Fokker-Planck representation of stochastic neural fields: derivation, analysis and application to grid cells

Exercises, solutions and references

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If not stated otherwise, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and ρ is a smooth solution of

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial s}\left[\left(\Phi\left(\int_{\mathbb{T}^d} W(x-y)\int_0^{+\infty} s\rho(y,s,t)\mathrm{d}s\mathrm{d}y + B\right) - s\right)\rho(x,s,t)\right] + \sigma\frac{\partial^2\rho}{\partial s^2},$$

with boundary condition

$$\left(\Phi\left(\int_{\mathbb{T}^d} W(x-y)\int_0^{+\infty} s\rho(y,s,t)dsdy + B(t)\right)\rho(x,s,t) - \sigma\frac{\partial\rho}{\partial s}(x,s,t)\right)\Big|_{s=0} = 0.$$

and

$$\forall t\in\mathbb{R}_+,\ \forall x\in\mathbb{T}^d,\qquad \int_0^{+\infty}\rho(x,s,t)ds = 1.$$

The function Φ is assumed to be smooth and globally Lipschitz, W is smooth on $\mathbb{T}^d$, and B and σ are positive constants.

Exercise 1. *Let*

$$\mathrm{d}s = -s\mathrm{d}t + \sqrt{2}\mathrm{d}W_t,$$

where W is a standard Brownian. Use Ito's lemma to check for yourself that the law of the SDE solves the Fokker-Planck equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial s}[-s\rho] - \frac{\partial^2\rho}{\partial s^2} = 0$$

in the weak sense.

Solution 1. *You arrive at the solution by first applying Itô's lemma to a test function and then taking the expectation.*

Exercise 2. *Let ρ be a smooth solution of the Fokker-Planck equation*

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial s}[-s\rho] - \frac{\partial^2\rho}{\partial s^2} = 0,\qquad s\in(0,+\infty),\qquad \frac{\partial\rho}{\partial s}(0,t)=0,\qquad \rho(s,0)=\rho^0(s). \tag{0.1}$$

- *Consider*

$$s = \frac{z}{\sqrt{2\tau+1}},\quad t=\frac{1}{2}\log(2\tau+1),\qquad u(z,\tau) = \frac{1}{\sqrt{2\tau+1}}\rho\left(\frac{z}{\sqrt{2\tau+1}}, \frac{1}{2}\log(2\tau+1)\right).$$

Denoting $\alpha(\tau) = \frac{1}{\sqrt{2\tau+1}} = e^{-t}$ in order to ease computations, check that u satisfies the heat equation

$$\frac{\partial u}{\partial\tau} - \frac{\partial^2 u}{\partial z^2} = 0,\qquad z\in(0,+\infty),\qquad \frac{\partial u}{\partial z}(0,\tau) = 0.$$

- *Using the heat kernel*

$$G(z,\tau,\xi,\eta) = \frac{1}{\sqrt{4\pi(\tau-\eta)}}e^{-\frac{(z-\xi)^2}{4(\tau-\eta)}}, \tag{0.2}$$

deduce an explicit form for the solution to (0.1).

Solution 2. • *Denoting* $\alpha(\tau) = \frac{1}{\sqrt{2\tau+1}} = e^{-t}$, *we have* $s = \alpha(\tau)z$, $z = e^t s$, $t = -\log(\alpha(\tau))$, $\tau = \frac{1}{2}(e^{2t}-1)$, $u(z,\tau) = \alpha(\tau)\rho(y\alpha(\tau), -\log(\alpha(\tau)))$. *We compute*

$$\begin{aligned}\frac{\partial u}{\partial \tau}(z,\tau) =& \alpha'(\tau)\rho\big(z\alpha(\tau), -\log(\alpha(\tau))\big) + z\alpha'(\tau)\alpha(\tau)\frac{\partial \rho}{\partial s}\big(z\alpha(\tau), -\log(\alpha(\tau))\big)\\ & - \alpha'(\tau)\frac{\partial \rho}{\partial t}\big(z\alpha(\tau), -\log(\alpha(\tau)),\\ \frac{\partial u}{\partial z}(z,\tau) =& \alpha(\tau)^2\frac{\partial \rho}{\partial s}\big(z\alpha(\tau), -\log(\alpha(\tau))\big),\\ \frac{\partial^2 u}{\partial z^2}(z,\tau) =& \alpha(\tau)^3\frac{\partial^2 \rho}{\partial s^2}\big(z\alpha(\tau), -\log(\alpha(\tau))\big) = -\alpha'(\tau)\frac{\partial^2 \rho}{\partial s^2}\big(z\alpha(\tau), -\log(\alpha(\tau))\big),\end{aligned}$$

