variance of a binomial random variable:

$$\sigma^2_{PD} = PD \times (1 - PD)$$

Further note, the EA term explicitly recognizes that only the risky portion of the asset is subject to default.

Professor's Note: Do not lose sight of the big picture here. We are merely applying the basic definition for standard deviation based on the terminal value of the risky asset on the horizon date.

It is also worthwhile to examine the multiplier (square root term) in more detail. Notice that each term is at most equal to one so the UL is a fraction of the exposure amount. In addition, in the extreme case where the default ($\sigma^2_{PD} = 0$) and recovery ($\sigma^2_{LR} = 0$) are known with certainty, the unexpected loss equals zero, which confirms that the EL is constant and also known with certainty.

Example: Computing expected and unexpected loss

Suppose XYZ bank has booked a loan with the following characteristics: total commitment of $2,000,000 of which $1,800,000 is currently outstanding. The bank has assessed an internal credit rating equivalent to a 1% default probability over the next year. The bank has additionally estimated a 40% loss rate if the borrower defaults. The standard deviation of PD and LR is 5% and 30%, respectively. **Calculate** the expected and unexpected loss for XYZ bank.

Answer:

We can calculate the expected and unexpected loss as follows:

$$EL = EA \times PD \times LR$$

Exposure amount = $1,800,000
Probability of default = 1%
Loss rate = 40%

$$EL = \$1,800,000 \times 0.01 \times 0.40 = \$7,200$$

$$UL = EA \times \sqrt{PD \times \sigma^2_{LR} + LR^2 \times \sigma^2_{PD}}$$

$$UL = \$1,800,000 \times \sqrt{0.01 \times 0.3^2 + 0.4^2 \times 0.05^2} = \$64,900$$

The unexpected loss represents 3.61% of the exposure amount: ($64,900 / $1,800,000).

Portfolio Expected and Unexpected Loss

LO 65.6: Calculate UL for a portfolio and the risk contribution of each asset.

As mentioned previously, expected loss on the portfolio, EL_P, is the sum of the expected losses of each asset:

$$EL_P = \sum_i EL_i = \sum_i (EA_i \times LR_i \times PD_i)$$

The calculation of portfolio unexpected loss (UL_P) is more complicated from the cross-terms in the variance formula for an N-asset portfolio:

$$UL_P = \sqrt{\sum_i \sum_j \rho_{ij} UL_i UL_j}$$

where each individual unexpected loss follows the unexpected loss equation discussed previously. In the special case where each $\rho_{ij} = 1$ for $i \neq j$, UL_P = sum of individual unexpected losses. In most cases, UL_P will be significantly less than the sum of individual UL_i.

This equation demonstrates that the risk of the portfolio is much less than the sum of the individual risk levels and illustrates that each asset contributes to only a portion of its unexpected loss in the portfolio. This effect is captured by the partial derivative of UL_P

with respect to UL_i. Hence, the **risk contribution** (RC), also known as the **unexpected loss contribution** (ULC), is defined as:

$$RC_i = UL_i \frac{\partial UL_P}{\partial UL_i}$$

After differentiation, we see that:

$$RC_i = \frac{UL_i \sum_j UL_j \rho_{ij}}{UL_P}$$

Thus, RC_i isolates the incremental risk of adding asset *i* to the existing portfolio.

For a two-asset portfolio, the risk contributions of each asset are calculated as follows:

$$RC_1 = UL_1 \times \frac{UL_1 + (\rho_{1,2} \times UL_2)}{UL_P}$$

$$RC_2 = UL_2 \times \frac{UL_2 + (\rho_{1,2} \times UL_1)}{UL_P}$$

Together, the two risk contributions will equal the unexpected loss on the portfolio (i.e., $RC_1 + RC_2 = UL_P$).

Professor's Note: We will demonstrate the calculation of risk contribution shortly.

Diversifiable and Undiversifiable Risk

An asset held in isolation bears both diversifiable (firm-specific) and undiversifiable (market) risk. In a portfolio context, the diversifiable risk is reduced effectively to zero in a large portfolio as the asset-specific risks, both good and bad, cancel each other out. On the other hand, undiversifiable risk is the residual credit risk (i.e., risk contribution) that cannot be diversified away.

The Effect of Correlation

The correlation between bank assets is critical to measuring the potential portfolio loss. Intuitively, as the correlation between assets increases, the bank suffers from **concentration risk**. In this scenario, default on one asset (due to, say, industry effects) spills over to other assets, exacerbating the problem.

It is, therefore, important to have reliable estimates of default correlations. Unfortunately, the practical implementation of this basic concept is challenging for two reasons. First, it is extremely difficult to assess the default correlation between unrelated obligors. Second, even if we were confident in our estimates, the sheer number of necessary calculations is unmanageable. For example, in a portfolio of only 20 assets, 190 correlation pairs are computed; in a portfolio of 100 assets, 4,950 pairs are computed {i.e, [n × (n – 1)] / 2 covariances}.

The following numerical example highlights the impact of the correlation coefficient on expected and unexpected loss. Note the lengthy process based only on a two-asset portfolio.

Example: Computing portfolio expected and unexpected loss

Bigger Bank has two assets outstanding. The features of the loans are summarized in Figure 1. Assuming a correlation of 0.3 between the assets, compute EL_P and UL_P as well as the risk contribution of each asset.

Figure 1: Loan Features

	Asset A	*Asset B*
EA	\$8,250,000	\$1,800,000
PD	0.50%	1.00%
LR	50.00%	40.00%
σ_{PD}	2.00%	5.00%
σ_{LR}	25.00%	30.00%

Answer:

Step 1: Compute EL for both assets.

$$\begin{aligned} EL_A &= EA \times PD \times LR \\ &= \$8{,}250{,}000 \times 0.005 \times 0.50 \\ &= \$20{,}625 \end{aligned}$$

$$\begin{aligned} EL_B &= EA \times PD \times LR \\ &= \$1{,}800{,}000 \times 0.01 \times 0.40 \\ &= \$7{,}200 \end{aligned}$$

Step 2: Compute UL for both assets.

$$UL = EA \times \sqrt{PD \times \sigma^2{}_{LR} + LR^2 \times \sigma^2{}_{PD}}$$

$$UL_A = \$8{,}250{,}000 \times \sqrt{0.005 \times 0.25^2 + 0.5^2 \times 0.02^2} = \$167{,}558$$

$$UL_B = \$1{,}800{,}000 \times \sqrt{0.01 \times 0.3^2 + 0.4^2 \times 0.05^2} = \$64{,}900$$

Step 3: Compute EL_P.

$$\begin{aligned} EL_P &= \$20{,}625 + \$7{,}200 \\ &= \$27{,}825 \end{aligned}$$

Step 4: Compute UL_P.

$$UL_P = \sqrt{(167{,}558)^2 + (64{,}900)^2 + (2)(0.3)(167{,}558)(64{,}900)} = \$197{,}009$$

Step 5: Compute RC for both assets.

$$RC_A = (167{,}558)\frac{(167{,}558 + 0.3 \times 64{,}900)}{197{,}009} = 159{,}070$$

$$RC_B = (64{,}900)\frac{(64{,}900 + 0.3 \times 167{,}558)}{197{,}009} = 37{,}939$$

$RC_A + RC_B = \$197{,}009 = UL_P$

Example: Computing portfolio expected and unexpected loss

Using the information provided in the previous example and assuming the correlation between the assets has decreased to 0.1, compute EL_P and UL_P.

Answer:

$EL_P = \$27{,}825$

$$UL_P = \sqrt{(167{,}558)^2 + (64{,}900)^2 + (2)(0.1)(167{,}558)(64{,}900)} = \$185{,}641$$

Because correlation does not impact each asset individually, the expected loss on the portfolio remains the same. However, the unexpected loss (variation) has decreased to \$185,641.

Economic Capital

LO 65.1: Evaluate a bank's economic capital relative to its level of credit risk.

LO 65.7: Describe how economic capital is derived.

The best estimate of the devaluation of a risky asset is expected loss. However, as previous examples illustrate, the unexpected loss can exceed the expected loss by a wide margin. If the bank holds inadequate reserves, there is a possibility the bank will become impaired. Therefore, it is imperative that the bank hold capital reserves to buffer itself from unexpected losses so it can absorb large losses and continue to operate.

Banks set aside credit reserves in preparation for expected losses. However, for unexpected losses, banks need to estimate the excess capital reserves needed to cover any unexpected losses. The excess capital needed to match the bank's estimate of unexpected loss is referred to as **economic capital**.

The amount of economic capital needed to absorb credit losses is the distance between the unexpected (negative) outcome and the expected outcome for a given confidence level. By knowing the shape of the loss distribution, EL_P, and UL_P, the difference between the expected outcome and the confidence level (typically 99.97%) can be estimated. This difference can then be represented as a multiple of portfolio unexpected loss, which is often referred to as the capital multiplier (CM). With this multiplier, economic capital for the portfolio can be derived as:

$$\text{economic capital}_P = UL_P \times CM$$

Modeling Credit Risk

LO 65.8: Explain how the credit loss distribution is modeled.

When estimating economic capital for credit risk, we are largely concerned with the tail of the chosen loss distribution. Credit risks are not normally distributed and tend to be highly skewed, because maximum gains are limited to receiving promised payments while extreme losses are very rare events. Therefore, in practice, a beta distribution is commonly applied in credit risk modeling. The mass of the beta distribution is located between zero and one, so when modeling credit events, losses are defined between 0% and 100%. This distribution is extremely flexible, and can be symmetric or skewed depending on the values of its shape parameters (i.e., β and α). When these shape parameters are equal, the distribution will be symmetric and its mean and variance will be characterized as EL_P and UL_P, respectively. The tail of the credit loss distribution, however, is more difficult to model. In practice, fitting the tail often involves combining the beta distribution with a Monte Carlo simulation.

LO 65.9: Describe challenges to quantifying credit risk.

The bottom-up risk measurement framework that attempts to quantify credit risk has several limitations:

1. Credits are presumed to be illiquid assets. With a bottom-up approach, credit losses are measured by their risk contribution to the credit portfolio and are not influenced by the correlation among risk factors as they are in liquid markets.
2. Credit risk models used in practice only use a one-year estimation horizon. Ideally, credit risk models should incorporate unexpected and expected changes in credit quality of borrowers that occur over several years; however, in practice, this approach is very difficult.
3. Other risk components (such as operational risk and market risk) are separated from credit risk and, thus, managed and measured in different departments within the bank.

Key Concepts

LO 65.1

A bank must hold capital reserves (i.e., economic capital) to buffer itself from unexpected losses so it can absorb large losses and continue to operate.

LO 65.2

Current measures used to evaluate credit risk are:

- The probability of default (PD), which is the likelihood that a borrower will default.
- The loss rate (LR), which represents the likely percentage loss if the borrower defaults.
- Exposure amount (EA), which can be stated as a dollar amount (e.g., the loan balance outstanding) or as a percentage of the nominal amount of the loan or the maximum amount available on a credit line.

LO 65.3

Expected loss is the product of exposure amount (EA), probability of default (PD), and loss rate (LR):

$$EL = EA \times PD \times LR$$

LO 65.4

Unexpected loss represents the variation in expected loss:

$$UL = EA \times \sqrt{PD \times \sigma_{LR}^2 + LR^2 \times \sigma_{PD}^2}$$

LO 65.5

The variance of the probability of default (PD) assuming a binominal distribution is:

$$\sigma_{PD}^2 = PD \times (1 - PD)$$

LO 65.6

In most cases, UL_P will be significantly less than the sum of individual UL_i.

$$UL_P = \sqrt{\sum_i \sum_j \rho_{ij} UL_i UL_j}$$

For a two-asset portfolio, the risk contributions of each asset are calculated as follows:

$$RC_1 = UL_1 \times \frac{UL_1 + (\rho_{1,2} \times UL_2)}{UL_P}$$

$$RC_2 = UL_2 \times \frac{UL_2 + (\rho_{1,2} \times UL_1)}{UL_P}$$

LO 65.7

The amount of economic capital needed to absorb credit losses is the distance between the unexpected (negative) outcome and the expected outcome for a given confidence level.

LO 65.8

In practice, a beta distribution is commonly used to model credit risk.

LO 65.9

The bottom-up risk measurement framework that attempts to quantify credit risk has several issues given that credits are presumed to be illiquid assets with one-year estimation horizons. In addition, other risk components, such as operational risks and market risks, are managed and measured in different departments within the bank.

Concept Checkers

1. XYZ Bank is trying to forecast the expected loss on a loan to a mid-size corporate borrower. It determines that there will be a 75% loss if the borrower does not perform the financial obligation. This risk measure is the:
   A. probability of default.
   B. loss rate.
   C. unexpected loss.
   D. exposure amount.

2. Which of the following statements about expected loss (EL) and unexpected loss (UL) is true?
   A. Expected loss always exceeds unexpected loss.
   B. Unexpected loss always exceeds expected loss.
   C. EL and UL are parameterized by the exact same set of variables.
   D. Expected loss is directly related to exposure.

3. If the recovery rate (RR) increases and the probability of default (PD) decreases, what will be the effect on expected loss (EL), all else equal?

	RR Increase	PD Decrease
A.	Increase	Increase
B.	Decrease	Increase
C.	Increase	Decrease
D.	Decrease	Decrease

4. Big Bank has contractually agreed to a $20,000,000 credit facility with Upstart Corp., of which $18,000,000 is currently outstanding. Upstart has very little collateral, so Big Bank estimates a one-year probability of default of 2%. The collateral is unique to its industry with limited resale opportunities, so Big Bank assigns an 80% loss rate. The expected loss (EL) for Big Bank is closest to:
   A. $68,000.
   B. $72,000.
   C. $272,000.
   D. $288,000.

5. Bigger Bank has two assets outstanding. The features of the loans are summarized in the following table. Assuming a correlation of 0.2 between the assets, what is the value of the unexpected loss of the portfolio (UL_P)?

