Hydrodynamic Limits of the Boltzmann Equation: A Rigorous Derivation of the Navier-Stokes system

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Second Part

Formal derivation of fluid limits

Hydrodynamic limit

Scope of the hydrodynamic limit: make the link between mesoscopic (kinetic) description of gas with its macroscopic description (fluid).

Convergence of solutions to (scaled) Boltzmann equation towards solutions related to Euler/Navier-Stokes equations.

Fundamental parameters: Knudsen number

We look at the scaled Boltzmann equation of the form

$$\mathrm{St}\partial_t F^\varepsilon(t,x,v) + v \cdot \nabla_x F^\varepsilon(t,x,v) = \frac{1}{\varepsilon}\mathcal{Q}(F^\varepsilon, F^\varepsilon)$$

Various choices of time scales $\mathrm{St} = \tau(\varepsilon)$ would lead to various fluid models: compressible Euler equation, incompressible Navier-Stokes system.

First limit: compressible Euler

We begin with assuming $\mathrm{St} = \tau(\varepsilon) = 1$ (of order 1).

$$\partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} \mathcal{Q}(F^\varepsilon, F^\varepsilon). \tag{1.1}$$

We focus here on the case $x \in \mathbb{T}^3$ or $x \in \mathbb{R}^3$ (to avoid boundary conditions issues).

Question

What happens in the fluid regime $\varepsilon \ll 1$? If $\varepsilon \to 0$, does $F^\varepsilon \to F^0$? If so, which is the equation solved by $F^0 = F^0(t, x, v)$.

Multiplying (1.1) with ε,

$$\varepsilon \left(\partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon\right) = \mathcal{Q}(F^\varepsilon, F^\varepsilon)$$

and assume $F^\varepsilon \to F^0$ as $\varepsilon \to 0$, then one expects

$$\mathcal{Q}(F^0, F^0) = 0.$$

Local Maxwellian

If $F^\varepsilon \to F^0$, then there exist $\varrho = \varrho(t,x)$, $\boldsymbol{u} = \boldsymbol{u}(t,x)$ and $\theta = \theta(t,x)$ such that

$$F^0(t,x,v) = \mathcal{M}_{(\varrho,\boldsymbol{u},\theta)}(v) = \frac{\varrho(t,x)}{(2\pi\theta(t,x))^{\frac{3}{2}}} \exp\left(-\frac{|v - \boldsymbol{u}(t,x)|^2}{2\theta(t,x)}\right)$$

The limit F^0 depends on t, x only through macroscopic quantities $\varrho, \boldsymbol{u}, \theta$.

Hilbert's expansion

To prove the convergence we consider the expansion of F^ε in terms of powers of ε:

Hilbert's expansion

Assume

$$F^\varepsilon(t,x,v) = \sum_{n\geqslant 0} \varepsilon^n F^n(t,x,v)$$

for suitable functions $F^n = F^n(t,x,v)$ (smooth and rapid decaying as $|v| \to \infty$).

Plug this expansion in (1.1) and balance the resulting coefficients of the successive powers of ε on each side of (1.1):

- ***Order*** ε^{-1}**.**

$$\mathcal{Q}(F^0,F^0) = 0, \qquad \text{i.e.} \qquad F^0 = \mathcal{M}_{(\varrho,\boldsymbol{u},\theta)}.$$

- ***Order*** ε^{0}**.**

$$\partial_t F^0 + v\cdot\nabla_x F^0 = \mathcal{Q}(F^0,F^1) + \mathcal{Q}(F^1,F^0)$$

- ***Order*** ε**.**

$$\partial_t F^1 + v\cdot\nabla_x F^1 = \mathcal{Q}(F^0,F^2) + \mathcal{Q}(F^2,F^0) + \mathcal{Q}(F^1,F^1)$$

and so on.....

Solvability condition

The second equation

$$\partial_t F^0 + v \cdot \nabla_x F^0 = \mathcal{Q}(F^0, F^1) + \mathcal{Q}(F^1, F^0)$$

will have a solution ***only if***

$$\int_{\mathbb{R}^3} \left(\partial_t F^0 + v \cdot \nabla_x F^0\right) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} \mathrm{d}v = \int_{\mathbb{R}^3} \left(\mathcal{Q}(F^0, F^1) + \mathcal{Q}(F^1, F^0)\right) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} \mathrm{d}v$$
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We will further be able to prove that this is an ***if and only if*** condition (Fredholm alternative for the linearized operator $\mathscr{L}_{F^0}(f) = \mathcal{Q}(F^0, f) + \mathcal{Q}(f, F^0)$).

Solvability condition

This implies

$$\begin{cases} \partial_t \int_{\mathbb{R}^3} \mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^3} v\mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \mathrm{d}v & = 0 \\ \partial_t \int_{\mathbb{R}^3} v\mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^3} v \otimes v\mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \mathrm{d}v & = 0 \\ \partial_t \int_{\mathbb{R}^3} \frac{1}{2}|v|^2\mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^3} v\frac{1}{2}|v|^2\mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \mathrm{d}v & = 0. \end{cases}$$

If $F^{\varepsilon} \to F^0 = \mathcal{M}_{(\varrho, \boldsymbol{u}, \theta)}$, then $\varrho, \boldsymbol{u}, \theta$ solve

Euler system for compressible fluids

$$\begin{cases} \partial_t \varrho(x,t) + \mathrm{div}_x \big(\varrho(x,t) \boldsymbol{u}(x,t) \big) = 0, \\ \\ \partial_t \big(\varrho \boldsymbol{u} \big) + \mathrm{div}_x (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x (\varrho \theta) = 0 \\ \\ \partial_t \left(\varrho \left(\frac{1}{2} |\boldsymbol{u}|^2 + \frac{3}{2} \theta \right) \right) + \mathrm{div}_x \left(\varrho \boldsymbol{u} \left(\frac{1}{2} |\boldsymbol{u}|^2 + \frac{5}{2} \theta \right) \right) = 0 \end{cases}$$

The term $p = \varrho \theta$ denotes the pressure in the case of a monoatomic perfect gas.