where $-\alpha'(\tau) = (2\tau+1)^{-\frac{3}{2}} = \alpha(\tau)^3$ *is used. Then, using the equation for* ρ,

$$\begin{aligned}\frac{\partial \rho}{\partial t}\big(z\alpha(\tau), -\log(\alpha(\tau))\big) =& z\alpha(\tau)\frac{\partial \rho}{\partial s}\big(z\alpha(\tau), -\log(\alpha(\tau))\big) + \rho\big(z\alpha(\tau), -\log(\alpha(\tau))\big)\\ & + \frac{\partial^2 \rho}{\partial s^2}\big(z\alpha(\tau), -\log(\alpha(\tau))\big).\end{aligned}$$

Combining the previous computations, we get the desired equation for u.

- *We can check that*

$$u(z,\tau) = \int_0^{+\infty} \left(G(z,\tau,\xi,0) + G(z,\tau,-\xi,0)\right) u^0(\xi)\mathrm{d}\xi.$$

is a solution, where $u^0(z) = \rho^0(\alpha(0)z) = \rho^0(z)$. *We then get*

$$\begin{aligned}\rho(s,t) &= e^t u\left(e^t s, \frac{1}{2}(e^{2t}-1)\right)\\ &= e^t\int_0^{+\infty}\left(G\left(e^t s, \frac{1}{2}(e^{2t}-1), \xi, 0\right) + G\left(e^t s, \frac{1}{2}(e^{2t}-1), -\xi, 0\right)\right) u^0(\xi)\mathrm{d}\xi.\end{aligned}$$

Exercise 3 (Weak formulation)**.**

- *Consider a solution* ρ *with fast-decay at* $s = +\infty$*; check that for any test function* $h \in C^2(\mathbb{R}_+)$ *with slow enough growth at* $s = +\infty$ *(including its derivatives),*

$$\begin{aligned}\forall x \in \mathbb{T}^d, \quad \frac{\partial}{\partial t}\int_0^{+\infty} h(s)\rho(x,s,t)ds =& \int_0^{+\infty}\left[(\Phi_{\bar{\rho}}(x,t) - s)\frac{dh}{ds}(s) + \sigma\frac{d^2h}{ds^2}(s)\right]\rho(x,s,t)ds\\ & + \sigma\frac{dh}{ds}(0)\rho(x,0,t),\end{aligned} \tag{0.3}$$

where

$$\Phi_{\bar{\rho}}(x,t) = \Phi(W * \bar{\rho}(x,t) + B).$$

- *Using a well-chosen* h, *prove that if* Φ *is non-positive, then*

$$\forall t \in \mathbb{R}_+,\ \forall x \in \mathbb{T}^d, \quad \bar{\rho}(x,t) \leqslant 1 + \sigma - \sigma e^{-2t} + e^{-2t}\sup_{x\in\mathbb{T}^d}\int_0^{+\infty} s^2\rho^0(x,s)ds.$$

Solution 3. • *Integration by part and applying boundary conditions.*

- *Let* $x \in \mathbb{T}^d$. *Applying (0.3) to* $h(s) = s^2$ *yields*

$$\frac{\partial}{\partial t}\int_0^{+\infty} s^2\rho(x,s,t)ds = 2\Phi_{\bar{\rho}}(x,t)\bar{\rho}(x,t) - 2\int_0^{+\infty} s^2\rho(x,s,t)ds + 2\sigma.$$

Denoting

$$M_2(x,t) = \int_0^{+\infty} s^2\rho(x,s,t)ds$$

and using non-negativity of $\bar\rho$ and non-positivity of Φ, we come to

$$\frac{\partial}{\partial t}\left(M_2(x,t)-\sigma\right) \leqslant -2\left(M_2(x,t)-\sigma\right).$$

