# Fokker-Planck rep. of stochastic neural fields: derivation, analysis and application to grid cells

Susanne Solem

Norwegian University of Life Sciences

Pierre Roux

École Centrale de Lyon

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## Lecture 1: Motivation, derivation and application

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Lecture 2: existence of solutions

Lecture 3-4: local stability and bifurcations


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# Fokker-Planck rep. of stochastic neural fields: derivation, analysis and application to grid cells

Tutorial: Exercises based on the lectures

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# Outline

1. Grid cells

2. Typical model for network of neurons/grid cells

3. Adding noise and mean-field limit

4. Fokker-Planck representation: results and challenges

# 7. Grid cells

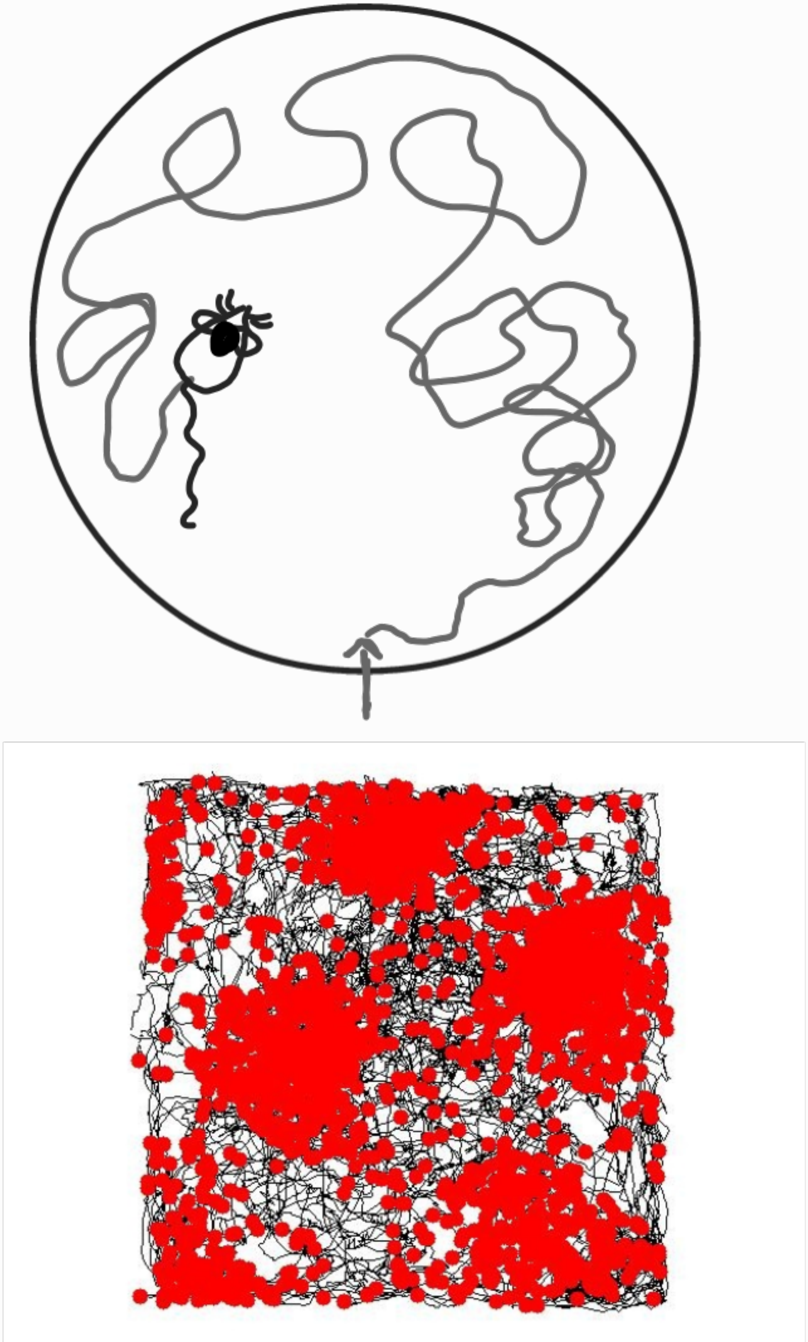
- Grid cells are neurons

- A neuron is said to "fire" when it emits an electrical signal.

- Grid cells fire in hexagonal patterns as a mammal traverses an open space.

- Mammals' internal "GPS".

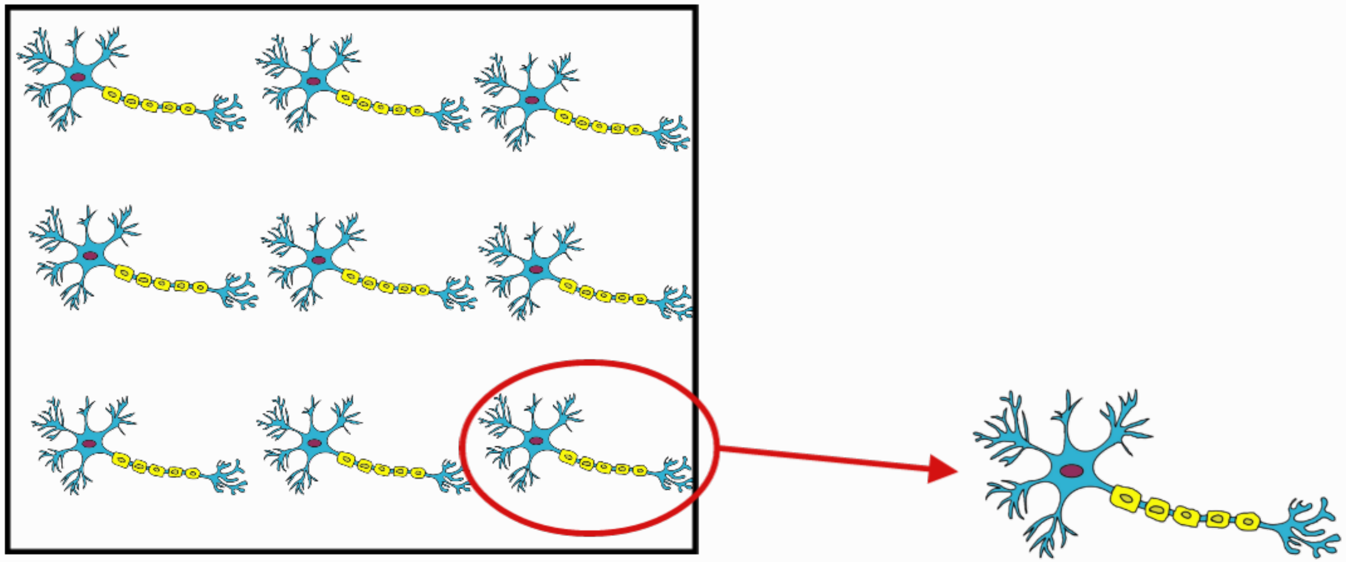
- Found in rats, mice, bats monkeys, humans...