Exercise

Computations of moments of $\mathcal{M}_{(\varrho,\boldsymbol{u},\theta)}$:

$$\int \mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \mathrm{d}v = \varrho, \int v\mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \mathrm{d}v = \varrho\boldsymbol{u}, \int v \otimes v\mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \mathrm{d}v = (\varrho\boldsymbol{u} \otimes \boldsymbol{u} + \varrho\theta \mathbf{Id}),$$

$$\int v|v|^2 \mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \mathrm{d}v = \varrho\boldsymbol{u}\left(\frac{1}{2}|\boldsymbol{u}|^2 + \frac{5}{2}\theta\right),$$

and

$$\int \mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \log \mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \mathrm{d}v = \varrho \log\left(\frac{\varrho}{\theta^{\frac{3}{2}}}\right) - \frac{3}{2}\big(1 + \log(2\pi)\big)\varrho,$$

$$\int v\mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \log \mathcal{M}_{(\varrho,\boldsymbol{u},\theta)} \mathrm{d}v = \varrho\boldsymbol{u} \log\left(\frac{\varrho}{\theta^{\frac{3}{2}}}\right) - \frac{3}{2}\big(1 + \log(2\pi)\big)\varrho\boldsymbol{u}.$$

Exercise

Let $\varrho_{\rm in} > 0$, $\boldsymbol{u}_{\rm in} \in \mathbb{R}^3$ and $\theta_{\rm in} > 0$ be continuous mappings on the torus $\mathbb{T}^3$. Assume that, for each $\varepsilon > 0$ the Boltzmann equation (1.1) has a solution F^ε such that

$$F^\varepsilon(t=0) = \mathcal{M}_{(\varrho_{\rm in}, \boldsymbol{u}_{\rm in}, \theta_{\rm in})}$$

with

$$\int_{\mathbb{T}^3} \left(\varrho_{\rm in} \log \varrho_{\rm in} - \varrho_{\rm in} + 1 + \frac{1}{2} \varrho_{\rm in} |\boldsymbol{u}_{\rm in}|^2 + \frac{3}{2} \varrho_{\rm in} \left(\theta_{\rm in} - 1 - \log \theta_{\rm in} \right) \right) \mathrm{d}x$$
$$= H\left(\mathcal{M}_{(\varrho_{\rm in}, \boldsymbol{u}_{\rm in}, \theta_{\rm in})}, \mathcal{M}_{(1,0,1)} \right) < \infty$$

Assume F^ε is rapidly decaying and that $\log F^\varepsilon$ has polynomial growth. Assume that F^ε converges uniformly towards F^0 as $\varepsilon \to 0$. Prove then that

$$F^0 = \mathcal{M}_{(\varrho, \boldsymbol{u}, \theta)}(v)$$

where $(\varrho, \boldsymbol{u}, \theta)$ solves the Euler system for compressible fluids

$$\partial_t \varrho(x,t) + \mathrm{div}_x \big(\varrho(x,t) \boldsymbol{u}(x,t) \big) = 0,$$

$$\partial_t \big(\varrho \boldsymbol{u} \big) + \mathrm{div}_x (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x (\varrho \theta) = 0$$

$$\partial_t \left(\varrho \left(\frac{1}{2} |\boldsymbol{u}|^2 + \frac{3}{2} \theta \right) \right) + \mathrm{div}_x \left(\varrho \boldsymbol{u} \left(\frac{1}{2} |\boldsymbol{u}|^2 + \frac{5}{2} \theta \right) \right) = 0.$$

Theorem (Bardos, Golse, 1984)

Let $F_{\rm in}(x,v) > 0$ be such that

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} F_{\rm in}(x,v)\left(1+|v|^2+|\log F_{\rm in}(x,v)|\right)\mathrm{d}v\mathrm{d}x < \infty\,.$$

For any $\varepsilon > 0$, let $F^\varepsilon(t,x,v)$ be a solution to (1.1) associated to the initial datum $F^\varepsilon(0) = F_{\rm in}$ and suitable decay at $|v| \to \infty$ (uniformly w.r.t. ε). Assume that

$$\lim_{\varepsilon\to 0} F^\varepsilon(t,x,v) = F^0(t,x,v) \qquad \textit{a. e. on } (0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3$$

then

$$F^0(t,x,v) = \mathcal{M}_{(\varrho,\boldsymbol{u},\theta)}(v) = \frac{\varrho(t,x)}{(2\pi\theta(t,x))^{\frac{3}{2}}}\exp\left(-\frac{|v-\boldsymbol{u}(t,x)|^2}{2\theta(t,x)}\right)$$

where $(\varrho,\boldsymbol{u},\theta)$ satisfying the compressible Euler system for perfect gas and

$$\int_{\mathbb{T}^3}\varrho(t,x)\log\frac{\varrho(t,x)}{\theta^{\frac{3}{2}}(t,x)}\mathrm{d}x \leqslant \int_{\mathbb{R}^3\times\mathbb{T}^3} F_{\rm in}(x,v)\log F_{\rm in}(x,v)\mathrm{d}x\mathrm{d}v\,.$$

Theorem (Continued...)

If moreover,

$$\lim_{\varepsilon\to 0}\int_0^T \mathrm{d}t\int_{\mathbb{T}^3\times\mathbb{R}^3}(1+|v|)F^\varepsilon(t,x,v)\log F^\varepsilon(t,x,v)\mathrm{d}x\mathrm{d}v$$
$$=\int_0^T \mathrm{d}t\int_{\mathbb{T}^3\times\mathbb{R}^3}(1+|v|)F^0(t,x,v)\log F^0(t,x,v)\mathrm{d}x\mathrm{d}v$$

then

$$\partial_t\big(\varrho S\big)+\mathrm{div}_x\big(\varrho \boldsymbol{u} S\big)\leqslant 0,$$

where

$$S=S(\varrho,\theta)=\ln\left(\frac{\varrho}{\theta^{\frac{3}{2}}}\right).$$

This last condition characterizes entropic solutions to the compressible Euler equations
Proof is an **Exercise**.

- Rigorous justification by Caflisch, 1980 up to the first singular time for solutions to Euler system; not clear that solutions to BE are nonnegative.
- Nishida, 1978, proof in the framework of analytical solution. Lifespan of solution not known to coincide with that of Euler system.

Emergence of incompressibility

For fluid models, incompressibility reads

$$\nabla_x \cdot \varrho \boldsymbol{u} = \mathrm{div}_x \varrho \boldsymbol{u} = 0.$$

This requires somehow $\mathrm{St} = \tau(\varepsilon) \to 0$ as $\varepsilon \to 0$ since

$$\mathrm{St}\, \partial_t \varrho_\varepsilon + \nabla_x \cdot (\varrho_\varepsilon \boldsymbol{u}_\varepsilon) = 0$$

where

$$\varrho_\varepsilon(t,x) = \int_{\mathbb{R}^d} F^\varepsilon(t,x,v)\mathrm{d}v, \qquad \varrho_\varepsilon(t,x)\boldsymbol{u}_\varepsilon(t,x) = \int_{\mathbb{R}^d} vF^\varepsilon(t,x,v)\mathrm{d}v.$$

If $\varrho_\varepsilon(t,x)\boldsymbol{u}_\varepsilon(t,x) \to \varrho(t,x)\boldsymbol{u}(t,x)$ then incompressibility requires $\mathrm{St}\partial_t \varrho_\varepsilon \to 0$.