Hence, by Grönwall's lemma applied to each point $x\in\mathbb{T}^d$ separately, we have

$$M_2(x,t)\leqslant \sigma(1-e^{-2t})+M_2(x,0)e^{-2t},$$

which translates into the uniform bound

$$\forall x\in\mathbb{T}^d, \forall t\in\mathbb{R}_+,\quad M_2(x,t)\leqslant \sigma(1-e^{-2t})+\left\|M_2(\cdot,0)\right\|_{L^\infty(\mathbb{T}^d)}e^{-2t}.$$

We eventually notice that

$$\begin{aligned}
\bar\rho(x,t) &= \int_0^1 s\rho(x,s,t)ds+\int_1^{+\infty} s\rho(x,s,t)ds\\
&\leqslant \int_0^1 \rho(x,s,t)ds+\int_1^{+\infty} s^2\rho(x,s,t)ds\\
&\leqslant \int_0^{+\infty} \rho(x,s,t)ds+\int_0^{+\infty} s^2\rho(x,s,t)ds\\
&\leqslant 1+\sigma(1-e^{-2t})+\left\|M_2(\cdot,0)\right\|_{L^\infty(\mathbb{T}^d)}e^{-2t}.
\end{aligned}$$

Exercise 4 (Universal bound for the boundary value)**.** *The existence theory provide us with the following:*

$$v(x,\tau)=2\int_0^{+\infty}G(\gamma(x,\tau),\tau,\xi,0)u^0(x,\xi)\mathrm{d}\xi+2\int_0^{\tau}\frac{\partial G}{\partial\xi}(\gamma(x,\tau),\tau,\gamma(x,\eta),\eta)v(x,\eta)\mathrm{d}\eta, \tag{0.4}$$

with G defined in (0.2) *and $\alpha(\tau)=\frac{1}{\sqrt{2\tau+1}}=e^{-t}$,*

$$\gamma(x,\tau)=-\int_0^{\tau}\Phi\big(\alpha(\eta)W*[\bar u(x,\eta)-\gamma(x,\eta)]+B\big)\alpha(\eta)d\eta=-\int_0^{\tau}\Psi(x,\eta)d\eta,$$

and

$$\rho(x,0,t)=\frac{1}{\sqrt{\sigma}}e^t v\left(x,\frac{1}{2}(e^{2t}-1)\right). \tag{0.5}$$

- *Prove that if Φ is non-negative, then*

$$\forall t\in\mathbb{R}_+,\ \forall x\in\mathbb{T}^d,\quad \rho(x,0,t)\leqslant\sqrt{\frac{2}{\pi\sigma}}\frac{1}{\sqrt{1-e^{-2t}}}; \tag{0.6}$$

  Start with $v(x,\tau)$ in modified variables and then switch back to original variables (x,t).

- *Assume Φ is an increasing function with $\Phi(x)=0$ for $x\leqslant 0$, the external input is constant and positive ($B(t)=B>0$), and the connectivity kernel is average inhibitory ($\int_{\mathbb{T}^d}W(x)\mathrm{d}x=W_0<0$). Prove that if*

$$\sigma>\frac{\pi B^2}{2|W_0|^2},$$

  then

$$\rho_\infty(s)=\sqrt{\frac{2}{\pi\sigma}}\,e^{-\frac{s^2}{2\sigma}}.$$

  defines a stationary state.

- *Deduce that the bound* (0.6) *is optimal in some sense.*

Solution 4. *Let us proceed in modified variables and prove a bound on $v(x,t)$. Note that the non-negativity of Φ implies non-negativity of Ψ. Then, building on the fact that*

$$\frac{\partial G}{\partial\xi}(\gamma(x,\tau),\tau,\gamma(x,\eta),\eta)=\frac{\gamma(x,\tau)-\gamma(x,\eta)}{4\sqrt{\pi}(\tau-\eta)^{\frac32}}e^{-\frac{(\gamma(x,\tau)-\gamma(x,\eta))^2}{4(\tau-\eta)}}=\frac{-\int_\eta^\tau\Psi(x,\zeta)d\zeta}{4\sqrt{\pi}(\tau-\eta)^{\frac32}}e^{-\frac{(\gamma(x,\tau)-\gamma(x,\eta))^2}{4(\tau-\eta)}},$$

we rewrite the expression (0.4) *for* v *in order to make signs salient:*

$$v(x,\tau) = 2\int_0^{+\infty} G(\gamma(x,\tau),\tau,\xi,0)u^0(x,\xi)d\xi$$
$$- 2\int_0^{\tau} \frac{\int_\eta^\tau \Psi(x,\zeta)\mathrm{d}\zeta}{4\sqrt{\pi}(\tau-\eta)^{\frac{3}{2}}} e^{-\frac{(\gamma(x,\tau)-\gamma(x,\eta))^2}{4(\tau-\eta)}} v(x,\eta)d\eta.$$