The root question:

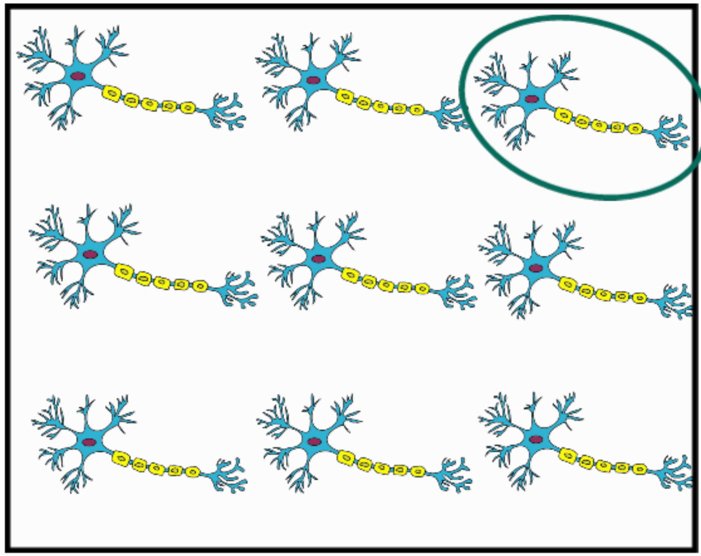
How is the grid cell system affected by noise?

# 2. A typical model for networks of neurons



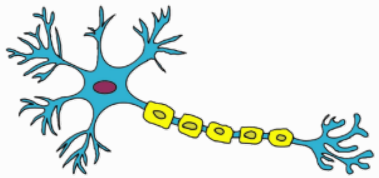
Network of neurons

Neuron at $x_i$
Described by its activity level $s_i$



Network of neurons

$W_{ij}$



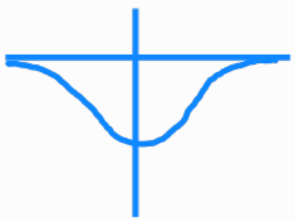
Neuron at $x_i$ Described by its activity level $s_i$

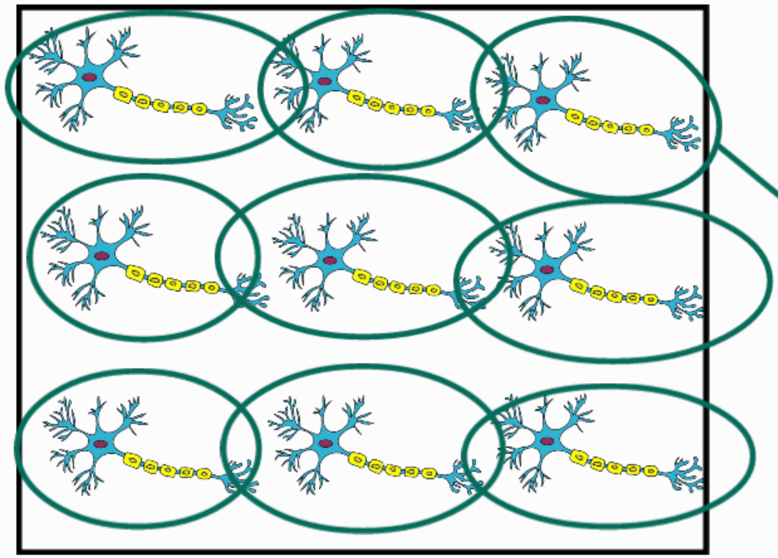
$$W_{ij} = W(x_i - x_j)$$

$$W \leq 0$$

$$x_i, x_j \in \Pi^2$$

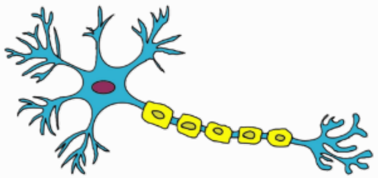
"in the brain"!





Network of neurons

$W_{ij}$



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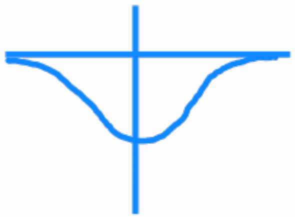
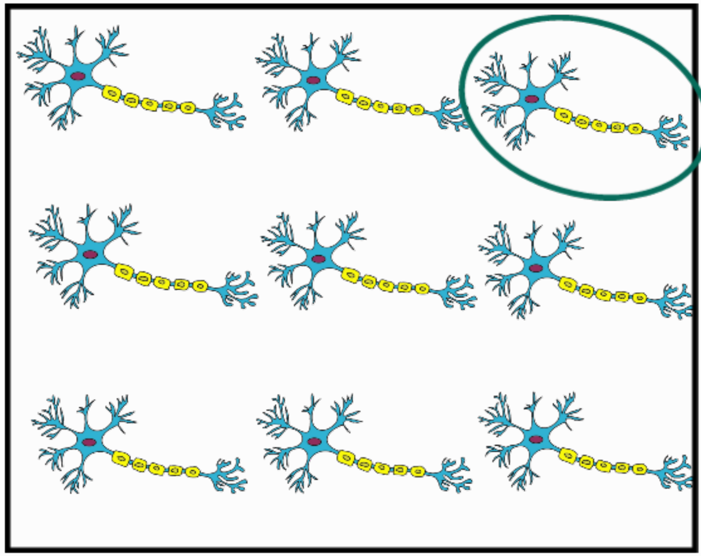
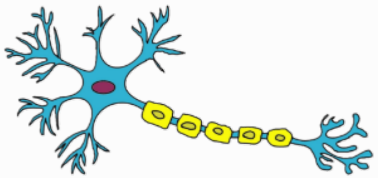




Network of neurons

$W_{ij}$



$$\frac{ds_i}{dt} = -s_i + \Phi\left(\sum_j W_{ij} s_j + B\right)$$

$\Phi$ is an increasing function

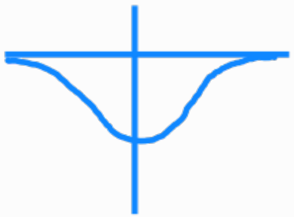
B - external input

$W_{ij} = W(x_i - x_j)$

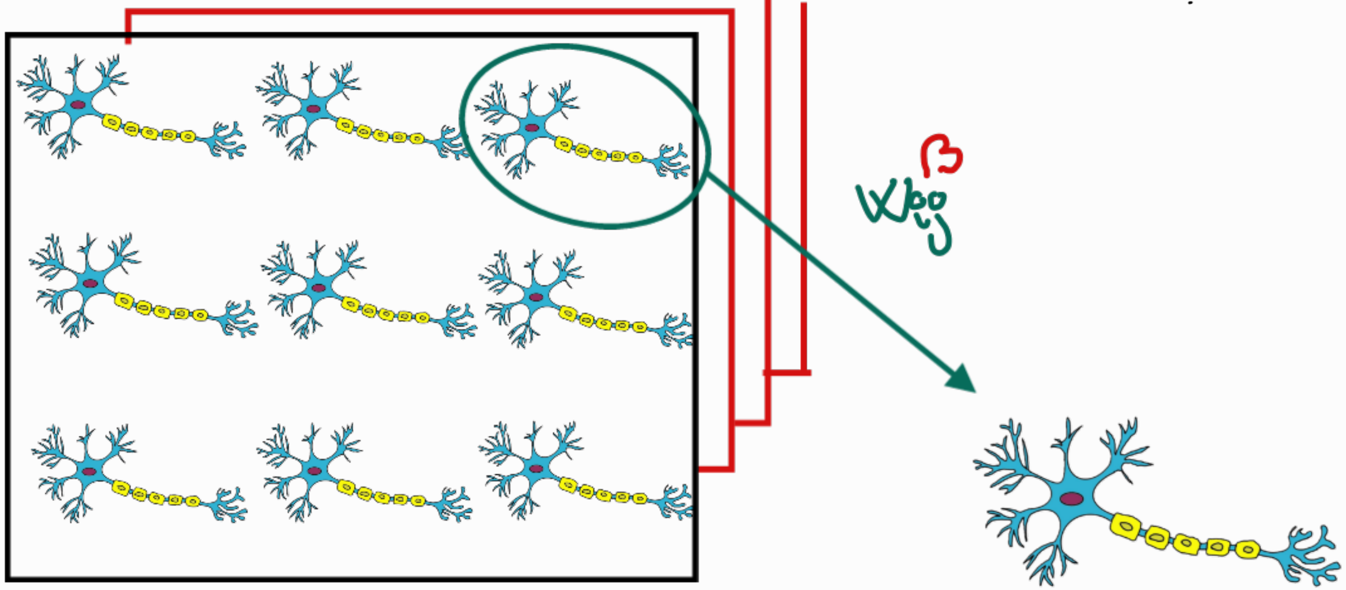
$W \leq 0$

$x_i, x_j \in T^2$

"in the brain"!



