Fokker-Planck représentation of stochastic neural fields: derivation, analysis and application to grid cells
Lecture 4 : bifurcations

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Vrije Universiteit Amsterdam,
From Microscopic Dynamics
to Continuum Models

Idea : vary the noise parameter

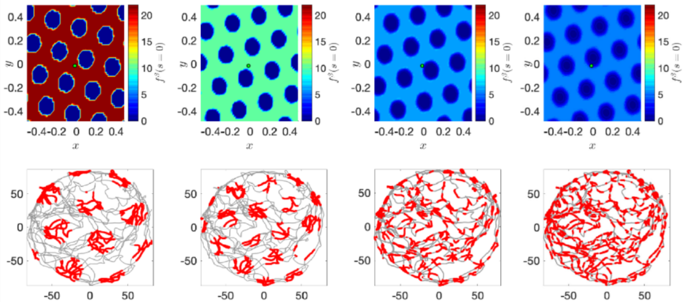

Figure – *Carrillo, Holden, Solem, 2022*

One population model

Let us consider the stationary pattern

$$0 = -\frac{\partial}{\partial s}\left(\left[\Phi_{\bar{\rho}}(\mathbf{x}) - s\right]\rho\right) + \sigma\frac{\partial^2 \rho}{\partial s^2},$$

where $\Phi(\mathbf{x})$ is given by

$$\Phi_{\bar{\rho}}(\mathbf{x}) = \Phi\left(W * \bar{\rho}\,(\mathbf{x}) + B\right),$$

where $B > 0$ is constant and

$$\bar{\rho}(\mathbf{x}) = \int_0^{\infty} s\rho(\mathbf{x}, s)\,\mathrm{d}s, \qquad \Phi_{\bar{\rho}}(\mathbf{x}, t)\rho(\mathbf{x}, 0) - \sigma\frac{\partial \rho}{\partial s}(\mathbf{x}, 0) = 0.$$

Step 1 : compute the stationary states

Stationary states satisfy

$$\sigma \partial_s \rho(x,s) = -\left(s - \Phi_{\bar{\rho}}\right)\rho(x,s).$$

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with conservation of unit mass we get Z :

$$Z = \int_0^{+\infty} \mathrm{e}^{-\frac{\left(s-\Phi_{\bar{\rho}}\right)^2}{2\sigma}} \,\mathrm{d}s.$$

Step 2 : focus on the average

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- Introduce $\kappa = \dfrac{1}{\sigma}$
- The average of any stationary state solves

$$\bar{\mathcal{G}}(\bar{\rho}, \kappa) = 0$$

where

$$\bar{\mathcal{G}}(\bar{\rho}, \kappa) = \bar{\rho} - \frac{1}{Z} \int_0^{+\infty} s \, \mathrm{e}^{-\kappa \frac{\left(s - \Phi_{\bar{\rho}}\right)^2}{2}} \, \mathrm{d}s, \qquad Z = \int_0^{+\infty} \mathrm{e}^{-\kappa \frac{\left(s - \Phi_{\bar{\rho}}\right)^2}{2}} \, \mathrm{d}s.$$

Step 3 : learn bifurcation theory (Oh no...)

Theorem (Crandall–Rabinowitz Theorem)

Grant technical Assumptions. Assume that

$$\ker\left(D_{\bar\rho}\mathcal{H}(0,\kappa_0)\right) = \mathrm{span}(\omega_0), \qquad \|\omega_0\| = 1$$

and

$$D^2_{\bar\rho\kappa}\mathcal{H}(0,\kappa_0)[\omega_0] \notin \mathrm{range}(D_{\bar\rho}\mathcal{H}(0,\kappa_0)).$$

Then there exists a nontrivial continuously differentiable curve

$$\{\ (\bar\rho(z),\kappa(z)) \mid z\in(-\delta,\delta),\ (\bar\rho(0),\kappa(0)) = (0,\kappa_0),\ \delta>0\ \}, \tag{1}$$

such that

$$\forall z\in(-\delta,\delta),\quad \mathcal{H}(\bar\rho(z),\kappa(z)) = 0,$$

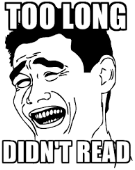
and in a neighbourhood of $(0,\kappa_0)$*, all the solutions to* $\mathcal{H}(\bar\rho,\kappa)=0$ *are either on the trivial solution line or on the nontrivial solution line* (1)*.*

Step 4 : choose a functional analysis framework

- If some assumptions on Φ and

$$W_0 = \int_{\mathbb{T}^d} W(x)\,\mathrm{d}x < 0,$$

then for all κ, $\exists!$ constant stationarry state $\bar{\rho}_\infty^\kappa$.



