Microscopic dynamics to continuum limits
Support class questions

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1 Agent-based models

Question 1: Convergence to an SDE system

In lectures we saw the convergence of an ABM with updates of the form

$$x_i(t+h) = \begin{cases} x_i(t) + \mu^h \left(x_j(t) - x_i(t)\right) & \text{with probability } p_{ij}(x) \\ x_i(t) & \text{with probability } 1 - p_{ij}(x)\,, \end{cases} \tag{1}$$

to a coupled ODE system given by

$$\frac{dx_i}{dt} = \frac{1}{N} \sum_{j=1}^{N} p_{ij}(x)\,(x_j - x_i)\,.$$

Now consider the following system with additional noise (called here adaptation noise)

$$x_i(t+h) = \begin{cases} x_i(t) + \mu^h \left(x_j(t) - x_i(t)\right) + \xi^h & \text{with probability } p_{ij}(x) \\ x_i(t) & \text{with probability } 1 - p_{ij}(x)\,. \end{cases} \tag{2}$$

a) By calculating the first and second moments of updates, repeat the derivation of the limiting system.

1. Using the theorem below, what conditions are required on the distribution of the noise ξ (thinking about how its moments should scale with h) to ensure convergence of the model? Similarly what conditions are required on $p_{ij}(x)$?
2. In what cases do we obtain a deterministic limit or a stochastic limit?
3. What conditions are required on ξ to ensure convergence with the same drift function as the deterministic system?

b) How might adaptation noise affect the long-term behaviour of the system? Is this the same in the ABM and limiting SDE model?

c) In (1) (and also in (2)) we assume that only individual i updates their opinion. How do these limits change if both agents i and j update their opinions?

Theorem: (adapted from Durrett [1])
For $1 \le i, j \le N$ and $p \in \mathbb{N}$, define

$$a^h_{ij}(x) = \frac{1}{h} \int_{\mathbb{R}^N} (y_i - x_i)(y_j - x_j)\, \Pi^h(x, dy)\,, \tag{3}$$

$$b^h_i(x) = \frac{1}{h} \int_{\mathbb{R}^N} (y_i - x_i)\, \Pi^h(x, dy)\,, \tag{4}$$

$$\gamma^h_p(x) = \frac{1}{h} \int_{\mathbb{R}^N} |y - x|^p\, \Pi^h(x, dy)\,. \tag{5}$$

where $\Pi^h(x, y)$ is the probability of transitioning from state x to state y in a timestep of size h.
Assume there exist Lipschitz continuous functions a and b such that for some $p \ge 2$ and for all $0 \le i, j \le N$, $R < \infty$, we have

a) $\lim_{h \searrow 0} \sup_{|x| \le R} |a^h_{ij}(x) - a_{ij}(x)| = 0$,

b) $\lim_{h \searrow 0} \sup_{|x| \le R} |b^h_i(x) - b_i(x)| = 0$,

c) $\lim_{h \searrow 0} \sup_{|x| \le R} \gamma^h_p(x) = 0$.

Then if $X^h_0 = x^h \to x$ we have $X^h_t \Rightarrow X_t$ as $h \to 0$, where X_t is the solution of

$$dX_t = b(X_t)\, dt + \sqrt{a(X_t)}\, d\beta_t. \tag{6}$$

Question 2: Numerical convergence study

a) For the ABM without additional noise (1) write code to simulate this system. At each timestep randomly select individuals, decide if they interact and update opinions accordingly. You should use the following interaction function for interaction probabilities

$$p_{ij}(x) = e^{-k(x_j - x_i)^2} \tag{7}$$

for some chosen $k > 0$.

b) Using this same interaction function write a solver for the corresponding ODE system. You can either use an ODE solver from a package (e.g. scipy) or write your own numerical scheme.

c) Compare these approaches. Run simulations of the ABM with different values of h and calculate the error between these realisations and the ODE model. Plot this error as you reduce h towards zero. Can you deduce anything from this about the convergence rate?

> **Note:** Small values of h will make the ABM very slow; I would look at a small population $N \leq 10$.