# 2. A typical model for grid cells




Network of neurons with orientation preference $\beta$, $\beta = n, e, s, w$

$$\frac{ds_i^{\beta}}{dt} = -s_i^{\beta} + \Phi\left( \sum_{\beta'} \sum_j W_{ij}^{\beta\beta'} s_j^{\beta'} + B \right)$$

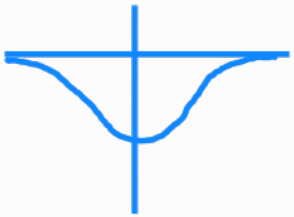
$B$ - external input

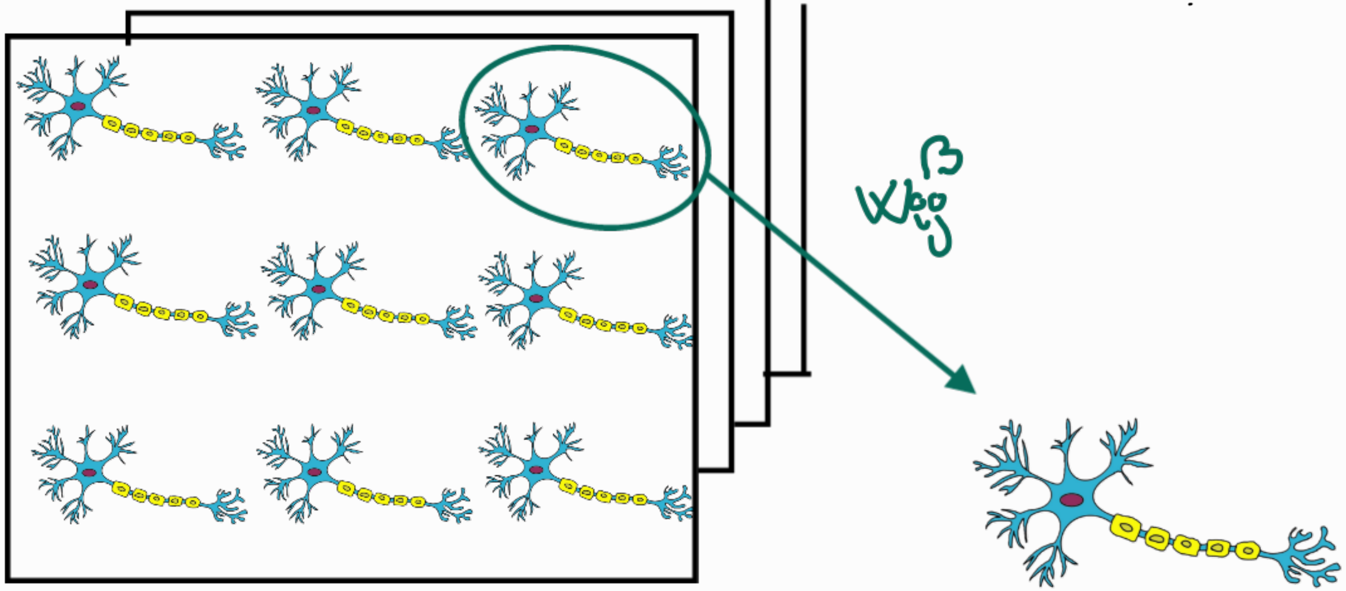
$W_{ij} = W(x_i - x_j)$

$W \leq 0$

$x_i, x_j \in \Pi^2$

"in the brain"!



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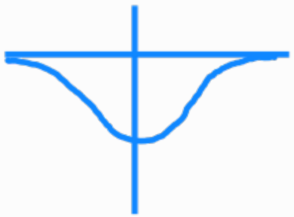
$$\frac{ds_i^{\beta}}{dt} = -s_i^{\beta} + \Phi\left(\sum_{\beta'}\sum_{j} W_{ij}^{\beta\beta'} s_j^{\beta'} + B\right)$$

$B$ - external input

$W_{ij}^{\beta} = W(x_i - x_j - r^{\beta})$

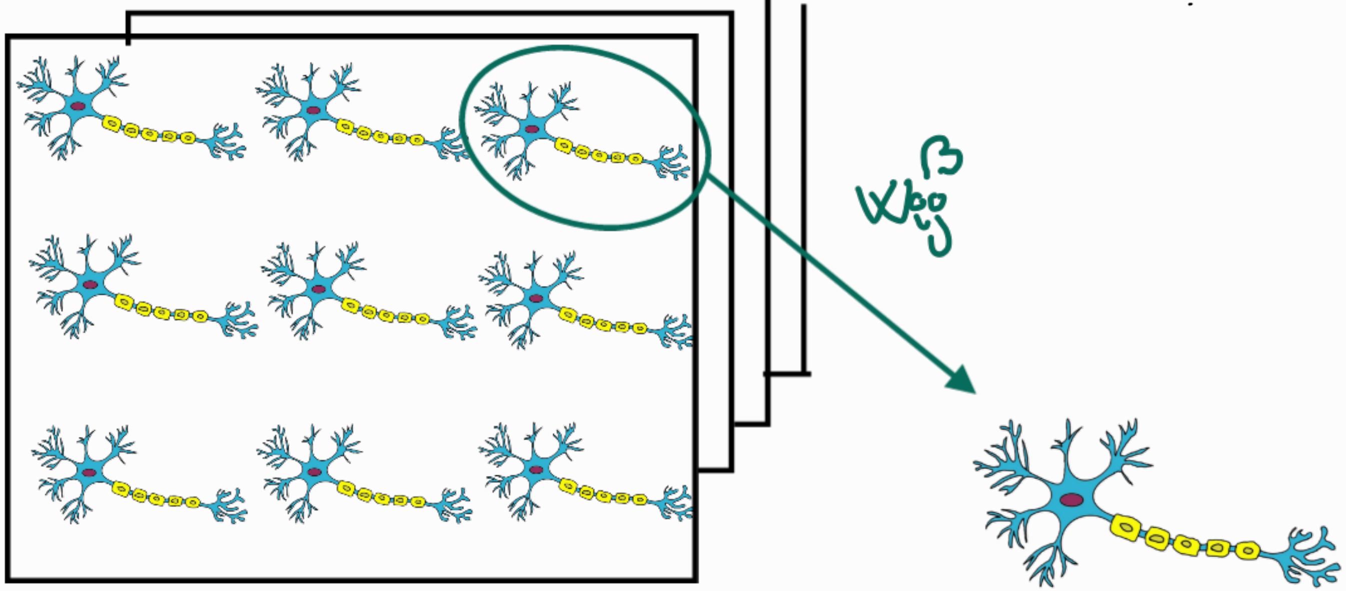
$W \leq 0$



$x_i, x_j \in \Pi^2$

"in the brain"!

