My research and future plans

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My research belongs to the fields of *differential geometry* and *geometric topology*, with a focus on the geometry and topology of *surfaces*, e.g. hyperbolic surfaces. More precisely, my research is in the domains of *Teichmüller-Thurston theory* and *higher Teichmüller-Thurston theory*.

My main achievements. During my PhD Thesis I worked on the compactification of character varieties using tropical geometry. The paper [3] is particularly interesting, see page 5. During my post-doc years I pioneered the theory of Teichmüller spaces for surfaces of infinite type, I wrote several papers on this topic, such as [18], described on page 6. The two papers [3, 18] are being particularly influential, with 45 citations each.

After moving to Heidelberg, I worked mainly in higher Teichmüller-Thurston theory. My most important contributions in this area are the papers [7], published in Geometry & Topology, [11] published in Journal of the European Mathematical Society, and [10] to appear in Memoirs of AMS. These papers are described on page 4-5. I also work on the classical Teichmüller-Thurston theory, see [9], published in International Mathematics Research Notices, and described on page 7.

Character varieties. At the center of my research interests lie the *character varieties* of a finitely generated group Γ,

$$X(\Gamma, G) = \mathrm{Hom}^*(\Gamma, G)/G\,,$$

the set of conjugacy classes of reductive representations of Γ into a reductive Lie group G. The most interesting case is when Γ is the fundamental group of a manifold, in which case $X(\Gamma, G)$ encodes interesting information about the manifold. For this reason, character varieties are important in several research areas in geometry and theoretical physics, e.g. *higher Teichmüller-Thurston theory*, *knot theory*, *integrable systems*, *gauge theory*, theory of *Higgs bundles*, *geometric quantization*, *supersymmetric quantum field theory*.

I usually consider the case when $\Gamma = \pi_1(S)$ is the fundamental group of a surface with negative Euler characteristic. In this case, character varieties are related with *Teichmüller-Thurston theory* and *higher Teichmüller-Thurston theory*, which are my main research areas.

Teichmüller-Thurston theory. The study of the character variety $X(\pi_1(S), PGL(2,\mathbb{R}))$ of a surface group in $PGL(2,\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)$, the isometry group of the *hyperbolic plane*, is part of *Teichmüller-Thurston theory*, a broad research area that can be seen from different points of view: Teichmüller space $\mathcal{T}(S)$ can be seen as a parameter space of *complex structures* on S or, equivalently, as a parameter space of *hyperbolic structures* on S. The space $\mathcal{T}(S)$ can also be identified with a connected component of the character variety $X(\pi_1(S), PGL(2,\mathbb{R}))$ consisting of special representations, called *Fuchsian representations*. Here, it is important to remark that only one connected component of the character variety can be identified with $\mathcal{T}(S)$, while the other connected components have different geometric properties. Teichmüller-Thurston theory is also related to *dynamical systems* (foliations, geodesic flow, Teichmüller flow), to *geometric group theory* (mapping class groups), and to *geometric analysis* (harmonic maps, quasi-conformal maps).

Higher Teichmüller-Thurston theory. If we replace $PGL(2,\mathbb{R})$ with a higher rank reductive Lie group G, we can study the character varieties $X(\pi_1(S),G)$ using ideas coming from the classical Teichmüller-Thurston theory. For an introduction to this theory, see Anna Wienhard's survey paper [62].

As in the case of $PGL(2,\mathbb{R})$, it is important to select special representations with good dynamical properties, that play the role of Fuchsian representations. In literature, there are several competing definitions of special representations, but recently one of them is emerging over the others: it is the notion of *positive representations*, introduced by Guichard-Wienhard [44] in 2018, see also [23, 29, 42, 28]. For a closed surface, they usually are a union of connected components of the character varieties. The set of positive representations is here denoted by $\mathrm{Pos}(S,G)$. There are four families of positive representations [44]:

(a) *Hitchin representations* in simple split Lie groups, originally introduced by Hitchin [46] in 1992, further studied by Labourie [49]). For a closed surface, they form connected components homeomorphic to $\mathbb{R}^{\dim(G)(2g-2)}$, called *Hitchin components* and here denoted by $\mathrm{Hit}(S,G)$.

(b) *Maximal representations* in Hermitian Lie groups of tube type. Originally introduced by Burger-Iozzi-Wienhard [33] in 2003, see also [31]. The set of maximal representations is here denoted by $\mathrm{Max}(S,G)$.