Emergence of incompressibility

Definition (Mach number)

The Mach number is the ratio

$$\mathrm{Ma} = \frac{|\boldsymbol{u}|}{c}$$

where c is the thermal speed of the gas, $|\boldsymbol{u}|$ is the magnitude of the typical macroscopic velocity.

Small Mach number

Incompressible limits are also small Mach number limits and correspond to study of fluctuations

$$F^{\varepsilon} = \mathcal{M}_{(1,0,1)} + \delta_{\varepsilon} f^{\varepsilon}$$

where δ_{ε} is proportional to the Mach number of the gas and $\delta_{\varepsilon} \to 0$ as $\varepsilon \to 0$.

Navier-Stokes scaling

Consider $\mathrm{St} = \varepsilon$ and the re-scaled Boltzmann equation

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} \mathcal{Q}(F^\varepsilon, F^\varepsilon)$$

This corresponds to the scaling

$$F^\varepsilon(t, x, v) = F\left(\varepsilon^{-2} t, \varepsilon^{-1} x, v\right)$$

where

$$\partial_t F + v \cdot \nabla_x F = \mathcal{Q}(F, F)\,.$$

Ansatz

$$F^\varepsilon = \mathcal{M} + \varepsilon f^\varepsilon$$

where

$$\mathcal{M} = \mathcal{M}_{(1,0,1)}$$

is some steady Maxwellian state where – with no loss of generality – we assumed here

$$\int_{\mathbb{R}^d \times \mathbb{T}^d} F^\varepsilon_{\mathrm{in}}(x, v) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} \mathrm{d}x\mathrm{d}v = \begin{pmatrix} 1 \\ 0 \\ \frac{d}{2} \end{pmatrix}.$$

$$\mathrm{Ma} \simeq \mathrm{St} \simeq \mathrm{Kn} = \varepsilon.$$

Ansatz

$$F^\varepsilon = \mathcal{M} + \varepsilon f^\varepsilon$$

where $\mathcal{M} = \mathcal{M}_{(1,0,1)}$ then

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \mathscr{L} f^\varepsilon + \frac{1}{\varepsilon} \mathcal{Q}(f^\varepsilon, f^\varepsilon) \tag{2.1}$$

where $\mathscr{L}$ is the linearized Boltzmann operator around some fixed $\mathcal{M}$

$$\mathscr{L} f = \mathcal{Q}(\mathcal{M}, f) + \mathcal{Q}(f, \mathcal{M})$$

Properties of the linearized operator

Natural space for $\mathscr{L}$ is the space $L^2(\mathbb{R}^d, \mathcal{M}^{-\frac{1}{2}}(v)\mathrm{d}v)$

Proposition (Hilbert, 1912)

On the space $L^2_v(\mathcal{M}^{-\frac{1}{2}})$, the linearized operator, with domain

$$\mathscr{D}(\mathscr{L}) = \{f \in L^2_v(\mathcal{M}^{-\frac{1}{2}})\,;\, \Sigma(\cdot)f \in L^2_v(\mathcal{M}^{-\frac{1}{2}})\}$$

splits as

$$\mathscr{L}f(v) = \Sigma(v)f - \mathcal{K}f(v)$$

where

$$\Sigma(v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathcal{M}_\star B(|v - v_*|, \sigma)\mathrm{d}v_*\mathrm{d}\sigma$$

and

$$\mathcal{K}f(v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v - v_*|, \sigma)\mathcal{M}\mathcal{M}_\star \left[\left(\frac{f}{\mathcal{M}}\right)' + \left(\frac{f}{\mathcal{M}}\right)'_\star - \left(\frac{f}{\mathcal{M}}\right)_\star\right] \mathrm{d}v_*\mathrm{d}\sigma.$$

Properties of the linearized operator

Proposition (Continued...)

It holds

1. *there is* $\nu_\star > 0$ *such that*

$$\nu_\star\,(1+|v|) \leqslant \Sigma(v) \leqslant \nu_\star^{-1}\,(1+|v|)\,, \qquad v \in \mathbb{R}^3\,.$$

2. $\mathcal{K}$ *is a compact operator on* $L^2_v(\mathcal{M}^{-\frac{1}{2}})$.
3. $(-\mathscr{L}, \mathscr{D}(\mathscr{L}))$ *is a self-adjoint nonnegative operator with*

$$\mathrm{Ker}\mathscr{L} = \mathrm{Span}\left\{\mathcal{M}, v_1\mathcal{M}, \ldots, v_d\mathcal{M}, |v|^2\mathcal{M}\right\}$$

Exercise ($L^2_v(\mathcal{M}^{-\frac{1}{2}})$ is the natural space thanks to H-Theorem)

Deduce that $-\mathscr{L}$ *is nonnegative from Boltzmann H-Theorem.*

Properties of the linearized operator

Corollary (Spectral gap)

There is $\lambda_\star > 0$ *such that*

$$\langle \mathscr{L} f, f\rangle \leqslant -\lambda \|f - \boldsymbol{\pi}_0 f\|^2_{L^2_v(\mathcal{M}^{-\frac{1}{2}})}, \qquad f \in \mathscr{D}(\mathscr{L})$$

where $\boldsymbol{\pi}_0$ *is the orthogonal projection over* $\mathrm{Ker}\mathscr{L}$.

Consequence of compactness of $\mathcal{K}$ (Weyl's Theorem) known since Hilbert. Quantitative estimate of $\lambda_\star$ very recent (Baranger & Mouhot, 2005).

The spectral projection

Exercise

The spectral projection $\boldsymbol{\pi}_0$ *on* $\mathrm{Ker}\mathscr{L}$ *is given by*

$$\boldsymbol{\pi}_0 g = \sum_{i=1}^{d+2} \left(\int_{\mathbb{R}^d} g \Psi_i \mathrm{d}v \right) \Psi_i \mathcal{M}$$

with

$$\Psi_1(v) = 1, \qquad \Psi_{i+1} = v_i \quad (i = 1, \ldots, d), \quad \Psi_{d+2}(v) = \frac{1}{\sqrt{2d}} \left(|v|^2 - d \right)$$

being an orthonormal basis of $\mathrm{Ker}\mathscr{L}$*. As a consequence, if* $g = g(x,v) \in L^2_x L^2_v(\mathcal{M}^{-\frac{1}{2}})$ *then*

$$\boldsymbol{\pi}_0 g(x,v) = \left[\varrho_g(x) + \boldsymbol{u}_g(x) \cdot v + \frac{1}{2} \theta_g(x) \left(|v|^2 - d \right) \right] \mathcal{M}(v)$$

with

$$\varrho_g(x) = \int_{\mathbb{R}^d} g(x,v) \mathrm{d}v, \qquad \boldsymbol{u}_g(x) = \int_{\mathbb{R}^d} v g(x,v) \mathrm{d}v$$

and

$$\theta_g(x) = \frac{1}{d} \int_{\mathbb{R}^d} \left(|v|^2 - d \right) g(x,v) \mathrm{d}v.$$