	Asset A	*Asset B*
Exposure	$5,100,000	$3,600,000
PD	2.00%	1.00%
LR	50.00%	40.00%
σ_{PD}	2.00%	5.00%
σ_{LR}	25.00%	20.00%

   A. Less than $100,000.
   B. Between $100,000 and $200,000.
   C. Between $200,000 and $300,000.
   D. Greater than $300,000.

Concept Checker Answers

1. B Current measures used to evaluate credit risk include the firm's probability of default, which is the likelihood that a borrower will default, the loss rate, which represents the likely percentage loss if the borrower defaults, the exposure amount, and the expected loss, which, for a given time horizon, is calculated as the product of the EA, PD, and LR. The stated 75% loss if the borrower defaults is the loss rate.

2. D EL increases with increases in the exposure amount. UL typically exceeds EL, but they are both equal to zero when probability of default is zero. UL has additional variance terms.

3. D If recovery rates increase, the loss rate will decrease, which will decrease expected loss. If the probability of default decreases, the expected loss will also decrease.

4. D
$$\begin{aligned} EL &= EA \times PD \times LR \\ &= \$18{,}000{,}000 \times 0.02 \times 0.8 \\ &= \$288{,}000 \end{aligned}$$

5. C The calculations below describe the steps to compute the unexpected loss of a portfolio.

Compute UL for both assets.

$$UL = EA \times \sqrt{PD \times \sigma^2_{LR} + LR^2 \times \sigma^2_{PD}}$$

$$UL_A = \$5{,}100{,}000 \times \sqrt{0.02 \times 0.25^2 + 0.5^2 \times 0.02^2} = \$187{,}386$$

$$UL_B = \$3{,}600{,}000 \times \sqrt{0.01 \times 0.2^2 + 0.4^2 \times 0.05^2} = \$101{,}823$$

Compute UL_P.

$$UL_P = \sqrt{(187{,}386)^2 + (101{,}823)^2 + (2)(0.2)(187{,}386)(101{,}823)} = \$230{,}464$$

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The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Operational Risk

Topic 66

Exam Focus

This topic introduces operational risk by defining operational risk and discussing the types of operational risk and bank business lines that must be considered when calculating operational risk capital requirements. Collecting data for loss frequency and loss severity distributions is an important component of allocating operational risk capital among various business lines. Methods for finding the necessary operational loss data points are based on both internal and external data and historical and forward-looking approaches.

Defining Operational Risk

Some firms define operational risk as all risk that is not credit or market risk. However, most people agree that this definition is far too broad. Past industry definitions of operational risk include:

- Financial risk that is not caused by market risk (i.e., unexpected asset price movements) or credit risk (i.e., the failure of a counterparty to meet financial obligations).
- Any risk developing from a breakdown in normal operations (e.g., system failures or processing mistakes).
- Any risk from internal sources (e.g., internal fraud), excluding the impact of regulatory action or natural disasters.
- Direct or indirect losses that result from ineffective or insufficient systems, personnel, or external events (e.g., natural disasters or political events), excluding business risk (the risk of earnings volatility resulting from business conditions).

In 2001, the Basel Committee on Banking Supervision attempted to incorporate industry views and build a consensus on the definition of operational risk. The Committee's statement of operational risk is as follows:

> *"The risk of direct and indirect loss resulting from inadequate or failed internal processes, people, and systems or from external events."*

The Basel Committee defines operational risk to include internal functions or processes, human factors, systems, firm infrastructure, and outside events. Problems with any of these areas can lead to direct and indirect losses, both expected and unexpected. The operational risk definition explicitly includes **legal risk**, but does not address **reputational risk** or **strategic risk**, presumably because they can be difficult to quantify. This definition concentrates on sources of losses and the impact of operational losses.

Sometimes operational risks interact with market and credit risks. For example, exceeding limits or misreporting positions may only result in losses if the market moves adversely (i.e., market risk). As another example, an error made during a loan documentation process may only become known if the counterparty defaults (i.e., credit risk).

Operational Risk Capital Requirements

LO 66.1: Compare three approaches for calculating regulatory capital.

The Basel Committee has proposed three approaches for determining the operational risk capital requirement (i.e., the amount of capital needed to protect against the possibility of operational risk losses): (1) the basic indicator approach, (2) the standardized approach, and (3) the advanced measurement approach. The **basic indicator approach** and the **standardized approach** determine capital requirements as a multiple of gross income at either the business line or institution level. The **advanced measurement approach** (AMA) offers institutions the possibility to lower capital requirements in exchange for investing in risk assessment and management technologies.

With the basic indicator approach, operational risk capital is based on 15% of the bank's annual gross income over a 3-year period. Gross income in this case includes both net interest income and noninterest income. For the standardized approach, the bank uses eight business lines with different **beta factors** to calculate the capital charge. With this approach, the beta factor of each business line is multiplied by the annual gross income amount over a 3-year period. The results are then summed to arrive at the total operational risk capital charge under the standardized approach. The beta factors used in this approach are shown as follows:

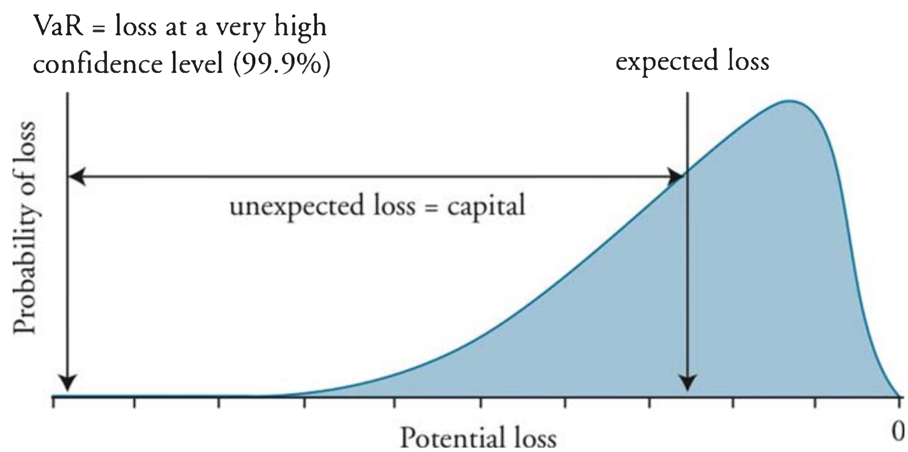
- Investment banking (corporate finance) 18%
- Investment banking (trading and sales) 18%
- Retail banking 12%
- Commercial banking 15%
- Settlement and payment services 18%
- Agency and custody services 15%
- Asset management 12%
- Retail brokerage 12%

Banks that want to take advantage of the possible lower capital requirements available by using the AMA will be required to determine the operational risk capital charge based on internal criteria that are both qualitative and quantitative in nature. The Basel Committee recommends that large banks move from the standardized approach to the AMA. In order to use either approach, banks must satisfy a number of conditions.

In order to use the standardized approach, banks must: (1) have an operational risk management function that is able to identify, assess, monitor, and control this type of risk, (2) document losses for each business line, (3) report operational risk losses on a regular basis, (4) have a system that has the appropriate level of documentation, and (5) conduct independent audits with both internal and external auditors.

In order to use the advanced measurement approach, banks must satisfy the above requirements in addition to being able to approximate unexpected losses. The calculation of unexpected losses is based on external and internal loss data as well as scenario analysis. With this estimate, the bank is able to find the necessary amount of capital to allocate to each business line based on the bank's operational value at risk (VaR) measure. The operational risk capital requirement currently proposed by the Basel Committee is equal to the unexpected loss in a total loss distribution that corresponds to a confidence level of 99.9% over a 1-year time horizon. This concept is illustrated in Figure 1.

Figure 1: Capital Requirement



Professor's Note: The calculation of operational risk capital using all three methods will be discussed in more detail in the FRM Part II curriculum.

Operational Risk Categories

LO 66.2: Describe the Basel Committee's seven categories of operational risk.

The Basel Committee on Banking Supervision disaggregates operational risk into seven types. A majority of the operational risk losses result from clients, products, and business practices.

1. *Clients, products, and business practices.* Failure (either intentional or unintentional) to perform obligations for clients. Examples include mishandling of confidential information, breaches in fiduciary duty, and money laundering.

2. *Internal fraud.* Disobeying the law, regulations, and/or company policy, or misuse of company property. Examples include misreporting data or insider trading. Well-known case studies dealing with internal fraud include Barings, Allied Irish Bank, and Daiwa.

3. *External fraud.* Actions by a third party that disobey the law or misuse property. Examples include robbery or computer hacking.

4. *Damage to physical assets.* Damage occurring from events, such as natural disasters. Examples include a terrorist attack, earthquakes, or fires.

5. *Execution, delivery, and process management.* Failure to correctly process transactions and the inability to uphold relations with counterparties. Examples include data entry errors or unfinished legal documents.

6. *Business disruption and system failures.* Examples include computer failures, both hardware- and software-related, or utility outages.

7. *Employment practices and workplace safety.* Actions that do not follow laws related to employment or health and safety. Examples include worker compensation, discrimination disputes, or disobeying health and safety rules.

Assuming that a bank was active in each of the eight business lines discussed previously, it would have 56 different measures of risk to aggregate into a single operational risk VaR measure. Institutions that are not active in each of the lines of business would require fewer risk measures.

Loss Frequency and Loss Severity

LO 66.3: Derive a loss distribution from the loss frequency distribution and loss severity distribution using Monte Carlo simulations.

Operational risk losses can be classified along two dimensions: loss frequency and loss severity. **Loss frequency** is defined as the number of losses over a specific time period (typically one year), and **loss severity** is defined as the value of financial loss suffered (i.e., the size of the loss). It can be reasonably assumed that these two dimensions are independent.

Loss frequency is most often modeled with a **Poisson distribution** (a distribution that models random events). The mean and variance of the Poisson distribution are equal to a single parameter, λ. Over a short time horizon, the probability of losses is then equal to $\lambda \times \Delta t$. Over a time horizon, T, the probability of n losses using this distribution is equal to:

$$e^{-\lambda T} \times \frac{(\lambda T)^n}{n!}$$

The parameter λ is equal to the average number of losses over a given time horizon. So if ten losses occurred over a 5-year period, λ would equal two per year (= 10 / 5).

Loss severity is often modeled with a **lognormal distribution**. This distribution is asymmetrical (the frequency of high-impact, low-frequency losses is not equal to the frequency of low-impact, high-frequency losses) and fat-tailed (rare events occur more often than would be indicated by a normal distribution). The mean and standard deviation are derived from the logarithm of losses.

Loss frequency and loss severity are combined in an effort to simulate an expected loss distribution (known as **convolution**). The best technique to accomplish this simulation is to use a **Monte Carlo simulation** process. With this process, we make random draws from the loss frequency data and then draw the indicated number of draws from the loss severity data. Each combination of frequency and severity becomes a potential loss event in our loss distribution. This process is continued several thousand times to create the potential loss distribution.

Having created the loss distribution, the desired percentile value can be measured directly. For example, the 99th percentile would correspond with the loss amount that is greater than 99% of the distribution's data. The difference between the losses at the selected percentile and the mean loss of the distribution equals the unexpected losses at the corresponding confidence level, as was illustrated in Figure 1.

Data Limitations

LO 66.4: Describe the common data issues that can introduce inaccuracies and biases in the estimation of loss frequency and severity distributions.

The historical record of operational risk loss data is currently inadequate. This creates challenges when trying to accurately estimate frequency and severity. Given the extreme risk that operational problems create, firms are beginning to build a database of potential loss events. Compared to credit risk losses, the data available for operational risk losses is clearly lacking. For example, firms can rely on credit rating agencies to get a clear view of default probabilities and expected losses when assessing credit risk.

It is recommended that banks use internal data when estimating the frequency of losses and utilize both internal and external data when estimating the severity of losses. Regarding external data, there are two data sources available to firms: sharing agreements with other banks and public data.

When incorporating both internal and external operational risk loss data, firms should adjust for inflation. In addition, when viewing external data from other banks it is necessary to use a scale adjustment that applies the loss event to your bank's situation. For instance, if Bank Z has a $5 million operational risk loss, how would this loss apply to your bank? A simple mathematical proportion will likely over or underestimate the actual loss. As a result, the accepted scale adjustment for firm size is as follows:

$$\text{estimated loss}_{\text{Bank Y}} = \text{external loss}_{\text{Bank Z}} \times \left(\frac{\text{revenue}_{\text{Bank Y}}}{\text{revenue}_{\text{Bank Z}}}\right)^{0.23}$$

Example: Firm size scale adjustment

If the observed loss for Bank Z is $5 million and it has $1 billion in revenue, what will be the estimated loss adjusted for firm size for Bank Y, which has revenue of $2 billion?

Answer:

$$\text{estimated loss}_{\text{Bank Y}} = \$5\text{ million} \times \left(\frac{\$2\text{ billion}}{\$1\text{ billion}}\right)^{0.23} = \$5{,}864{,}175$$

Notice that this loss is much less than the proportional estimate of a $10 million loss given that Bank Y has twice the revenue.

Scale-adjusted loss data and other data obtained through sharing agreements are useful when constructing a firm's loss severity distribution. Public data, however, is less reliable given the inherent reporting biases. Public loss data likely only contains relatively large losses from firms that have weak internal controls. As a result, public data is more appropriate when used relative to internal losses. This would involve assigning a multiple to internal data estimates (i.e., mean and standard deviation) that reflects the severity of public external data.

LO 66.5: Describe how to use scenario analysis in instances when data is scarce.

Another method for obtaining additional operational risk data points is to use **scenario analysis**. Regulators encourage the use of scenarios since this approach allows management to incorporate events that have not yet occurred. This has a positive effect on the firm since management is actively seeking ways to immunize against potential operational risk losses. The drawback is the amount of time spent by management developing scenarios and contingency plans.

Forward-Looking Approaches

LO 66.6: Describe how to identify causal relationships and how to use risk and control self assessment (RCSA) and key risk indicators (KRIs) to measure and manage operational risks.