(c) Positive representations in $SO(p,q)$. This is a new family of positive representations discovered in [44].

(d) Positive representations in some exceptional Lie group. Their properties are not yet completely understood.

The positive representations are discrete and faithful. For a closed surface S, they are *Anosov representations*, a notion introduced by Guichard-Wienhard [43] in 2012. This guarantees that they have good dynamical properties and makes them suitable for playing the role of Fuchsian representations for higher rank Lie groups.

My goals and methods. In my research work, I study the character varieties $X(\pi_1(S),G)$, for a higher rank Lie group G, with the aim of building a theory as rich as the classical Teichmüller-Thurston theory. On the one hand there are several interesting questions about the topology and the geometry of the spaces of positive representations $\mathrm{Pos}(S,G)$: we suppose that their geometry is as rich as the geometry of Teichmüller spaces, or even richer. On the other hand there are questions about how the spaces $\mathrm{Pos}(S,G)$ can be used to parametrize geometric structures on manifolds, as it happens for Teichmüller spaces. Some of these questions have an answer, for other questions we have only conjectures.

In order to achive these goals, I use a variety of tools, coming from different areas of mathematics. As expected, I am using the theory of Riemann surfaces and geometric topology; but I am also using tools coming from dynamical systems, gauge theory, algebraic geometry, tropical geometry, algebraic combinatorics (cluster algebras), analysis of PDEs. A very notable tool is given by *Higgs bundles* and the non-abelian Hodge correspondence, a theory developed by Hitchin [45, 46] and many others. This theory comes from theoretical physics and it uses algebraic geometry to parametrize solutions of a system of global elliptic PDEs (Hitchin's equations [45]). This tool can be used to describe character varieties and geometric structures on manifolds, see also my survey paper [5].

Geometric structures. The theory of *geometric structures* on manifolds originated from the work of Klein, Cartan and Ehresmann. A *geometry* (X, G) is given by the transitive action of a Lie group G on a homogeneous space X, and a *geometric structure* on a manifold M is a way to locally mirror on M the geometry of X, in other words it is an atlas of M with charts in X and transition functions locally in G. An important example is given by the eight 3-dimensional geometries that Thurston introduced in the statement of the geometrization conjecture, later proved by Perelman. In higher Teichmüller-Thurston theory, a special place is taken by the *parabolic geometries*, e.g. the real and complex *projective geometries*, the geometry of *Grassmannians*, or the geometry of *flag manifolds*.

Teichmüller spaces arise naturally as parameter spaces of hyperbolic structures on surfaces, and a natural question in higher Teichmüller-Thurston theory is whether the connected components of positive representations $\mathrm{Pos}(S, G)$ may also serve as parameter spaces of geometric structures on some compact manifold M. This is a key step to understand the sets $\mathrm{Pos}(S, G)$ as generalizations of Teichmüller spaces.

In some sense, we know that this is possible, because, for closed surfaces, positive representations are Anosov. In their groundbreaking work [43], Guichard-Wienhard found conditions for the existence of co-compact *domains of discontinuity* for Anosov representations on some parabolic homogeneous spaces. The quotient of the domain is a closed manifold carrying a parabolic geometric structure. Their results were then improved by Kapovich-Leeb-Porti [48].

In this way, we can see every connected component of $\mathrm{Pos}(S, G)$ as a parameter space of parabolic geometric structures on a closed manifold M. But this construction doesn't give any information about the topology of M, and this is a source of open questions.

In a recent work about domains of discontinuity for Anosov representations into complex Lie groups, Dumas-Sanders [38] conjectured that, when the representation is an Anosov deformation of a Hitchin representation, such quotients are always fiber bundles over the surface S. I first proved some special cases of this conjecture, both for complex and for real Lie groups, see for example the paper with Davalo and Li [8] where we prove this for domains in $\mathbb{RP}^n$ and $\mathbb{CP}^n$. Recently, in the paper in preparation with Maloni, Tholozan and Wienhard [19] we proved the theorem not only in the cases conjectured by Dumas-Sanders, but in wider generality. This theorem is a crucial step in understanding the topology of M: we show that M is a fiber bundle over S, with a certain fiber F. We also show that the fiber carries an $SO(2)$-action that serves as structure group of the bundle, and we compute the topological invariants of the bundle. We still cannot describe the fiber F in general, but in the cases where F can be determined, our theorem completely describes the topology of M. In several special cases, we can also describe the topology of F, but in general this is still an open question.