Properties of the linearized operator

Corollary (Fredholm alternative)

On the space $L^2_v(\mathcal{M}^{-\frac{1}{2}})$, one has

$$\mathrm{Range}\mathscr{L} = (\mathrm{Ker}\mathscr{L})^\perp = \left\{ g \in L^2_v(\mathcal{M}^{-\frac{1}{2}})\ ;\ \int_{\mathbb{R}^3} g \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} \mathrm{d}v = 0 \right\}$$

and $\mathscr{L}_{|\mathrm{Ker}\mathscr{L}^\perp}$ is invertible: for any $f \in \mathrm{Im}\mathscr{L}$, the equation

$$\mathscr{L} g = f$$

has a unique solution $g \in \mathrm{Range}(\boldsymbol{Id} - \boldsymbol{\pi}_0)$.

Technical tools

Let

$$\boldsymbol{A} = \boldsymbol{A}(v) := v \otimes v - \frac{1}{d}|v|^2 \mathbf{Id}, \qquad \boldsymbol{b}(v) = \boldsymbol{b}(v) := \frac{1}{2}\left(|v|^2 - (d+2)\right) v.$$

Lemma (1)

One has that $\boldsymbol{A}\mathcal{M}$, $\boldsymbol{b}\mathcal{M} \in (\mathrm{Ker}\mathscr{L})^\perp$ in $L^2_v(\mathcal{M}^{-\frac{1}{2}})$ and there exists two radial functions $\chi_i = \chi_i(|v|)$, $i = 1, 2$, such that

$$\widetilde{\boldsymbol{A}}(v) = \chi_1(|v|)\boldsymbol{A}(v) \in \mathscr{M}_d(\mathbb{R}) \qquad \textit{and} \qquad \widetilde{\boldsymbol{b}}(v) = \chi_2(|v|)\boldsymbol{b}(v) \in \mathbb{R}^d,$$

satisfy

$$\mathscr{L}(\widetilde{\boldsymbol{A}}\,\mathcal{M}) = -\boldsymbol{A}\,\mathcal{M}, \qquad \mathscr{L}(\widetilde{\boldsymbol{b}}\,\mathcal{M}) = -\boldsymbol{b}\,\mathcal{M}. \tag{2.2}$$

Moreover,

$$\int_{\mathbb{R}^d} \widetilde{\boldsymbol{A}}^{i,j}\mathscr{L}(\widetilde{\boldsymbol{A}}^{k,\ell}\mathcal{M})\mathrm{d}v = -\nu\left(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}\right)$$

$$\int_{\mathbb{R}^d} \widetilde{\boldsymbol{b}}_i\mathscr{L}(\widetilde{\boldsymbol{b}}_j\mathcal{M})\mathrm{d}v = -\frac{d+2}{2}\gamma\,\delta_{ij}, \qquad \forall\, i,j,k,\ell \in \{1,\dots,d\}, \tag{2.3}$$

with

$$\nu := -\frac{1}{(d-1)(d+2)}\int_{\mathbb{R}^d} \widetilde{\boldsymbol{A}} : \mathscr{L}(\widetilde{\boldsymbol{A}}\mathcal{M})\mathrm{d}v \geqslant 0, \qquad \gamma := -\frac{2}{d(d+2)}\int_{\mathbb{R}^d} \widetilde{\boldsymbol{b}}\cdot\mathscr{L}(\widetilde{\boldsymbol{b}}\mathcal{M})\mathrm{d}v \geqslant 0.$$

Technical tools

Lemma (2)

Given $g \in \mathrm{Ker}\mathscr{L}$ *given by*

$$g(x,v) = \pi_0 g(x,v) = \left[\varrho(x) + u(x)\cdot v + \frac{1}{2}\theta(x)\left(|v|^2 - d\right)\right]\mathcal{M}(v), \tag{2.4}$$

it holds that

$$\int_{\mathbb{R}^d} \widetilde{\boldsymbol{A}}\, \mathcal{Q}(g,g)\mathrm{d}v = \left(u\otimes u - \frac{2}{d}|u|^2\mathbf{Id}\right), \qquad \forall\, i,j = 1,\ldots,d\,.$$

Proof comes from a symmetric property Cercignani, 1967, of $\mathscr{L}$

Exercise

Prove that, if $f\mathcal{M} \in \mathrm{Ker}(\mathscr{L})$ *then* $\mathcal{Q}(f\mathcal{M}, f\mathcal{M}) = -\frac{1}{2}\mathscr{L}(f^2\mathcal{M})$. *Deduce the above Lemma.*

Hint: start from $\mathcal{Q}(\mathcal{M}_{(\varrho,u,\theta)}, \mathcal{M}_{\varrho,u,\theta}) = 0$ *and differentiate this up to second order in* (ϱ, u, θ).

Technical tools

Lemma (3)

Let g be given by (2.4). For any $i, j = 1, \dots, d$ it holds that

$$\int_{\mathbb{R}^d} v_\ell \, \widetilde{\boldsymbol{A}}^{i,j} \, g \mathrm{d}v = \begin{cases} \nu \, u_j & \text{if } i \neq j, \ \ell = i, \\ \nu \, u_i & \text{if } i \neq j, \ \ell = j, \\ -\frac{2}{d}\nu \, u_\ell + 2\nu \, u_i \delta_{i\ell} & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Moreover, for any $i = 1, \dots, d$, it holds that

$$\int_{\mathbb{R}^d} \widetilde{\boldsymbol{b}}_i \, \mathcal{Q}(g, g) \mathrm{d}v = \frac{d+2}{2} (\theta \, u_i),$$

and, if moreover $\varrho + \theta = 0$, then

$$\mathrm{div}_x \left(\int_{\mathbb{R}^d} \widetilde{\boldsymbol{b}}_i \, g \, v \mathrm{d}v \right) = \gamma \frac{d+2}{2} \partial_{x_i} \theta \, .$$

Proof is a simple Exercise.