It is important for management to use forward-looking approaches in an attempt to prepare for future operational risk losses. One way to accomplish this objective is to learn from the mistakes of other companies. For example, in Book 1 we learned about a number of financial disasters, including Barings and Allied Irish Bank, which both suffered losses due to rogue traders. Another example is the Hammersmith and Fulham case, where banks took note of a court ruling dealing with counterparty risk.

In the Hammersmith and Fulham case, two traders entered into 600 interest rate swaps totaling 6 billion British pounds over the span of two years. The traders had a low level of understanding of these derivative contracts, and losses quickly grew to millions of pounds. Counterparties became very concerned about the level of credit risk. The auditor at Hammersmith and Fulham was able to void the swap agreements by convincing the court that the traders and, in turn, the company did not have the authority to enter into these transactions. The court agreed with the company and as a result, the swap counterparties were left with unhedged positions and were unable to collect payments from Hammersmith and Fulham.

Causal relationships are a convenient method of identifying potential operational risks. Relationships are analyzed to check for a correlation between firm actions and operational risk losses. For example, if employee turnover or the use of a new computer system demonstrates a strong correlation with losses, the firm should investigate the matter. It is necessary to conduct a cost-benefit analysis if significant relationships are discovered.

One of the most frequently used tools in operational risk identification and measurement is the **risk and control self assessment** (RCSA) program. The basic approach of an RCSA is to survey those managers directly responsible for the operations of the various business lines. It is presumed that they are the closest to the operations and are, therefore, in the best position to evaluate the risks. The problem with this assumption is that you cannot reasonably expect managers to disclose risks that are out of control. Also, a manager's perception of an appropriate risk-return tradeoff may be different than that of the institution. A sound risk management program requires that risk identification and measurement be independently verified.

The identification of appropriate **key risk indicators** (KRIs) may also be very helpful when attempting to identify operational risks. Examples of KRIs include employee turnover and the number of transactions that ultimately fail. In order to be valuable as risk indicators, the factors must (1) have a predictive relationship to losses and (2) be accessible and measurable in a timely fashion. The idea of utilizing KRIs is to provide the firm with a system that warns of possible losses before they happen.

Scorecard Data

LO 66.7: Describe the allocation of operational risk capital to business units.

Allocating operational risk capital to each business unit encourages managers to improve their management of operational risks. Less capital will be allocated to those business units that are able to reduce the frequency and severity of risks. The reduction in capital will increase the unit's return on invested capital measure; however, reducing capital may not be ideal if the costs of reducing certain risks outweigh the potential benefits. It is, therefore, necessary for each business line manager to be allocated the optimal amount of operational risk capital.

One method for allocating capital is the **scorecard approach**. This approach involves surveying each manager regarding the key features of each type of risk. Questions are formulated, and answers are assigned scores in an effort to quantify responses. The total score for each business unit represents the total amount of risk. Scores are compared across business units and validated by comparison with historical losses.

Examples of survey questions include: (1) the ratio of supervisors to staff, (2) employee turnover rate, (3) average number of open positions in the business unit at one time, and (4) the presence of confidential information. The objective of the scorecard approach is to make business line managers more aware of operational risks and the potential for losses from those risks. It also encourages senior management to become more involved with the risk management process.

The Power Law

LO 66.8: Explain how to use the power law to measure operational risk.

The power law is useful in extreme value theory (EVT) when we evaluate the nature of the tails of a given distribution. The use of this law is appropriate since operational risk losses are likely to occur in the tails. The law states that for a range of variables:

$$P(V > X) = K \times X^{-\alpha}$$

where:
V = loss variable
X = large value of V
K and α = constants

The probability that *V* is greater than *X* equals the right side of the equation. The parameters on the right side are found by using operational risk loss data to form a distribution and then using a maximum-likelihood approach to estimate the constants. The power law makes the calculation of VaR at high confidence levels possible since low values of α will represent the extreme tails and, hence, the value at risk from potential operational risks.

Insurance

LO 66.9: Explain the risks of moral hazard and adverse selection when using insurance to mitigate operational risks.

Managers have the option to insure against the occurrence of operational risks. The important considerations are how much insurance to buy and which operational risks to insure. Insurance companies offer polices on everything from losses related to fire to losses related to a rogue trader. A bank using the AMA for calculating operational risk capital requirements can use insurance to reduce its capital charge. Two issues facing insurance companies and risk managers are moral hazard and adverse selection.

A **moral hazard** occurs when an insurance policy causes an insured company to act differently with the presence of insurance protection. For example, if a firm is insured against a fire, it may be less motivated to take the necessary fire safety precautions. To help protect against the moral hazard issue, insurance companies use deductibles, policy limits, and coinsurance provisions. With coinsurance provisions, the insured firm pays a percentage of the losses in addition to the deductible.

An interesting dilemma exists for rogue trader insurance. A firm with a rogue trader has the potential for profits that are far greater than potential losses, given the protection of insurance. As a result, insurance companies that offer these policies are careful to specify trading limits, and some may even require the insured firm not to reveal the presence of the policy to traders. These insurance companies are also banking on the fact that the discovery of a rogue trader would greatly increase the firm's insurance premiums and greatly harm the firm's reputation.

Adverse selection occurs when an insurance company cannot decipher between good and bad insurance risks. Since the insurance company offers the same polices to all firms, it will attract more bad risks since those firms with poor internal controls are more likely to desire insurance. To combat adverse selection, insurance companies must take an active role in understanding each firm's internal controls. Like auto insurance, premiums can be adjusted to adapt to different situations with varying levels of risk.

Key Concepts

LO 66.1

The Basel definition of operational risk is "the risk of direct and indirect loss resulting from inadequate or failed internal processes, people, and systems or from external events."

The three methods for calculating operational risk capital requirements are: (1) the basic indicator approach, (2) the standardized approach, and (3) the advanced measurement approach (AMA). Large banks are encouraged to move from the standardized approach to the AMA in an effort to reduce capital requirements.

LO 66.2

Operational risk can be divided into seven types: (1) clients, products, and business practices, (2) internal fraud, (3) external fraud, (4) damage to physical assets, (5) execution, delivery, and process management, (6) business disruption and system failures, and (7) employment practices and workplace safety.

LO 66.3

Operational risk losses can be classified along two dimensions: loss frequency and loss severity. Loss frequency is defined as the number of losses over a specific time period, and loss severity is defined as the size of a loss, should a loss occur.

LO 66.4

Banks should use internal data when estimating the frequency of losses and utilize both internal and external data when estimating the severity of losses. Regarding external data, banks can use sharing agreements with other banks (which includes scale-adjusted data) and public data.

LO 66.5

Scenario analysis is a method for obtaining additional operational risk data points. Regulators encourage the use of scenarios since they allow management to incorporate events that have not yet occurred.

LO 66.6

Forward-looking approaches are also used to discover potential operational risk loss events. Forward-looking methods include: (1) causal relationships, (2) risk and control self assessment (RCSA), and (3) key risk indicators.

LO 66.7

Allocating operational risk capital can be accomplished by using the scorecard approach. This approach involves surveying each manager regarding the key features of each type of risk. Answers are assigned scores in an effort to quantify responses.

LO 66.8

The power law is useful in extreme value theory (EVT) when we evaluate the nature of the tails of a given distribution. The use of this law is appropriate since operational risk losses are likely to occur in the tails.

LO 66.9

Two issues facing insurance companies that provide insurance for operational risks are moral hazard and adverse selection. A moral hazard occurs when an insurance policy causes a company to act differently with insurance protection. Adverse selection occurs when an insurance company cannot decipher between good and bad insurance risks.

Concept Checkers

1. In constructing the operational risk capital requirement for a bank, risks are aggregated for:
   A. commercial and retail banking.
   B. investment banking and asset management.
   C. each of the seven risk types and eight business lines that are relevant.
   D. only those business lines that generate at least 20% of the gross revenue of the bank.

2. According to current Basel Committee proposals, banks using the advanced measurement approach must calculate the operational risk capital charge at a:
   A. 99 percentile confidence level and a 1-year time horizon.
   B. 99 percentile confidence level and a 5-year time horizon.
   C. 99.9 percentile confidence level and a 1-year time horizon.
   D. 99.9 percentile confidence level and a 5-year time horizon.

3. Which of the following is not one of the seven types of operational risk identified by the Basel Committee?
   A. Failed business strategies.
   B. Clients, products, and business practices.
   C. Employment practices and workplace safety.
   D. Execution, delivery, and process management.

4. The Basel definition of operational risk focuses on the risk of losses due to inadequate or failed processes, persons, and systems that cannot protect a company from outside events. The definition has been subject to criticism because it excludes:
   A. market and credit risks.
   B. indirect losses.
   C. failure of information technology operations.
   D. impacts of natural disasters.

5. Which of the following measurement approaches for assessing operational risk would be most appropriate for small banks?
   A. Loss frequency approach.
   B. Basic indicator approach.
   C. Standardized approach.
   D. Advanced measurement approach (AMA).

Concept Checker Answers

1. **C** The construction of the operational risk capital for a bank requires that risks be aggregated over each of the seven types of risk and each of the eight business lines that are relevant for the particular bank.

2. **C** Current Basel Committee proposals require that operational risk capital be calculated at the 99.9th percentile level over a 1-year horizon.

3. **A** Failed business strategies are not included in the definition of operational risk, which includes (1) clients, products, and business practices; (2) internal fraud; (3) external fraud; (4) damage to physical assets; (5) execution, delivery, and process management; (6) business disruption and system failures; and (7) employment practices and workplace safety.

4. **A** The Basel definition excludes credit or market risks. All of the other choices are incorporated in the definition of operational risk.

5. **B** The basic indicator approach is more common for less-sophisticated, typically smaller banks. There is only one indicator of operational risk: gross income.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Governance Over Stress Testing

Topic 67

Exam Focus

Oversight of stress testing activities forms a critical component of an institution's governance and controls. In this topic, we look at the responsibilities of the board of directors and senior management, who must ensure that there are sufficient internal controls (including policies, procedures, and documentation) around how stress tests are conducted, reviewed, validated and monitored. For the exam, know the various roles of the board and management, who not only must understand and interpret the stress testing results, but also must actively challenge them, review their limitations, and recommend remedial actions. You should understand the purpose and features of robust policies, procedures, and documentation, as well as the importance of validation and independent review. Also, understand the role of the internal audit, which is necessary to enhance the integrity of the stress testing process. Finally, be able to discuss other key elements of the stress testing process, including stress testing coverage, stress testing types and approaches, and capital and liquidity stress testing.

Effective Governance and Controls Over Stress Testing

LO 67.1: Describe the key elements of effective governance over stress testing.

Effective governance and controls are critical to ensuring that stress tests are conducted appropriately and are subject to adequate oversight. Proper governance and controls are especially important for stress tests that are highly technical in nature and where stress tests require a large number of assumptions or create uncertainties in estimating stressed events and conditions. In general, institutions should aim for oversight that is tailored to the complexity and characteristics (e.g., size, profile, and culture) of the specific institution. Key elements of effective governance and controls over stress testing include the governance structure, policies and procedures, documentation, validation and independent review, and internal audit.

Responsibilities of the Board and Senior Management

LO 67.2: Describe the responsibilities of the board of directors and senior management in stress testing activities.

In order to ensure there is adequate oversight, institutions should have separation of duties between the board of directors and senior management. This separation of duties also applies to stress testing. Although the board and management share several common responsibilities, they each have distinct responsibilities within an institution.

Board of Directors

The board of directors has oversight for an organization's key strategies and decisions and is responsible and accountable for the entire organization. It is important that the board discuss and evaluate information received from senior management and review it with a critical eye. The board should be sufficiently knowledgeable about the organization's stress testing activities to ask informed questions, even if it is not directly involved with and does not possess expert knowledge in the stress testing activities or their technical details. Board members should be critical of stress tests by actively challenging assumptions.

Stress testing results are important because they are used to inform the board of the institution's risk appetite and risk profile, as well as its operating and strategic decisions. Stress test results provide forward-looking assessments and are especially important in decisions regarding capital and liquidity adequacy and capital funding plans. However, boards should view results with some degree of skepticism and should not rely on a single stress test exercise, but rather supplement it with other tests and quantitative and qualitative information. Stress testing can serve as an early warning sign, especially in nonstress times, allowing the board to take actions that include adjusting capital levels, increasing liquidity, adjusting risks, or engaging in or withdrawing from certain activities.

Senior Management

Senior management is accountable to the board and is responsible for the satisfactory implementation of the stress testing activities authorized by the board. This entails establishing robust policies and procedures to ensure compliance with these activities, reviewing and coordinating stress test activities, and remedying any issues. Management should ensure that the staff is competent and adequately compensated. Senior management should also ensure that there is appropriate buy-in throughout the institution. To avoid inconsistencies, gaps, or problems, management must ensure that stress test assumptions remain transparent and are used in a clear manner.

It is senior management's responsibility to ensure that stress testing activities do not simply rely on a single test, but instead rely on a series of stress tests to evaluate risks. Stress tests should aid management's decision making relating to business strategies, risk limits, and the institution's capital, liquidity, and risk profile. It is prudent practice to benchmark results against an adequate benchmark to aid comparison and allow management to properly evaluate the stress test results. Senior management should actively challenge these results; therefore, it is critical that management remain knowledgeable of the details of the stress testing activities.

Stress tests should be appropriately aggregated, remedial actions should be carried out appropriately, and results should be adequately documented. Senior management should also consider the effectiveness of mitigation techniques and the possibility that these remedial actions could break down during stressed times. Stress tests can be used to supplement other risks as well as capital and adequacy measures.

Senior management should regularly report back to the board on stress testing results and developments and on the adequacy of compliance procedures. Reports should be clear and concise, and they should explain the main elements of the stress testing activities, their assumptions, and any limitations. Senior management should also report on the governance, validation, and independent review of stress tests and stress testing results. In addition, senior management should ensure that stress testing activities are reviewed by an independent, unbiased party (e.g., through an internal audit).