As a complement to our results that describe the topology of M, another important open question is about giving a characterization to the geometric structures parametrized by a connected component of $\mathrm{Pos}(M, S)$. This is usually only a small subset of all the geometric structures carried by M, and it would be interesting to understand which geometric structures appear in the parametrization. This is a question I am addressing in an ongoing project.

It is remarkable how, also for these geometric and topological questions, the tool of *Higgs bundles* can be powerful, see my survey paper [5]. The idea comes from Baraglia's Thesis [26] under Hitchin's supervision, where he constructed some real projective structures using Higgs bundles for the Hitchin components of $SL(3,\mathbb{R})$ and $Sp(4,\mathbb{R})$. I generalized this technique: in collaboration with Li [12] we used this idea to construct Anti de Sitter structures on 3-manifolds, and with Davalo and Li [8] we applied it to construct real and complex projective structures for the Hitchin components of $SL(2n,\mathbb{R})$, giving a proof of the Dumas-Sanders conjecture in this case. To do this, it is necessary to use the non-abelian Hodge correspondence between Higgs bundles and representations obtained by solving Hitchin's equations [45], a system of elliptic global PDEs. This idea seems to be very promising, and we will apply it to other cases.

Topology. The problem of describing the topology of the spaces $\mathrm{Pos}(S,G)$ of positive representations is hard and interesting. I approached this problem using both geometric methods and analytic methods.

The geometric methods consists of choosing some topological data on the surface, and use it to define coordinates describing the topology of $\mathrm{Pos}(S,G)$. In a collaboration with Wienhard, our co-supervised PhD student Rogozinnikov and Guichard [10], we describe new coordinates on $\mathrm{Max}(S,Sp(2n,\mathbb{R}))$, when S is a surface with punctures. We call them the *non-commutative coordinates*. This work generalizes the Fock-Goncharov coordinates, that are suitable for describing Hitchin representations, to the setting of maximal representations. The coordinates are associated to a choice of an ideal triangulation and they induce a structure of non-commutative cluster variety on $X(\pi(S),Sp(2n,\mathbb{R}))$ (see Berenstein-Retakh [27]). These coordinates give a precise description of the space $\mathrm{Max}(S,Sp(2n,\mathbb{R}))$: we explicitly determine its topology and its homotopy type and, for $Sp(4,\mathbb{R})$, we describe the singularities of the space.

In a project in progress, we also show that the non-commutative coordinates can also be constructed with *Spectral Networks*, special graphs on branched coverings of surfaces coming from the work of Gaiotto-Moore-Neitzke [40] on *supersymmetric quantum field theories*.

We are continuing the study of the non-commutative coordinates. In a project in progress, we can generalize the results of [10] to $\mathrm{Max}(S,G)$ for G any classical Hermitian group of tube type. This uses the results we obtained in collaboration with Berenstein and Retakh [6]. Our aim is to generalize this even further and obtain non-commutative coordinates for all the spaces of positive representations $\mathrm{Pos}(S,G)$.

The analytical methods consist of using *Higgs bundles* and the *non-abelian Hodge correspondence*. An example of this is my paper with Collier [7] about maximal representations in $PSp(4,\mathbb{R})$ and $Sp(4,\mathbb{R})$, where we obtained a very precise description of the topology of the maximal components of the character varieties for a closed surface, with special attention to their singularities. See below for more detail about this work.

In my paper with Lee and Schaffhauser [11] we study the topology of the character variety $X(\Gamma,G)$, where Γ is a 2-dimensional orbifold group and G is split. Orbifold groups are a large family of groups including fundamental group of non-orientable surfaces, surfaces with boundary and 2-dimensional hyperbolic Coxeter groups. We prove that $X(\Gamma,G)$ always contains a connected component, the *Hitchin component*, homeomorphic to a ball, and we compute its dimension. We give applications to the Hitchin components of surfaces with boundary [52], to the pressure metric [30] on Hitchin components of ordinary surfaces, and to parameter spaces of geometric strucutres on Seifert fibered 3-manifolds. Our proof uses equivariant Higgs bundles as a technical tool. In work in progress, we are generalizing this to other types of positive representations.