Technical tools from fluid dynamics

We set

$$L^2_0(\mathbb{T}^d) := \left\{ \phi \in L^2_x(\mathbb{T}^d) \; ; \; \int_{\mathbb{T}^d} \phi(x)\,\mathrm{d}x = 0 \right\}.$$

Then, for any $\phi \in L^2_0(\mathbb{T}^d)$ there is a unique solution $f \in \mathbb{W}^{2,2}_x(\mathbb{T}^d) \cap L^2_0(\mathbb{T}^d)$ to the equation

$$-\Delta_x f = \phi \,, \qquad x \in \mathbb{T}^d \,.$$

We denote then by $(-\Delta_x)^{-1}$ the *bounded operator*

$$(-\Delta_x)^{-1} \; : \; \phi \in L^2_0(\mathbb{T}^d) \mapsto f \in \mathbb{W}^{2,2}_x(\mathbb{T}^d) \cap L^2_0(\mathbb{T}^d)\,.$$

Definition (Leray projection)

For a smooth vector field $\boldsymbol{u}$, we set

$$\mathbb{P}\boldsymbol{u} = \boldsymbol{u} - \nabla_x \, \Delta_x^{-1}(\nabla_x \cdot \boldsymbol{u}).$$

It holds that $\mathbb{P}\boldsymbol{u}$ is divergence-free,

$$\nabla_x \mathbb{P}\boldsymbol{u} = 0\,.$$

Exercise

Find the exact form of $\mathbb{P}\boldsymbol{u}$ in terms of Fourier expansion of $\boldsymbol{u}$.

Formal derivation of the Navier-Stokes system

Recall

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \mathscr{L} f^\varepsilon + \frac{1}{\varepsilon} \mathcal{Q}(f^\varepsilon, f^\varepsilon)$$

and assume $f^\varepsilon \to f$ as $\varepsilon \to 0$. Multiply the equation with ε^2 and let $\varepsilon \to 0$,

$$\mathscr{L} f^\varepsilon \to 0 \implies \mathscr{L} f = 0.$$

Thus $f \in \mathrm{Ker}\mathscr{L}$, i.e. $f = \boldsymbol{\pi}_0 f$

$$f(t,x,v) = \left[\varrho(t,x) + \boldsymbol{u}(t,x) \cdot v + \frac{1}{2}\theta(t,x)\left(|v|^2 - d\right)\right] \mathcal{M}(v)\,. \tag{3.1}$$

Again, f depends on t, x only through *macroscopic quantities*.

Compute suitable velocity average on the equation for f^ε

$$\left\langle g \right\rangle = \int_{\mathbb{R}^3} g(x,v)\mathrm{d}v$$

From $f^\varepsilon \to f$ we expect

$$\left\langle \psi f^\varepsilon \right\rangle \to \left\langle \psi f \right\rangle \qquad \text{in } \mathscr{D}'_{t,x}$$

for suitable ψ.

Set

$$\varrho_\varepsilon(t,x) = \left\langle f^\varepsilon \right\rangle, \qquad u_\varepsilon(t,x) = \left\langle v f^\varepsilon \right\rangle$$

$$\varrho(t,x) = \left\langle f \right\rangle, \qquad \boldsymbol{u}(t,x) = \left\langle v f \right\rangle, \qquad \theta(t,x) = \left\langle |v|^2 f \right\rangle.$$

Incompressibility condition

$$\varepsilon \partial_t \varrho_\varepsilon + \operatorname{div}_x (\boldsymbol{u}_\varepsilon) = 0\,,$$
$$\varepsilon\, \partial_t \boldsymbol{u}_\varepsilon + \operatorname{Div}_x (\boldsymbol{J}_\varepsilon) = 0\,,$$

where $\boldsymbol{J}_\varepsilon = \boldsymbol{J}_\varepsilon(t,x)$ denotes the tensor

$$\boldsymbol{J}_\varepsilon(t,x) := \left\langle v \otimes v\, f^\varepsilon \right\rangle,$$

since both $\mathscr{L}$ and $\mathcal{Q}$ conserve mass and momentum.

Incompressibility

$$\varepsilon \partial_t \varrho_\varepsilon \to 0 \qquad \operatorname{div}_x (\boldsymbol{u}_\varepsilon) \to \operatorname{div}_x \boldsymbol{u}$$

Thus

$$\operatorname{div}_x \boldsymbol{u} = 0.$$

Boussinesq relation

$$\mathrm{Div}_x\left(\boldsymbol{J}_\varepsilon\right) \to \mathrm{Div}_x\left(\left\langle v \otimes vf \right\rangle\right)$$

with

$$\left\langle v \otimes vf \right\rangle = \left\langle v \otimes v \left[\varrho(t,x) + \boldsymbol{u}(t,x)\cdot v + \frac{1}{2}\theta(t,x)\left(|v|^2 - d\right)\right]\mathcal{M}(v)\right\rangle = (\varrho + \theta)\boldsymbol{Id}$$

Thus

$$\nabla_x(\varrho + \theta) = 0.$$

Moreover

$$\int_{\mathbb{R}^3} \varrho_\varepsilon(t,x)\mathrm{d}x = 0 \implies \int_{\mathbb{R}^3} \varrho(t,x)\mathrm{d}x = 0,$$

idem with $\left\langle \frac{1}{2}|v|^2 f^\varepsilon \right\rangle$ and θ.

Boussinesq relation

$$\varrho(t,x) + \theta(t,x) = 0\,.$$

Conservation laws

Mass

$$\partial_t \varrho_\varepsilon + \frac{1}{\varepsilon}\mathrm{div}_x\left(u_\varepsilon\right) = 0\,, \tag{3.2}$$

Velocity

$$\partial_t u_\varepsilon + \frac{1}{\varepsilon}\mathrm{Div}_x\left(\boldsymbol{J}_\varepsilon\right) = \partial_t u_\varepsilon + \frac{1}{\varepsilon}\mathrm{Div}_x\left\langle \boldsymbol{A} f^\varepsilon \right\rangle + \frac{1}{\varepsilon}\nabla_x p_\varepsilon = 0\,, \tag{3.3}$$

where $\boldsymbol{A}(v) = v \otimes v - \frac{1}{d}|v|^2 \boldsymbol{Id}$ is traceless and $p_\varepsilon = \frac{1}{d}\left\langle |v|^2 f^\varepsilon \right\rangle$.