It is also senior management's responsibility to regularly update stress testing activities given changing risks, data sources, and internal or external operating environments. Stress testing activities and their underlying models should be reviewed, adjusted, and refined on an ongoing basis; senior management should confirm that the models and activities remain appropriate for the institution. Finally, management is also responsible for conducting stress tests to ensure that the institution is sufficiently flexible to withstand new risks and vulnerabilities.

Policies, Procedures, and Documentation

LO 67.3: Identify elements of clear and comprehensive policies, procedures, and documentations on stress testing.

Clear and comprehensive policies, procedures, and documentation are critical in codifying an institution's risk practices, including its stress testing activities. Policies and procedures should be clear and concise and reviewed and approved annually.

Appropriate documentation of stress testing policies and procedures is also critical. Documentation should include a description of the stress tests used, their results, main assumptions, any limitations and constraints, and the appropriate remedies. Appropriate documentation allows senior management to both track and analyze results over time and is useful to stress test developers. Since documentation is a complex and time-consuming process, institutions should provide incentives to create effective documentation. Stress test developers should be given responsibility for documentation, and the institution should ensure that other market participants (e.g., management, vendors, and reviewers) also document their efforts related to stress testing. Institutions that use stress tests from external parties should ensure these tests are appropriately documented.

Policies, procedures, and documentation should address the implementation of stress testing activities, including:

- Describing the overall purpose of stress tests.
- Establishing consistent and adequate stress testing practices.
- Describing roles and responsibilities, including influences over external resources (e.g., vendors and data providers).
- Determining frequency and priority of stress testing activities.
- Outlining the stress test process, including scenario design and selection.
- Disclosing the validation and review process.
- Providing transparency to third parties regarding the stress testing process, which allows third parties to evaluate the tests and their components.
- Indicating the use of stress tests and their users and any remedial actions.
- Updating and reviewing policies and procedures to remain consistent with the institution's risk appetite, risk exposures, and changing market conditions.

Validation and Independent Review

LO 67.4: Identify areas of validation and independent review for stress tests that require attention from a governance perspective.

Prudent governance should also incorporate ongoing validation and independent review of stress testing activities. These should be done in an unbiased manner using a critical review to ensure stress tests were conducted appropriately. Validation and independent review of stress tests should be incorporated into an institution's overall validation and review processes.

To enhance the usefulness of validating stress tests, institutions should include nonstress periods (i.e., the "good times" periods) in their models to test their predictive power. A failure of the stress tests to perform well in a data-rich environment would lead to questions about the tests' usefulness. Note, however, that even if the tests performed well, the positive results should be interpreted with caution. For example, the correlation among market factors may have changed, and market participants may behave differently in different environments. In addition, it takes time to confirm whether models with recent enhancements to stress tests perform better. As a result, models used in nonstress periods could require a different set of assumptions.

Challenges with model validation may also be addressed through expert-based judgment, sensitivity analysis, simulation techniques, putting greater emphasis on ensuring that stress tests remain sound, and benchmarking to either internal or external models. Benchmarks should be carefully selected to ensure they appropriately reflect the institution's risks and exposures.

It is not necessary for institutions to fully validate stress tests in the same manner as other tests. However, any limitations on test validation results should be communicated and disclosed in a transparent manner. Where validation and independent review identifies major deficiencies, institutions should have a remediation plan.

Validation and independent review may also include the following areas:

- Independent and effective challenge of the review process.
- Independent and effective challenge of the qualitative components of the stress test.
- Implementing proper development standards for stress tests.
- Recognizing methodology, data, and data quality limitations.
- Validating and accurately implementing stress tests that are appropriate for their use.
- Monitoring performance.
- Accounting for uncertainty in stress results through factors, including confidence bands around estimates.

Role of Internal Audits

LO 67.5: Describe the important role of the internal audit in stress testing governance and control.

The internal audit forms a crucial component of an institution's governance and controls. It is intended to assess the integrity and reliability of an institution's policies and procedures, including those pertaining to stress tests. Auditors should be independent with sufficient knowledge and technical expertise to conduct their reviews.

The internal audit should verify that stress tests are conducted thoroughly and as intended, and that the staff in charge of these activities possesses the necessary expertise and adheres to the appropriate policies and procedures. An internal audit should also review the procedures pertaining to the documentation, review, and approval of stress tests. Any deficiencies in stress tests should also be identified.

Key Aspects of Stress Testing Governance

LO 67.6: Identify key aspects of stress testing governance, including stress testing coverage, stress testing types and approaches, and capital and liquidity stress testing.

As part of its overall stress testing governance framework, institutions should also consider stress testing coverage, stress testing types and approaches, and capital and liquidity stress testing. While these aspects will differ by institutions, senior management should review and evaluate them and present them for board review.

Stress Testing Coverage

- Stress testing results may exclude important factors, including portfolios, liabilities, and exposures. As a result, it is important that institutions provide appropriate coverage for stress testing and document what factors are and are not covered.
- Stress testing coverage can be applied to individual exposures, to the entire institution, or to various sublevels within an institution.
- Stress testing should incorporate the relationship between various risks and exposures, and detect risk concentrations and causes of risks that could negatively impact the institution.
- Coverage should be applied on both a short-term and long-term basis.

Stress Testing Types and Approaches

- When stress testing includes scenario analysis, the scenarios selected should be sufficiently robust to be credible to stakeholders (internal and external).
- Scenarios should consider the firm-specific and system-wide impacts of stresses both based on historical analysis and on hypothetical scenarios.
- Cumulative and knock-on effects should be carefully considered.
- For stresses that are done on a firm-wide basis, it is necessary for all business lines to use the same stress assumptions to ensure consistency.
- Stress testing types should include those that extend beyond traditional risk expectations (these include reverse stress tests that "break the bank") and challenge the entire institution's viability. This is true even if estimation may be problematic.

Capital and Liquidity Stress Testing

- Capital and liquidity stress testing should be harmonized with overall strategy and planning, and the results should be updated for all material results and events.

- Stress tests should consider the impact of multiple simultaneous risks on earnings, losses, cash flows, capital, and liquidity.
- Stress testing should aid in contingency planning by identifying excess exposures and areas where liquidity and capital positions can be strengthened, or by identifying actions that are otherwise not possible during stressed times (i.e., raising capital).
- Stress testing should consider the effect on subsidiaries that encounter liquidity and capital issues.
- Stress testing should also look at capital and liquidity problems that can arise simultaneously and thereby magnify risks. For example, an institution may need to sell assets at depressed market prices or incur funding costs at above-market rates.
- With respect to capital and liquidity funding costs, institutions should clearly articulate their objectives.

Key Concepts

LO 67.1

Stress tests should be conducted appropriately under adequate oversight as part of an institution's governance and controls.

Key elements of effective governance and controls over stress testing include the governance structure, policies and procedures, documentation, validation and independent review, and internal audit.

LO 67.2

The board of directors is accountable for the entire organization and must be sufficiently knowledgeable about the organization's stress testing activities.

Stress testing results inform the board of the institution's risk appetite and profile and operating and strategic decisions, and they may serve as early warning signs of upcoming pressures.

Boards should actively challenge the results of stress tests and supplement them with other tests as well as both quantitative and qualitative information.

Senior management, with oversight from the board, is responsible for establishing robust policies for stress tests, which could supplement other risk, capital, and adequacy measures. Management is also responsible for reviewing and coordinating stress test activities, assigning competent staff, challenging results and assumptions, and incorporating remedies to potential problems.

Senior management should ensure that there is a sufficient range of stress testing activities to evaluate risks. Results should be benchmarked and regularly updated given that risks, data sources, and the operating environment can change. An independent auditor should verify test results.

Senior management should regularly report to the board of directors on stress testing results in a clear and concise way, highlighting the key elements of the stress testing activities and any limitations.

LO 67.3

Stress testing activities should be governed by clear and comprehensive policies and procedures that are updated annually and documented in an appropriate manner. Appropriate documentation allows senior management to track and analyze results over time.

Documentation should include a description of the stress tests, main assumptions, results, limitations, and any proposed remedies. Those responsible for documentation should be given incentives given the complexity and time-consuming nature of preparing documents. Documentation received from external parties should be reviewed.

Stress test policies, procedures, and documentation should address a range of issues, including describing the purpose and process of stress test activities, establishing consistent and adequate practices, defining roles and responsibilities, determining the frequency and priority of stress testing activities, and describing proposed remedies.

LO 67.4

Ongoing validation and independent review of stress testing activities is an important component of an institution's governance. Validation and independent review should be unbiased and critical, and they should form part of the institution's overall validation and review processes.

Results should be compared with appropriate benchmarks that reflect the institution's risks and exposures.

Stress tests for nonstress periods (i.e., "good times") should be incorporated into an institution's model to test their predictive power and usefulness. Models used in nonstress periods may require a different set of assumptions given changing risks, correlations, and behavior by market participants.

Institutions do not need to fully validate stress tests, but limitations, challenges, and proposed remedies should be communicated and disclosed in a transparent manner.

Validation and independent review should challenge the review process and the qualitative components of the stress test, implement development standards, validate and implement stress tests, and monitor performance.

LO 67.5

The internal audit assesses the integrity and reliability of policies and procedures. It should verify that stress tests are conducted thoroughly and as intended, by staff with the relevant expertise.

The internal audit should also review the procedures relating to the documentation, review, and approval of stress tests. Any deficiencies should be identified.

LO 67.6

An institution's governance framework should incorporate stress testing coverage, stress testing types and approaches, and capital and liquidity stress testing.

Stress testing coverage should incorporate the relationship between risks and exposures in the short term and long term, and detect risk concentrations that could negatively impact an institution. Coverage can be applied to individual exposures, the entire institution, or various sublevels within an institution.

Stress testing types and approaches should consider the firm-specific and system-wide impacts of stresses from historical and hypothetical scenarios, and should include stresses that challenge the entire institution's viability. When stress testing is complemented with scenario analysis, scenarios should be robust and credible.

Capital and liquidity stress testing should be harmonized with overall strategy and planning and updated for all material results and events. The impact of multiple simultaneous risks should be considered, including capital and liquidity problems that could arise simultaneously and magnify risks. Objectives for capital and liquidity funding costs should be clearly articulated. Excess exposures and areas should be identified where liquidity and capital positions can be strengthened.

Concept Checkers

1. Which of the following statements about governance structure is accurate?
   A. Senior management has ultimate oversight responsibility and accountability for an entire institution.
   B. The board of directors has responsibility for implementing authorized stress testing activities.
   C. The board of directors can change an institution's capital levels and exposures following a review of stress test results.
   D. Senior management should use scenario analysis, not stress testing, to evaluate an institution's risk decisions.

2. Which of the following statements about documentation of stress tests is most appropriate?
   A. Institutions are not concerned if their vendors document stress testing activities.
   B. Institutions should incentivize documenting stress tests to increase efficiency.
   C. Documentation is not useful for stress test developers, but it is important to senior management.
   D. Documentation should not include a description of the types of stress tests and methodologies, but it should include a description of the key assumptions and limitations.

3. Which of the following actions is least likely a component of the validation and independent review of stress tests?
   A. Using expert-based judgment.
   B. Testing data during nonstress periods.
   C. Communicating stress test results to all stress test users.
   D. Reviewing the qualitative but not the judgmental aspects of stress tests.

4. Which of the following statements best reflects the responsibilities of an internal audit?
   A. An internal audit should not assess the staff involved in stress testing activities.
   B. An internal audit must independently assess each stress test used.
   C. An internal audit should review the manner in which stress testing efficiencies are identified and tracked.
   D. The internal audit function needs to be impartial but does not need to be independent.

5. Which of the following reasons best explains why institutions use reverse stress tests?
   A. To identify liquidity risk.
   B. To identify risk concentrations.
   C. To assess where multiple risks occur simultaneously.
   D. To test events that threaten the viability of the institution.

Concept Checker Answers

1. C Stress testing can serve as an early warning sign of upcoming pressures and risks. The board of directors can take actions that include adjusting capital levels, increasing liquidity, adjusting risks, or engaging in or withdrawing from certain activities.

   The board of directors has ultimate oversight responsibility and accountability for an entire institution. Senior management is responsible for implementing authorized stress testing activities. Senior management should use stress testing, complemented with scenario analysis, to evaluate an institution's risk decisions.

2. B Institutions should offer incentives for documenting stress tests to ensure that documentation is effective and complete.

   Institutions should ensure that other market participants, including management, vendors, and reviewers, adequately document their stress testing activities. Documentation is useful for both stress test developers and senior management. Documentation should include a description of the types of stress tests and methodologies, and a description of the key assumptions and limitations.

3. D Validation and independent review of stress tests includes a review of both the qualitative and judgmental aspects of stress tests.

   Validation and independent review of stress tests should also use expert-based judgment, test data during nonstress periods, and communication of stress test results to all stress test users.

4. C An internal audit should review the manner in which stress testing efficiencies are identified, tracked, and remedied.

   An internal audit should assess not only the stress testing activities, but also the staff involved in stress testing activities. An internal audit does not need to independently assess each stress test used. The internal audit function needs to be independent and objective.

5. D Institutions use reverse stress tests that "break the bank" in order to assess the events that are outside of normal business expectations and could threaten the institution's viability.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Stress Testing and Other Risk Management Tools

Topic 68

Exam Focus

Stress testing focuses on the infrequent but large-scale events that occur in the left tail of the return distribution. These are precisely the events that traditional value at risk (VaR) models cannot accommodate. The basic idea with stress testing is to shock key input variables by large amounts and measure the impact on portfolio value. For the exam, understand the similarities and differences between stress tests and VaR measures. Also, be able to explain the importance of a stressed VaR.

The Role of Stress Testing

LO 68.1: Describe the relationship between stress testing and other risk measures, particularly in enterprise-wide stress testing.

Prior to the 2007–2009 financial crisis, enterprise-wide stress testing tended to be very basic, especially for banks. **Stress testing**, specifically supervisory stress tests for regulatory purposes, assists in obtaining an enterprise-wide understanding of risk. With stress testing and value at risk (VaR) measures, using different data and scenarios could result in different results.

