Mean field limits for interacting particle systems, their inference, and applications

Part 1 – From Agent-Based Models to Interacting Particle Systems

Susana N. Gomes
University of Warwick
Susana.Gomes@warwick.ac.uk

UNIVERSITY OF WARWICK

From microscopic dynamics to continuum limits - ACDC Autumn School
23/10/2025

Outline

Motivation

Agent based models and their properties

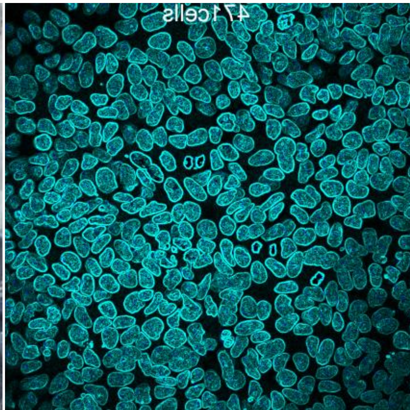
UNIVERSITY OF WARWICK

Brief introduction to collective dynamics

Interacting particle systems are ubiquitous in the real-world, appearing in several application areas:

- Biology and Life Sciences
  - Flocks of birds, schools of fish, herds of sheep, …
  - Cell dynamics
- Social Sciences
  - Crowd/Pedestrian dynamics
  - Opinion dynamics, …
- Physics and Engineering
  - Drones, robots, …
  - Molecular dynamics
  - Movement of galaxies
- Many other examples



How to model these problems

Depending on the application, or what we want to be able to say from our models, we can take different approaches to model collective behaviour. Today, I will go over three types of models and discuss connections between them. In particular, I will go over:

- Agent-based models – one-to-one interactions

How (and when) pairs of individuals interact *matters.*

↓ Consider correct time-scales

- Interacting particle systems – individual interactions within a group

All pairs of individuals interact in a "similar" way, and it doesn't matter which specific pair is interacting

↓ Large population limit

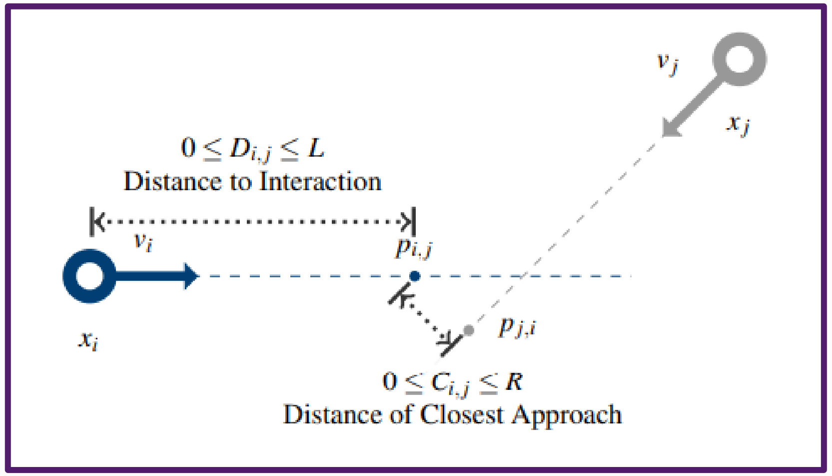
- Mean-field limits – modelling the group as a whole

We care mostly about the whole group behaviour, rather than individual interactions.

Examples of agent-based models

These models are used when we want to describe interactions accurately, usually resulting from a *rational behaviour* point of view. Who each agent interacts with matters.

Example 1: Two pedestrians interacting to avoid collisions.
In the figure, pedestrians at positions x_i, x_j with velocity v_i, v_j see each other and update their path to avoid a collision.



Example 2: Two people interact and change their opinions based on their conversation

Figure: from Bailo, Carrillo, and Degond (2018), “Pedestrian models based on rational behaviour”, in Crowd Dynamics, Vol. 1

Agent-based models

These are based on one-to one interactions, where at each time step, we

1. Select an individual i whose current state x_i will be updated.
2. Select agent j, who they will interact with, and has state x_j.
3. Agents i and j interact with probability p_{ij} and update the state of one (or both) individuals according to some rule to be determined by the model.

Note that:
We can include noise at any of these points: selection of agent to interact with, shown or observed state, updated state, external influences, …

These models are usually ***Markov chains/processes***: the next state only depends on the current state of the system, i.e., the system has no memory. It is possible to include memory, but models become more complicated to analyse.