Energy

$$\partial_t \left\langle \frac{1}{2}|v|^2 f^\varepsilon \right\rangle + \frac{1}{\varepsilon}\mathrm{div}_x\left\langle \frac{1}{2}|v|^2 v\, f^\varepsilon \right\rangle = 0 \tag{3.4}$$

Conservation laws

We know $\mathrm{div}_x \boldsymbol{u}_\varepsilon \to 0$, so what about

$$\frac{1}{\varepsilon}\mathrm{div}_x \boldsymbol{u}_\varepsilon \qquad ?$$

In the same way,

$$\left\langle \boldsymbol{A} f^\varepsilon \right\rangle \to \left\langle \boldsymbol{A} f \right\rangle$$

and $f \in \mathrm{Ker}\mathscr{L}$ so that Exercise

$$\left\langle \boldsymbol{A} f \right\rangle = 0.$$

Now, what about

$$\frac{1}{\varepsilon}\left\langle \boldsymbol{A} f^\varepsilon \right\rangle \qquad ?$$

Emergence of diffusion

Compute the limit of

$$\frac{1}{\varepsilon}\mathrm{Div}_x\left\langle \boldsymbol{A} f^\varepsilon \right\rangle$$

and

$$\frac{1}{\varepsilon}\mathrm{div}_x\left\langle \frac{1}{2} v|v|^2 f^\varepsilon \right\rangle.$$

$\mathscr{L}$ is self-adjoint

It holds that

$$\left\langle \boldsymbol{A} f^\varepsilon \right\rangle = \int_{\mathbb{R}^d} \boldsymbol{A}\mathcal{M} f^\varepsilon \mathcal{M}^{-1} \mathrm{d}v = \left\langle \mathcal{A}\mathcal{M}, f^\varepsilon \right\rangle_{L^2_v(\mathcal{M}^{-\frac{1}{2}})}$$

$$= -\left\langle \mathscr{L}(\widetilde{\boldsymbol{A}}\mathcal{M}), f^\varepsilon \right\rangle_{L^2_v(\mathcal{M}^{-\frac{1}{2}})} = -\left\langle \widetilde{\boldsymbol{A}}\mathscr{L} f^\varepsilon \right\rangle.$$

Emergence of diffusion

Now, recalling

$$\varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} \mathscr{L} f^\varepsilon + \mathcal{Q}(f^\varepsilon, f^\varepsilon)$$

we deduce

$$-\frac{1}{\varepsilon} \left\langle \widetilde{\boldsymbol{A}} \mathscr{L} f^\varepsilon \right\rangle = \left\langle \widetilde{\boldsymbol{A}}\, \mathcal{Q}(f^\varepsilon, f^\varepsilon) \right\rangle - \left\langle \widetilde{\boldsymbol{A}} \left(\varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon \right) \right\rangle$$

and we got

$$-\frac{1}{\varepsilon} \left\langle \widetilde{\boldsymbol{A}}^{i,j} \mathscr{L} f^\varepsilon \right\rangle \longrightarrow \left\langle \widetilde{\boldsymbol{A}}^{i,j} \mathcal{Q}(f, f) \right\rangle - \mathrm{div}_x \left\langle v \widetilde{\boldsymbol{A}}^{i,j} f \right\rangle$$

Use of the technical Lemmas

$$-\frac{1}{\varepsilon}\left\langle \widetilde{\boldsymbol{A}}^{i,j}\mathscr{L}f^{\varepsilon}\right\rangle \longrightarrow \left\langle \widetilde{\boldsymbol{A}}^{i,j}\mathcal{Q}(f,f)\right\rangle - \mathrm{div}_x\left\langle v\widetilde{\boldsymbol{A}}^{i,j}f\right\rangle$$

with

$$\left\langle \widetilde{\boldsymbol{A}}^{i,j}\mathcal{Q}(f,f)\right\rangle = \left(\boldsymbol{u}\otimes\boldsymbol{u} - \frac{2}{d}|\boldsymbol{u}|^2\mathbf{Id}\right)$$

and

$$\left\langle v_\ell\,\widetilde{\boldsymbol{A}}^{i,j}\,f\right\rangle = \begin{cases} \nu\,\boldsymbol{u}_j & \text{if } i\neq j\,,\ \ell = i\,,\\ \nu\,\boldsymbol{u}_i & \text{if } i\neq j\,,\ \ell = j\,,\\ -\frac{2}{d}\nu\,\boldsymbol{u}_\ell + 2\nu\,\boldsymbol{u}_i\delta_{i\ell} & \text{if } i = j\,,\\ 0 & \text{else}\,. \end{cases}$$

Therefore

$$\mathrm{div}_x\left\langle v\widetilde{\boldsymbol{A}}^{i,j}f\right\rangle = \begin{cases} \nu(\partial_{x_i}\boldsymbol{u}_j + \partial_{x_j}\boldsymbol{u}_i) & \text{if } i\neq j\\ 2\nu\partial_{x_i}\boldsymbol{u}_i & \text{if } i = j\,, \end{cases}$$

where we used the incompressibility condition. Thus

$$\mathrm{div}_x\left\langle v\widetilde{\boldsymbol{A}}^{i,j}f\right\rangle = \nu(\partial_{x_i}\boldsymbol{u}_j + \partial_{x_j}\boldsymbol{u}_i) \qquad \forall i,j.$$

Emergence of diffusion

$$\frac{1}{\varepsilon}\left\langle \boldsymbol{A}^{i,j} f^{\varepsilon} \right\rangle \longrightarrow \boldsymbol{u}_i \boldsymbol{u}_j - \frac{2}{d}|\boldsymbol{u}|^2 \delta_{ij} - \nu(\partial_{x_i} \boldsymbol{u}_j + \partial_{x_j} \boldsymbol{u}_i)$$

and

$$\mathrm{Div}_x^i(\partial_{x_i} \boldsymbol{u}_j + \partial_{x_j} \boldsymbol{u}_i) = \Delta_x \boldsymbol{u}_i$$

thanks to the incompressibility condition.

Summary

$$\frac{1}{\varepsilon}\mathrm{Div}_x \left\langle \boldsymbol{A} f^{\varepsilon} \right\rangle \longrightarrow \mathrm{Div}_x \left(\boldsymbol{u} \otimes \boldsymbol{u} - \frac{2}{d}|\boldsymbol{u}|^2 \mathbf{Id} \right) - \nu \Delta_x \boldsymbol{u}.$$

Limiting velocity equation

Recall from (3.3)

$$\partial_t u_\varepsilon + \frac{1}{\varepsilon}\mathrm{Div}_x\left(\boldsymbol{J}_\varepsilon\right) = \partial_t u_\varepsilon + \frac{1}{\varepsilon}\mathrm{Div}_x\left\langle \boldsymbol{A} f^\varepsilon \right\rangle + \frac{1}{\varepsilon}\nabla_x p_\varepsilon = 0$$

Apply the Leray projection to kill $\nabla_x p_\varepsilon$. Set $\mathbb{P}u_\varepsilon = \tilde{u}_\varepsilon$, it holds

$$\partial_t \tilde{u}_\varepsilon + \frac{1}{\varepsilon}\mathbb{P}\mathrm{Div}_x\left\langle \boldsymbol{A} f^\varepsilon \right\rangle = 0$$

and

$$\partial_t \tilde{u}_\varepsilon \to \partial_t \mathbb{P}\boldsymbol{u} = \partial_t \boldsymbol{u}$$

while

$$\frac{1}{\varepsilon}\mathbb{P}\mathrm{Div}_x\left\langle \boldsymbol{A} f^\varepsilon \right\rangle \longrightarrow \mathbb{P}\mathrm{Div}_x\left(\boldsymbol{u}\otimes\boldsymbol{u}\right) - \mathbb{P}\left(\frac{2}{d}\nabla_x|\boldsymbol{u}|^2\right) - \nu\mathbb{P}\Delta_x\boldsymbol{u}$$
$$= \mathbb{P}\mathrm{Div}_x\left(\boldsymbol{u}\otimes\boldsymbol{u}\right) - \nu\Delta_x\boldsymbol{u}.$$