I am also working on $X(\pi(S), PSL(2,\mathbb{C}))$, where $PSL(2,\mathbb{C})$ can be seen as the oriented isometry group of the hyperbolic space $\mathbb{H}^3$. In a work in progress with Li and Sanders we are using *Higgs bundles* to describe the space of equivariant minimal surfaces in $\mathbb{H}^3$. This space is closely related with the *nilpotent cone* of the *Higgs bundles moduli spaces*, a space that is a deformation retract of $X(\pi(S), PSL(2,\mathbb{C}))$. Hence this work will give information on the homotopy type of the character variety. A stratification of the nilpotent cone is known, but the open question that we are answering is how the different pieces intersect. In the future, I plan to use the techniques developed in this project to understand the topology of other character varieties for some real simple Lie groups.

Complex geometry. Teichmüller space is a complex manifold, and it can be seen as a parameter space for Riemann surfaces. Similarly, we can ask whether the spaces $\mathrm{Pos}(S, G)$ have a mapping class group invariant complex structure, even if G is a real Lie group. Moreover, we can ask whether, in some cases, $\mathrm{Pos}(S, G)$ parametrizes Riemann surfaces with additional structure.

In my paper with Collier [7], we answer all these questions in the affirmative for the maximal representations in $PSp(4,\mathbb{R})$ and $Sp(4,\mathbb{R})$. We also describe the topology of the maximal components, with special attention to their singularities. We use *Higgs bundles* as a technical tool to prove these results.

In [7] we show that, if an open subset of $X(\pi_1(S), G)$ satisfies *Labourie's conjecture*, then that subset has a mapping class group invariant structure of complex analytic space. Labourie's conjecture [50] states that every Hitchin representation admits a unique equivariant *minimal surface* in the symmetric space of the target group G, but it is natural to extend this conjecture to every maximal representation. In [7] we prove Labourie's conjecture for all maximal representations in $PSp(4,\mathbb{R})$ and $Sp(4,\mathbb{R})$ (for Hitchin representations in $Sp(4,\mathbb{R})$ it was proved by Labourie [51], and other special cases by Collier [35]).

Labourie's conjecture is now well understood for all the semi-simple Lie groups of rank 2 (see Labourie [51] and Collier-Tholozan-Toulisse [36]), but it has recently been shown to be false for the groups of rank greater or equal than 3 (see Sagman-Smillie [57]). It may still be possible to generalize the results about complex structures on $\mathrm{Pos}(S, G)$ for groups of rank greater than 2, but this will require different ideas.

Compactifications and Tropical Geometry. Character varieties are usually not compact, so it is natural to ask what happens when a sequence of elements goes to infinity. The theory of *Thurston's compactification* of Teichmüller spaces [58], a fundamental tool in the classification of homeomorphisms of surfaces, gives a boundary for Teichmüller space $\mathcal{T}(S)$ called *Thurston's boundary*, which can be identified with the parameter space of *measured laminations* on S. The boundary is homeomorphic to a sphere, and Thurston used *train tracks* to give it a mapping class group invariant *piecewise-linear structure*.

Thurston's theory was generalized by Morgan-Shalen [53] in a way that can be applied to all character varieties $X(\pi_1(S), G)$. Anyway, the general theory is much less understood, and questions about the topology of the boundary, its structure and the interpretation of the boundary points are still open in general. Many people worked at the general theory including myself [1, 4, 3, 2], Parreau [54, 55], Cooper-Delp [37], Burger-Pozzetti [34], Burger-Iozzi-Parreau-Pozzetti [32].

In my PhD Thesis [1], I studied this construction using *tropical geometry*, a construction that transforms a complex *algebraic set* in a *polyhedral complex*. I proved in [3] that also a real *semi-algebraic* set is always transformed into a polyhedral complex. The proof is quite technical and uses tools from logic and model theory, since the methods from commutative algebra that were used over the complex field don't suffice any more in the case of real semi-algebraic sets. This work was very influential, it became a foundational paper in the field of computational real algebraic geometry, it was cited 45 times and it still attracts new citations every year.

I then applied this theorem to suitable connected components of the character varieties for the Lie group $SL(n, \mathbb{R})$, which are real semi-algebraic sets: the tropical polyhedral complex is closely related with the boundary. In this way, I proved that the topological dimension of the Morgan-Shalen boundary is strictly smaller than the dimension of the character variety (contrary to a conjecture of Wolff [60]), and the boundary is always homeomorphic to a compact subset of a Euclidean space.