Agent-based models in opinion dynamics

In the case of opinion dynamics,[1] the state of each agent is their opinion, and interactions depend on how different opinions are. So we:

1. Select an individual i whose current **opinion** $x_i \in [-1,1]$ will be updated.

2. Select who they will interact with, call it j, who has **opinion** x_j.

3. They interact with probability p_{ij} and update the state of one (or both) individuals according to some rule. For example, the opinions change if x_i and x_j are close enough, i.e.

$$d(x_i, x_j) \leq R.$$

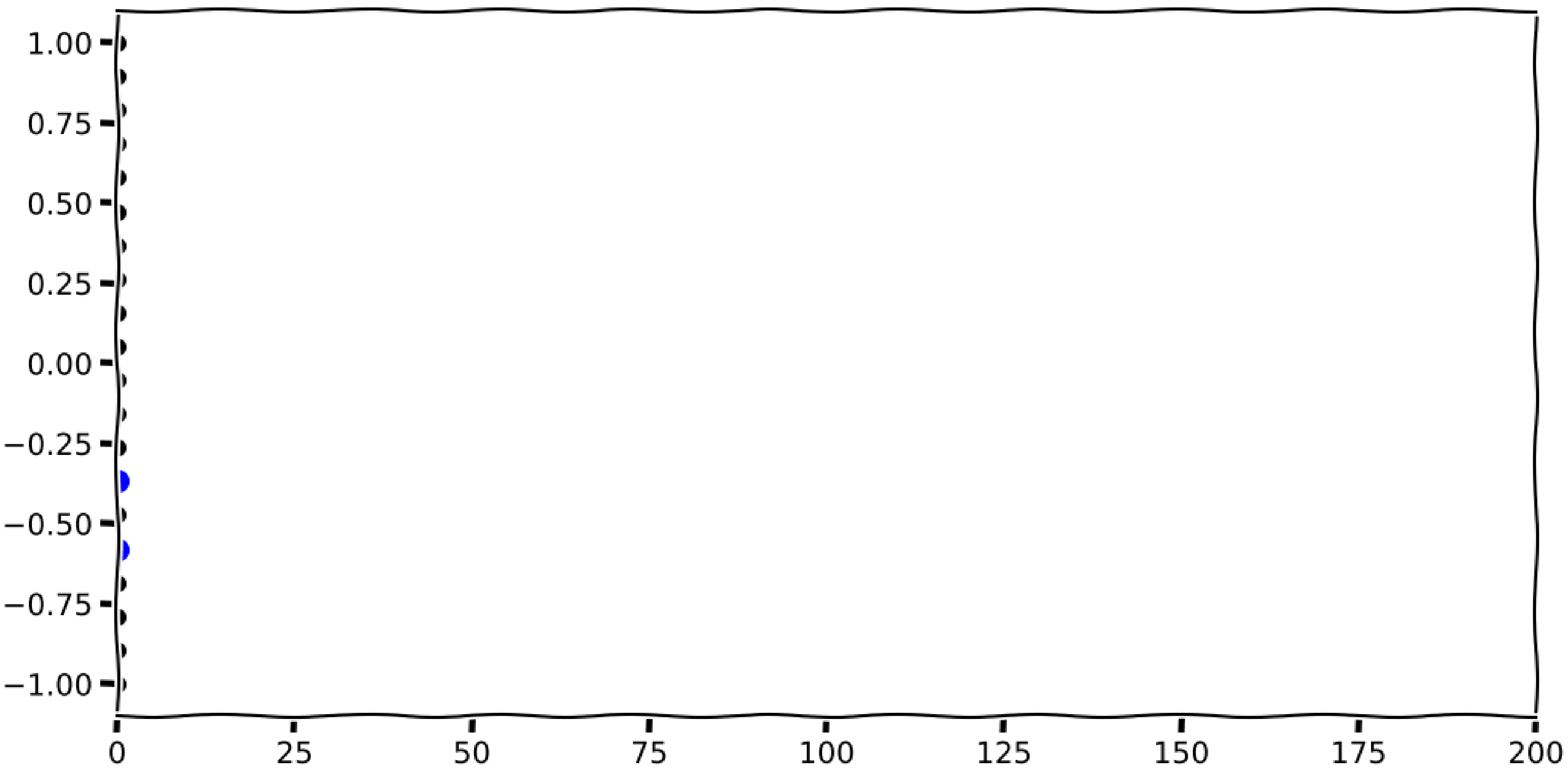
The resulting update is:

$$x_i(t+h) = \begin{cases} x_i(t) + \mu^h \left(x_j(t) - x_i(t)\right) & \text{with probability } p_{ij}(x) \\ x_i(t) & \text{with probability } 1 - p_{ij}(x)\,. \end{cases}$$

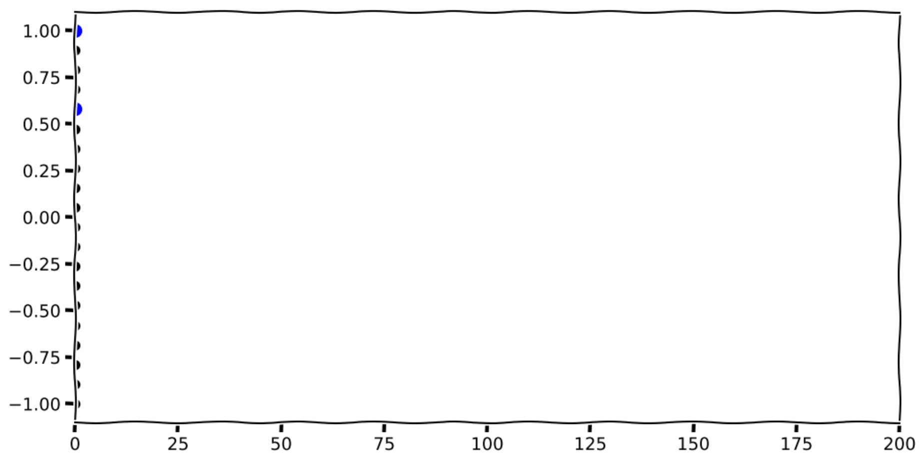
1 - I will look into an opinion dynamics model for which we can write simple rules, but similar results to those I will show are possible to obtain by taking limits of the generators of the associated Markov chains (ask me, or Andrew, if you are interested!)