$r^{\beta}$ - orientation preference shift




Network of neurons with orientation preference $\beta$, $\beta = n, e, s, w$

$$\frac{ds_i^\beta}{dt} = -s_i^\beta + \Phi\left( \sum_{\beta'} \sum_j W_{ij}^{\beta\beta'} s_j^{\beta'} + B^\beta \right)$$

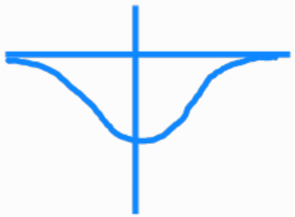
$B^\beta$ - external input dep. on velocity and orientation. pref.

$W_{ij}^{\beta\beta} = W(x_i - x_j - r^\beta)$

$W \leq 0$

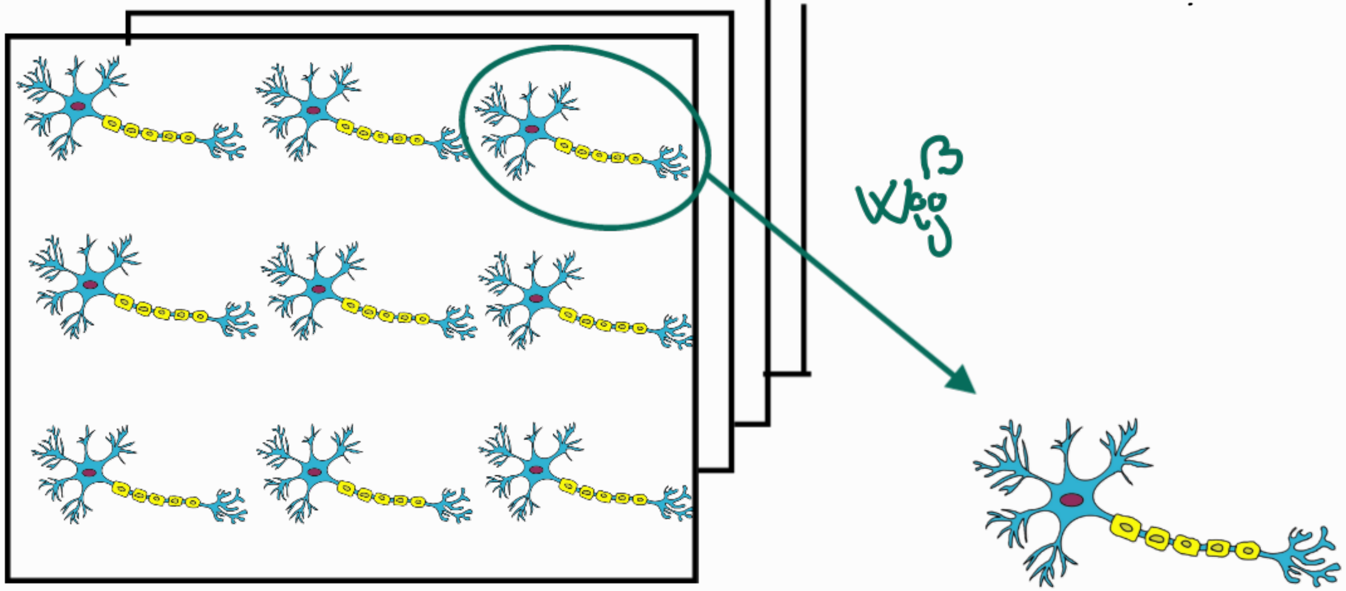
$x_i, x_j \in \mathbb{T}^2$

"in the brain"!



$r^\beta$ - orientation preference shift

Burak & Fiete '09, Couey et al. '13, ...




Network of neurons with orientation preference $\beta$, $\beta = n, e, s, w$

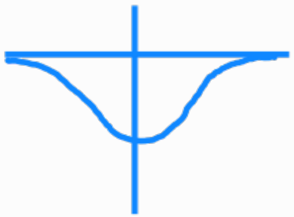
$$\frac{ds_i^{\beta}}{dt} = -s_i^{\beta} + \Phi\left( \sum_{\beta'} \sum_j W_{ij}^{\beta\beta'} s_j^{\beta'} + B^{\beta} \right)$$

$B^{\beta}$ - external input dep. on velocity and orientation pref.

but not position!

$$W_{ij}^{\beta\beta'} = W(x_i - x_j - r^{\beta})$$

$W \leq 0$



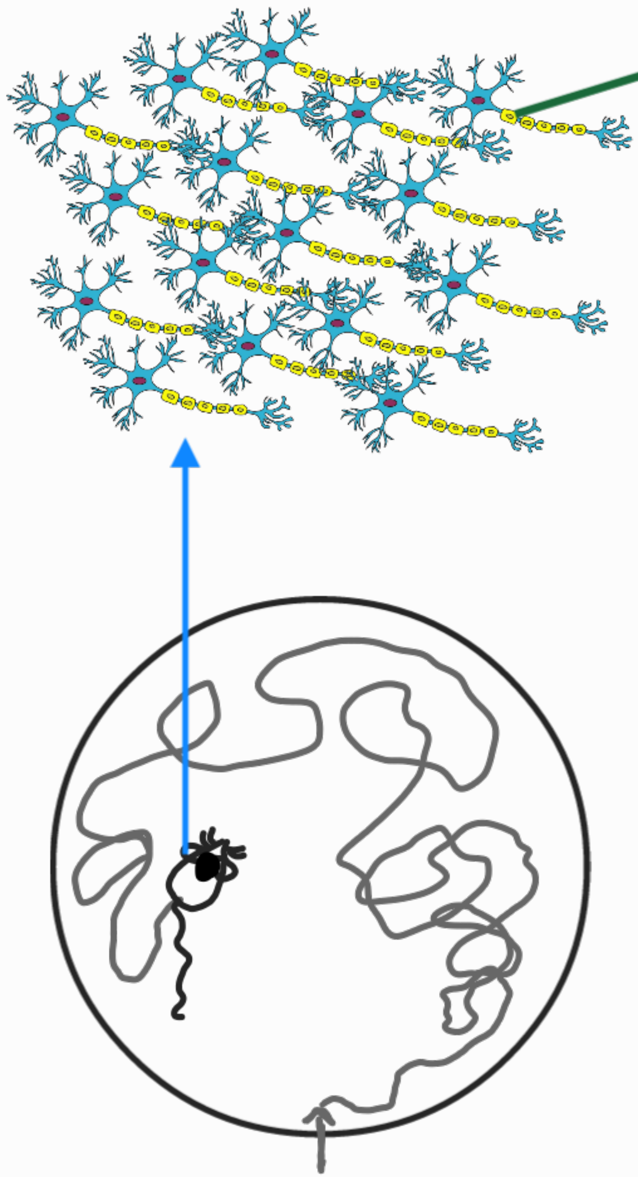
$x_i, x_j \in \mathbb{T}^2$

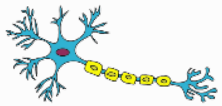
"in the brain"!