Both stress tests and VaR measures [including economic capital (EC) measures] attempt to transform scenarios into loss estimates. The loss estimates' distributions provide the basis to compute VaR at a very high confidence level. Stress tests tend to look at far fewer scenarios compared to VaR measures. Practically speaking, there could be up to three major differences in the definition of loss estimates between stress tests and VaR/EC measures:

1. Stress tests usually define losses from an accounting perspective, while VaR/EC measures usually take a market view of losses (i.e., deducting the opportunity cost of equity capital).

2. Historically, stress tests have looked at longer time horizons, whereas VaR/EC measures have looked at point-in-time losses only.

3. Stress tests do not focus on probabilities, but instead focus on ordinal rankings such as "severe," "very severe," and "extremely severe." In contrast, VaR/EC measures focus on cardinal probabilities when interpreting the results of using Monte Carlo simulation (i.e., complicated statistical models) or historical simulation (i.e., actual past results). For example, a 99% VaR loss is interpreted as a 1-in-100 event.

Additionally, stress tests usually develop scenarios that are conditional. For example, regulatory stress tests tend to use the current period as a departure point in order to develop several hypothetical scenarios. In contrast, VaR/EC measures tend to develop unconditional scenarios.

Complementing Stress Tests with VaR Models

LO 68.2: Describe the various approaches to using VaR models in stress tests.

VaR/EC models compute expected losses using the following formula:

expected loss = PD × LGD × EAD

where:
PD = probability of default
LGD = loss given default
EAD = exposure at default

In applying such models, financial institutions usually use a Merton model to simulate defaults and credit quality. With this model, asset returns are simulated using a factor model and default would occur when the simulated asset value falls short of a threshold (related to the borrower's leverage) at the end of one year.

In an attempt to assign a probability to a hypothetical (i.e., prospective) or historical stress scenario, one could determine where the stress test losses fall within the VaR/EC loss distribution. For example, a stress scenario may suggest a 95th percentile VaR loss, so one would use the 95th percentile EC model loss as an approximation for the stressed loss resulting from market risk. By assigning probabilities to outcomes, the calculated probability from the loss distribution facilitates the implementation of stress test results.

All financial institutions must operate with some risk, so not all of those risks can be avoided. Unacceptably large losses have a very low probability of occurring and are, therefore, not usually considered. Most of the time, the calculated output is simply interpreted as the magnitude and probability of loss for a given scenario. An alternate approach to measuring loss and likelihood of a particular scenario is to use stressed inputs.

Stressed Inputs and Stressed VaR

LO 68.3: Explain the importance of stressed inputs and their importance in stressed VaR.

Stressed inputs have been used in analyzing market risk and for both supervisory and internal purposes within financial institutions. The revised Basel market risk capital framework mandates the use of stressed inputs—for example, a **stressed value at risk** (SVaR) measure. Such a measure is meant to mimic a period of market stress and would be computed using a 10-day, 99th percentile, one-tailed confidence interval VaR measure of the investment portfolio.

The revised framework also mandates the use of thorough and wide-ranging *stress tests* for financial institutions that employ internal models to compute the necessary market risk capital amounts. The Basel III revisions mandate the use of *stressed parameters* to calculate the default risk capital charge for counterparty credit risk.

Stressed VaR can be used to analyze the potential losses for an investment portfolio. In addition, it can be used to compute the capital charge for credit valuation adjustments (CVAs). The CVA represents the expected value or price of counterparty credit risk. For the VaR of CVA, it is important to select the correct stress period so as to compute a stressed CVA VaR that is higher than an unstressed CVA VaR.

Stressed Risk Metrics Advantages and Disadvantages

LO 68.4: Identify the advantages and disadvantages of stressed risk metrics.

A key advantage of using stressed risk metrics is that they are conservative. In other words, the nature of the calculations should allow more-than-sufficient capital to be set aside for unexpected losses for future stressed events.

On the other hand, a key disadvantage is that risk metrics are stressed and will not necessarily respond to current market conditions. Instead, they will be impacted mainly by portfolio assets.

Key Concepts

LO 68.1

Stress tests and value at risk (VaR) and economic capital (EC) measures attempt to transform scenarios into loss estimates. The loss estimates' distributions provide the basis to compute VaR at a very high confidence level. Stress tests tend to look at far fewer scenarios compared to VaR measures. Additionally, stress tests usually develop scenarios that are conditional, while VaR/EC measures tend to develop unconditional scenarios.

LO 68.2

VaR/EC models compute expected losses using the following general formula:

$$\text{expected loss} = \text{PD} \times \text{LGD} \times \text{EAD}$$

In an attempt to assign a probability to a hypothetical or historical stress scenario, one could determine where the stress test losses fall within the VaR/EC loss distribution. By assigning probabilities to outcomes, the calculated probability from the loss distribution facilitates the implementation of stress test results.

LO 68.3

Stressed inputs have been used in analyzing market risk and for both supervisory and internal purposes within financial institutions. There is a mandated use of stressed inputs, stress tests, and stressed parameters by the Basel Accords. Stressed VaR can be used to analyze the potential losses for an investment portfolio. In addition, it can be used to compute the capital charge for credit valuation adjustments (CVAs).

LO 68.4

A key advantage of using stressed risk metrics is that they are conservative. A key disadvantage is that risk metrics will not necessarily respond to current market conditions.

Concept Checkers

1. Which of the following statements regarding differences between stress tests and economic capital (EC) methods is correct?
   A. Stress tests tend to analyze a shorter period of time compared to EC methods.
   B. Stress tests tend to use ordinal rank arrangements, while EC methods use cardinal probabilities.
   C. Stress tests tend to focus on unconditional scenarios, while EC methods tend to focus on conditional scenarios.
   D. Stress tests tend to compute losses from the perspective of the market as opposed to EC methods that compute losses from an accounting perspective.

2. Which of the following statements regarding stress testing and value at risk (VaR) methods is correct?
   A. Cardinal probabilities are a key feature of stress testing.
   B. Practically speaking, stress tests focus on many scenarios.
   C. Both stress tests and VaR methods attempt to transform a scenario into a loss estimate.
   D. For regulatory stress tests, generating hypothetical scenarios uses past history as a departure point.

3. Which of the following statements regarding VaR models in stress tests is correct?
   A. The use of stressed inputs has been especially notable in the area of credit risk.
   B. Financial institutions usually use a binomial model to simulate defaults and credit quality.
   C. Assigning probabilities to outcomes often allows the results of stress tests to be generated.
   D. If a scenario's loss magnitude corresponds to a 95th percentile loss on a VaR loss distribution, then one would take a much higher loss in the economic capital (EC) model as a proxy for the stressed loss resulting from market risk.

4. Which of the following items is not required to be used as a result of the changes to the Basel market risk capital framework and the changes outlined in Basel III?
   A. Stress tests.
   B. Stressed inputs.
   C. Stressed parameters.
   D. Stressed simulations.

5. Which of the following statements most likely describes an advantage of using stressed risk metrics?
   A. The risk metric will be more realistic.
   B. The risk metric will be more conservative.
   C. The risk metric will mirror the portfolio returns.
   D. The risk metric will respond to current market conditions.

Concept Checker Answers

1. **B** Stress tests tend to use ordinal rank arrangements, while EC methods use cardinal probabilities. Stress tests tend to focus on longer periods of time (e.g., several years) compared to EC methods (e.g., point in time). Stress tests tend to focus on conditional scenarios, while EC methods tend to focus on unconditional scenarios. Stress tests tend to compute losses from an accounting perspective while EC methods tend to compute them from a market perspective.

2. **C** Both stress tests and VaR methods attempt to transform a scenario into a loss estimate. Cardinal probabilities are a key feature of VaR methods, not stress testing. Stress tests often focus on a few scenarios (while VaR methods often focus on many scenarios). For regulatory stress tests, generating hypothetical scenarios uses the current period, not past history, as a departure point.

3. **C** Assigning probabilities to outcomes often allows the results of stress tests to be generated. The use of stressed inputs has been especially notable in the area of market risk, not credit risk. Financial institutions usually use a Merton model, not a binomial model, to simulate defaults and credit quality. The same percentile loss on the VaR loss distribution would be taken in the EC model as a proxy for the stressed loss resulting from market risk.

4. **D** Changes to the Basel market risk capital framework require the use of stressed inputs and stress tests. The changes in Basel III outline the use of stressed parameters.

5. **B** A key advantage of using stressed risk metrics is that they are conservative. In examining capital adequacy for unexpected losses and considering stressed metrics, the amount of capital is likely to be more than sufficient. In other words, a risk metric that is stressed is likely to be more conservative. A more conservative risk metric does not necessarily mean it is more realistic. One of the disadvantages of using stressed inputs is that the risk metric becomes unresponsive to current market conditions and is more dependent on the investments within the portfolio.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. This topic is also covered in:

Principles for Sound Stress Testing Practices and Supervision

Topic 69

Exam Focus

The recent financial crisis has revealed numerous weaknesses in stress testing practices as employed by banks, and by doing so, has intensified the need for improvement in stress testing methodologies, scenarios, and the handling of specific risks. Learning from the identified weaknesses of stress testing practices, banks should introduce numerous improvements. Perhaps, most importantly, banks should incorporate into their stress testing models feedback effects arising from initial shocks (such as mortgage delinquencies) that can spill over to other segments and markets, further increasing correlations among various risks and eventually generating severe consequences for banks, markets, and other entities, both nationally and globally.

Stress Testing in Risk Management

LO 69.1: Describe the rationale for the use of stress testing as a risk management tool.

Stress testing is an important risk management tool that enables a bank to identify the potential sources of risk, evaluate the magnitude of risk, develop tolerance levels for risk, and generate strategies to mitigate risk. For example, stress testing, as a complementary tool to other risk measures, can be valuable for a bank to determine its potential needs for capital to absorb losses in the event of "stress" conditions (i.e., large shocks). Such advance recognition, evaluation, and planning could be crucial for a bank's effective risk management or even survival.

Moreover, Pillar 1 of Basel II related to minimum capital requirements mandates that banks undertake stress testing for assessing capital adequacy if they are using the Internal Models Approach (IMA) to determine market risk or advanced or foundation internal ratings-based (IRB) approaches to determine credit risk. Pillar 2 of Basel II related to the supervisory review process (SRP) mandates that banks undertake general stress tests. Compliance with Basel II requirements would help banks to correctly assess risk and develop plans to reduce their actual losses. Recent financial turmoil has substantially enhanced the significance of comprehensive, flexible, and forward-looking stress testing, as many banks discovered that they were ill-prepared to identify, assess, and mitigate the severe shock they experienced. Their models and strategies, based on past statistical relationships, did not work well under new and rapidly changing conditions.

Professor's Note: The three Pillars of the Basel II Accord, as well as methods for calculating capital requirements for both credit and market risk, will be addressed in the FRM Part II curriculum.

LO 69.2: Describe weaknesses identified and recommendations for improvement in:
- **The use of stress testing and integration in risk governance**
- **Stress testing methodologies**
- **Stress testing scenarios**
- **Stress testing handling of specific risks and products**

LO 69.3: Describe stress testing principles for banks regarding the use of stress testing and integration in risk governance, stress testing methodology and scenario selection, and principles for supervisors.

Stress Testing and Risk Governance

Weaknesses

Lack of involvement of board and senior management. Banks that were able to deal with recent financial crisis relatively well were the ones that attracted active and comprehensive involvement from the board and senior management in the entire stress testing process. The process involved everything from developing and implementing the plan to applying the results as a continually acceptable strategic planning process. On the other hand, banks that did not have the active involvement of senior management and the board were likely hit hard by the recent financial turmoil.

Lack of overall organizational view. Prior to the recent crisis, stress testing in many banks was conducted by separate units concentrating on a risk function or a business line without taking into consideration the overall impact on the bank under stress conditions. Separate testing created organizational barriers making it difficult to integrate insights and perspectives obtained on a bank-wide basis. Moreover, stress tests were conducted based on routine, standard, and mechanical processes and methodologies which lacked the ability to fully encompass changing business environments and insights gained from various areas of the bank.

Lack of fully developed stress testing. Stress testing for market risks has been conducted for years but stress testing for credit risk is quite recent, and stress testing for other risks (e.g., operational) is still in its infancy. Moreover, prior to the crisis, there was no mechanism in place to identify the correlations among various risks. Therefore, stress testing did not adequately identify correlated exposures and risk concentrations across the bank.

Lack of adequate response to crisis. Stress testing methods were not flexible or effective enough to respond swiftly and comprehensively to the changing conditions as the crisis developed. Investment in information technology was not sufficient, which made it difficult to obtain timely information in order to evaluate and generate new and changing test scenarios.

Recommendations

Stress testing, overall governance, and risk management. A stress testing program should constitute a critical component of a bank's governance and risk management planning. To achieve this goal, the board and senior management should be actively and fully engaged in the stress testing process. The board has the ultimate responsibility for the stress testing program, while senior management should be responsible for managing, implementing, and reviewing the program. Senior management should ensure that the results of stress tests are used in decision making processes, such as setting the risk tolerance and exposure limits, assessment of strategic options, and planning for liquidity and capital adequacy.

Comprehensive stress testing program. A bank should operate a comprehensive stress testing program, focusing on four major areas:

- Risk identification and management.
- Alternative risk perspectives.
- Liquidity and capital management.
- Communication (both internal and external).

To promote effective risk identification and control, stress testing should be used at various levels and activities, such as risk concentrations, business strategies, credit and investment portfolios, and individual and group lending. To achieve a better understanding of risk, stress testing should provide an independent and complementary assessment to other risk measures, such as **economic capital** and **value at risk** (VaR). Stress tests should simulate unprecedented extreme events (shocks). Use of such stress testing would help to discover correlations among various types of risks on a firm-wide basis. Stress testing should be rigorous and forward-looking in order to identify bank specific or market-wide events that may produce a negative impact on a bank's capital and liquidity positions. Stress testing results and risk management strategies should be communicated internally as well as externally. Such disclosure provides an opportunity for market participants to develop a better understanding of a bank's risk exposure and risk management strategies.