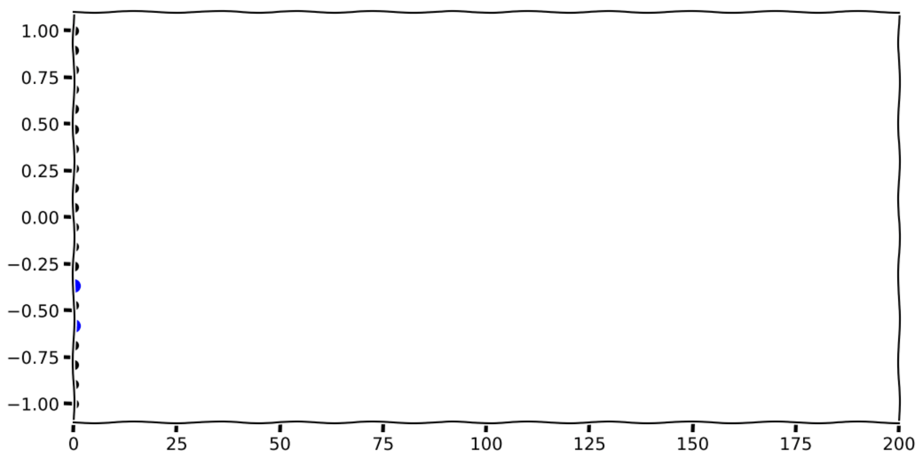
Simulating agent-based models

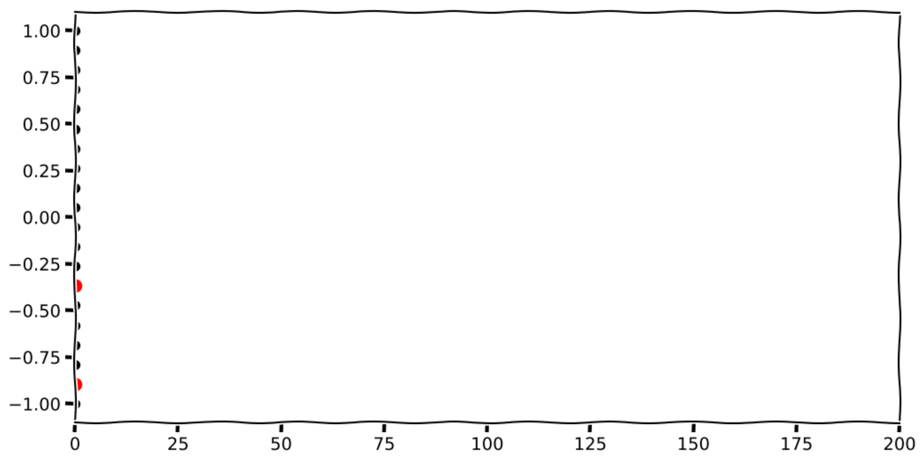
To simulate this process, we simply need to define μ^h and $p_{ij}(x)$.

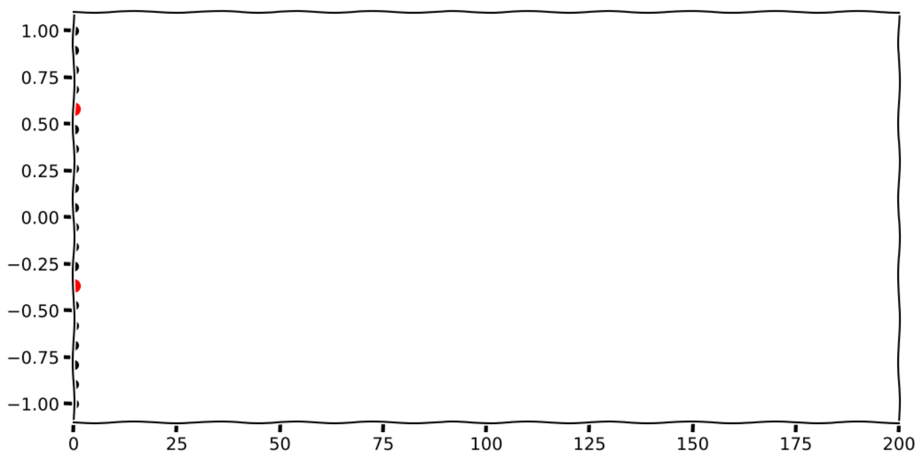


Multiple realisations of the simulations









Discussion

Agent-based models are an excellent tool for modelling one to one interactions. However, they come with many advantages, disadvantages, and challenges

Advantages:

- Can model interactions realistically:
  - we can include as many effects as we want, including noise at various stages
  - each agent can have their own update rule, including how to interact with their environment
- Can be good for targeting very specific scenarios (e.g. evacuating a specific venue, modelling the social rules in a specific situation, etc.)