$r^{\beta}$ - orientation preference shift

Burak & Fiete '09, Couey et al. '13, ...

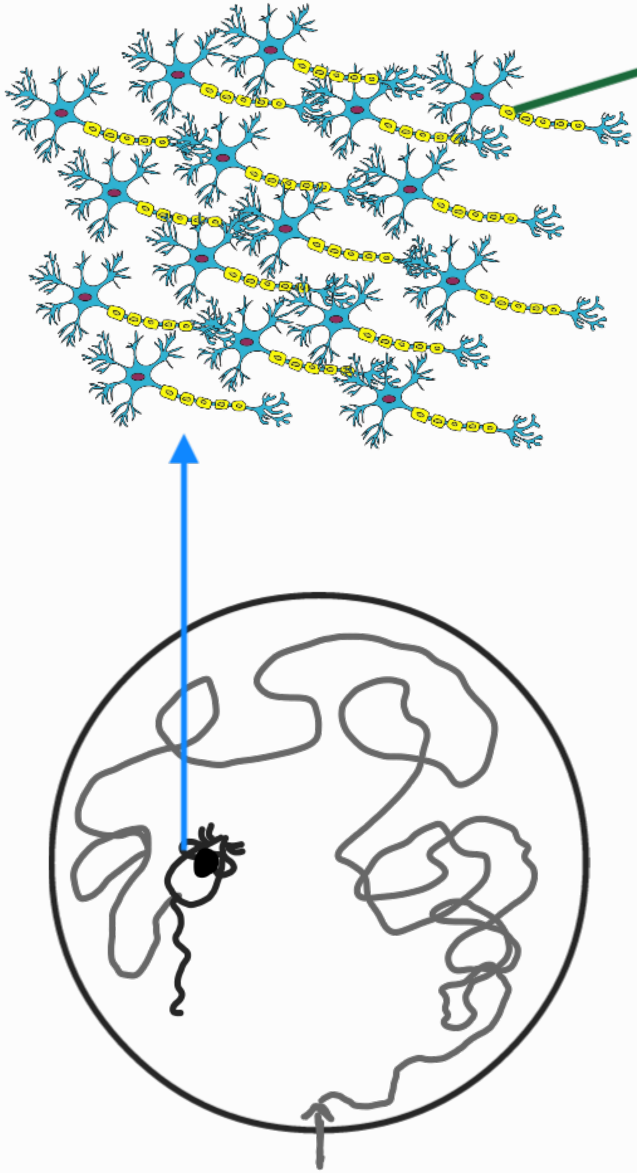
Connecting
to the
mammals
movements

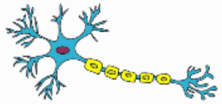




activity described by:

$$\frac{ds_i^\beta}{dt} + s_i^\beta = \Phi\left(\sum_{\beta'}\sum_j W_{ij}^{\beta'} s_j^{\beta'} + B_i^\beta(t)\right)$$

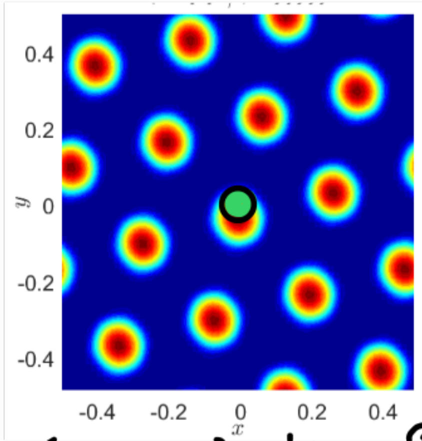




activity described by:

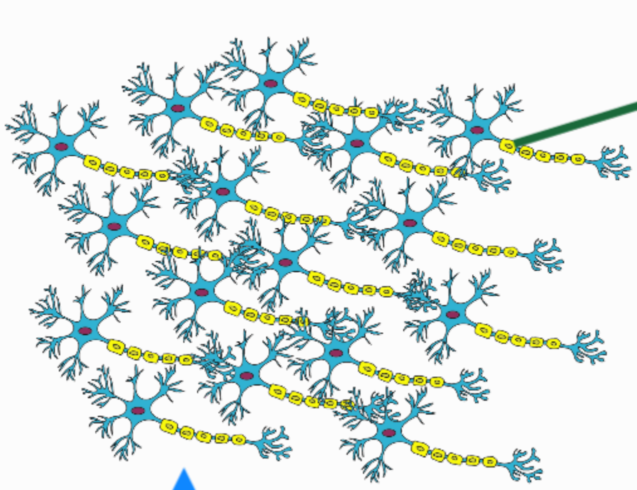
$$\frac{ds_i^\beta}{dt} + s_i^\beta = \Phi\left(\sum_{\beta'}\sum_j W_{ij}^{\beta'} s_j^{\beta'} + B_i^\beta(t)\right)$$

and visualized as:




snapshot $s_i^3$

Activity in the brain



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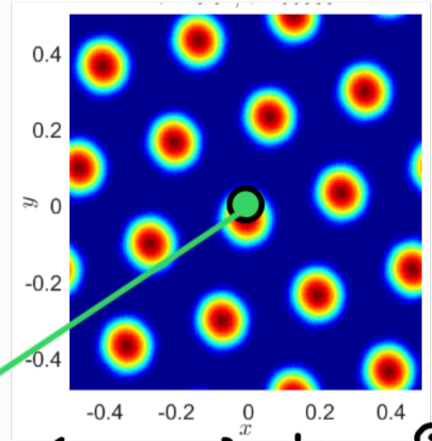


