Hydrodynamic Limits of the Boltzmann Equation: A Rigorous Derivation of the Navier-Stokes system

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First Part

Introduction to Boltzmann Equation

History & Motivation : Hilbert's 6th problem

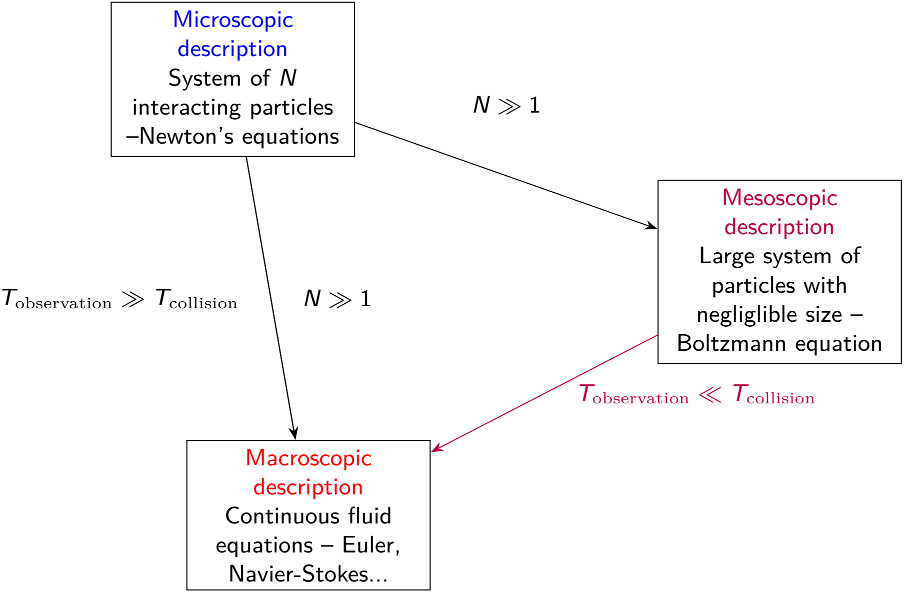
Different levels of description of a gas:

- **Microscopic description**: tracking each gas particle, whose dynamics are described by Newton's laws.
- **Macroscopic description**: considering the gas as a fluid and focusing on the evolution of macroscopic observables (temperature, velocity, etc.) which leads to NAVIER-STOKES/EULER TYPE OF EQUATIONS.

Mesoscopic description

Intermediate level of description - of statistical nature - in which we look at the typical behaviour of a particle: STATISTICAL description of the gas.

Pioneers of kinetic theory of gases: Daniel Bernoulli (1738, Newton's laws); Rudolf Clausius (1865, entropy, mean free path); J. C. Maxwell (intermolecular forces, 1867), etc.


Microscopic description
System of N interacting particles –Newton's equations
$N \gg 1$
Mesoscopic description
Large system of particles with negligible size – Boltzmann equation
$T_{\text{observation}} \gg T_{\text{collision}}$
$N \gg 1$
$T_{\text{observation}} \ll T_{\text{collision}}$
Macroscopic description
Continuous fluid equations – Euler, Navier-Stokes...

Microscopic Description

A gas is a cloud of N particles described by their positions and velocities:

$$(\mathbf{x}_i(t); v_i(t))_{i=1\dots N} \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$$

governed by classical mechanics laws of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_i(t) = v_i(t), \quad m\frac{\mathrm{d}}{\mathrm{d}t}v_i(t) = \mathbf{F}_i(t)$$

where $\mathbf{F}_i$ describes all forces acting on particle i (external forces – gravity, electric fields – plus interaction forces with other particles).

Orders of magnitude

- Monoatomic gas at room temperature and atmospheric pressure: approximately $N = 10^{20}$ particles with radius $R \simeq 10^{-8}\mathrm{cm}$ in a volume of $1\mathrm{cm}^3$. In practice, solving Newton's equations numerically is impossible.
- Excluded volume (total volume occupied by the gas if particles packed):

$$\mathrm{vol} = \frac{4\pi}{3}NR^3 \simeq 5.10^{-4}\mathrm{cm}^3 \ll 1\mathrm{cm}^3.$$

Excluded volume is negligeable (perfect rarefied gas).

Need for an intermediate level of description:
A coarser description than Newton's equations, containing all macroscopic information of the gas.

Mesoscopic scale

Considering a small volume around a point x in space. The number of particles is large enough to estimate the average behavior of the gas but small enough (at our scale) to treat the gas density as exactly at x.

Kinetic Description

L. Boltzmann's idea:

Describe a gas by a distribution function

$$F(t, x, v)$$

which represents the density of gas particles at position x, with velocity v, at time t ($x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, $t > 0$).

