Fokker-Planck representation of stochastic neural fields: derivation, analysis and application to grid cells

Tutorial session

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If not stated otherwise, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and ρ is a smooth solution of

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial s}\left[\left(\Phi\left(\int_{\mathbb{T}^d} W(x-y)\int_0^{+\infty} s\rho(y,s,t)\mathrm{d}s\mathrm{d}y + B\right) - s\right)\rho(x,s,t)\right] + \sigma\frac{\partial^2 \rho}{\partial s^2},$$

with boundary condition

$$\left(\Phi\left(\int_{\mathbb{T}^d} W(x-y)\int_0^{+\infty} s\rho(y,s,t)dsdy + B(t)\right)\rho(x,s,t) - \sigma\frac{\partial \rho}{\partial s}(x,s,t)\right)\Big|_{s=0} = 0.$$

and

$$\forall t \in \mathbb{R}_+,\ \forall x \in \mathbb{T}^d, \quad \int_0^{+\infty}\rho(x,s,t)ds = 1.$$

The function Φ is assumed to be smooth and globally Lipschitz, W is smooth on $\mathbb{T}^d$, and B and σ are positive constants.

Exercise 1. *Let*

$$\mathrm{d}s = s\mathrm{d}t + \sqrt{2}\mathrm{d}W,$$

where W is a standard Brownian. Use Ito's lemma to argue that the law of the SDE solves the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial s}[-s\rho] - \frac{\partial^2 \rho}{\partial s^2}$$

in the weak sense.

Exercise 2. *Let ρ be a smooth solution of the Fokker-Planck equation*

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial s}[-s\rho] - \frac{\partial^2 \rho}{\partial s^2} = 0, \qquad s \in (0,+\infty), \qquad \frac{\partial \rho}{\partial s}(0,t) = 0, \qquad \rho(s,0) = \rho^0(s). \tag{0.1}$$

- *Consider*

$$s = \frac{z}{\sqrt{2\tau+1}}, \quad t = \frac{1}{2}\log(2\tau+1), \qquad u(z,\tau) = \frac{1}{\sqrt{2\tau+1}}\rho\left(\frac{z}{\sqrt{2\tau+1}}, \frac{1}{2}\log(2\tau+1)\right).$$

  Check that u satisfies the heat equation

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial z^2} = 0, \qquad z \in (0,+\infty), \qquad \frac{\partial u}{\partial z}(0,\tau) = 0.$$

- *Using the heat kernel*

$$G(z,\tau,\xi,\eta) = \frac{1}{\sqrt{4\pi(\tau-\eta)}}e^{-\frac{(z-\xi)^2}{4(\tau-\eta)}}, \tag{0.2}$$

  deduce an explicit form for the solution to (0.1).

Exercise 3 (Weak formulation)**.**

- *Consider a solution ρ with fast-decay at $s = +\infty$; check that for any test function $h \in C^2(\mathbb{R}_+)$ with slow enough growth at $s = +\infty$ (including its derivatives),*

$$\forall x \in \mathbb{T}^d, \quad \frac{\partial}{\partial t}\int_0^{+\infty} h(s)\rho(x,s,t)ds = \int_0^{+\infty}\left[(\Phi_{\bar\rho}(x,t)-s)\frac{dh}{ds}(s)+\sigma\frac{d^2h}{ds^2}(s)\right]\rho(x,s,t)ds + \sigma\frac{dh}{ds}(0)\rho(x,0,t),$$

where

$$\Phi_{\bar\rho}(x,t) = \Phi(W * \bar\rho(x,t) + B).$$

- *Using a well-chosen h, prove that if Φ is non-positive, then*

$$\forall t \in \mathbb{R}_+,\ \forall x \in \mathbb{T}^d, \quad \bar\rho(x,t) \leqslant 1+\sigma-\sigma e^{-2t}+e^{-2t}\sup_{x\in\mathbb{T}^d}\int_0^{+\infty} s^2\rho^0(x,s)ds.$$

Exercise 4 (Universal bound for the boundary value)**.** *The existence theory provide us with the following:*

$$v(x,\tau) = 2\int_0^{+\infty} G(\gamma(x,\tau),\tau,\xi,0)u^0(x,\xi)\mathrm{d}\xi + 2\int_0^{\tau}\frac{\partial G}{\partial\xi}(\gamma(x,\tau),\tau,\gamma(x,\eta),\eta)v(x,\eta)\mathrm{d}\eta,$$

with G defined in (0.2)*,*

$$\gamma(x,\tau) = -\int_0^{\tau}\Phi\big(\alpha(\eta)W * [\bar u(x,\eta)-\gamma(x,\eta)]+\beta\big)\alpha(\eta)d\eta = -\int_0^{\tau}\Psi(x,\eta)d\eta,$$

and

$$\rho(x,0,t) = \frac{1}{\sqrt{\sigma}}e^t v\left(x, \frac{1}{2}(e^{2t}-1)\right).$$