Multiple perspectives and techniques. Stress testing programs involve various phases and steps, including identification of risk events, application of modeling techniques, implementation of test procedures, and use of the results for refining the risk mitigation strategies. In this entire process, there should be extensive consultations, collaborations, and interactions among various senior experts within a bank in order to collect multiple perspectives and synergies. Moreover, banks should use multiple techniques, ranging from sensitivity analysis (examining impact of one risk factor on a bank's performance holding all other factors constant), to scenario analysis (examining various scenarios one at a time), to risk simulation (examining the impact of multiple risk factors and interactions simultaneously).

Written policies and documentation. Procedures and written policies should be adequately documented to govern a stress testing program. Documentation is particularly important for firm-wide stress testing. Documentation should include fundamental items such as test types, methodologies, scenarios, underlying assumptions, results, and mitigation strategies.

Sound infrastructure. To implement the stress testing program effectively under times of stress, a bank should have in place a sound, adequate, and flexible infrastructure as well as data collection arrangements. Such infrastructure should enable a bank to aggregate risk exposures quickly, alter methodologies, generate new scenarios, and conduct ad-hoc stress testing under extreme conditions.

Regular assessment. Evaluation of the effectiveness and robustness of stress testing programs should be conducted on a regular basis, on both a quantitative and qualitative basis. Quantitative evaluation should compare the effectiveness of a bank's testing program with other stress testing programs within and outside the bank, and qualitative evaluation should assess those elements of the program that are based on expert opinions and subjective judgments.

Stress Testing Methodologies

Weaknesses

Inadequate infrastructure. Banks did not have adequate infrastructure (and data collection systems) in place that would enable them to quickly identify risk exposures and aggregate them at an institutional level.

Inadequate risk assessment approaches. Stress testing methods were based on an underlying assumption that risk is generated by known and non-stochastic processes, which would mean that future risk events could be forecasted reasonably well. However, recent turmoil clearly revealed the fallacy of such an assumption and demonstrated that risk assessment approaches that resulted from these methods were ineffective. Historical financial correlations broke down once the crisis started and stress testing models based only on historical data failed to predict the possibility of severe shocks, much less how to cope with these shocks.

Inadequate recognition of interactive effects. The recent financial crisis has generated strong examples of feedback, spillover, and system-wide interactional effects. Stress testing models were unable to capture these effects and did not perform well under a rapidly changing risk characteristics environment.

Initial mortgage default shocks caused an adverse impact on the prices of mortgage-backed securities (MBS), in particular, collateralized mortgage obligations (CMOs). Rising uncertainty about the value of underlying investments (i.e., the pool of mortgages) brought the securitization market to an almost complete halt as activity in origination and distribution of CMOs significantly declined, substantially reducing liquidity. Pipeline risk increased as banks were forced to warehouse loans that they intended to securitize. Funding liquidity risk concerns increased, causing the interbank lending activity to decline significantly. In sum, initial difficulties in the mortgage market caused a decline in funding liquidity, which forced market participants to sell the securities at a significant loss.

Inadequate firm-wide perspective. Stress testing prior to the crisis was mostly geared toward individual business lines, products, and risk exposures without having a comprehensive firm-wide perspective. Firm-wide testing can generate synergistic effects, enabling a bank to better identify and manage risk. For example, a comprehensive stress test based on interactions among various experts across the bank would have enabled traders to recognize in a timely manner the increasing risk of mortgage-backed securities as retail lending units were curtailing their exposure to mortgage lending.

Recommendations

Comprehensive stress testing. Stress testing should be comprehensive, covering business areas and risk exposures as individual entities and on a firm-wide level. It should examine the impact of stress events (shocks) on risk factors while taking into consideration feedback and spillover effects due to correlations. All stress testing activities, in the end, should produce a comprehensive and complete firm-wide risk view.

Risk concentrations. A bank may develop risk concentrations along different dimensions, including concentrations in name, industry, region, single or correlated risk factors, off-balance sheet, contractual or non-contractual (reputational) exposures. Stress testing should enable a bank to identify and control risk concentrations. To achieve this goal, stress testing methodology should be comprehensive and firm-wide as well as focused on a wide array of concentrations, including on-balance sheet and off-balance sheet exposures.

Multiple measures. In order to develop an adequate understanding of the impact of stress events on a bank's overall performance, profitability, operations, and viability, numerous measures should be used. For example, a bank should measure the likely impact of stress conditions on asset and portfolio values, accounting and economic profits (losses), funding gaps, and capital requirements.

Stress Testing Scenarios

Weaknesses

Lack of depth and breadth. Prior to the recent financial turmoil, banks used stress testing scenarios that were ineffective in capturing extreme shocks. Banks employed scenarios based on shocks with mild intensity, shorter duration, and smaller spillover or feedback effects among various markets, assets, and positions. Even when banks used somewhat extreme scenarios, results produced a decline in earnings of 25% or less, however, banks can certainly lose (and have indeed lost) more than 25% in severe stress conditions.

Lack of adequate techniques. Banks have employed numerous techniques, including sensitivity analysis to generate testing scenarios. **Sensitivity analysis** does not take into consideration the feedback effect resulting from correlations among various risk factors, positions, and markets, since it focuses only on the impact of a shock on a single factor at a point in time while holding all other factors constant. Other scenario-generating approaches involve multiple factors, but they are historical or hypothetical in nature.

Lack of forward-looking scenarios. Historical and hypothetical scenarios as used by banks turned out to be less effective in the context of the recent crisis because they were based on shocks with smaller intensity, a shorter length of time, and insignificant risk correlations. Banks rarely discussed the possibility of extreme scenarios, and when it was discussed, such a possibility was quickly ruled out.

Recommendations

A variety of events. Stress testing should cover scenarios encompassing a variety of events and varying severity levels, both at micro and macro levels. That is, scenarios should involve testing both at the single entity (a specific product or business line) level and at the entire firm level.

Futuristic outlook. Scenarios should be developed based on potential future events, emerging risks, new products, and asset and liability composition, rather than historical relationships, which may not continue in the future due to changing risk dynamics and market characteristics.

Synergy effect. In order to develop effective future scenarios, opinions and forecasts should be collected and synthesized from experts and senior management across the bank. The discussion processes should be comprehensive, participative, and well-integrated among various units, products, and business lines.

Time horizon. Stress testing should cover various time horizons along with liquidity conditions. As we have witnessed, liquidity can deteriorate quickly and recessions can continue longer than anticipated. A bank should assess its coping strategies for such occurrences during various time horizons. Underlying assumptions and scenarios can change if the length of time of stress testing has increased. Therefore, a bank should recognize that time horizons can play a critical role in scenario development and testing methodology.

Reverse stress testing. Reverse stress testing involves three phases: outcome, events, and hedging. First, reverse stress testing starts from a scenario with a known outcome, such as severe capital inadequacy, panic deposit withdrawals, or insolvency. Second, an assessment is made as to what kind of events, isolated or correlated, firm or market specific, or other events, can lead to the outcome. Lastly, a bank evaluates the effectiveness of its risk management (e.g., hedging) strategies to cope with the events likely to produce the outcome, including an extreme outcome, such as insolvency. The recent crisis has intensified the need for banks to undertake reverse stress testing in order to enhance the overall effectiveness of their risk management plans to cope with events such as extreme events that may have a low probability of occurrence.

Stress Test Handling

Risks Arising From Complex Structured Products

Complex structured products, such as mortgage-backed securities (MBS), asset-backed securities (ABS), and collateralized debt obligations (CDOs), offer cash flows to investors based on an underlying pool of mortgages, loans, account receivables, corporate bonds, or other financial instruments. For example, in the case of MBSs, mortgage loans are packaged into mortgage pools, and tranches are then issued on the underlying pool where each tranche offers a different risk and return profile.

The use of structured securities has significantly increased over the past decade as a risk management (or investment) tool. Given the characteristics of the structured products, it should be obvious that the nature, extent, and sources of risk for these securities will be different from non-structured investments that do not have a complex structure, such as regular bonds.

Stress testing of these complex products was not properly conducted based on "severe scenarios." More importantly, banks ascertained the risk of these complex structured securities based on the credit rating of apparently similar cash instruments, such as bonds. The fact of the matter is that these products have a different risk profile, and the credit rating for regular bonds should not have been used to determine the riskiness of these more complex instruments.

There are several recommendations for improving the stress testing of risk arising from the use of complex structured products.

- *A stress test should use all the relevant information about an underlying asset pool.* For example, quality of loans, creditworthiness of the borrower, maturity, and interest rates.
- *Impact of market conditions.* For example, investors are subject to prepayment risk if market mortgage rates decline below the rates on existing mortgages.
- *Contractual obligations.* For example, contingent funding agreements in which firms ensure timely payment of interest and principal if certain agreed upon conditions occur.
- *Subordination level of a specific tranche.* For example, a tranche may offer cash flows only after payment has been made to other tranches, meaning that such a tranche exhibits higher risk, particularly under stress conditions.

Basis Risk

Banks are exposed to various risks that can adversely impact their earnings, asset values, and even solvency. For example, an increase in market interest rates can produce a decline in asset values and interest margins, which in turn can pose challenges to a bank's overall capital adequacy. Banks use numerous risk management tools, including futures contracts, to hedge against potential losses arising from directional risks, such as an inverse relationship between a bank's security portfolio (bonds) and an increase in market interest rates. A bank can engage in a short hedge (selling a futures contract) in order to protect the bond portfolio value in the event of an increase in interest rates. Losses in cash instruments (bonds) are offset by gains in futures markets in the event of an increase in interest rates.

However, successful futures hedging is based on certain underlying assumptions, including that the basis does not change between opening and closing of a futures position. As shown in the futures material in Book 3, **basis** is the difference between prices or interest rates between the cash market and the futures markets. Changes in basis between opening and closing of futures position is called **basis risk**, which can yield an ineffective hedge. Due to basis risk, instead of completely offsetting the cash market losses through the use of futures contract, a bank may experience a partial offsetting, or in worst case scenarios, no offsetting of losses. Basis risk can arise from various sources, including low correlation between the price movement of underlying cash instruments and the futures contracts, cross hedging, and relative illiquidity of futures contracts.

Banks tend to focus on directional risks but ignore the essence of basis risk while conducting stress testing. Therefore, they could not adequately ascertain the effectiveness of their hedging strategies utilizing futures contracts. Banks can improve stress test handling by taking basis risk into consideration. The disconnect between futures and cash prices may increase illiquidity under market stress conditions, which can significantly reduce hedging effectiveness. Also, under stress conditions, hedging may turn out to be less successful when multiple entities are trying to pursue the same hedging techniques. Banks should include all of these scenarios, under stressed conditions, to evaluate the effectiveness of hedging strategies.

Counterparty Credit Risk

Banks use numerous tools to effectively manage risk exposures, including swaps, forwards, options, and insurance contracts. For example, banks and dealers can purchase default insurance on their structured credit products, such as collateralized debt obligations. This insurance protection generates cost savings for the issuers. The recent financial crisis, however, produced an unprecedented event. Entities expected to provide protection sustained severe losses in the wake of chaotic market conditions and in turn lost credibility. For example, in the case of **monoline insurers** (which provide default protection insurance to issuers of various securities in a specific industry), there had been no default or downgrading prior to 2007, but the recent crisis produced downgrading or even the default of some of these insurers.

Downgrading of some monoline insurers during the recent financial crisis immediately caused downgrading of numerous issuers that had received protection from these insurers, creating a **wrong-way risk**. A wrong-way risk emerges when the probability of default of counterparties increases as a result of general market conditions (general wrong-way risk). Another example of a wrong-way risk is when a company writes an option on its own stock. So, if the probability of default of the writer increases, the risk exposure of the bank increases as well (specific wrong-way risk). Prior to the recent crisis, bank holding companies engaged in diversified business activities did not focus on wrong-way risk in their stress testing practices. That was certainly a weakness in their stress test handling program. Based on the lessons learned from the recent global financial crisis, it is recommended that banks include the potential risks, arising from financial conditions of counterparties, market spillover, and feedback effects under severe stress conditions.

Pipeline Risk

Securitization, creating investment securities from a pool of underlying assets, is used by banks to expand sources of funding, free-up balance sheet space for higher yielding or safer assets, and generate extra revenue. However, securitization is not without risks. During market stress, a bank may not be able to complete the entire process of selling the securities to the public through the issuer-special purpose entities (SPEs). Consequently, market conditions may force the bank to warehouse underlying assets longer than planned and incur financing costs. In addition, a bank can be exposed to many other risks, including liquidity risk due to lack of access to the securitization market. Such risk is called **pipeline risk** (a.k.a. warehouse risk).

Prior to the recent crisis, banks conducted stress tests based on the assumption that pipeline risk would be minimal or non-existent. That is, banks believed that securitization markets would continue to operate smoothly, and if there were any disruption, it would not exist for long. That was certainly not the case with recent subprime securitization. Given the recent experience, a bank should include pipeline risk into its scenario stress tests. Regardless of the probability of the securitization of assets, banks should include such exposures in stress testing procedures for effective management of pipeline risk.

Contingent Risk

There are several sources of contingent risk, including the potential risk arising from the process of securitization and creation of off-balance sheet vehicles, such as special purpose entities (SPEs). As mentioned, securitization enables a bank to free-up balance sheet space (by selling loans to SPEs). However, banks are obligated to inject credit and liquidity to off-balance sheet entities due to contractual agreements or reputational concerns. Banks provide support to off-balance sheet entities, even when they are not obligated to avoid a materially adverse impact on their reputation. Therefore, banks expose themselves to the potential risk involved in fulfilling their commitments, contractual or otherwise, to off-balance sheet vehicles. Such risk may intensify if banks are facing tough times themselves.