- The quantity $F(t, x, v)\mathrm{d}x\mathrm{d}v$ represents the number of particles in a volume element centered at x with radius $\mathrm{d}x$, whose velocities lie within a volume element centered at v with radius $\mathrm{d}v$, at time $t > 0$.
- The macroscopic information of the gas is "contained" in $F(t, x, v)$: the local temperature $\theta(t, x)$, the density $\varrho(t, x)$ and velocity $u(t, x)$ are average quantities derived from $F(t, x, v)$.

The Boltzmann Equation (1872)

Evolution of $F(t,x,v)$: In the absence of interactions with other particles, the motion of a particle located at point x with velocity v is rectilinear (we neglect external forces here): *Free transport*

$$\partial_t F(t,x,v) + v \cdot \nabla_x F(t,x,v) = 0.$$

Problem: How to account for the interactions between particles ("collisions")?

$$\partial_t F(t,x,v) + v \cdot \nabla_x F(t,x,v) = \left(\frac{\partial F}{\partial t}\right)_{\text{coll}} = \text{Gain} - \text{Loss}$$

References:

- Cercignani, 1988, Cercignani, Illner, Pulvirenti, 1994.
- Glassey, 1991.
- Villani, 2002, mathematically oriented survey.

Hypotheses concerning the collision phenomena

- **Rarefied gas:** Collisions involving more than two particles ($k > 2$) can be neglected; this leads to **binary collisions**.
- Collisions are localized and instantaneous: two particles entering into collision at time $t > 0$ at point x depart immediately from x; collisions only modify velocities of particles.
- Collisions are **elastic** (conservation of energy and momentum).
- **Molecular chaos hypothesis (Stosszahlansatz):** The velocities of two colliding gas particles are uncorrelated and independent of their positions.

Collision operator

We describe collisions though the collision operator $\mathcal{Q}(F,F)$; $\mathcal{Q}$ is quadratic (binary collisions) and acts only on velocities (collisions localized and instantaneous).

$$\mathcal{Q}(F,F) = \mathrm{Gain} - \mathrm{Loss} = \mathcal{Q}^{+}(F,F) - \mathcal{Q}^{-}(F,F).$$

$\mathcal{Q}^{+}(F,F)(t,x,v)$ is the density of particles with velocity v produced at time t in position x by a collision between two particles (with different particles, say $v', v'_\star$).

Gain part

$$\mathcal{Q}^{+}(F,F)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{p}([v',v'_*] \to [v,v_*])F_2(v',v'_*)\mathrm{d}v'\mathrm{d}v'_*$$

where $\mathbf{p}([v',v'_*] \to [v,v_*])$ is the probability that two particles with respective velocity v', v'_* undergo a collision resulting in new velocities v, v_* while $F_2(v,v_*)$ is the *joint* distribution of the pair of particles with velocities v', v'_*).

Molecular Chaos $\iff$ $F_2(v',v'_*) = F(v)F(v'_*)$.

Gain operator

$$\mathcal{Q}^{+}(F,F)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbf{p}([v',v'_*] \to [v,v_*])F(v')F(v'_*)\mathrm{d}v'\mathrm{d}v'_*.$$

$\mathcal{Q}^-(F,F)(t,x,v)$ is the density of particles with velocity v which change velocity due to a collision at time t in the position x with another particles (with velocity say v_*)

Loss operator

$$\begin{aligned}\mathcal{Q}^-(F,F)(v) &= \int_{\mathbb{R}^3} \mathbf{p}([v,v_*]\to[v',v'_*])F_2(v,v_*)\mathrm{d}v'\mathrm{d}v'_* \\ &= F(v)\int_{\mathbb{R}^3} \mathbf{p}([v,v_*]\to[v',v'_*])f(v_*)\mathrm{d}v_*\mathrm{d}v'_*.\end{aligned}$$

We need to compute $\mathbf{p}([v,v_*]\to[v',v'_*])$.

Elastic collision

(v', v'_*) pre-collisional velocities; (v, v_*) post-collisional velocities.

- Reversible collision:

$$\mathbf{p}([v, v_*] \to [v', v'_*]) = \mathbf{p}([v', v'_*] \to [v, v_*])$$

  for all choices (v, v_*, v', v'_*).

- Conservation of kinetic energy:

$$m\frac{|v|^2}{2} + m\frac{|v_*|^2}{2} = m\frac{|v'|^2}{2} + m\frac{|v'_*|^2}{2}.$$

- Conservation of momentum

$$mv + mv_* = mv' + mv'_*.$$

No loss of generality $m = 1$.