Disadvantages / Challenges:

- Computational cost: usually only model one interaction at a time
  - need a large number of time steps / updates to reach an equilibrium
  - this is made worse as the number of agents increases
- Due to the random nature of the system, we might need to run a large number of simulations to compute statistics / average behaviour.

The limit of fast interactions

From Agent-Based Models to interacting particle systems

For the opinion dynamics case, see
A. Nugent, SNG, M.-T. Wolfram, Physica A 651, 129886, 2024

UNIVERSITY OF WARWICK

Agent-based models as a Markov process

As mentioned earlier, we can write the agent-based model as a discrete-time Markov process with state space $\mathbb{R}^N$, time step h, and transition function given by

$$\Pi^h(x,y) = \begin{cases} \frac{1}{N^2} p_{ij}(x) & \text{if } y = x + e_i \mu^h (x_j - x_i) \text{ for some } i \neq j, \\ \frac{1}{N} + \frac{1}{N^2} \sum_{i \neq j} \big(1 - p_{ij}(x)\big) & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

From this, we can calculate the mean and second moments of increments for each agent:

$$\underbrace{b_i^h(x) = \frac{1}{h} \mathbb{E}\left[x_i(t+h) - x_i(t) \,\middle|\, x(t) \right]}_{\text{Expected change for agent } i}$$

$$\underbrace{a_{ij}^h(x) = \frac{1}{h} \mathbb{E}\left[\Big(x_i(t+h) - x_i(t)\Big) \Big(x_j(t+h) - x_j(t)\Big) \,\middle|\, x(t) \right]}_{\text{Correlation between changes for agents } i \text{ and } j}$$

A detour into generators of Markov processes...

Using Ito's formula, you can use the moments of a stochastic process to characterise the generator of the Markov process associated with this model:

$$(\mathcal{L}f)(x) = \lim_{h\to 0}\frac{1}{h}\mathbb{E}\left[f\big(X(h)\big) - f(x) \mid X_0 = x\right]$$

$$\approx \lim_{h\to 0}\left(f'(x)\frac{1}{h}\mathbb{E}[X(h) - x \mid X_0 = x] + \frac{f''(x)}{2}\frac{1}{h}\mathbb{E}[(X(h) - x)^2 \mid X_0 = x]\right)$$

$$\approx \lim_{h\to 0}\left(f'(x)\frac{1}{h}\int_{\mathbb{R}} (y - x)\Pi^h(x, dy) + \frac{f''(x)}{2}\frac{1}{h}\int_{\mathbb{R}} (y - x)^2\Pi^h(x, dy)\right)$$

$$\approx \lim_{h\to 0}\left(f'(x)b^h(x) + \frac{f''(x)}{2}a^h(x)\right).$$

The quantity in brackets is the generator of the stochastic process solving the SDE

$$dX(t) = b^h\big(X(t)\big)dt + \sqrt{a^h(X(t))}\, dW(t)$$

A detour into generators of Markov processes...

Using Ito's formula, you can use the moments of a stochastic process to characterise the generator of the Markov process associated with this model:

$$(\mathcal{L}f)(x) = \lim_{h\to 0}\frac{1}{h}\mathbb{E}\left[f\big(X(h)\big) - f(x) \,|\, X_0 = x\right]$$

$$\approx \lim_{h\to 0}\left(f'(x)\frac{1}{h}\mathbb{E}[X(h) - x|X_0 = x] + \frac{f''(x)}{2}\frac{1}{h}\mathbb{E}[(X(h) - x)^2 \,|\, X_0 = x]\right)$$

$$\approx \lim_{h\to 0}\left(f'(x)\frac{1}{h}\int_{\mathbb{R}} (y - x)\Pi^h(x, dy) + \frac{f''(x)}{2}\frac{1}{h}\int_{\mathbb{R}} (y - x)^2\Pi^h(x, dy)\right)$$

$$\approx \lim_{h\to 0}\left(f'(x)b^h(x) + \frac{f''(x)}{2}a^h(x)\right).$$

Taking the limit, we obtain the SDE

$$dX(t) = b\big(X(t)\big)dt + \sqrt{a(X(t))}\, dW(t)$$

The limit of fast, small interactions

Replacing the calculations on the previous slide with our expressions for $b_i^h(x)$ and $a_{ij}^h(x)$, we have

$$b_i^h(x) = \frac{1}{h}\int_{\mathbb{R}^N}(y_i - x_i)\Pi^h(x, dy), \qquad a_{ij}^h(x) = \frac{1}{h}\int_{\mathbb{R}^N}(y_i - x_i)\big(y_j - x_j\big)\Pi^h(x, dy).$$

By setting $\mu^h = Nh$, we obtain:

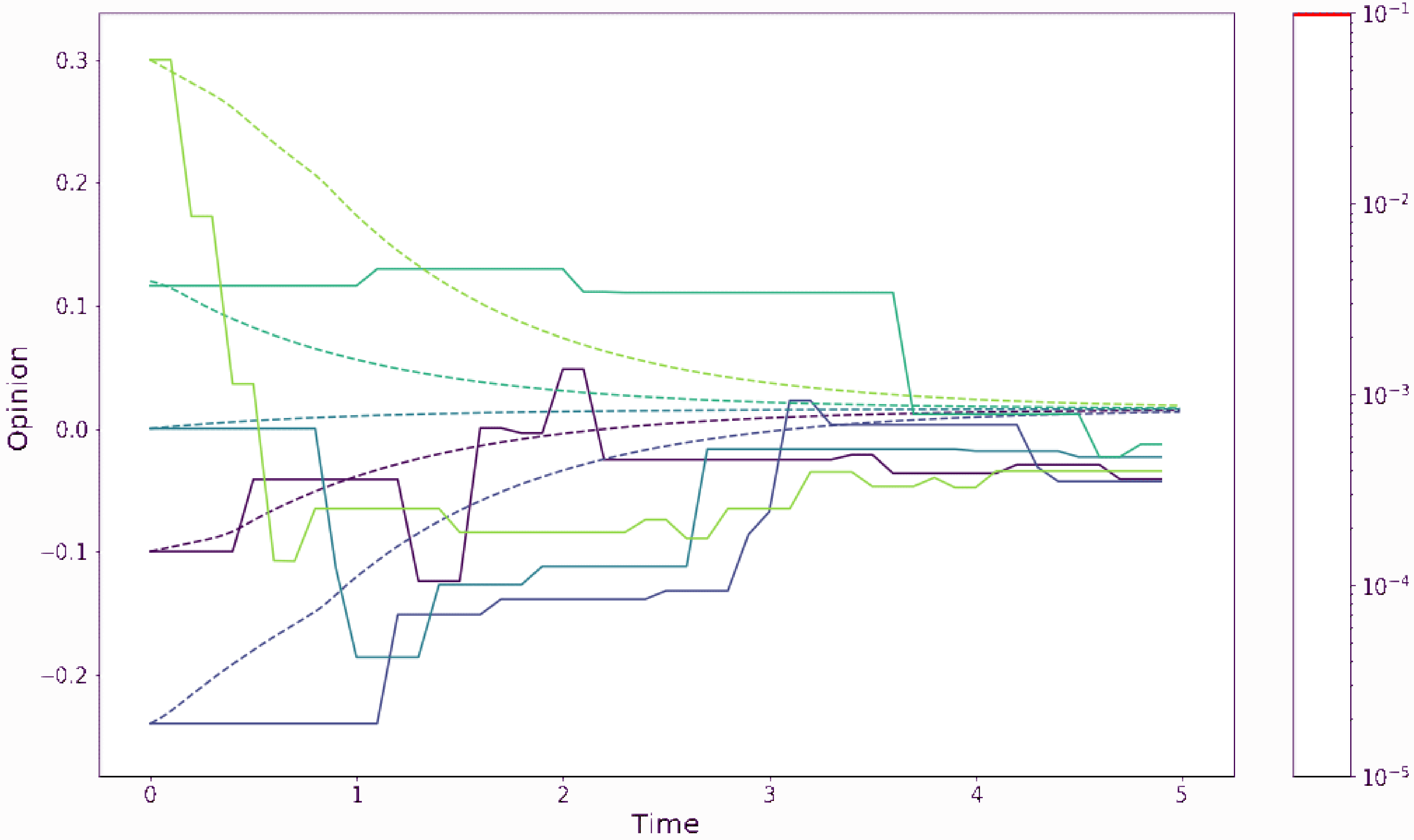
$$b_i^h(x) = \frac{1}{N}\sum_{j=1}^{N} p_{ij}(x)(x_j - x_i), \qquad a_{ii}^h(x) = h\left(\sum_{j=1}^{N} p_{ij}(x)\big(x_j - x_i\big)^2\right), \qquad a_{ij}^h(x) = 0, i \neq j.$$

Using some results from stochastic calculus[2], we have that as $h \to 0$, the ABM converges in probability to the solution of the system of ODEs

$$\frac{dx_i}{dt} = \frac{1}{N}\sum_{j=1}^{N} p_{ij}(x)\big(x_j - x_i\big).$$

2 – See Durrett, Stochastic Calculus: A practical introduction, 2018.

From agent-based models to continuum models



Modifications to the standard model

This approach still works if we have other things happening within the model. For example…

- If the agent j that interacts with agent i is chosen from the neighbours of i in a social network, then $p_{ij} = 0$ if i and j are not neighbours; in this case, the dynamics is normalised by k_i, the degree of agent i.
- Adding noise at different stages of the process results in different models, including systems of SDEs instead of ODEs. For example,
  - If the update is of the form $x_i(t+1) = x_i(t) + \mu^h \left(x_j(t) - x_i(t) \right) + \xi^h$ (adaptation noise) then it is possible that $a_{ii}(x)$ converges to a constant and we have an SDE with additive noise.
  - Similarly, if μ^h is random, depending on its variance, we can obtain an SDE with multiplicative noise.
- In both cases, we might need to worry about what happens at the boundaries (if $x_i(t+1)$ is close to 1 or -1).
- More on this in the exercises.