Limiting velocity equation

Writing

$$\mathbb{P}\mathrm{Div}_x\,(\boldsymbol{u} \otimes \boldsymbol{u}) = \mathrm{Div}_x\,(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p$$

it holds

Proposition

The limit velocity $\boldsymbol{u}$ *satisfies*

$$\partial_t \boldsymbol{u} - \nu\,\Delta_x \boldsymbol{u} + \mathrm{Div}_x\,(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p = 0 \tag{3.5}$$

The incompressibility condition implies

$$\mathrm{Div}_x^i\,(\boldsymbol{u} \otimes \boldsymbol{u}) = \boldsymbol{u} \cdot \nabla_x \boldsymbol{u}_i$$

and (3.5) reads as usual

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla_x \boldsymbol{u} + \nabla_x p = \nu\,\Delta_x \boldsymbol{u}$$

with (semi)-explicit *viscosity*

$$\nu := -\frac{1}{(d-1)(d+2)}\left\langle \widetilde{\boldsymbol{A}} : \mathscr{L}(\widetilde{\boldsymbol{A}}\mathcal{M}) \right\rangle \geqslant 0\,.$$

Limiting temperature

Same method. Starting from (3.4)

$$\partial_t\left\langle \frac{1}{2}|v|^2 f^\varepsilon\right\rangle + \frac{1}{\varepsilon}\mathrm{div}_x\left\langle \frac{1}{2}|v|^2 v\, f^\varepsilon\right\rangle = 0\,.$$

Observe again

$$\left\langle \frac{1}{2}|v|^2 v\, f^\varepsilon\right\rangle \longrightarrow \left\langle \frac{1}{2}|v|^2 v f\right\rangle = \frac{d+2}{2}\boldsymbol{u}$$

so that

$$\mathrm{div}_x\left\langle \frac{1}{2}|v|^2 v\, f^\varepsilon\right\rangle \longrightarrow \frac{d+2}{2}\mathrm{div}_x\boldsymbol{u} = 0.$$

What about $\frac{1}{\varepsilon}\mathrm{div}_x\left\langle \frac{1}{2}|v|^2 v\, f^\varepsilon\right\rangle$?

One writes

$$\left\langle \frac{1}{2}|v|^2 v\, f^\varepsilon\right\rangle = \left\langle \boldsymbol{b} f^\varepsilon\right\rangle + \frac{d+2}{2}\left\langle v\, f^\varepsilon\right\rangle = \left\langle \boldsymbol{b} f^\varepsilon\right\rangle + \frac{d+2}{2}u_\varepsilon\,,$$

where we recall

$$\boldsymbol{b}(v) = \frac{1}{2}\left(|v|^2 - (d+2)\right)v.$$

Limiting temperature

As before

$$\left\langle \boldsymbol{b} f^{\varepsilon} \right\rangle = -\left\langle \widetilde{\boldsymbol{b}} \mathscr{L} f^{\varepsilon} \right\rangle$$

and, from

$$\varepsilon \partial_t f^{\varepsilon} + v \cdot \nabla_x f^{\varepsilon} = \frac{1}{\varepsilon} \mathscr{L} f^{\varepsilon} + \mathcal{Q}(f^{\varepsilon}, f^{\varepsilon})$$

we deduce

$$\frac{1}{\varepsilon} \left\langle \boldsymbol{b} f^{\varepsilon} \right\rangle = -\frac{1}{\varepsilon} \left\langle \widetilde{\boldsymbol{b}} \mathscr{L} f^{\varepsilon} \right\rangle = \left\langle \widetilde{\boldsymbol{b}} \mathcal{Q}(f^{\varepsilon}, f^{\varepsilon}) \right\rangle - \left\langle \widetilde{\boldsymbol{b}} \left(\varepsilon \partial_t f^{\varepsilon} + v \cdot \nabla_x f^{\varepsilon} \right) \right\rangle$$

and

$$\frac{1}{\varepsilon} \left\langle \boldsymbol{b} f^{\varepsilon} \right\rangle \longrightarrow \left\langle \widetilde{\boldsymbol{b}} \mathcal{Q}(f, f) \right\rangle - \mathrm{div}_x \left\langle v \widetilde{\boldsymbol{b}} f \right\rangle$$

Use of the technical Lemmas

$$\left\langle \widetilde{\boldsymbol{b}}\mathcal{Q}(f,f)\right\rangle = \frac{d+2}{2}\,(\theta\,\boldsymbol{u})$$

and

$$\mathrm{div}_x\left\langle v\widetilde{\boldsymbol{b}}f\right\rangle = \gamma\frac{d+2}{2}\nabla_x\theta\,.$$

Therefore

$$\frac{1}{\varepsilon}\mathrm{div}_x\left\langle \boldsymbol{b}f^{\varepsilon}\right\rangle \longrightarrow \frac{d+2}{2}\,(\mathrm{div}_x(\theta\boldsymbol{u}) - \gamma\Delta_x\theta) = \frac{d+2}{2}\,(\boldsymbol{u}\cdot\nabla_x\theta - \gamma\Delta_x\theta)\,.$$

Limiting temperature

Recalling

$$\partial_t\left\langle \frac{1}{2}|v|^2 f^\varepsilon\right\rangle + \frac{1}{\varepsilon}\mathrm{div}_x\left\langle \boldsymbol{b}\, f^\varepsilon\right\rangle + \frac{d+2}{2}\frac{1}{\varepsilon}\mathrm{div}_x \boldsymbol{u}_\varepsilon = 0\,.$$

and

$$\left\langle \frac{1}{2}|v|^2, f^\varepsilon\right\rangle \longrightarrow \frac{d}{2}\left(\varrho+\theta\right) = 0$$

we get

$$\frac{d+2}{2}\left(\boldsymbol{u}\cdot\nabla_x\theta - \gamma\Delta_x\theta\right) = -\frac{d+2}{2}\lim_{\varepsilon}\frac{1}{\varepsilon}\mathrm{div}_x\boldsymbol{u}_\varepsilon$$

where this last limit now exists. But conservation of mass gives

$$\partial_t\varrho_\varepsilon + \frac{1}{\varepsilon}\mathrm{div}_x\boldsymbol{u}_\varepsilon = 0$$

so that

$$\frac{d+2}{2}\left(\boldsymbol{u}\cdot\nabla_x\theta - \gamma\Delta_x\theta\right) = \frac{d+2}{2}\partial_t\varrho = -\frac{d+2}{2}\partial_t\theta$$

thanks to Boussinesq condition.