Elastic Collisions

Parameterization of velocities in the center of mass reference frame:

$$V = \frac{v + v_\star}{2}.$$

Note that:

$$V = V'.$$

Let

$$u = v - v_\star$$

be the post-collisional relative velocity, and u' the pre-collision relative velocity. Then

$$|u|^2 = |\ u'|^2.$$

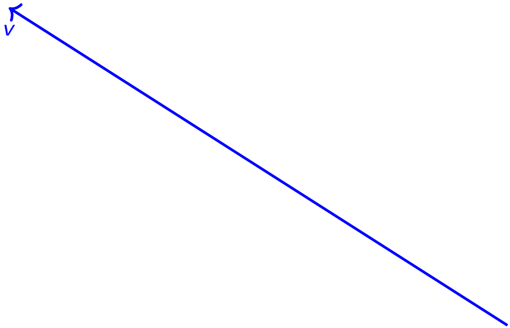
From this, we deduce the following parameterization:

$$v' = \frac{v + v_\star}{2} + \frac{|v - v_\star|}{2}\sigma, \quad v'_\star = \frac{v + v_\star}{2} - \frac{|v - v_\star|}{2}\sigma,$$

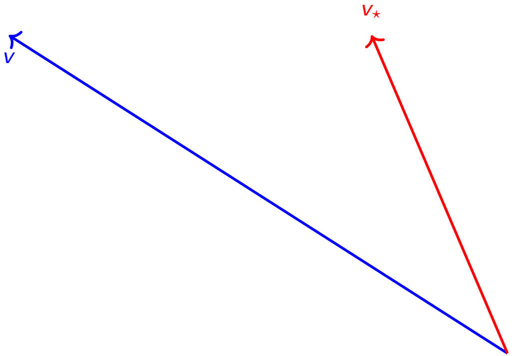
where $\sigma \in \mathbb{S}^2$. In particular:

$$\mathbf{p}([v, v_\star] \to [v', v'_\star]) = \mathbf{p}(v, v_\star, \sigma)$$

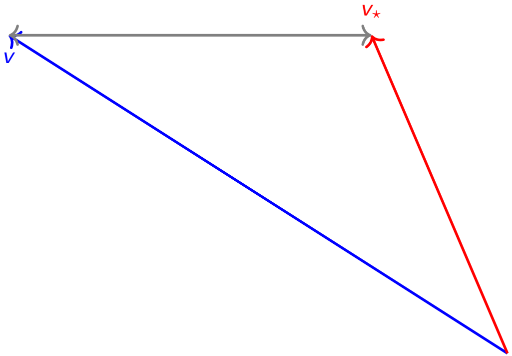
Geometry of Elastic Collision



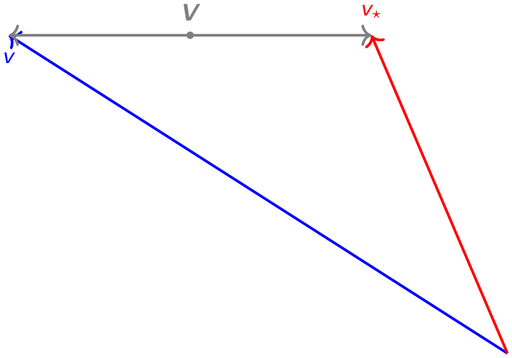
Geometry of Elastic Collision



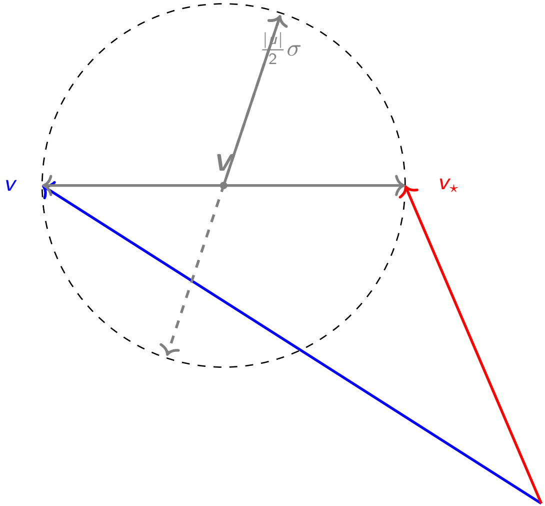
Geometry of Elastic Collision



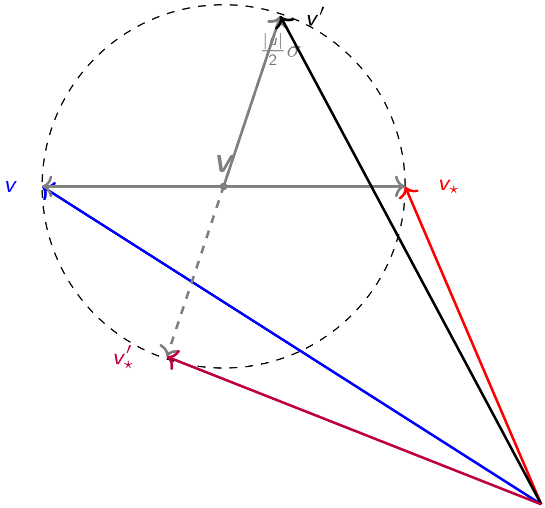
Geometry of Elastic Collision



Geometry of Elastic Collision



Geometry of Elastic Collision



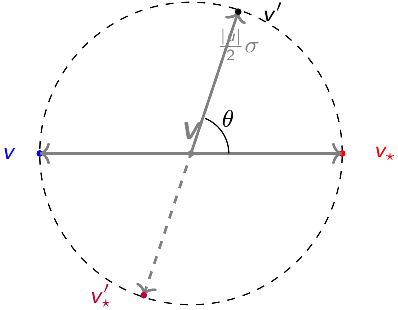
Geometry of Elastic Collision

In particular,

$$\mathbf{p}([v, v_\star] \to [v', v'_\star]) = \mathbf{p}(v, v_\star, \sigma)$$

depends only on the magnitude of the relative velocity $|u| = |v - v_\star|$, and the deviation angle θ such that:

$$\cos\theta = \frac{u \cdot \sigma}{|u|}.$$



$$\mathbf{p}([v, v_\star] \to [v', v'_\star]) = B(|v - v_\star|, \cos\theta).$$

Summary

Boltzmann Equation

$$\partial_t F(t,x,v) + v \cdot \nabla_x F(t,x,v) = \mathcal{Q}(F,F)(t,x,v)$$

plus boundary and initial conditions,

with

$$\mathcal{Q}(F,F)(t,x,v) = \int_{\mathbb{S}^2 \times \mathbb{R}^3} B(|v - v_\star|, \cos\theta) \left(F' F'_\star - F F_\star \right) \mathrm{d}v_\star \mathrm{d}\sigma$$

where $F = F(t,x,v)$, $F' = F(t,x,v')$, $F_\star = F(t,x,v_\star)$, $F'_\star = F(t,x,v'_\star)$

and

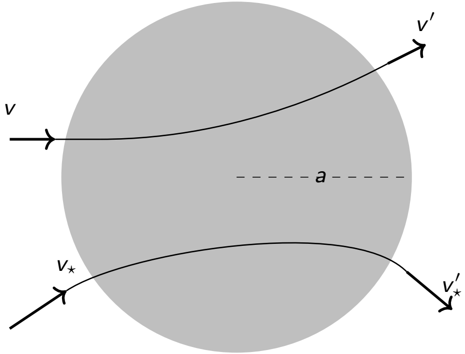
$$v' = \frac{v + v_\star}{2} + \frac{|v - v_\star|}{2}\sigma, \quad v'_\star = \frac{v + v_\star}{2} - \frac{|v - v_\star|}{2}\sigma.$$

Derivation of $B(|v - v_\star|, \cos\theta)$

Suppose particles interact due to a repulsive force derived from a potential U:

$$\mathbf{F} = -\nabla U$$

with $U = U(\varrho)$ depending only on the distance ϱ between particles. The force is more or less long-range, i.e., $U(\varrho) = 0$ for $\varrho > a > 0$.



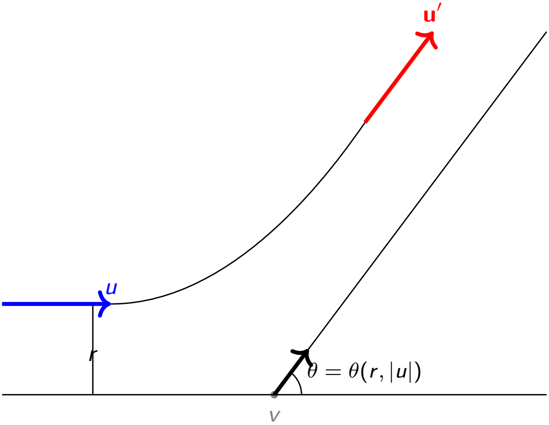
Hard sphere model: $a = 0$ (particles as billiard balls).

Derivation of $B(|v - v_\star|, \cos\theta)$

The derivation of $B(|v - v_\star|, \cos\theta)$ is related to the computation of the differential cross section for particle scattering under the potential U:

$$B(|v - v_\star|, \cos\theta) = |v - v_\star| \frac{r}{\sin\theta} \frac{\mathrm{d}r}{\mathrm{d}\theta}$$

where r is the **impact parameter** and θ the deflection angle.



Two-body problem in the center of mass frame

Explicit calculation in the case of **hard spheres**:

$$B(|v - v_*|, \cos\theta) = c_0|v - v_*|, \qquad c_0 > 0$$

i.e., the collision kernel does not depend on the deviation angle.