All the factors under the second integral being non-negative, and u^0 *having unit mass, we claim*

$$0 \leqslant v(x,\tau) \leqslant 2\int_0^{+\infty} G(\gamma(x,\tau),\tau,\xi,0)u^0(x,\xi)\mathrm{d}\xi \leqslant \frac{1}{\sqrt{\pi\tau}}.$$

In original variables (formula (0.5)*), this translates into the uniform bound*

$$0 \leqslant \rho(x,0,t) \leqslant \frac{e^t}{\sqrt{\sigma}\sqrt{2\pi\left(e^{2t}-1\right)}} = \sqrt{\frac{2}{\pi\sigma}}\frac{1}{\sqrt{1-e^{-2t}}}.$$

- *If*

$$\sigma > \frac{\pi B^2}{2|W_0|^2}, \tag{0.7}$$

  then we have

$$\sqrt{\frac{2\sigma}{\pi}} \geqslant \frac{B}{|W_0|} \qquad \text{and} \qquad \Phi\left(W_0\sqrt{\frac{2\sigma}{\pi}}+B\right) = 0.$$

  Then, defining

$$\rho_\infty(s) = \sqrt{\frac{2}{\pi\sigma}}e^{-\frac{s^2}{2\sigma}},$$

  we observe that

$$\bar{\rho}_\infty = \sqrt{\frac{2}{\pi\sigma}}\int_0^{+\infty} s\mathrm{e}^{-\frac{1}{2\sigma}s^2}\mathrm{d}s = \sqrt{\frac{2}{\pi\sigma}}\sigma = \sqrt{\frac{2\sigma}{\pi}}.$$

  Hence, if (0.7) *holds, then* ρ_∞ *is a stationary state.*

- *In the limit* $t \to +\infty$*, the bound* (0.6) *is reached by the stationary state.*

Exercise 5. *Another possible statistical description of a neural network is to consider the probability density* $\rho(v,t)$ *of finding at time* t *a neuron with electric potential* $v \in (-\infty, V_F]$*. When a neuron reaches the firing potential* V_F*, it is reset to* V_R*. In the case of balanced excitation and inhibition, the (rescaled) model reads*

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial v}[-v\rho] - \sigma\frac{\partial^2\rho}{\partial v^2} = 0, \qquad v \in (-\infty, V_R)\cup(V_R,V_F),$$

with boundary conditions

$$\rho(V_F,t) = 0, \qquad \rho(V_F^-,t) = \rho(V_R^+,t), \qquad \frac{\partial\rho}{\partial v}(V_R^-,t) - \frac{\partial\rho}{\partial v}(V_R^+,t) = \frac{N(t)}{\sigma}.$$

and where the flux of firing neurons at time t *is*

$$N(t) = -\sigma\frac{\partial\rho}{\partial v}(V_F,t) > 0,$$

and ρ *must remain a probability density*

$$\int_{-\infty}^{V_F}\rho(v,t)\mathrm{d}v = 1.$$

- *Given a stationary firing rate* N_∞*, find the form of the corresponding stationary state* ρ_∞*. Then, prove that there exists a unique possible value for the stationary firing rate* $N_\infty > 0$*.*

- *We admit that there exists a unique stationary state* (ρ_∞, N_∞) *and that for any convex and* C^2 *function* $G : \mathbb{R} \to \mathbb{R}_+$, *the following entropy dissipation holds*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{V_F} G\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty \mathrm{d}v \quad &= \quad -\sigma\int_{-\infty}^{V_F}\left[\frac{\partial}{\partial v}\left(\frac{\rho}{\rho_\infty}\right)\right]^2 G''\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty \mathrm{d}v \\ &\quad -N_\infty\Big[G\left(\frac{N}{N_\infty}\right) - G\left(\frac{\rho}{\rho_\infty}\right) - \Big(\frac{N}{N_\infty} - \frac{\rho}{\rho_\infty}\Big)G'\left(\frac{\rho}{\rho_\infty}\right)\Big]\Big|_{v=V_R}.\end{aligned}$$