Quesiton 3: Inference setup

Imaging you had data generated by the process with updates of the form (1) (i.e. you have the complete timeseries for all x_i).

a) How could you determine μ^h and how many interactions would you need to observe to do so?

b) Assume interaction probabilities have the form $p_{ij}(x) = \phi(x_j - x_i)$. How could you go about inferring the function ϕ? (There is no single answer to this question). What problems might you encounter?

c) How would you adapt your method to account for additional sources of noise, for adaptation noise as in (2)? Could your method also estimate the parameters of this noise?

2 Coupled ODE models

Question 1: Methods for proving convergence

Consider the following ODE system for a finite population of N individuals,

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{1}{N} \sum_{j=1}^{N} \phi(x_j - x_i)\,(x_j - x_i)\,, \\ x_i(0) &= x_{i,0}\,. \end{aligned}$$

Assume that $\phi : \mathbb{R} \to [0, 1]$ is a globally Lipschitz function satisfying $\phi(r) = \phi(-r)$ and $\phi(r) \geq c$ for all $r \in \mathbb{R}$ and some constant $c > 0$. Let $[-M, M]$ be some interval containing all $x_{i,0}$.

a) The first method to prove consensus considers

$$x_m(t) = \min_{i \in \{1,\ldots,N\}} x_i(t)\,, \quad x_M(t) = \max_{i \in \{1,\ldots,N\}} x_i(t)\,.$$

Write down ODEs for each of these values and use the assumptions on ϕ to show that the diameter

$$D(t) = x_M(t) - x_m(t)$$

is exponentially decreasing, and thus the population approaches consensus, at some rate you should determine.

b) The second method follows an energy argument. Define

$$\mathcal{E}(t) = \sum_{i,j} \varphi\big(x_j(t) - x_i(t)\big), \quad \varphi(r) := \int_0^r s\phi(s)\,ds\,.$$

To save some time in the following computations you may find the following helpful:

If $A_{ij} = A_{ji}$ then $\sum_{i,j} A_{ij}(y_j - y_i) = 0$.

If $A_{ij} = -A_{ji}$ then $\sum_{i,j} A_{ij}(y_j - y_i) = 2\sum_{i,j} A_{ij} y_j$.

Show that

$$\frac{d\mathcal{E}}{dt} = -2N \sum_{i=1}^{N} \left(\frac{dx_i}{dt}\right)^2 ,$$

and that for some constant $C > 0$

$$\left(\frac{1}{2}\sum_{i,j} \phi\big(x_j - x_i\big)\,(x_j - x_i)^2\right)^2 \leq C\left(\sum_{j=1}^{N}\left(\frac{dx_j}{dt}\right)^2\right).$$

Combine these inequalities and the boundedness of $\mathcal{E}$ to deduce for all pairs of individuals i, j

$$\phi\big(x_j - x_i\big)\,(x_i - x_j)^2 \to 0\,,$$

and so the population must approach consensus.

Note: It is also possible to show convergence in this case using a spectral analysis. If you're interested this is covered in Section 2.2 of [2] (which also includes versions of the other methods in this question).

Question 2: Exploring parameters

This one is fairly open: code up your favourite coupled ODE or SDE model from the examples and examine the effect of various parameters. Of particular interest are the parameters of the interaction functions in each model. You might consider

- Whether the interactions stop after some distance and if both short-range and long-range interactions are permitted.
- The effect of including short-range repulsion.
- If interactions decay with distance, what effect does the decay rate have?

3 Mean-field PDE models

In this question we'll examine the mean-field PDE for a different model (described in [3]) which extends the opinion dynamics model introduced above to include age-structure. We consider the following ODEs:

$$\frac{dx_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} K(a_i, a_j)\,\phi(x_j - x_i)\,(x_j - x_i)\,, \quad \frac{da_i}{dt} = \tau\,,$$

where $x_i \in [-1, 1]$ and $a_i \in [0, A]$ are the opinions and ages respectively of individual i. The constant $\tau > 0$ is a fixed ageing rate and $K(\cdot, \cdot)$ is some kernel describing interactions between individuals of different ages.