More generally, if $U(\varrho) = \frac{1}{\varrho^{s-1}}$ with $s > 2$, then

$$B(|v - v_\star|, \cos\theta) = b(\cos\theta)\ |v - v_\star|^\gamma, \quad \text{where} \quad \gamma = \frac{s-5}{s-1}$$

and $b(\cos\theta)$ is a (non-explicit) function.

Remark: The model of hard spheres corresponds to the choice $s = \infty$ in the interaction potential.

$$U(\varrho) = \frac{1}{\varrho^{s-1}} \quad \Longrightarrow \quad B(|u|, \cos\theta) = |u|^{\gamma}\ b(\cos\theta)$$

The function $b(\cos\theta)$ has a non-integrable singularity at $\theta \sim 0$:

$$\sin\theta\ b(\cos\theta) \sim K\theta^{-1-\nu}, \quad \text{with} \quad \nu = \frac{2}{s-1}.$$

This singularity poses a serious problem for the analysis of the Cauchy problem. It is usually remedied by replacing B with an integrable kernel – this is called the GRAD ANGULAR CUTOFF HYPOTHESIS:

$$\int_0^{\pi} B(|u|, \cos\theta) \sin\theta \mathrm{d}\theta < \infty.$$

- **Hard potentials:** $\gamma = \frac{s-5}{s-1} > 0$;
- **Soft potentials:** $\gamma = \frac{s-5}{s-1} < 0$.

The case $s = 2$ corresponds to Coulomb interaction and Boltzmann equation is meaningless in this case (LANDAU EQUATION),

Model Validation

The validity of the equation proposed by Boltzmann was long disputed (irreversibility, etc.). A rigorous mathematical justification was only provided in 1973 by Oscar E. LANFORD III, who demonstrated the derivation from microscopic to macroscopic.

We consider the microscopic system of $N \gg 1$ identical particles with radius $\sigma > 0$. We solve the Newton equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_i(t) = v_i(t), \quad \frac{\mathrm{d}}{\mathrm{d}t}v_i(t) = \mathbf{F}_i(t)$$

in phase space:

$$\Lambda = \left\{(\mathbf{x}_i, v_i) \in \mathbb{R}^{6N} \;:\; |\mathbf{x}_i - \mathbf{x}_j| > \sigma \quad \text{for } i \neq j\right\}.$$

The first marginal of the system's distribution converges (for small times) to a solution of the Boltzmann equation for **hard spheres** when:

Boltzmann-Grad Limit

$$N \to \infty, \quad \sigma \to 0, \quad \text{and} \quad N\sigma^2 \to \lambda > 0$$

Remark: The gas volume, of order $N\sigma^3$, tends to zero in this limit.

This limiting property characterizes **rarefied gases**, i.e.,

- Infinite number of particles;
- Point particles ($\sigma \to 0$);
- Non-zero surface density and zero volume.
- λ^{-1} measures the sparsity of the gas (proportional to the mean free path).

Mean free path

Average distance between two successive collisions.

$$\text{mean free path} \simeq \frac{1}{\mathcal{N} \times \mathcal{A}}$$

with $\mathcal{N}$ is the number of gas particles per unit volume, $\mathcal{A}$ area of the section of any individual particle.

Previous example: Monoatomic gas at room temperature and atmospheric pressure: approximately $N = 10^{20}$ particles with radius $R \simeq 10^{-8}\mathrm{cm}$ in a volume of $1\mathrm{cm}^3$.

$$\mathcal{N} = 10^{20}\text{particles}/\mathrm{cm}^3, \quad \mathcal{A} = \pi R^2 \simeq 3.10^{-16}\mathrm{cm}^2$$

so

$$\text{mean free path} \simeq \frac{1}{3}10^{-4}\mathrm{cm} \ll 1\mathrm{cm}.$$

Model Validation

- O. Landford, 1973, pioneering rigorous validation for hard-spheres interactions. Validity up to some finite time $T_0 < \lambda$.
- Illner, Pulvirenti, 1989, long-time result but for the near-vacuum case
- Gallagher, Saint Raymond, Texier, 2011; Pulvirenti, Saffirio, Simonella, 2011, more general interactions kernels and explicit convergence rates;
- Pulvirenti, Simonella, 2020, explicit decay rates for cumulants associated with the hard sphere system;
- Bodineau, Gallagher, Saint Raymond, Simonella, 2020-2024, deriving the equation for fluctuations around equilibrium;
- Extension of the validity time up to existence time of global existence of solutions to BE Deng, Hani, Ma, 2024.