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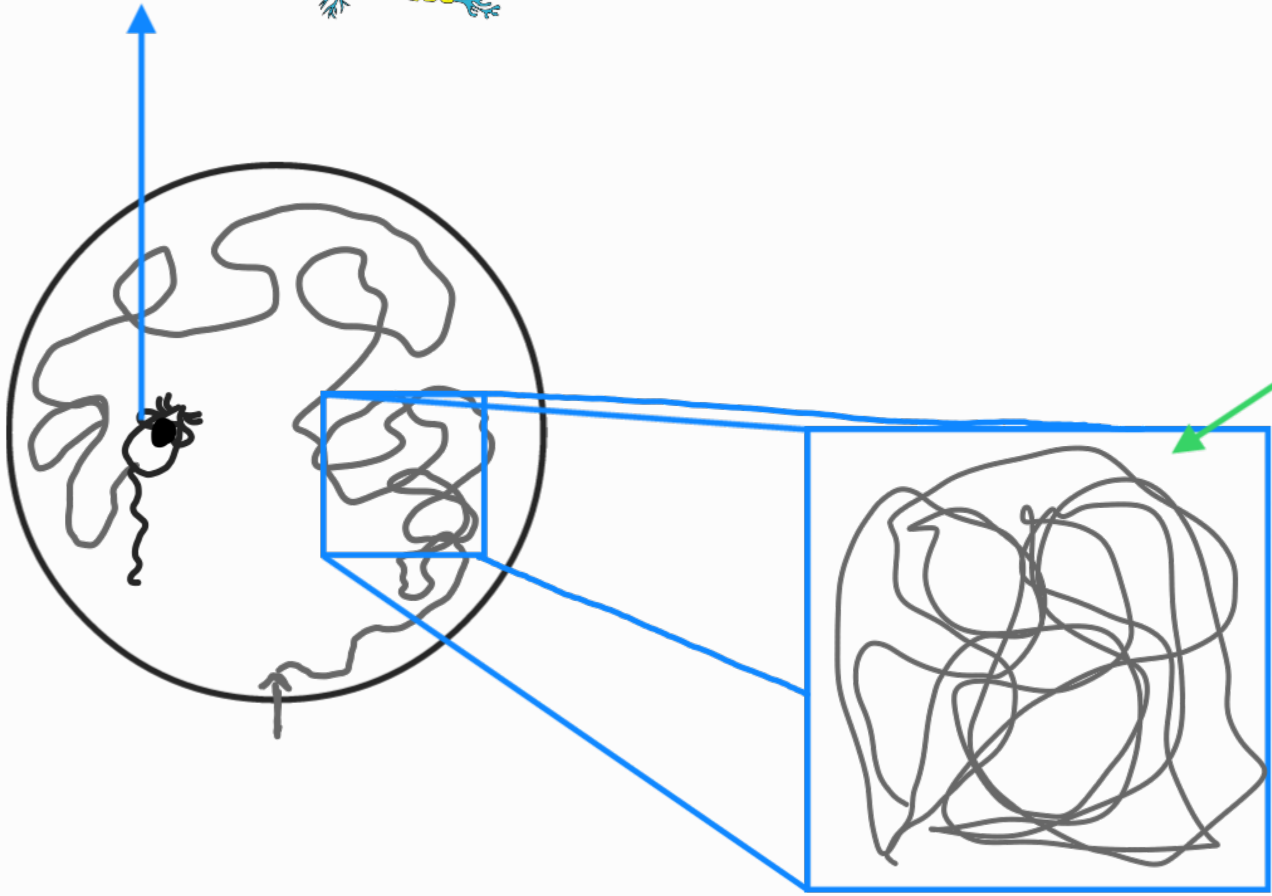
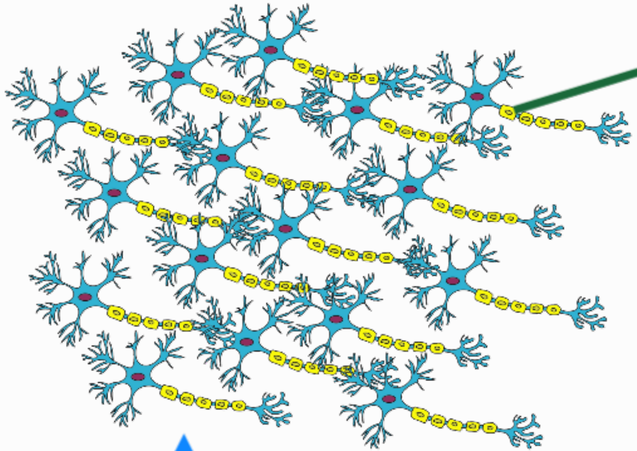
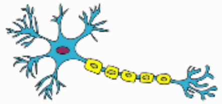


$\Phi_i > \varepsilon > 0$

$\Rightarrow$ • on path





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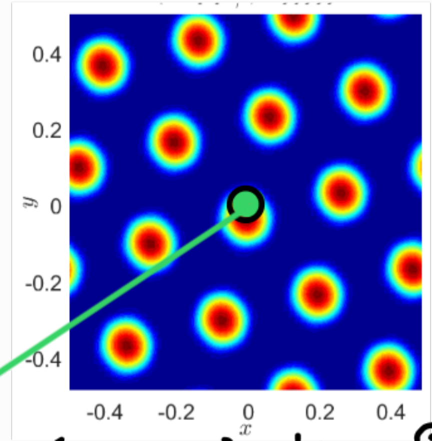
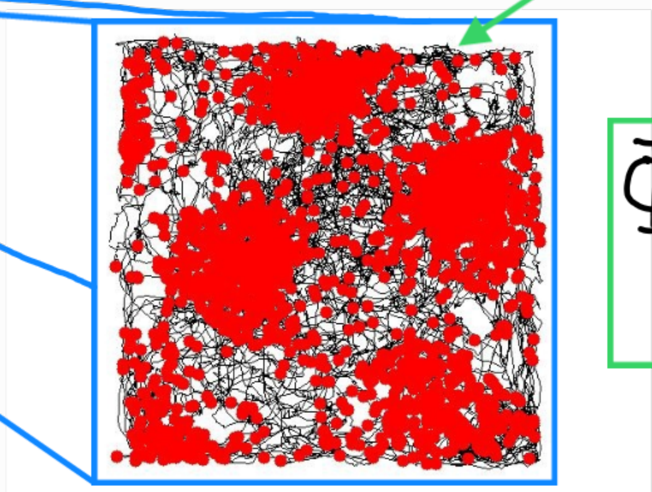