We also admit that there exists $\nu > 0$ *such that for all* q *smooth enough on* $(-\infty, V_R) \cup (V_R, V_F]$ *and with sufficient decay, satisfying also* $\int_{-\infty}^{V_F} q(v)\mathrm{d}v = 0$ *(*q *is extended by continuity at* V_R*), then*

$$\nu\int_{-\infty}^{V_F}\left(\frac{q}{\rho_\infty}\right)^2\rho_\infty \mathrm{d}v \leqslant \int_{-\infty}^{V_F}\left[\frac{\partial}{\partial v}\left(\frac{q}{\rho_\infty}\right)\right]^2\rho_\infty \mathrm{d}v.$$

Prove, choosing a suitable G *and assuming all smoothness and decay required, that the relative entropy*

$$\mathcal{E}(t) = \int_{-\infty}^{V_F} G\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty \mathrm{d}v$$

decays to 0 exponentially fast.

Solution 5. • *The main stationary equation is*

$$\frac{\mathrm{d}}{\mathrm{d}v}[-v\rho_\infty] - \sigma\frac{\mathrm{d}^2\rho_\infty}{\mathrm{d}v^2} = 0, \qquad N_\infty = -\sigma\frac{\mathrm{d}\rho_\infty}{\mathrm{d}v}(V_F).$$

Choose $v \in (-\infty, V_R)$. *We integrate the equation first on* (v, V_R), *which yields*

$$-V_R\rho_\infty(V_R^-) - \sigma\frac{\mathrm{d}\rho_\infty}{\mathrm{d}v}(V_R^-) + v\rho_\infty(v) + \sigma\frac{\mathrm{d}\rho_\infty}{\mathrm{d}v}(v) = 0.$$

Then, we integrate on (V_R, V_F), *which gives*

$$-V_F\underbrace{\rho_\infty(V_F)}_{=0}\underbrace{-\sigma\frac{\mathrm{d}\rho_\infty}{\mathrm{d}v}(V_F)}_{=N_\infty} + V_R\underbrace{\rho_\infty(V_R^+)}_{=\rho_\infty(V_R^-)} + \sigma\frac{\mathrm{d}\rho_\infty}{\mathrm{d}v}(V_R^+) = 0.$$

Summing the two previous equations we get

$$\underbrace{\sigma\left(\frac{\mathrm{d}\rho}{\mathrm{d}v}(V_R^+, t) - \frac{\mathrm{d}\rho}{\mathrm{d}v}(V_R^-, t)\right)}_{=-N_\infty} + N_\infty + v\rho_\infty(v) + \sigma\frac{\mathrm{d}\rho_\infty}{\mathrm{d}v}(v) = 0.$$

It yields (be careful it is valid only for $v \in (-\infty, V_R)$*),*

$$\rho_\infty(v) = C_1 e^{-\frac{v^2}{2\sigma}},$$

for some positive constant C_1.
Now, fix $v \in (V_R, V_F)$ *and integrate the main equation first on* $(-\infty, V_F)$ *assuming decay at infinity,*

$$-V_R\rho_\infty(V_R^-) - \sigma\frac{\mathrm{d}\rho_\infty}{\mathrm{d}v}(V_R^-) = 0,$$

and then on (V_R, v),

$$-v\rho_\infty(v) - \sigma\frac{\mathrm{d}\rho_\infty}{\mathrm{d}v}(v) + V_R\underbrace{\rho_\infty(V_R^+)}_{=\rho_\infty(V_R^-)} + \sigma\frac{\mathrm{d}\rho_\infty}{\mathrm{d}v}(V_R^+) = 0.$$

Combining the two gives

$$-v\rho_\infty(v) - \sigma\frac{\mathrm{d}\rho_\infty}{\mathrm{d}v}(v) = \sigma\left(\frac{\mathrm{d}\rho}{\mathrm{d}v}(V_R^+, t) - \frac{\mathrm{d}\rho}{\mathrm{d}v}(V_R^-, t)\right) = N_\infty.$$