Fundamental properties of the collision operator $\mathcal{Q}$

Here, we focus only on the collision operator $\mathcal{Q}$. It is local in x, t, so we ignore the dependence on these variables:

$$\mathcal{Q}(f,f)(v) = \int_{\mathbb{S}^2\times\mathbb{R}^3} B(|u|,\cos\theta)\left(f'f'_\star - ff_\star\right)\mathrm{d}v_\star\mathrm{d}\sigma$$
$$= \int_{\mathbb{R}^3}\mathrm{d}v_\star\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi} B(|u|,\cos\theta)\left(f'f'_\star - ff_\star\right)\sin\theta\,\mathrm{d}\theta$$

where $f = f(v)$, $f' = f(v')$, $f_\star = f(v_\star)$, $f'_\star = f(v'_\star)$, and

$$v' = \frac{v+v_\star}{2} + \frac{|v-v_\star|}{2}\sigma, \quad v'_\star = \frac{v+v_\star}{2} - \frac{|v-v_\star|}{2}\sigma.$$

Change of variables at pre-post collision: $(v, v_\star) \to (v', v'_\star)$ is an involution. Its Jacobian is:

$$\frac{\partial(v, v_\star)}{\partial(v', v'_\star)} = 1.$$

Let $\psi(v)$ be an arbitrary test function. We compute the observable:

$$\int_{\mathbb{R}^3} \mathcal{Q}(f, f)(v)\psi(v)\mathrm{d}v.$$

We have:

$$\int_{\mathbb{R}^3} \mathcal{Q}(f, f)\psi \, dv = -\frac{1}{4} \int_{\mathbb{S}^2 \times \mathbb{R}^3 \times \mathbb{R}^3} B(|u|, \cos\theta) \left(f' f'_\star - f f_\star\right) \times$$
$$\times \left(\psi' + \psi'_\star - \psi - \psi_\star\right) \mathrm{d}v \mathrm{d}v_\star \mathrm{d}\sigma$$

In particular, if $\psi = 1$, or $\psi(v) = v_i$, or $\psi(v) = |v|^2$, then we obtain:

$$\int_{\mathbb{R}^3} \mathcal{Q}(f,f)(v)\,\mathrm{d}v = 0,$$

$$\int_{\mathbb{R}^3} \mathcal{Q}(f,f)(v) v_i\,\mathrm{d}v = 0, \quad \forall i = 1,2,3, \text{(conservation of linear momentum)},$$

$$\int_{\mathbb{R}^3} \mathcal{Q}(f,f)(v)|v|^2\,\mathrm{d}v = 0, \text{(conservation of kinetic energy)}.$$

H-Theorem of Boltzmann

For $f = f(v) > 0$.

$$\int_{\mathbb{R}^3} \mathcal{Q}(f,f) \log f \,\mathrm{d}v = -\frac{1}{4} \int_{\mathbb{S}^2 \times \mathbb{R}^3 \times \mathbb{R}^3} B(|u|, \cos\theta) \big(f' f'_\star - f f_\star\big) \log \left(\frac{f' f'_\star}{f f_\star} \right) \mathrm{d}v_\star \mathrm{d}v \mathrm{d}\sigma.$$

$$\mathscr{D}(f) := \int_{\mathbb{R}^3} \mathcal{Q}(f,f) \log f \,\mathrm{d}v \leqslant 0$$

Furthermore, the following conditions are equivalent:

1. $\mathcal{Q}(f,f)(v) = 0$ for almost every $v \in \mathbb{R}^3$;
2. $\displaystyle\int_{\mathbb{R}^3} \mathcal{Q}(f,f) \log f(v) \,\mathrm{d}v = 0$;
3. f is a **Maxwellian**, i.e.,

$$f(v) = \mathcal{M}_{(\varrho,u,\Theta)}(v) = \frac{\varrho}{(2\pi\Theta)^{3/2}} \exp\left(-\frac{|v-u|^2}{2\Theta} \right),$$

where $\varrho, \Theta > 0$ and $u \in \mathbb{R}^3$.

Exercise (Perthame, 1990 - use Fourier transform)

If $\int_{\mathbb{R}^3} (1+|v|^2) f(v)\,\mathrm{d}v < \infty$ *and*

$$f(v)f(v_\star) = f(v')f(v'_\star)$$

for all v, v_*, σ *then* f *is a Maxwellian.*

Solutions to the Boltzmann equation satisfy

$$\partial_t \int_{\mathbb{R}^3} F \log F \mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^3} vF \log F \mathrm{d}v = -\mathscr{D}(F) \leqslant 0.$$

Irreversibility and arrow of time.

Relative entropy

- Given a Maxwellian state $\mathcal{M}$, we can measure the local density fluctuation around the equilibrium state in terms of relative entropy

$$\mathcal{H}(F|\mathcal{M}) = \int_{\mathbb{R}^3} \left[F \log \frac{F}{\mathcal{M}} - F + \mathcal{M} \right] \mathrm{d}v$$

  which depends on (t, x).