snapshot $s_i^\beta$

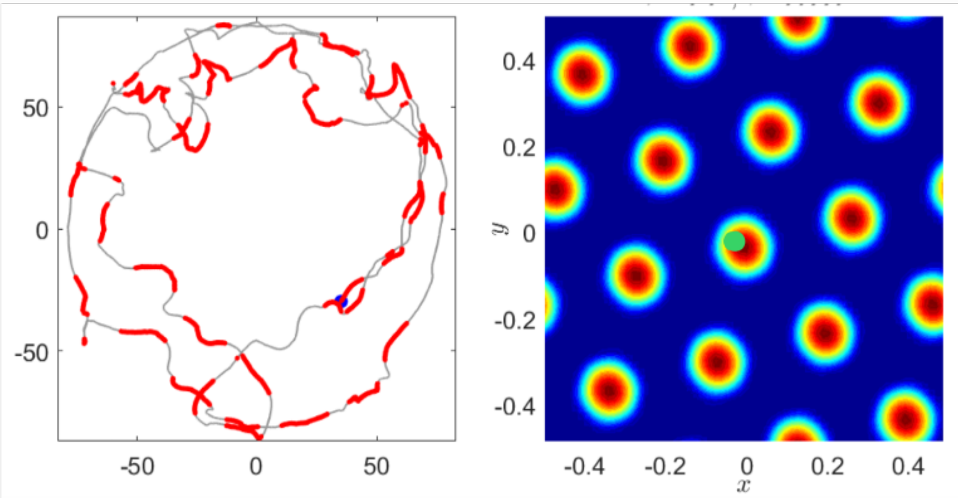
Activity in the brain



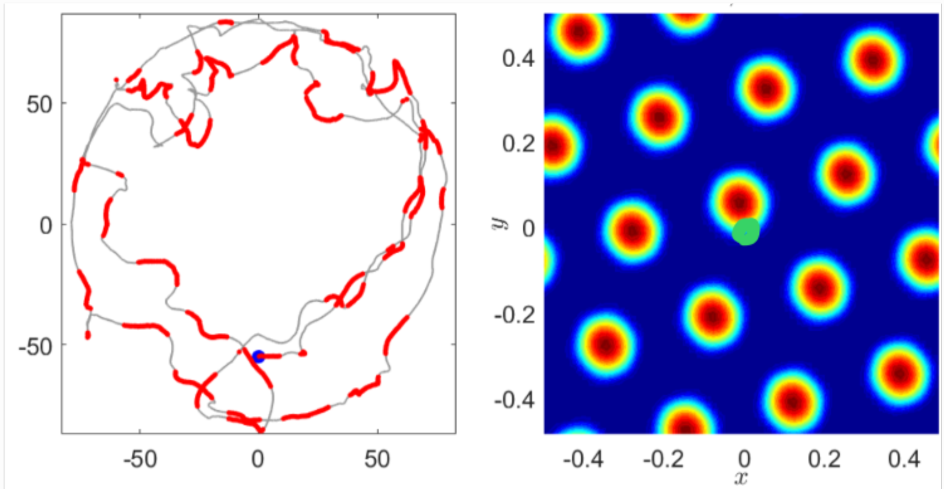
$\Phi_i > \varepsilon > 0$

$\Rightarrow$ ● on path

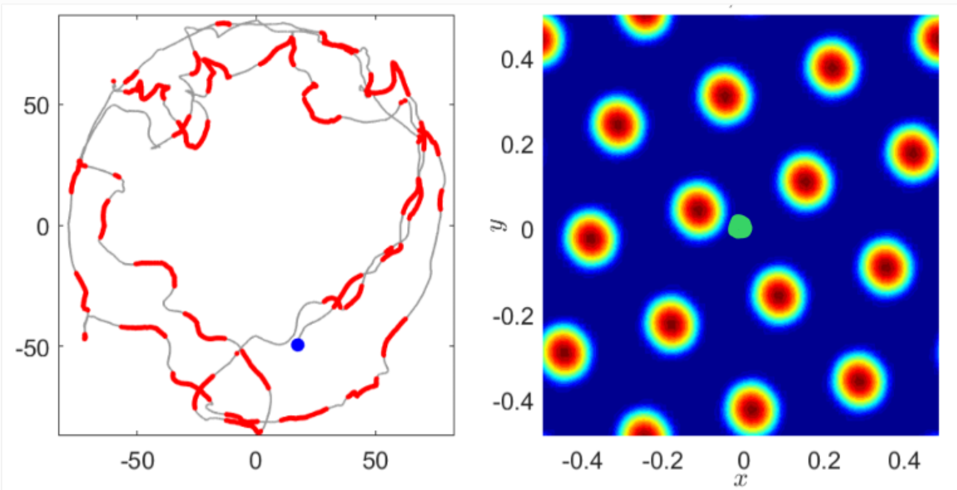
t = 860s




t = 880s

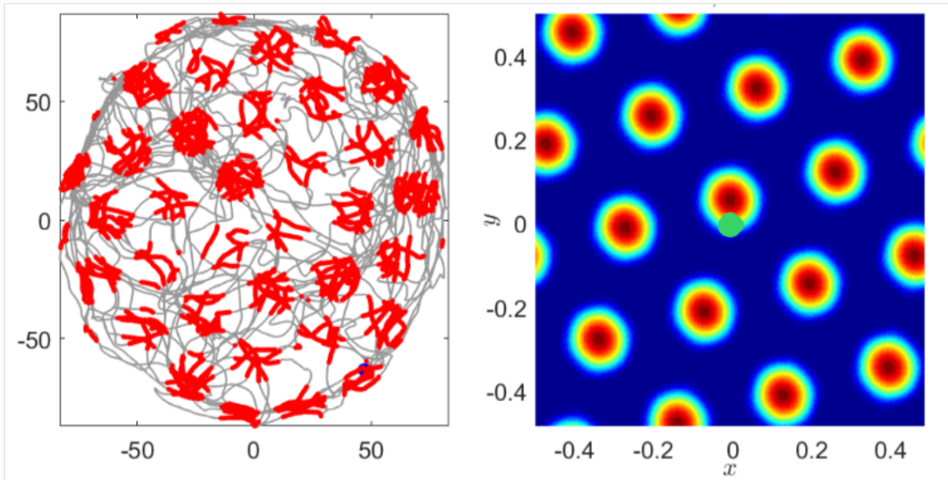



t = 870s




⋮

t = 6000s



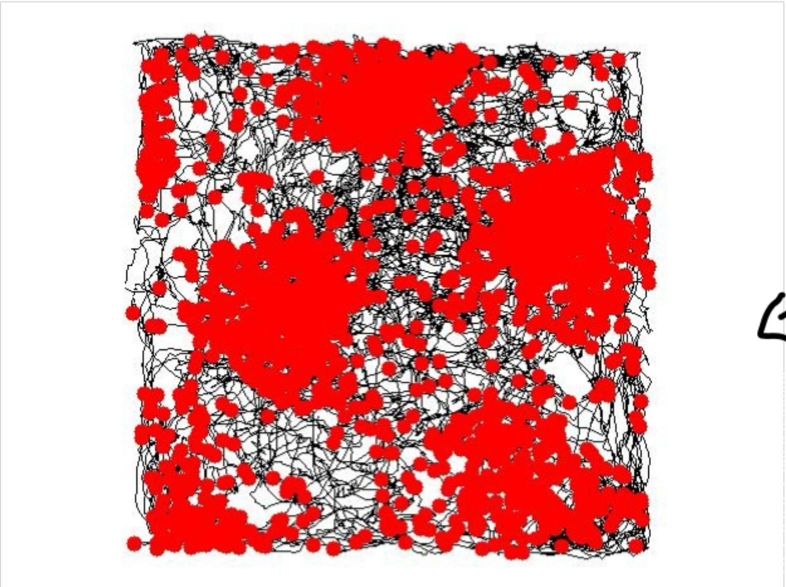
# 3. Adding noise

The root question:

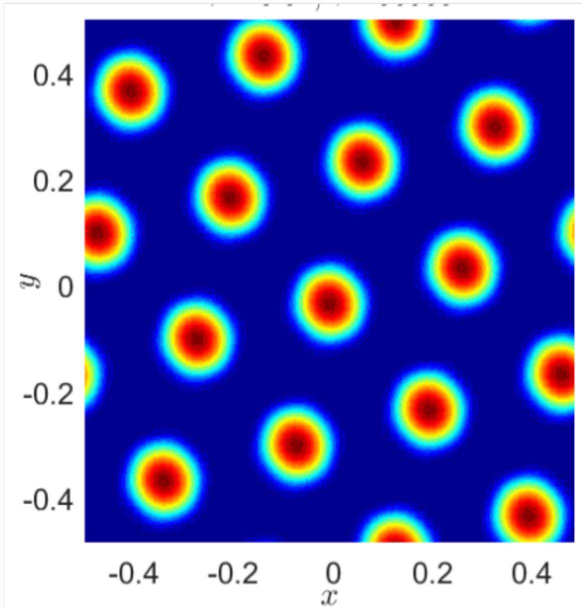
How is the grid cell system affected by noise?