- *Prove that if Φ is non-negative, then*

$$\forall t \in \mathbb{R}_+,\ \forall x \in \mathbb{T}^d, \quad \rho(x,0,t) \leqslant \sqrt{\frac{2}{\pi\sigma}}\frac{1}{\sqrt{1-e^{-2t}}}; \tag{0.3}$$

  Start with $v(x,\tau)$ in modified variables and then switch back to original variables (x,t).

- *Assume Φ is an increasing function with $\Phi(x) = 0$ for $x \leqslant 0$, the external input is constant and positive ($B(t) = B > 0$), and the connectivity kernel is average inhibitory ($\int_{\mathbb{T}^d} W(x)\mathrm{d}x = W_0 < 0$). Prove that if*

$$\sigma > \frac{\pi B^2}{2|W_0|^2},$$

  then

$$\rho_\infty(s) = \sqrt{\frac{2}{\pi\sigma}}\,e^{-\frac{s^2}{2\sigma}}.$$

  defines a stationary state.

- *Deduce that the bound* (0.3) *is optimal in some sense.*

Exercise 5. *Another possible statistical description of a neural network is to consider the probability density $\rho(v,t)$ of finding at time t a neuron with electric potential $v \in (-\infty, V_F]$. When a neuron reaches the firing potential V_F, it is reset to V_R. In the case of balanced excitation and inhibition, the (rescaled) model reads*

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial v}[-v\rho]-\sigma\frac{\partial^2\rho}{\partial v^2} = 0, \qquad v \in (-\infty, V_R)\cup(V_R,V_F),$$

with boundary conditions

$$\rho(V_F,t) = 0, \qquad \rho(V_F^-,t) = \rho(V_R^+,t), \qquad \frac{\partial\rho}{\partial v}(V_R^+,t)-\frac{\partial\rho}{\partial v}(V_R^-,t) = \frac{N(t)}{\sigma}.$$

and where the flux of firing neurons at time t is

$$N(t) = -\sigma\frac{\partial\rho}{\partial v}(V_F,t) > 0,$$

and ρ must remain a probability density

$$\int_{-\infty}^{V_F}\rho(v,t)\mathrm{d}v = 1.$$

- *Given a stationary firing rate N_∞, find the form of the corresponding stationary state ρ_∞. Then, prove that there exists a unique possible value for the stationary firing rate $N_\infty > 0$.*
- *We admit that there exists a unique stationary state (ρ_∞, N_∞) and that for any convex and C^2 function $G : \mathbb{R} \to \mathbb{R}_+$, the following entropy dissipation holds*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{V_F} G\left(\frac{\rho}{\rho_\infty}\right) \rho_\infty \mathrm{d}v \quad &= \quad -\sigma \int_{-\infty}^{V_F} \left[\frac{\partial}{\partial v}\left(\frac{\rho}{\rho_\infty}\right)\right]^2 G''\left(\frac{\rho}{\rho_\infty}\right) \rho_\infty \mathrm{d}v \\ &\quad -N_\infty \Big[G\left(\frac{N}{N_\infty}\right) - G\left(\frac{\rho}{\rho_\infty}\right) - \Big(\frac{N}{N_\infty} - \frac{\rho}{\rho_\infty}\Big) G'\left(\frac{\rho}{\rho_\infty}\right)\Big]\Big|_{v=V_R}. \end{aligned}$$

We also admit that there exists $\nu > 0$ such that for all q smooth enough on $(-\infty, V_R) \cup (V_R, V_F]$ and with sufficient decay, satisfying also $\int_{-\infty}^{V_F} q(v)\mathrm{d}v = 0$ (q is extended by continuity at V_R), then

$$\nu \int_{-\infty}^{V_F} \left(\frac{q}{\rho_\infty}\right)^2 \rho_\infty \mathrm{d}v \leqslant \int_{-\infty}^{V_F} \left[\frac{\partial}{\partial v}\left(\frac{q}{\rho_\infty}\right)\right]^2 \rho_\infty \mathrm{d}v.$$

Prove, choosing a suitable G and assuming all smoothness and decay required, that the relative entropy

$$\mathcal{E}(t) = \int_{-\infty}^{V_F} G\left(\frac{\rho}{\rho_\infty}\right) \rho_\infty \mathrm{d}v$$

decays to 0 exponentially fast.