- In spatially homogeneous case $F_{\mathrm{in}}(x, v) = f_{\mathrm{in}}(v)$, the relative entropy is decreasing

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}(F(t)|\mathcal{M}) \leqslant 0$$

  where $\mathcal{M}$ is the Maxwellian state associated to f_{in}.

Exercise (Csiszàr-Kullback inequality)

For $f = f(v)$ depending only on v (for simplicity) and $\mathcal{M}$ the Maxwellian with same mass as f

$$\int_{\mathbb{R}^3} f(v)\mathrm{d}v = \int_{\mathbb{R}^3} \mathcal{M}(v)\mathrm{d}v = 1\,.$$

Prove that

$$\|f - \mathcal{M}\|^2_{L^1(\mathbb{R}^3)} \leqslant 2\mathcal{H}(f|\mathcal{M}).$$

Conservation Laws

Let F be a solution of the Boltzmann equation:

$$\partial_t F(t,x,v) + v \cdot \nabla_x F(t,x,v) = \mathcal{Q}(F,F)(t,x,v),$$

for $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, $t \geqslant 0$.
From the identities

$$\int_{\mathbb{R}^3} \mathcal{Q}(F,F) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0,$$

we deduce **conservation laws:**

$$\partial_t \int_{\mathbb{R}^3} F(t,x,v)\mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^3} vF(t,x,v)\mathrm{d}v = 0,$$

$$\partial_t \int_{\mathbb{R}^3} vF(t,x,v)\mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^3} v \otimes vF(t,x,v)\mathrm{d}v = 0,$$

$$\partial_t \int_{\mathbb{R}^3} |v|^2 F(t,x,v)\mathrm{d}v + \nabla_x \cdot \int_{\mathbb{R}^3} v|v|^2 F(t,x,v)\mathrm{d}v = 0.$$

We will come back to these conservation laws for the hydrodynamic limit.

We have:

$$\int_{\mathbb{R}^3} v \otimes v F(t,x,v)\,\mathrm{d}v = (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \mathbb{P}_F$$

where

$$\mathbb{P}_F(t,x) = \int_{\mathbb{R}^3} (v - \boldsymbol{u}(t,x)) \otimes (v - \boldsymbol{u}(t,x))\, F(t,x,v)\,\mathrm{d}v \in \mathbb{R}^{3\times 3}.$$

We obtain:

$$\partial_t(\varrho \boldsymbol{u}) + \nabla_x \cdot (\varrho \boldsymbol{u} \otimes \boldsymbol{u} + \mathbb{P}_F) = 0$$

$\mathbb{P}_F$ is interpreted as the stress tensor (responsible for the variation of the mass flux).

The energy density is given by:

$$\mathsf{E}(t,x) = \frac{1}{2}\int_{\mathbb{R}^3} |v|^2\, F(t,x,v)\,\mathrm{d}v.$$

Noting that $|v|^2 = |(v-\boldsymbol{u}) + \boldsymbol{u}|^2$, we get:

$$\mathsf{E}(t,x) = \frac{1}{2}\varrho|\boldsymbol{u}|^2 + \frac{1}{2}\int_{\mathbb{R}^3} |v - \boldsymbol{u}(t,x)|^2\, F(t,x,v)\,\mathrm{d}v.$$

The energy density is given by

$$\mathsf{E}(t,x) = \frac{1}{2}\int_{\mathbb{R}^3} |v|^2\, F(t,x,v)\,\mathrm{d}v.$$

Noting that $|v|^2 = |(v - \boldsymbol{u}) + \boldsymbol{u}|^2$, we get

$$\mathsf{E}(t,x) = \underbrace{\frac{1}{2}\varrho|\boldsymbol{u}|^2}_{\text{average kinetic energy}} + \underbrace{\frac{1}{2}\int_{\mathbb{R}^3} |v - \boldsymbol{u}(t,x)|^2\, F(t,x,v)\,\mathrm{d}v}_{\text{internal energy}=:\varrho\boldsymbol{e}}.$$

with $\mathrm{trace}(\mathbb{P}_F) = 2\varrho\boldsymbol{e}$.

The energy flux is described by:

$$\mathrm{q}(t,x) = \int_{\mathbb{R}^3} (v - \boldsymbol{u}(t,x))|v - \boldsymbol{u}(t,x)|^2 F(t,x,v)\,\mathrm{d}v,$$

which represents the heat flux.
The energy balance equation reads:

$$\partial_t \mathrm{E}(t,x) + \nabla_x \cdot \left(\varrho \boldsymbol{u}\left(\frac{1}{2}|\boldsymbol{u}|^2 + \boldsymbol{e}\right)\right) + \frac{1}{2}\nabla_x \cdot (\mathbb{P}_F \boldsymbol{u}) = -\operatorname{div}_x \mathrm{q}(t,x),$$

where the heat flux is:

$$\mathrm{q}(t,x) = \int_{\mathbb{R}^3} (v - \boldsymbol{u}(t,x))|v - \boldsymbol{u}(t,x)|^2 F(t,x,v)\,\mathrm{d}v.$$