Question 1: Derivation

We now define the empirical distribution over both opinions and ages by

$$\rho^N(t, a, x) = \frac{1}{N}\sum_{j=1}^{N} \delta_{x_i(t)}(x)\,\delta_{a_i(t)}(a)\,,$$

where $\delta_z(x)$ is a Dirac-delta distribution centred at z.

a) From this empirical density integrate against a continuous, compactly supported test function $\zeta(a, x)$ and differentiate with respect to time to formally derive the PDE satisfied by ρ^N.

> **Note:** I can go through this on the blackboard if people would find this useful, just let me know.

b) To interpret ρ as a probability density we include no-flux boundary conditions at the opinion boundaries $x = \pm 1$. We must also derive an appropriate boundary condition on a. Assume the following: when individual i reaches $a_i = A$ they're removed from the population and replaced with a new individual (keeping index i) with age $a_i = 0$ and an opinion chosen at random from a given distribution μ. For this setup, what are the appropriate boundary conditions for the mean-field PDE?

Question 2: Model behaviour

To skip Question 1 (or check your answer before continuing) the PDE and boundary conditions can be found on the last page. In this question we derive ODEs/PDEs describing features of interest.

a) Define the distribution

$$\pi(t, a) = \int_{-1}^{1} \rho(t, a, x)\, dx\,,$$

which describes the total density of individuals with age $a \in [0, A]$. How do you expect this distribution to evolve? Can you write a PDE you expect it to satisfy? Calculate $\partial_t \pi$ to see if you were right!

> **Note:** Don't worry too much here about technical details such as swapping integrals and derivatives, you can assume that everything is sufficiently nice. Alternatively you can worry about the technical details if you prefer...

b) We make the additional assumption that $M(a, b) = M(b, a)$ and consider next the mean opinion

$$m(t) = \int_0^A \int_{-1}^{1} x\, \rho(t, a, x)\, dx\, da\,.$$

Again, how do you expect this mean to evolve, and can you write an ODE that captures this? (I think this one is a bit trickier, don't forget the boundary conditions!) As above, calculate the time derivative to see if you were right.

> You may use the fact that when $M(a, b) = M(b, a)$,
>
> $$\int_0^A \int_{-1}^{1} F[\rho](t, a, x)\, dx\, da = 0\,.$$
>
> (If you want to very this write the above as a quadruple integral and compare the two changes of variables $(x, a) \to (-x, a)$, $(y, b) \to (-y, b)$ and $(x, a) \to (-y, b)$, $(y, b) \to (-x, a)$).

PDE for opinion dynamics with age structure

Section 3 Question 1 gives the following PDE

$$\partial_t \rho(t,a,x) + \tau \partial_a \rho(t,a,x) + \partial_x F[\rho](t,a,x) = 0 \tag{10a}$$

$$F[\rho](t,a,-1) = F[\rho](t,a,1) = 0 \tag{10b}$$

$$\rho(t,0,x) = \mu(x) \int_{-1}^{1} \rho(t,A,y)\,dy \tag{10c}$$

$$\rho(0,a,x) = \rho_0(a,x)\,, \tag{10d}$$

where the flux is given by

$$F[\rho](t,a,x) = \rho(t,a,x)\left(\int_{-1}^{1} \int_0^A K(a,b)\,\varphi(y-x)\,\rho(t,b,y)\,db\,dy \right).$$

The equation (10a) is derived from the empirical density. The condition (10b) is a no-flux boundary condition at the opinion boundaries, while the condition (10c) is the age boundary condition accounting for the ageing process described in part b). Finally (10d) is the initial condition.

If we were to consider the model with additive noise we would instead have the flux

$$F[\rho](t,a,x) = \rho(t,a,x)\left(\int_{-1}^{1} \int_0^A K(a,b)\,\varphi(y-x)\,\rho(t,b,y)\,db\,dy \right) - \frac{\sigma^2}{2}\partial_x \rho(t,a,x)\,.$$

References

[1] Richard Durrett. *Stochastic calculus: a practical introduction.* CRC press, 2018.

[2] Sebastien Motsch and Eitan Tadmor. Heterophilious dynamics enhances consensus. *SIAM review*, 56(4):577–621, 2014.

[3] Andrew Nugent, Susana N Gomes, and Marie-Therese Wolfram. Opinion dynamics with continuous age structure. *arXiv preprint arXiv:2503.04319*, 2025.