The Hegselmann-Krause model

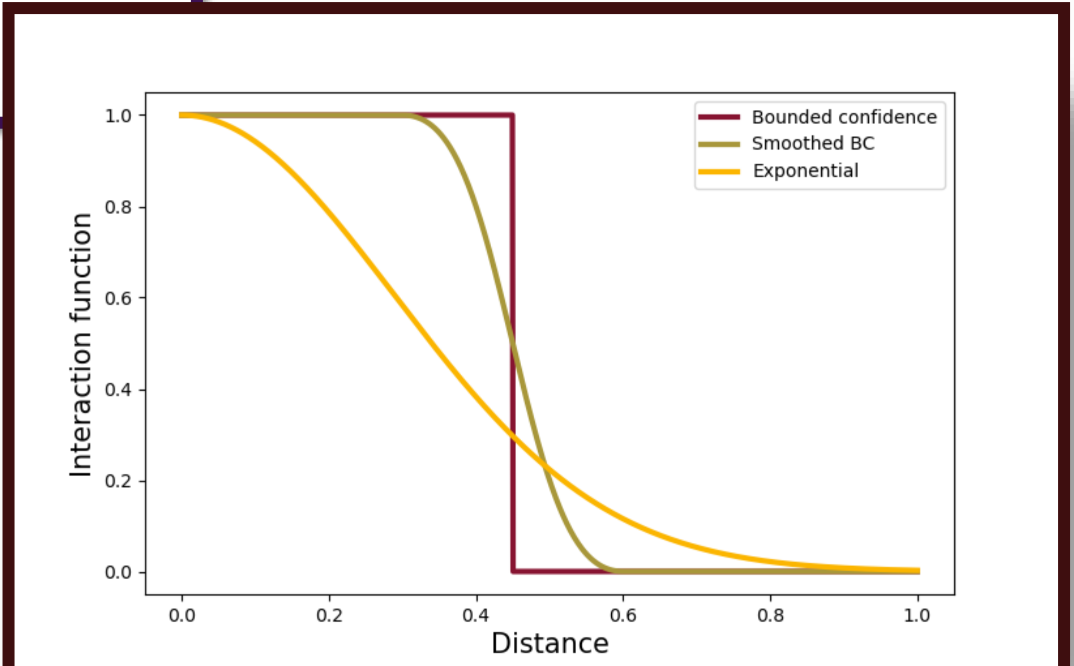
The most well-known model for opinion dynamics is the Hegselmann-Krause model. This corresponds to agents i and j interacting if their opinions are close enough:

Individual's opinion

Interaction function

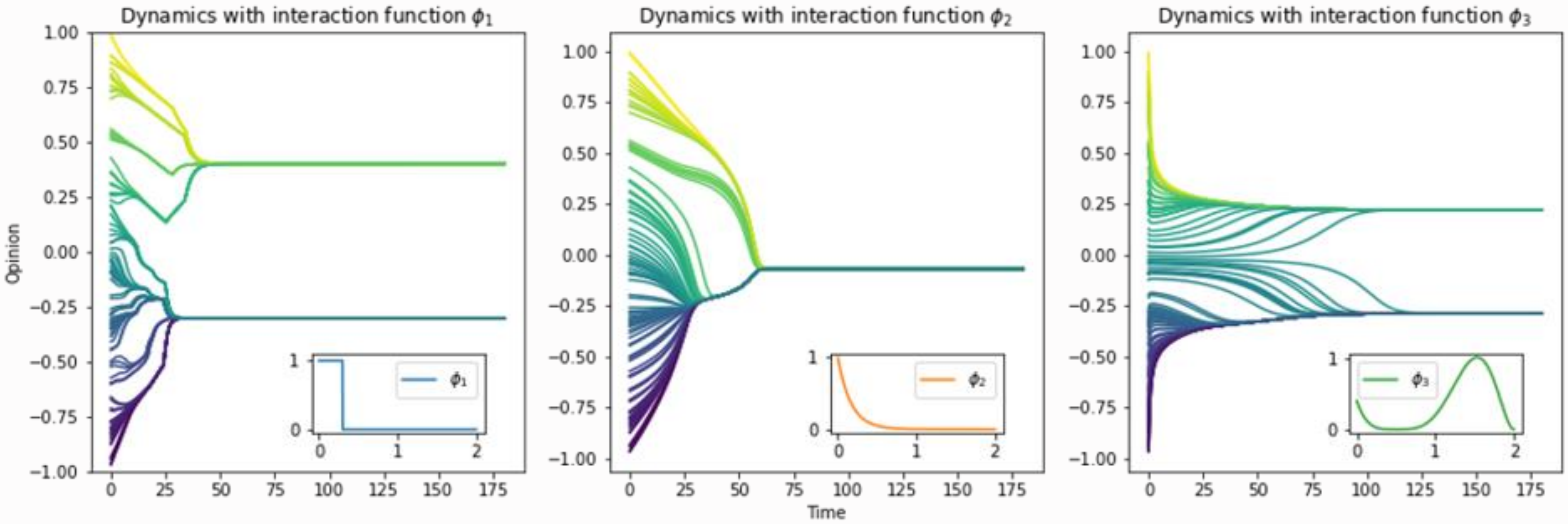
$$\frac{dx_i}{dt} = \frac{1}{N} \sum_{j=1}^{N} \phi(|x_j - x_i|)\,(x_j - x_i)$$

Population size



The Hegselmann-Krause model

The most well-known model for opinion dynamics is the Hegselmann-Krause model. This corresponds to agents i and j interacting if their opinions are close enough:



Now we can have some analytical results...