Moreover, this result can be applied to the Teichmüller space, and I proved that the piecewise-linear structure found by Thurston is the same that comes from the tropicalization construction. This shows the relationship between the piecewise-linear structure on the boundary and the structure of semi-algebraic set of the character variety.

In my paper [4], I describe the boundary points. The character variety for $SL(n, \mathbb{R})$ parametrizes actions of the group on real projective spaces. The idea for this paper is that the boundary points should parametrize the tropicalization of such actions. I gave a definition of tropical projective spaces, and proved that the boundary points parametrize actions of the group on the tropical projective spaces I defined. As sets, these tropical projective spaces are objects usually called buildings, and I showed that they carry structures coming from tropical geometry, the analog of projective structures.

The philosophy is that given a parameter space that parametrizes certain objects, its tropicalization is a parameter space that parametrizes the tropicalization of the said objects. Another situation where this philosophy is even more clear is in my paper [20], where we consider the Hilbert schemes. We prove that the tropicalization of the Hilbert scheme with Hilbert polynomial p parametrizes the tropical varieties with Hilbert polynomial p.

Teichmüller spaces for surfaces of infinite type. In some papers in collaboration with Papadopoulos, Liu and Su [18, 14, 13, 15, 16], we pioneered the study of Teichmüller Spaces for surfaces of *infinite topological type*, i.e. surfaces of infinite genus and/or with infinitely many punctures. Surfaces of infinite type, arise naturally in several areas of geometry, for example, they can appear as Fatou sets of complex dynamical systems, they can be leaves of foliations, or they appear in the theory of Schottky uniformization. In all the cited examples, they carry complex structures, and an understanding of these complex structures is essential to understand the mathematical problem at hand. When we started our work on Riemann surfaces of infinite type we were motivated by these examples, and by the curiosity to understand how far Teichmüller theory can be pushed.

In the case of surfaces of infinite type, there are many different definitions of Teichmüller spaces and their distances. For surfaces of finite type, all the usual definitions of the space agree, and all the distances give the same topology, hence there is essentially only one Teichmüller space. This is very important: when studying surfaces of finite type, it is possible to choose the definition of the space and distance that best suits the problem, and then it is easy to switch from one definition to another. For surfaces of infinite topological type, this is much harder, and in our papers we studied the relationships between the possible different definitions of the space and of its distance, we found conditions when they agree, and we understood how different they are when they disagree.

Our most important paper on this topic is [18], introducing several techniques to study these Teichmüller spaces using geometric and topological tools. It became an influential paper, and received 45 citations, with new citations coming every year.

The theory of surfaces of infinite type is attracting interest in the field of geometric group theory, with the theory of *big mapping class groups* (see e.g. [24, 39, 25]), and in the field of dynamical systems, see e.g. [47, 56, 22]. Our results on their Teichmüller space contributed to this development, for example they are used by [39, 25, 22, 21] and others.

Surfaces with boundary. Thurston [59] introduced *Thurston's asymmetric metric*, also known as the *Lipschitz metric*, on Teichmüller space. He constructed a family of geodesics called *stretch lines*, and showed how this beautiful theory gives plenty of geometric information about the relationships between hyperbolic structures and geodesic laminations. Thurston's theory only addresses surfaces without boundary.

The analog theory for *surfaces with boundary* is more complicated, and it can be also more interesting, see for example the wonderful applications to Margulis spacetimes given by Guéritaud-Kassel [41]. In a paper with Papadopoulos, Liu and Su [17], we studied this case, and we showed that the *horofunction compactification* of $\mathcal{T}(S)$ for the asymmetric metric is the Thurston boundary (for closed surfaces, see Walsh [61]). We had many open questions about the geometry of the asymmetric metric for surfaces with boundary, but we didn't make further progress because it is quite hard to construct enough geodesics.

In the paper with Disarlo [9], we solve this problem by constructing a large family of geodesics that we call *generalized stretch lines*. Using this tool, we can prove several geometric properties of the distance, for example that every two points can be connected by a geodesic segment, and that the distance is induced by a Finsler metric. This gives us a good understanding of the geometry of this distance and of its relationship with the geometry of the hyperbolic structures.

With these new insights on the geometry of Teichmüller spaces for surfaces with boundary, it will be possible to continue the study of this metric, and to answer other open questions. For example, I am now writing a paper with Papadopoulos, Liu and Su where we prove that the metric is rigid, i.e. that the isometry group is the extended mapping class group. For closed surfaces, this was proved by Walsh [61].

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