Prior to the recent crisis, banks' stress testing mechanisms did not take into consideration the contingent risks related to off-balance sheet exposures. Adequate recognition of such risks would have helped banks to avoid concentrations in such exposures. Stress testing should include scenarios to assess the bank's exposure to off-balance sheet commitments, both contractual and reputational. In addition, stress testing should focus on the potential impact of contingent risk on other risk exposures, such as liquidity, credit, and market risks.

Funding Liquidity Risk

Stress testing conducted by banks did not capture the nature, size, duration, and intensity of the recent crisis. It did not assess funding liquidity risk adequately and also failed to recognize the interrelationship between funding liquidity risk and market (or trading) liquidity risk.

Future stress tests should focus on the correlation of various factors under stress conditions, which may increase the risk exposure for banks. For example, a decline in an asset value (or category) may dry up its liquidity, liquidity pressures may intensify due to contractual or reputational concerns, and damage to a bank's financial condition may diminish access to funding markets.

Recommendations to Supervisors

Assess stress testing methods. Supervisors should make frequent and comprehensive assessments of a bank's stress testing procedures. This involves evaluating a bank's compliance with sound stress testing practices and understanding how stress testing impacts strategic decision making at various levels of management. It is also recommended that supervisors share their views on the direction of global financial markets and how a bank can develop forward-looking stress tests to combat the impact of future market crises.

Take corrective actions. In the event that stress testing procedures or analysis is deemed inadequate, a supervisor should push for corrective actions. It is important to continually verify the effectiveness of key stress testing assumptions and their relevance going forward.

Should a deficiency arise, a supervisor may propose a revision to bank policies and/or a reduction of global risk exposures.

Challenge firm-wide scenarios. It is necessary for supervisors to question the use of stress tests that produce unrealistic results or are inconsistent with a bank's risk appetite. Supervisors should also encourage banks to evaluate scenarios that could harm its reputation or strategic planning effectiveness. It is also recommended to test scenarios that could impact individual business lines within the bank.

Evaluate capital and liquidity needs. Under the Basel II Accord, banks should conduct an analysis of their stress tests when assessing both capital requirements and liquidity. It is, therefore, recommended that supervisors consider the potential capital needs of a bank under times of stress. For a robust analysis, supervisors should utilize capital ratios in their assessment of capital adequacy and determine the mobility of capital across business lines. The ability to meet capital requirements during stress scenarios is crucial to ensure that a bank will remain solvent if such an event occurs. If capital appears low, the supervisor could recommend that the bank increase capital above the requirements outlined by the Basel Committee. Liquidity buffers should also be evaluated for times of stress. If liquidity appears inadequate, contingencies should be discussed with senior management.

Apply additional stress scenarios. It is prudent for supervisors to conduct additional stress tests using common scenarios within a bank's jurisdiction. These additional scenarios would complement the bank's existing stress scenarios and should be relatively easy to implement. It should be clear to bank management that these suggested stress exercises are not a substitute to the existing stress tests designed by senior management.

Consult additional resources. In order to expand their knowledge of stress testing, supervisors should consult with other experts to identify potential stress vulnerabilities. Discussing the behavior of other banks within the industry could provide a greater understanding of imbalances created by banks and how those imbalances may impact the financial markets. It is also important for supervisors to evaluate their own performance and acquire new skills if necessary, such as knowledge of updated quantitative models.

Key Concepts

LO 69.1

Stress testing is an important tool that enables a bank to identify, assess, monitor, and manage risk. Recent financial turmoil has substantially increased the need for flexible, comprehensive, and forward-looking stress testing.

LO 69.2

Major weaknesses and recommendations for stress testing and integration in risk governances are as follows. Weaknesses: lack of involvement of board and senior management, lack of overall organizational view, lack of fully developed stress testing, lack of adequate response to crisis. Recommendations: Stress testing should form an essential ingredient of overall governance of risk management plan, encompass multiple techniques and perspectives, involve a sound infrastructure and regular assessment, produce written policies and recommendations, and generate comprehensive firm and market-wide scenario testing.

Stress testing methodologies were based on inadequate infrastructure, inadequate risk assessment approaches, inadequate recognition of correlation, and inadequate firm-wide perspectives. Given these weaknesses, recommendations for improvement include development of a comprehensive stress testing approach, identification and control of risk concentrations, and multiple measurements of stress impact.

Stress testing scenarios lacked depth and breadth because they were based on mild shocks, shorter duration, and smaller correlation effects among various markets, portfolios, and positions.

Banks evaluated the risk of complex structured products based on the credit rating of similar cash instruments. However, the nature, magnitude, and sources of risk for these products are different from non-structured products. In order to identify, assess, monitor, and control risk exposure of complex structured products, stress testing plans should utilize all the relevant information about the underlying asset pool, market conditions, contractual obligations, and subordination levels.

Historically, stress testing has not fully recognized funding liquidity risk and its correlation with other risks in times of crises. Future stress tests should focus more on correlations of various factors and risks, including funding liquidity risk.

LO 69.3

Principles for sound stress testing supervision include: assessing stress testing methods, taking corrective actions, challenging firm-wide scenarios, evaluating capital and liquidity needs, applying additional stress scenarios, and consulting additional resources.

Concept Checkers

1. Prior to the recent crisis, stress testing was primarily based on which of the following characteristics?

   I. Historical or hypothetical scenarios.
   II. Significant system-wide correlations.

   A. I only.
   B. II only.
   C. Both I and II.
   D. Neither I nor II.

2. Stress testing for a bank's securitized exposures should primarily consider which of the following features?
   A. Credit ratings of issuers' bonds.
   B. Credit ratings of issuers' bonds and quality of underlying asset pool.
   C. Quality of underlying asset pool, subordination level of tranches, and systematic market conditions.
   D. Credit ratings of issuers' bonds, quality of underlying asset pool, and systematic market conditions.

3. Which of the following statements related to stress test handling is correct?
   A. Hedging, through futures contracts, can result in significant loss if basis changes between the opening and closing of futures position.
   B. Pipeline risk emerges only due to market conditions.
   C. Reputational risk is not as important as contractual risk.
   D. A bank with large exposures to counterparties should not be concerned about counterparties' exposure to market conditions or specific assets.

4. Which of the following statements related to conducting stress tests is incorrect?
   A. Basel II requires banks to undertake stress tests for assessing capital adequacy at least once a month.
   B. Results of stress testing should be used for strategic business planning purposes.
   C. Stress testing can use sensitivity analysis to assess risk.
   D. Stress testing should be used to identify risk concentrations.

5. Which of the following statements is(are) correct?

   I. Stress testing for credit risk has been conducted for years, whereas stress testing for interest rate risks is quite recent.
   II. During the recent crisis, pipeline risk decreased as banks were able to warehouse loans which they intended to securitize.

   A. I only.
   B. II only.
   C. Both I and II.
   D. Neither I nor II.

Concept Checker Answers

1. **A** Recent turmoil revealed numerous weaknesses in banks' stress taking practices, such as lack of proper recognition of extreme shocks and presence of significant system-wide correlations (feedback and spillover effects) between different markets, risks, and portfolio positions. Shorter test durations and historical or hypothetical scenario-based testing were key weaknesses in stress testing practices. Actual events showed longer duration of stress conditions and breakdown of historical statistical relationships.

2. **C** Banks made a major mistake in assessing the riskiness of securitized products by relying on credit ratings of apparently similar issues, like corporate bonds. Securitized products are complex and possess different characteristics and risk exposures. Therefore, external rating for non-structured products should not be applicable to these products.

3. **A** Future contracts are used to offset cash market losses against the gains in the futures market. Hedging will be effective if the basis does not change between the opening and closing of the futures position. Changes in basis can reduce effectiveness of hedging and can produce significant loss.

4. **A** Basel II does not impose monthly requirements for stress testing.

5. **D** Stress testing for credit risk is quite recent, whereas stress testing for market and interest rate risks has been conducted for years. Pipeline risk increased, not decreased, as banks were forced to warehouse loans on their balance sheet due to deteriorating conditions in securitization markets.

Self-Test: Valuation and Risk Models

15 questions: 36 minutes

1. As a junior quantitative analyst, you have been assigned to research coherent risk measures. Which of the following properties of coherent risk measures explicitly takes into the account the diversification benefits of holding assets in a portfolio with less-than-perfect correlation of returns?
   A. Monotonicity.
   B. Positive homogeneity.
   C. Subadditivity.
   D. Translation invariance.

2. Assume that stock Y is a non-dividend-paying stock priced at $60 per share, and that options on this stock that expire in one year are trading with an exercise price of $55. Given a risk-free rate of 2%, what are the Black-Scholes-Merton (BSM) model values for the corresponding European call and put options assuming $N(d_1) = 0.9767$ and $N(d_2) = 0.9732$?
   A. European call is $6.136 and European put is $0.047.
   B. European call is $5.735 and European put is $0.035.
   C. European call is $5.076 and European put is $0.076.
   D. European call is $3.517 and European put is $1.483.

3. The current stock price of Heart, Inc., is $80. Call and put options with exercise prices of $50 and 15 days to maturity are currently trading. Which of these scenarios is most likely to occur if the stock price falls by $1?

	Call value	Put value
A.	Decrease by $0.94	Increase by $0.08
B.	Decrease by $0.76	Increase by $0.96
C.	Decrease by $0.07	Increase by $0.89
D.	Decrease by $0.76	Increase by $0.89

4. A put option with an exercise price of $45 is trading for $3.50. The current stock price is $45. What is the most likely effect on the option's delta and gamma if the stock price increases to $50?
   A. Both delta and gamma will increase.
   B. Both delta and gamma will decrease.
   C. One will increase and the other will decrease.
   D. Both delta and gamma will stay the same.

5. From the Black-Scholes-Merton model, $N(d_1) = 0.42$ for a 3-month call option on Panorama Electronics common stock. If the stock price falls by $1.00, the price of the call option will:
   A. decrease by less than the increase in the price of the put option.
   B. increase by more than the decrease in the price of the put option.
   C. decrease by the same amount as the increase in the price of the put option.
   D. increase by more than the increase in the price of the put option.

6. Consider the following three bonds that all have par values of $100,000.

   I. A 10-year zero coupon bond priced at 48.20.
   II. A 5-year 8% semiannual-pay bond priced with a YTM of 8%.
   III. A 5-year 9% semiannual-pay bond priced with a YTM of 8%.

   Rank the three bonds in terms of how important reinvestment income is to an investor who wishes to realize the stated YTM of the bond at purchase by holding it to maturity.
   A. III, II, I.
   B. I, II, III.
   C. II, III, I.
   D. I, III, II.

Use the following information to answer Questions 7 through 9.

A bond dealer provides the following selected information on a portfolio of fixed-income securities.

Par Value	*Mkt. Price*	*Coupon*	*Modified Duration*	*Effective Duration*	*Convexity*
$2 million	100	6.5%	8	8	308
$3 million	93	5.5%	6	1	100
$1 million	95	7%	8.5	8.5	260
$4 million	103	8%	9	5	–70

7. What is the effective duration for the portfolio?
   A. 4.81.
   B. 5.63.
   C. 7.17.
   D. 7.88.

8. What is the price value of a basis point for this portfolio?
   A. $5,551.18.
   B. $7,026.60.
   C. $3,234.08.
   D. $4,742.66.

9. What is the approximate price change for the 7% bond if its yield to maturity increases by 25 basis points?
   A. –$19,415.63.
   B. –$17,864.11.
   C. –$20,181.85.
   D. –$16,748.53.

10. Which of the following differences between key rate and forward bucket analysis is(are) true?
    I. Estimating portfolio volatility with both methods is similar except the forward bucket technique requires fewer inputs and correlations.
    II. The key rate shift approach assumes changes in rates in and around the chosen key rates.
    A. I only.
    B. II only.
    C. Both I and II.
    D. Neither I nor II.

11. Sarah Johnson is a risk manager at the hedge fund International Management, Inc. She is analyzing the debt levels of several emerging countries and is relying on bond rating agencies to draw conclusions regarding the ability and willingness of the various countries to service debt. The risk management division of her firm has prepared its own analysis and there are several discrepancies between the agency ratings and the firm's own ratings. A colleague of Johnson recommends that she ignore the agency ratings and rely solely on the firm ratings. Which of the following statements correctly describes a reason Johnson may not want to rely solely on rating agency opinions regarding debt repayment?
    A. Ratings are not influenced by politics and governments or regimes.
    B. Ratings are often delayed relative to the dynamic business and political environments.
    C. When one rating agency upgrades or downgrades a country, the other agencies do not follow suit.
    D. Rating agencies are not optimistic when it comes to rating sovereigns and corporations.

12. You are an associate at a rating agency reviewing a research report compiled by one of the new analysts. Which of the following statements in the report is correct?
    A. For a given rating category, default rates show statistically significant variation based on geographic location.
    B. For a given rating category, default rates show statistically significant variation based on industry.
    C. The cumulative default rate is generally more dramatic for a bond rated Baa3 than for a bond rated Ba1.
    D. The cumulative default rate is generally less dramatic for a bond rated BB than for a bond rated BBB.

13. Global Bank has made a loan with the following characteristics: total commitment of $5 million, of which $4.1 million is currently outstanding. Global has assessed an internal credit rating equivalent to a 1.5% default probability over the next year. Global has additionally estimated a 35% loss rate. What is the expected loss for the loan?
A. $21,525.
B. $24,596.
C. $26,250.
D. $27,735.

14. Loss frequency and loss severity are combined in an effort to simulate an expected loss distribution. Loss frequency is most often modeled with which of the following distributions?
A. Bernoulli distribution.
B. Binomial distribution.
C. Lognormal distribution.
D. Poisson distribution.

15. As an associate risk manager at a bank, you are concerned about the various risks faced by the bank's securitization transactions. Which of the following risks refers to a bank having to hold onto assets for longer than planned and incurring financing costs as a result?
A. Contingent risk.
B. Funding liquidity risk.
C. Pipeline risk.
D. Wrong-way risk.

Self-Test Answers: Valuation and Risk Models

1. **C** Subadditivity refers to the concept that the risk of a portfolio is at most equal to the risk of the assets within the portfolio. This suggests that portfolio risk would be less than the sum of the individual risks of the assets due to diversification.