Proposition

The limit temperature θ satisfies

$$\partial_t\theta - \gamma\,\Delta_x\theta + \boldsymbol{u}\cdot\nabla_x\theta = 0. \tag{3.6}$$

Formal result

Theorem

Let f^ε be a solution to

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \mathscr{L} f^\varepsilon + \frac{1}{\varepsilon} \mathcal{Q}(f^\varepsilon, f^\varepsilon)$$

with initial datum $f^\varepsilon_{\rm in} \in \mathrm{Range}(\boldsymbol{Id} - \boldsymbol{\pi}_0)$. *Assume that* $f^\varepsilon \to f$ *(in some suitable sense). Then*

$$f = f(t,x,v) = \left[\varrho(t,x) + \boldsymbol{u}(t,x) \cdot v + \frac{1}{2}\theta(t,x)\left(|v|^2 - d\right)\right] \mathcal{M}(v)$$

where $(\varrho, \boldsymbol{u}, \theta)$ *satisfy the the following* incompressible Navier-Stokes-Fourier system

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla_x \boldsymbol{u} + \nabla_x p = \nu\, \Delta_x u\,, \\ \partial_t\, \theta + \boldsymbol{u} \cdot \nabla_x \theta = \gamma\, \Delta_x \theta\,, \\ \mathrm{div}_x \boldsymbol{u} = 0\,, \qquad \varrho + \theta = 0\,, \end{cases} \tag{3.7}$$

where the viscosity $\nu > 0$ *and heat conductivity* $\gamma > 0$ *are given by*

$$\nu := -\frac{1}{(d-1)(d+2)}\left\langle \widetilde{\boldsymbol{A}} : \mathscr{L}(\widetilde{\boldsymbol{A}}\mathcal{M})\right\rangle, \qquad \gamma := -\frac{2}{d(d+2)}\left\langle \widetilde{\boldsymbol{b}} \cdot \mathscr{L}(\widetilde{\boldsymbol{b}}\mathcal{M})\right\rangle.$$

How to make all this rigorous ?

First problem: well-posedness of BE

- Difficult problem to prove global existence of solutions to

$$\partial_t F + v \cdot \nabla_x F = \mathcal{Q}(F, F)$$

  using *only* the physical natural quantities

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} F_{\rm in}(1 + |v|^2)\mathrm{d}v\mathrm{d}x < \infty, \qquad \int_{\mathbb{T}^d \times \mathbb{R}^d} F_{\rm in}|\log F_{\rm in}|\mathrm{d}v\mathrm{d}x < \infty.$$

  Problem answered by Di Perna & Lions 1989 with the notion of *renormalized solutions* (solutions in a very weak sense). No uniqueness known.

- Rather standard argument allows to prove existence and uniqueness for close to equilibrium solutions: for suitable norm $\|\cdot\|$, there is $\delta > 0$ such that

$$\|F_{\rm in} - \mathcal{M}\| \leqslant \delta$$

  implies the existence and uniqueness of solution $F(t, x, v)$ with

$$\|F(t, x, v) - \mathcal{M}\| \leqslant C\delta, \qquad \forall t \geqslant 0.$$

Well-posedness of BE

Close-to-equilibrium

- First attempt is norm of $L^\infty_{x,v}(\mathcal{M}^{-\frac{1}{2}}\mathrm{d}v\mathrm{d}x)$, Ukai 1974.
- Extension to $H^\ell_x L^2_v(\mathcal{M}^{-\frac{1}{2}}\mathrm{d}v\mathrm{d}x)$ through spectral analysis or energy methods (Ukai, Bardos 1989; Guo 2002–2005; etc..) with $\ell > \frac{d}{2}$.
- Extension to more natural L^1-based spaces through enlargement/factorisation methods Gualdani, Mischler, Mouhot, 2010 in space $W^{1,1}_x L^1_v(\varpi_q)$ where

$$\varpi_q(v) = (1+|v|)^q, \qquad q > q_0.$$

This is true typically for $\varepsilon = 1$. For the rescaled BE, need of estimates *uniform in* ε.

How to make all this rigorous ?

Need for *a priori* estimates

Renormalized solutions

- First important breakthrough: *road map* towards the proof of convergence of *renormalized* solutions of BE towards Leray solutions for Navier-Stokes: Bardos, Golse, Levermore, 1991-1993 program.
- Answer to Bardos, Golse, Levermore program by Golse & Saint-Raymond, 2001-2004, extension to general collision kernels Levermore, Masmoudi, 2003.

Close-to-equilibrium solutions

- First important breakthrough is the convergence of spectral modes of linearized Bolzmann equation towards those of Navier-Stokes Ellis & Pinsky 1974.
- *Strong* convergence of close-to-equilibrium strong solutions by Bardos, Ukai 1991. Important extensions of strong convergence solutions by Gallagher, Tristani 2020 and Carrapatoso, Gallagher, Tristani 2024. Unified theory for several models Gervais, L. 2023.
- A priori estimates uniform w.r.t. ε by enlargment/factorization in L^1-spaces by Briant, Merino-Aceituno, Mouhot 2019.
- *Weak* convergence and a priori estimates by energy methods (for inelastic gas) in Alonso, L., Tristani 2021.

Other possible limits

Incompressible Euler Limit

For this model, $\mathrm{St} = \mathrm{Ma} = \varepsilon \ll 1$ but $\mathrm{Kn} = \varepsilon^a$, $a > 1$, i.e.

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon^a} \mathcal{Q}(F^\varepsilon, F^\varepsilon)$$

where

$$F^\varepsilon = \mathcal{M} + \varepsilon f^\varepsilon.$$

Then, f_ε converges towards $(u \cdot v)\mathcal{M}$ where $u = u(t,x)$ is a solution to the compressible Euler equation

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = 0, \qquad \nabla_x \cdot u = 0.$$

- Bardos, Golse, Levermore, 1991, formal derivation.
- Saint-Raymond, 2002, first rigorous proof for dissipative solutions of Euler eqs. Improvement with relative entropy method Saint-Raymond, 2009.

Other possible limits

Stokes Limit

For this model, $\mathrm{St} = \mathrm{Kn} = \varepsilon \ll 1$ but $\mathrm{Ma} = \varepsilon^a$, $a > 1$, i.e.

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} \mathcal{Q}(F^\varepsilon, F^\varepsilon)$$

where

$$F^\varepsilon = \mathcal{M} + \varepsilon^a f^\varepsilon.$$

- Bardos, Golse, Levermore, 1991, formal derivation.
- Golse, Levermore, 2002, first rigorous proof.