The new question:

How is the grid cell model affected by noise?



firing fields of one neuron in physical space




activity pattern of neurons

Approach: study noise-induced behaviour in a deterministic framework

or the hope

Approach: study noise-induced behaviour in a deterministic framework

or the hope

Meaning: derive a PDE description which is amenable to analytic treatment

$$\frac{ds_i}{dt} + s_i = \Phi\left( \sum_j W_{ij}\, s_j + B\,(t) \right)$$

remove spatial preference
↳ for simplicity

$$\frac{ds_{in}}{dt} + s_{in} = \Phi\left(\sum_{m}\sum_{j} W_{ij}\,\frac{s_{jm}}{NJ} + B\,(t)\right)$$

reinterpretation:
at each $x_j \in \mathbb{T}^2$ you have N neurons

$$\frac{ds_{in}}{dt} + s_{in} = \Phi\left(\sum_m \sum_j \frac{W_{ij}\, s_{jm}}{NJ} + B(t)\right)$$

reinterpretation: at each $x_j \in \mathbb{T}^2$ you have $N$ neurons

Crucial if you want to end up with a PDE description later

$$ds_{in} + s_{in}\,dt = \Phi\left(\sum_{m}\sum_{j} W_{ij}\,\frac{s_{jm}}{NJ} + B\,(t)\right)dt \quad + \sqrt{2\sigma}\,dW_{ni}$$

added noise

$$ds_{in} + s_{in}\,dt = \Phi\left(\frac{\sum_m \sum_j W_{ij}\, s_{jm}}{NJ} + B\,(t)\right)dt \quad + \sqrt{2\sigma}\, dW_{ni}$$

added noise

$\sigma$ could be constant, or for example depend on the network activity

$$ds_{in} + s_{in}\,dt = \Phi\left(\sum_{m}\sum_{j} W_{ij}\,\frac{s_{jm}}{NJ} + B\,(t)\right)dt \quad + \sqrt{2\sigma}\,dW_{ni} - dl_{in}$$

a reflection term

Prevents $s_{in}$ from becoming negative

$$ds_{in} + s_{in}\,dt = \Phi\left(\sum_m \sum_j W_{ij} \frac{s_{jm}}{NJ} + B(t)\right) dt + \sqrt{2\sigma}\, dW_{ni} - dl_{in}$$

with empirical measure

$$\rho_{NJ}(dx, ds) = \frac{1}{NJ} \sum_{j=1}^{J} \sum_{n=1}^{N} \delta_{(x_j, s_{jn})}$$

$$ds_{in} + s_{in}\,dt = \Phi\left(\frac{\sum_m \sum_j W_{ij}\, s_{jm}}{NJ} + B(t)\right)dt + \sqrt{2\sigma}\,dW_{ni} - dl_{in}$$

Send $J, N \to \infty$ *

Then $\sup_t \mathbb{E}\left(W_1\left(\rho_{NJ}(t), \rho(t)\right)\right) \to 0$

($W_1(\mu, \nu)$ denotes the Wasserstein distance between $\mu$ and $\nu$.)

* Carillo, Clini, S., 2023

$$ds_{in} + s_{in}\,dt = \Phi\left(\sum_m \sum_j \frac{W_{ij}\, s_{jm}}{NJ} + B(t)\right)dt + \sqrt{2\sigma}\, dW_{ni} - dl_{in}$$

Send $J, N \to \infty$*

Then $\sup_t \mathbb{E}\left(W_1\left(\rho_{NJ}(t), \rho(t)\right)\right) \to 0$

the law of

$$ds(x) + s(x)\,dt = \Phi\left(\int_{\mathbb{T}^d} W(x-y)\,\bar{\rho}(y)\,dy\right)dt + \sqrt{2\sigma}\, dW(x) - dl(x)$$

$$\bar{\rho}(x) = \int_0^\infty s\,\rho(s,x)\,ds$$

* Carillo, Clini, S., 2023

$$ds_{in} + s_{in}\,dt = \Phi\left(\sum_m \sum_j \frac{W_{ij}\, s_{jm}}{NJ} + B(t)\right)dt + \sqrt{2\sigma}\, dW_{ni} - dl_{in}$$

Send $J, N \longrightarrow \infty$ *

**The mean-field limit:**

- $\sigma$ can depend on $s$
- Sznitman coupling (aggregation, related neuron models, ...)

* Carillo, Clini, S., 2023

# 4. Fokker-Planck representation: results and challenges

# PDE description

The probability density at time $t$, $\rho(t,x,s)$, of finding a neuron at $x \in \mathbb{T}^d$ with activity level $s \geqslant 0$ solves

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial s}\left(\left(\Phi_{\bar{\rho}} - s\right)\rho\right) + \sigma \frac{\partial^2 \rho}{\partial s^2},$$

where

$$\Phi_{\bar{\rho}}(t,x) = \Phi\left(\int_{\mathbb{T}^d} W(x-y)\bar{\rho}(t,y)\,\mathrm{d}y + B(t)\right),$$

with $\bar{\rho}(t,x) = \int_0^\infty s\rho(t,x,s)\,\mathrm{d}s$, and $\left(\Phi_{\bar{\rho}}\rho - \sigma\frac{\partial\rho}{\partial s}\right)(t,x,0) = 0.$

# PDE description

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A deterministic setting with noise as a parameter

# PDE description

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Nonlinear, non-local, and has a tricky BC

Tricky? Then why do we keep it?

Disregarding the BC, the mean activity is described by

$$\frac{d\bar{\rho}}{dt} = \Phi\left(W * \bar{\rho} + B\right) - \bar{\rho}$$

Disregarding the BC, the mean activity is described by

$$\frac{d\bar{\rho}}{dt} = \Phi(w * \bar{\rho} + B) - \bar{\rho}$$

- "Back to start"
- well-studied
- No dependence on $\sigma$ !

While, with the BC:

$$\frac{d\bar{\rho}}{dt} = \Phi(W * \bar{\rho} + B) - \bar{\rho} + \sigma \rho(x, s=0, t)$$

The rest of the day:
tools, results and challenges for

The probability density at time $t$, $\rho(t,x,s)$, of finding a neuron at $x \in \mathbb{T}^d$ with activity level $s \geqslant 0$ solves

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial s}\left(\left(\Phi_{\bar\rho} - s\right)\rho\right) + \sigma \frac{\partial^2 \rho}{\partial s^2},$$

where

$$\Phi_{\bar\rho}(t,x) = \Phi\left(\int_{\mathbb{T}^d} W(x-y)\bar\rho(t,y)\,\mathrm{d}y + B(t)\right),$$

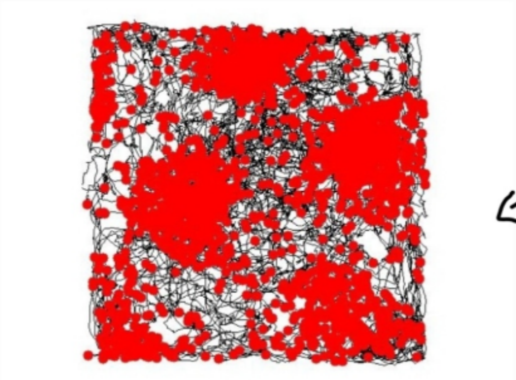
with $\bar\rho(t,x) = \int_0^\infty s\rho(t,x,s)\,\mathrm{d}s$, and $\left(\Phi_{\bar\rho}\rho - \sigma\frac{\partial\rho}{\partial s}\right)(t,x,0) = 0.$

...and similar equations

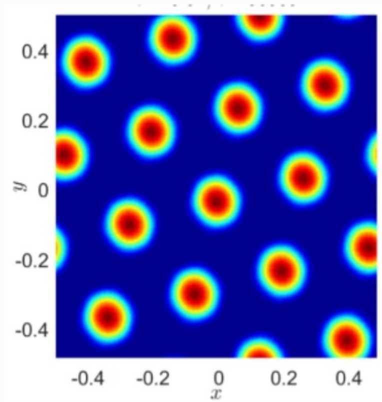
# After lunch :

# Results related to the "main" question

How is the grid cell model affected by noise?



firing fields of one neuron in physical space




activity pattern of neurons

on the torus, not physical space