   (See Topic 54)

2. **A** The value of the call is computed as follows: $c = [S \times N(d_1)] - [Xe^{-rT} \times N(d_2)]$.

   Thus, $c = (60 \times 0.9767) - (55e^{-0.02} \times 0.9732) = 58.602 - 52.466 = \$6.136$.

   Using put-call parity, we can find the value of the put option as follows:

   $p = c - S + Xe^{-rT} = 6.136 - 60 + 55e^{-0.02} = \$0.047$

   (See Topic 56)

3. **A** The call option is deep in-the-money and must have a delta close to one. The put option is deep out-of-the-money and will have a delta close to zero. Therefore, the value of the in-the-money call will decrease by close to $1 (e.g., $0.94), and the value of the out-of-the-money put will increase by a much smaller amount (e.g., $0.08). The call price will fall by more than the put price will increase.

   (See Topic 57)

4. **C** The put option is currently at-the-money since its exercise price is equal to the stock price of $45. As stock price increases, the put option's delta (which is less than zero) will increase toward zero, becoming less negative. The put option's gamma, which measures the rate of change in delta as the stock price changes, is at a maximum when the option is at-the-money. Therefore, as the option moves out-of-the-money, its gamma will fall.

   (See Topic 57)

5. **A** If $\Delta S = -\$1.00$, $\Delta C \approx 0.42 \times (-1.00) = -\$0.42$, and $\Delta P \approx (0.42 - 1) \times (-1.00) = \$0.58$.

   The call will decrease by less ($0.42) than the increase in the price of the put ($0.58).

   (See Topic 57)

6. **A** Reinvestment income is most important to the investor with the 9% coupon bond, followed by the 8% coupon bond and the zero-coupon bond. In general, reinvestment risk increases with the coupon rate on a bond.

   (See Topic 60)

7. **A** Portfolio effective duration is the weighted average of the effective durations of the portfolio bonds.

Numerators in weights are market values (par value × price as percent of par). Denominator is total market value of the portfolio.

$$\frac{2}{9.86}(8)+\frac{2.79}{9.86}(1)+\frac{0.95}{9.86}(8.5)+\frac{4.12}{9.86}(5)=4.81 \text{ (weights are in millions)}$$

(See Topic 61)

8. **D** Price value of a basis point can be calculated using effective duration for the portfolio and the portfolio's market value, together with a yield change of 0.01%. Convexity can be ignored for such a small change in yield.

$4.81 \times 0.0001 \times 9{,}860{,}000 = \$4{,}742.66$

(See Topic 61)

9. **A** Based on the effective duration and convexity of the 7% bond, the approximate price change is:

$(-8.5 \times 0.0025) + (0.5 \times 260 \times 0.0025^2) \times 950{,}000 = -\$19{,}415.63$

(See Topic 61)

10. **B** Estimating portfolio volatility with both methods is similar except the bucket technique requires more inputs and correlations. The key rate shift approach assumes changes in rates in and around the chosen key rates.

(See Topic 62)

11. **B** Investors need rating agencies to update ratings in a timely fashion. Some market participants feel that the agencies take too long to change ratings, leaving investors unprotected in the event of a crisis. Also, some argue that rating agencies are too optimistic when it comes to rating sovereigns and corporations. In addition, when one agency upgrades or downgrades a country, the other agencies tend to follow suit. Finally, agencies rely on information from governments. If governments hide the truth and reveal only positive information, ratings will be incorrect.

(See Topic 63)

12. **B** Empirical evidence suggests that for a given rating category, default rates can vary from industry to industry to a statistically significant degree.

(See Topic 64)

13. **A** EL = exposure amount × probability of default × loss rate

EL = $4.1 million × 0.015 × 0.35

EL = $21,525

(See Topic 65)

14. **D** Loss frequency is most often modeled with a Poisson distribution (a distribution that models random events). Loss severity is often modeled with a lognormal distribution.

(See Topic 66)

15. **C** Pipeline risk originates from market stress when a bank may not be able to complete the entire process of selling the securities to the public through the issue-special purpose entities. Consequently, pipeline risk arises because market conditions may force the bank to warehouse underlying assets for longer than planned and therefore, incurring financing costs.

(See Topic 69)

FORMULAS

Valuation and Risk Models

VaR Methods

$$\text{VaR}(X\%) = z_{X\%}\sigma$$

where:

VaR (X%) = the X% probability value at risk

$z_{X\%}$ = the critical *z*-value based on the normal distribution and the selected X% probability

σ = the standard deviation of daily returns on a percentage basis

$$\text{VaR}(X\%)_{\text{dollar basis}} = \text{VaR}(X\%)_{\text{decimal basis}} \times \text{asset value}$$

$$= (z_{X\%}\sigma) \times \text{asset value}$$

$$\text{VaR}(X\%)_{\text{J-days}} = \text{VaR}(X\%)_{\text{1-day}}\sqrt{J}$$

$$\text{VaR} = \left[\hat{R}_P - (z)(\sigma)\right]V_P$$

Topic 52

GARCH(1,1):

$$\sigma_t^2 = a + br_{t-1,t}^2 + c\sigma_{t-1}^2$$

Topic 53

Taylor Series approximation (order two): $f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$

Topic 55

risk-neutral valuation:

$$U = \text{size of the up-move factor} = e^{\sigma\sqrt{t}}$$

$$D = \text{size of the down-move factor} = e^{-\sigma\sqrt{t}} = \frac{1}{e^{\sigma\sqrt{t}}} = \frac{1}{U}$$

$$\pi_u = \text{probability of an up move} = \frac{e^{rt} - D}{U - D}$$

$$\pi_d = \text{probability of a down move} = 1 - \pi_u$$

Topic 56

expected value: $E(S_T) = S_0 e^{\mu T}$

Black-Scholes-Merton Option Pricing Model:

$$c_0 = \left[S_0 \times N(d_1)\right] - \left[X \times e^{-R_f^c \times T} \times N(d_2)\right]$$

$$p_0 = \left\{X \times e^{-R_f^c \times T} \times \left[1 - N(d_2)\right]\right\} - \left\{S_0 \times \left[1 - N(d_1)\right]\right\}$$

where:

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left[R_f^c + \left(0.5 \times \sigma^2\right)\right] \times T}{\sigma \times \sqrt{T}}$$

$$d_2 = d_1 - \left(\sigma \times \sqrt{T}\right)$$

continuously compounded returns: $u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$

put-call parity:

$$c_0 = p_0 + S_0 - \left(X \times e^{-R_f^c \times T}\right)$$

or

$$p_0 = c_0 - S_0 + \left(X \times e^{-R_f^c \times T}\right)$$

Topic 57

$$\text{delta} = \Delta = \frac{\partial c}{\partial s}$$

$$\text{portfolio delta} = \Delta_p = \sum_{i=1}^{n} w_i \Delta_i$$

$$\text{gamma: } \Gamma = \frac{\partial^2 c}{\partial s^2}$$

relationship among delta, theta, and gamma: $r\Pi = \Theta + rS\Delta + 0.5\sigma^2 S^2 \Gamma$

$$\text{vega} = \frac{\partial c}{\partial \sigma}$$

$$\text{rho} = \frac{\partial c}{\partial r}$$

Topic 58

accrued interest:

$$AI = c\left(\frac{\text{number of days from last coupon to the settlement date}}{\text{number of days in coupon period}}\right)$$

where:

c = coupon payment

clean price = dirty price – accrued interest

Topic 59

$$\text{spot rate: } z(t) = 2\left[\left(\frac{1}{d(t)}\right)^{1/2t} - 1\right]$$

$$FV_n = PV_0 \times \left[1 + \frac{r}{m}\right]^{m \times n}$$

where:
r = annual rate
m = number of compounding periods per year
n = number of years

$$\text{HPR: } r = m\left[\left(\frac{FV_n}{PV_0}\right)^{\frac{1}{m \times n}} - 1\right]$$

$$\text{par rate, } C_T\text{: } \frac{C_T}{2} \times \sum_{t=1}^{2T} d\left(\frac{t}{2}\right) + d(T) = 1$$

Topic 60

$$\text{PV of a perpetuity} = \frac{C}{y}$$

$$\text{realized return: } R_{t-1,t} = \frac{BV_t + C_t - BV_{t-1}}{BV_{t-1}}$$

bond price:

$$P = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \ldots + \frac{C_N}{(1+y)^N}$$

where:
P = the price of the security
C_k = the annual cash flow in year k
N = term to maturity in years
y = the annual yield or YTM on the security

Topic 61

$$DV01 = -\frac{\Delta BV}{10{,}000 \times \Delta y}$$

where:
ΔBV = change in bond value
Δy = change in yield

$$\text{modified duration} = \frac{\text{Macaulay duration}}{(1 + \text{periodic market yield})}$$

$$\text{modified duration} = \frac{1}{BV}\frac{\Delta BV}{\Delta y}$$

$$\text{effective duration} = \frac{BV_{-\Delta y} - BV_{+\Delta y}}{2 \times BV_0 \times \Delta y}$$

where:
$BV_{-\Delta y}$ = estimated price if yield decreases by a given amount, Δy
$BV_{+\Delta y}$ = estimated price if yield increases by a given amount, Δy
BV_0 = initial observed bond price
Δy = change in required yield, in decimal form

DV01 = duration × 0.0001 × bond value

$$\text{convexity} = \frac{BV_{-\Delta y} + BV_{+\Delta y} - 2 \times BV_0}{BV_0 \times \Delta y^2}$$

percentage price change ≈ duration effect + convexity effect

$$= [-\text{duration} \times \Delta y \times 100] + \left[\left(\tfrac{1}{2}\right) \times \text{convexity} \times (\Delta y)^2 \times 100\right]$$

$$\text{duration of portfolio} = \sum_{j=1}^{K} w_j \times D_j$$

Topic 62

key rate '01: $DV01^k = -\frac{1}{10{,}000}\frac{\Delta BV}{\Delta y^k}$

key rate duration: $D^k = -\frac{1}{BV}\frac{\Delta BV}{\Delta y^k}$

Topic 65

expected loss: $EL = EA \times PD \times LR$

unexpected loss: $UL = EA \times \sqrt{PD \times \sigma^2_{LR} + LR^2 \times \sigma^2_{PD}}$

portfolio expected loss: $EL_P = \sum_i EL_i = \sum_i (EA_i \times LR_i \times PD_i)$

portfolio unexpected loss: $UL_P = \sqrt{\sum_i \sum_j \rho_{ij} UL_i UL_j}$

risk contribution: $RC_i = \frac{UL_i \sum_j UL_j \rho_{ij}}{UL_P}$

Using the Cumulative Z-Table

Probability Example

Assume that the annual earnings per share (EPS) for a large sample of firms is normally distributed with a mean of \$5.00 and a standard deviation of \$1.50. What is the approximate probability of an observed EPS value falling between \$3.00 and \$7.25?

If EPS = x = \$7.25, then $z = (x - \mu)/\sigma = (\$7.25 - \$5.00)/\$1.50 = +1.50$

If EPS = x = \$3.00, then $z = (x - \mu)/\sigma = (\$3.00 - \$5.00)/\$1.50 = -1.33$

For z-value of 1.50: Use the row headed 1.5 and the column headed 0 to find the value 0.9332. This represents the area under the curve to the left of the critical value 1.50.

For z-value of –1.33: Use the row headed 1.3 and the column headed 3 to find the value 0.9082. This represents the area under the curve to the left of the critical value +1.33. The area to the left of –1.33 is 1 – 0.9082 = 0.0918.

The area between these critical values is 0.9332 – 0.0918 = 0.8414, or 84.14%.

Hypothesis Testing – One-Tailed Test Example

A sample of a stock's returns on 36 non-consecutive days results in a mean return of 2.0%. Assume the population standard deviation is 20.0%. Can we say with 95% confidence that the mean return is greater than 0%?

H_0: $\mu \leq 0.0\%$, H_A: $\mu > 0.0\%$. The test statistic = z-statistic = $\frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$ = (2.0 – 0.0) / (20.0 / 6) = 0.60.

The significance level = 1.0 – 0.95 = 0.05, or 5%.

Since this is a one-tailed test with an alpha of 0.05, we need to find the value 0.95 in the cumulative z-table. The closest value is 0.9505, with a corresponding critical z-value of 1.65. Since the test statistic is less than the critical value, we fail to reject H_0.

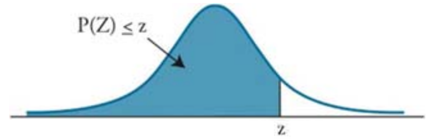
Hypothesis Testing – Two-Tailed Test Example

Using the same assumptions as before, suppose that the analyst now wants to determine if he can say with 99% confidence that the stock's return is not equal to 0.0%.

H_0: $\mu = 0.0\%$, H_A: $\mu \neq 0.0\%$. The test statistic (z-value) = (2.0 – 0.0) / (20.0 / 6) = 0.60. The significance level = 1.0 – 0.99 = 0.01, or 1%.

Since this is a two-tailed test with an alpha of 0.01, there is a 0.005 rejection region in both tails. Thus, we need to find the value 0.995 (1.0 – 0.005) in the table. The closest value is 0.9951, which corresponds to a critical z-value of 2.58. Since the test statistic is less than the critical value, we fail to reject H_0 and conclude that the stock's return equals 0.0%.

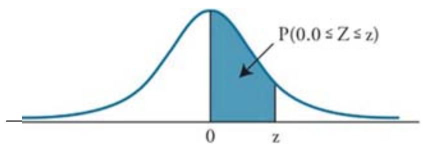
Cumulative Z-Table



$P(Z \leq z) = N(z)$ for $z \geq 0$

$P(Z \leq -z) = 1 - N(z)$

z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990



Alternative Z-Table

$P(Z \leq z) = N(z)$ for $z \geq 0$

$P(Z \leq -z) = 1 - N(z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3356	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4939	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

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