With the HK model, we can make predictions about the system. For example, it is possible to prove...

- The opinion diameter $D(t) = \max |x_i(t) - x_j(t)|$ is a non-increasing function, and therefore opinions always remain in the interval $[-1,1]$.
- As $t \to \infty$, the diameter $D(t)$ converges to a fixed value, leading to a stationary state. If $x_i(t) = x_j(t)$ for all i, j, we say that the system has **reached consensus**.
- For all $\varepsilon > 0$ there exists a time t^* such that for all i, j we have

$$\phi(|x_j(t^*) - x_i(t^*)|)\,|x_j(t^*) - x_i(t^*)|^2 < \varepsilon$$

- This means that if there exists $c > 0$ such that $\phi(r) > c$ for all $r \in [0,2]$ then consensus is guaranteed.
- The value of R in the interaction function for which there is consensus can be quantified using an order parameter[1]

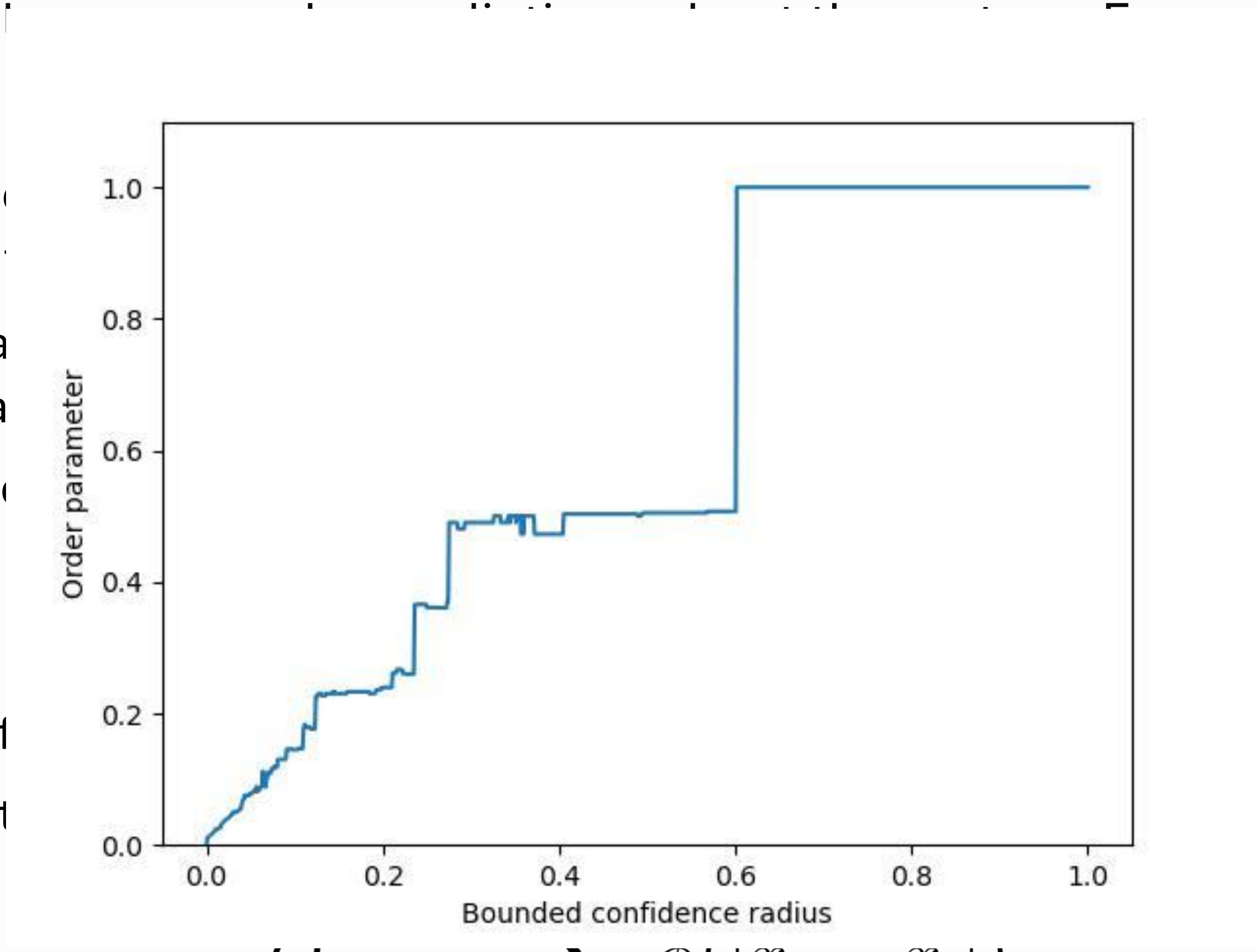
$$Q = \frac{1}{N^2} \sum_{i,j} \phi(|x_j - x_i|)$$

1 – Order parameter first introduced by Wang, Li, Chazelle et al. (2017)

Now we can have some analytical results...