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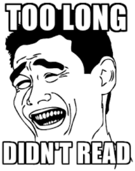
$$\Phi_0 = \Phi(W_0 \bar{\rho}_\infty + B), \quad \Phi_0' = \Phi'(W_0 \bar{\rho}_\infty + B), \quad \Phi_0'' = \Phi''(W_0 \bar{\rho}_\infty + B).$$



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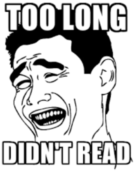
- We work in the space

$$L^2_S(\mathbb{T}^d) = \left\{ u \in L^2(\mathbb{T}^d) \mid u(\dots, -x_i, \dots) = u(\dots, x_i, \dots) \ \text{ a.e. in } \mathbb{T}^d \right\}.$$



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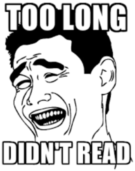
We use the Hilbert basis

$$(\omega_k)_{k\in\mathbb{N}^d}, \qquad \omega_k(x) = \Theta(k)\prod_{i=1}^d \cos(2\pi k_i x_i)$$



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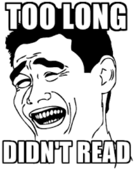

Fourier coefficients $\tilde{W}(k) = \langle\, W \mid \omega_k \,\rangle_{L^2},\ k \in \mathbb{N}^d$.

Step 5 : Get work done

Define the functional

$$\begin{array}{rccl} \mathcal{H}: & L^2_S(\mathbb{T}^d)\times\mathbb{R}^*_+ & \to & L^2_S(\mathbb{T}^d) \\ & (\bar\rho,\kappa) & \mapsto & \bar{\mathcal{G}}(\bar\rho^{\kappa}_{\infty}+\bar\rho,\kappa). \end{array}$$

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Compute Fréchet derivative

$$D_{\bar{\rho}}\mathcal{H}(0, \kappa)[h_1] \; = \; h_1 - \Phi_0' \left(1 - \bar{\rho}_\infty \left(\bar{\rho}_\infty - \Phi_0\right) \kappa\right) W * h_1,$$

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Look ! a cute hedgehog !

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Step 6 : Eat the fruit of the labor of your hands

Theorem (Carrillo, R., Solem)

Let $\kappa \in \left(\frac{2|W_0|^2}{\pi B^2}, +\infty\right)$. Assume second order things and $\exists! k^* \in \mathbb{N}^d$ such that

$$\frac{\tilde{W}(k^*)}{\Theta(k^*)} = \frac{1}{\Phi_0' \left(1 - \bar{\rho}_\infty \left(\bar{\rho}_\infty - \Phi_0\right) \kappa\right)} =: F_{bifurc}(\kappa).$$

Then, in a neighbourhood of $(\bar{\rho}_\infty, \kappa)$ in $L^2_S(\mathbb{T}^d) \times \mathbb{R}_+^*$, all the stationary states are either of the form (ρ_∞, κ) or in the curve

$$\{ \ (\rho_{\kappa(z)}, \kappa(z)) \mid z \in (-\delta, \delta), \ (\rho_{\kappa(0)}, \kappa(0)) = (\rho_\infty^\kappa, \kappa), \ \delta > 0 \ \},$$

defined by,

$$\bar{\rho}_{\kappa(z)}(x) = \bar{\rho}_\infty^{\kappa(z)} + z\omega_{k^*}(x) + o(z), \qquad x \in \mathbb{T}^d.$$

Satisfying the bifurcation condition

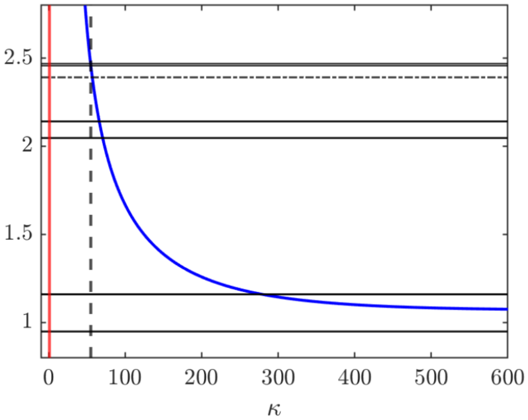

Figure – $B = 3$, $\Phi(x) = 0.5x\left(1 + \frac{x}{\sqrt{x^2+0.1}}\right)^+$, *and* $W(x,y) = -0.005 \cdot 2^{14}\left(1 + \tanh\left(10 - 50\sqrt{x^2+y^2}\right)\right)$.