Conservation Laws

Euler equations for compressible fluids

$$\partial_t \varrho(t,x) + \operatorname{div}_x(\varrho(t,x)\boldsymbol{u}(t,x)) = 0$$

$$\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u} + \mathbb{P}_F) = 0$$

$$\partial_t \mathrm{E}(t,x) + \operatorname{div}_x(\boldsymbol{u}\mathrm{E}) + \frac{1}{2}\operatorname{div}_x(\mathbb{P}_F \boldsymbol{u}) = -\operatorname{div}_x \mathrm{q}(t,x).$$

where

$$\mathrm{q}(t,x) = \int_{\mathbb{R}^3} (v - \boldsymbol{u}(t,x))|v - \boldsymbol{u}(t,x)|^2 F(t,x,v)\,\mathrm{d}v,$$

$$\mathbb{P}_F(t,x) = \int_{\mathbb{R}^3} (v - \boldsymbol{u}(t,x)) \otimes (v - \boldsymbol{u}(t,x))\, F(t,x,v)\,\mathrm{d}v.$$

We observe that the system is not closed, because q is a third-order moment of f.

Remark

If $\mathbb{P}_F = p\,\mathbb{I}$ and $\mathrm{q} = 0$, then we recover the compressible Euler system for the pressure of ideal gases.

Knudsen and the others

Boltzmann-Grad Limit

A microscopic system of $N \gg 1$ hard spheres of radius $\sigma > 0$.

$$N \to \infty, \quad \sigma \to 0, \quad \text{and} \quad N\sigma^2 \to \lambda,$$

where $\lambda > 0$ measures the sparsity of the gas, and $1/\lambda$ is proportional to the mean free path ℓ, with

$$\ell = \mathcal{O}\left(\frac{\mathcal{V}}{\lambda}\right),$$

where $\mathcal{V}$ is the characteristic volume.

The mean free path is **the average distance a particle travels between two collisions**.

Knudsen number

The Knudsen number is defined as the ratio between the mean free path and a macroscopic length scale:

$$\mathrm{Kn} = \frac{\text{mean free path}}{\text{characteristic length}}.$$

- L characteristic macroscopic length;
- T characteristic time scale;
- Θ the reference temperature.

Then

$$\mathrm{Kn} = \frac{\ell}{L} \simeq \frac{\mathcal{V}}{\lambda L}$$

Thermal speed

$$c = \sqrt{\frac{5}{3} k\Theta},$$

where k is the Boltzmann constant. This c corresponds to the thermal speed (speed of sound in a monatomic gas at temperature Θ).

Dimensionless Boltzmann Equation

Define the dimensionless variables:

$$\hat{t} = \frac{t}{T}, \quad \hat{x} = \frac{x}{L}, \quad \hat{v} = \frac{v}{c}.$$

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The scaled distribution:

$$\hat{F}(\hat{t}, \hat{x}, \hat{v}) = \frac{L^3 c^3}{N} F(t, x, v),$$

where N is the number of molecules in the volume L^3.

Define the scaled collision kernel:

$$\widehat{B}(|\hat{v}|, \cos\theta) = \frac{1}{\sigma^2 c} B(|v|, \cos\theta),$$

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If F is a solution of the Boltzmann equation, then:

$$\frac{L}{cT} \partial_{\hat{t}} \hat{F}(\hat{t}, \hat{x}, \hat{v}, \hat{t}) + \hat{v} \cdot \nabla_{\hat{x}} \hat{F}(\hat{t}, \hat{x}, \hat{v}) = \frac{N \times \sigma^2}{L^2} \widehat{\mathcal{Q}}(\hat{F}, \hat{F}).$$

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Note that:

$$\frac{N \times \sigma^2}{L^2} = L \times \frac{N \times \sigma^2}{L^3} \simeq L \times \frac{\lambda}{\mathcal{V}} \simeq \frac{L}{\ell} = \frac{1}{\mathrm{Kn}}.$$

Dimensionless Boltzmann Equation

Strouhal number

Define the kinetic Strouhal number:

$$\frac{L}{cT} =: \mathsf{St}.$$

Hats off....

Dimensionless - rescaled Boltzmann equation

$$\mathsf{St}\,\partial_t F + v \cdot \nabla_x F = \frac{1}{\mathrm{Kn}} \mathcal{Q}(F, F).$$

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Several scalings are possible for the Strouhal number: $\mathsf{St} = \tau_\varepsilon$. Depending on the size of τ_ε, solutions of the Boltzmann equation exhibit different hydrodynamic features.