A solution that has value 0 at V_F must be of the form

$$\rho_\infty(v) = C_2 e^{-\frac{v^2}{2\sigma}} + e^{-\frac{v^2}{2\sigma}} \int_v^{V_F} \frac{N_\infty}{\sigma} e^{\frac{w^2}{2\sigma}} \mathrm{d}w,$$

for some constant $C_2 \in \mathbb{R}$. Using $\rho_\infty(V_F) = 0$ again, we get $C_2 = 0$, which means that on (V_R, V_F),

$$\rho_\infty(v) = \frac{N_\infty}{\sigma} e^{-\frac{v^2}{2\sigma}} \int_v^{V_F} e^{\frac{w^2}{2\sigma}} \mathrm{d}w.$$

Matching $\rho_\infty(V_R^-) = \rho_\infty(V_R^+)$, we get

$$C_1 = \int_{V_R}^{V_F} e^{\frac{w^2}{2\sigma}} \mathrm{d}w$$

and thus, for any $v \in (-\infty, V_F]$ (the whole domain plus extension by continuity at V_R and V_F), we have the final form

$$\rho_\infty(v) = \frac{N_\infty}{\sigma} e^{-\frac{v^2}{2\sigma}} \int_{\max(v,V_R)}^{V_F} e^{\frac{w^2}{2\sigma}} \mathrm{d}w.$$

Since ρ_∞ must be a probability density, integrating on $(-\infty, V_F)$ gives the constraint

$$N_\infty = \frac{1}{I_\infty},$$

with

$$I_\infty = \frac{1}{\sigma} \int_{-\infty}^{V_F} e^{-\frac{v^2}{2\sigma}} \int_{\max(v,V_R)}^{V_F} e^{\frac{w^2}{2\sigma}} \mathrm{d}w > 0.$$

- *We can choose $G(x) = (x-1)^2$, apply a Poincaré-like inequality with $q = \rho - \rho_\infty$ and notice that, by convexity of G,*

$$-N_\infty \Big[G\left(\frac{N}{N_\infty}\right) - G\left(\frac{\rho}{\rho_\infty}\right) - \Big(\frac{N}{N_\infty} - \frac{\rho}{\rho_\infty}\Big) G'\left(\frac{\rho}{\rho_\infty}\right) \Big] \Big|_{v=V_R} \leqslant 0,$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{V_F} \left(\frac{\rho(v,t) - \rho_\infty(v)}{\rho_\infty(v)}\right)^2 \rho_\infty(v) \mathrm{d}v \leqslant -2\sigma\nu \int_{-\infty}^{V_F} \left(\frac{\rho(v,t) - \rho_\infty(v)}{\rho_\infty(v)}\right)^2 \rho_\infty(v) \mathrm{d}v,$$

where ν is the Poincaré constant. By Grönwall's lemma,

$$\int_{-\infty}^{V_F} \left(\frac{\rho(v,t) - \rho_\infty(v)}{\rho_\infty(v)}\right)^2 \rho_\infty(v) \mathrm{d}v \leqslant e^{-2\sigma\nu t} \int_{-\infty}^{V_F} \left(\frac{\rho^0(v) - \rho_\infty(v)}{\rho_\infty(v)}\right)^2 \rho_\infty(v) \mathrm{d}v,$$

which is the convergence in relative entropy of $\rho(\cdot, t)$ towards ρ_∞ at exponential speed.

The following is a list of the papers on which the lectures were based. The works listed below, of course, build on the work of many others, as you can see in the reference lists within each paper.

References

[1] J. A. Carrillo, A. Clini, and S. Solem. The mean field limit of stochastic differential equation systems modelling grid cells. *SIAM Journal on Mathematical Analysis*, 2022.

[2] J. A. Carrillo, H. Holden, and S. Solem. Noise-driven bifurcations in a neural field system modeling networks of grid cells. *J. Math. Biol.*, 85(42), 2022.

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[4] J. A. Carrillo, P. Roux, and S. Solem. Noise-driven bifurcations in a nonlinear Fokker-Planck system describing stochastic neural fields. *Phys. D*, 449:Paper No. 133736, 20, 2023.

[5] J. A. Carrillo, P. Roux, and S. Solem. Well-posedness and stability of a stochastic neural field in the form of a partial differential equation. *Journal de Mathématiques Pures et Appliquées*, 193:103623, 2025.