Bifurcation condition = Linear stability condition

- Bifurcation condition :
$\frac{\tilde{W}(k^*)}{\Theta(k^*)} = F_{bifurc}(\kappa) = \left[\Phi_0' \left(1 - \bar{\rho}_\infty \left(\bar{\rho}_\infty - \Phi_0\right)\kappa\right)\right]^{-1}.$

1. Carrillo, Holden, Solem, 2022

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- Linear stability condition [1] :
  $\forall k \in \mathbb{N}^d, \; \frac{\tilde{W}(k)}{\Theta(k)} < F_{linear}(\kappa) = \left[\Phi_0' \kappa \int_0^{+\infty} (s - \bar{\rho}_\infty)^2 \rho_\infty(s) \, \mathrm{d}s\right]^{-1}.$

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Theorem (Carrillo, R., Solem)

Under mild assumptions,

$$\forall \kappa \in \left(\frac{2|W_0|^2}{\pi B^2}, +\infty\right), \qquad F_{bifurc}(\kappa) = F_{linear}(\kappa) =: F(\kappa).$$

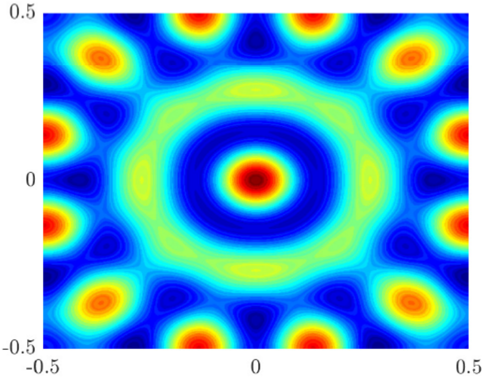
and

$$\lim_{\kappa \to \frac{2|W_0|^2}{\pi B^2}} F(\kappa) = +\infty, \qquad \lim_{\kappa \to +\infty} F(\kappa) = F^* > 0.$$

1. Carrillo, Holden, Solem, 2022

Superposition of the first Fourier modes

$$\omega(x) = \cos(8\pi x)\cos(2\pi y) + \cos(2\pi x)\cos(8\pi y) + \cos(6\pi x)\cos(6\pi y).$$



Time evolution

$$\omega^{\text{hex}}(x) = \cos(6\pi x)\cos(6\pi y) + \cos(8\pi x)\cos(2\pi y) + \cos(2\pi x)\cos(8\pi y) + \sin(6\pi x)\sin(6\pi y) + \sin(8\pi x)\sin(2\pi y) + \sin(2\pi x)\sin(8\pi y).$$

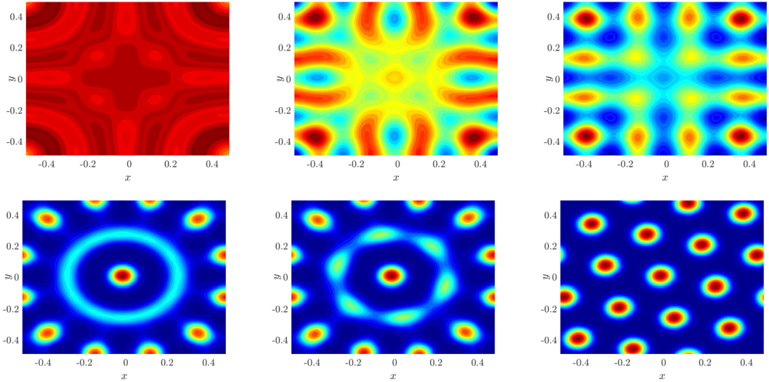


Figure – $t = 40, 220, 1500, 1810, 2190,$ *and* 2400 *ms*.

Thanking people

Thank you !