With the HK model … ample, it is possible to prove...

- The opinion diam… and therefore opinions always remain in …
- As $t \to \infty$, the dia… state. If $x_i(t) = x_j(t)$ for all i, j, we say tha…
- For all $\varepsilon > 0$ ther…



- This means that i… onsensus is guaranteed.
- The value of R in … uantified using an order parameter[1]

$$Q = \frac{\quad}{N^2} \sum_{i,j} \varphi(|x_j - x_i|)$$

1 – Order parameter first introduced by Wang, Li, Chazelle et al. (2017)

Adding networks

In a more realistic situation, agents' interactions happen within a social network and the model can include this as follows:

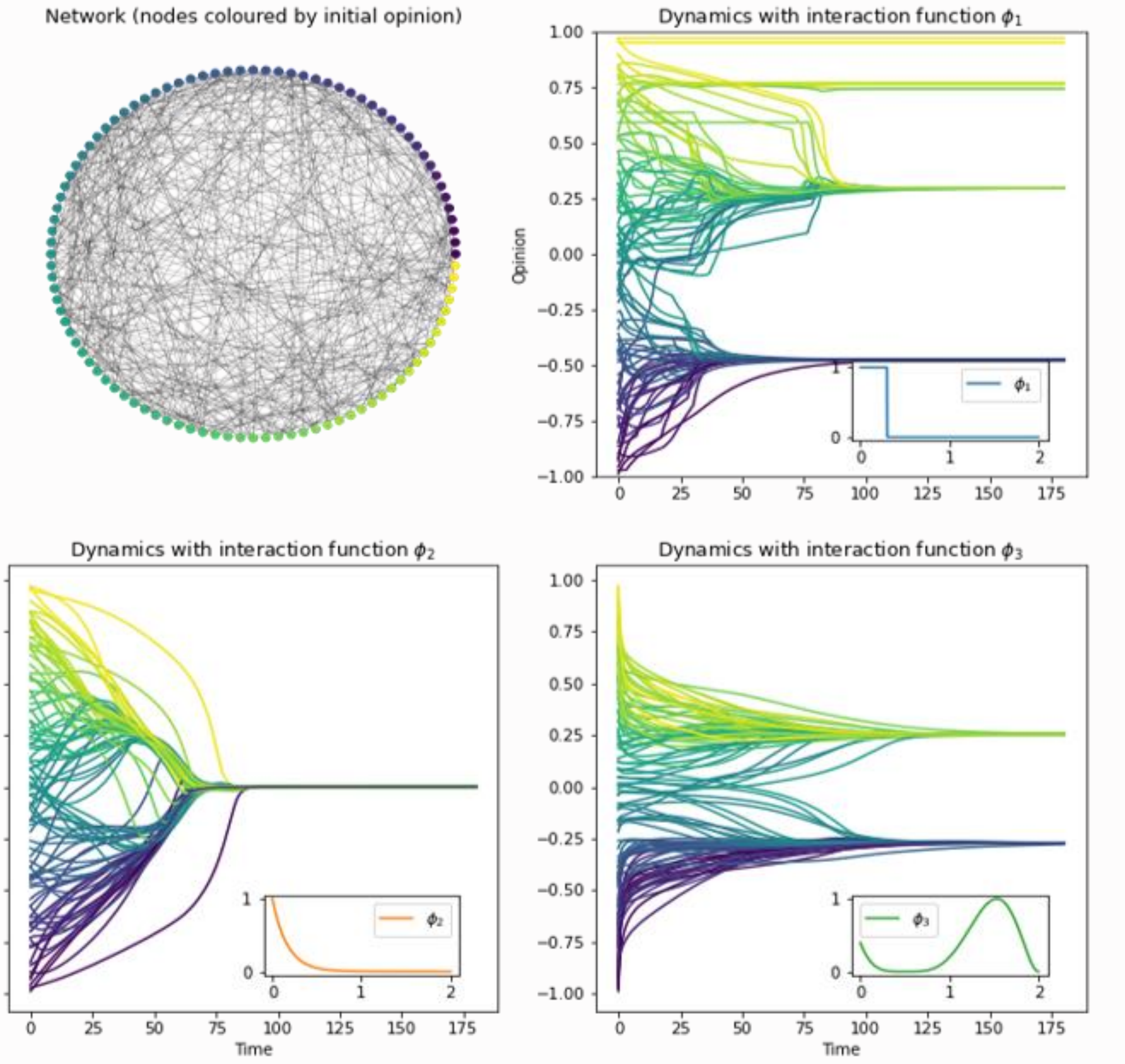
$$\frac{dx_i}{dt} = \frac{1}{k_i}\sum_{j=1}^{N} w_{ij}\phi\left(\left|x_j - x_i\right|\right)\left(x_j - x_i\right), \qquad k_i = \sum_{j=1}^{N} w_{ij}\,.$$

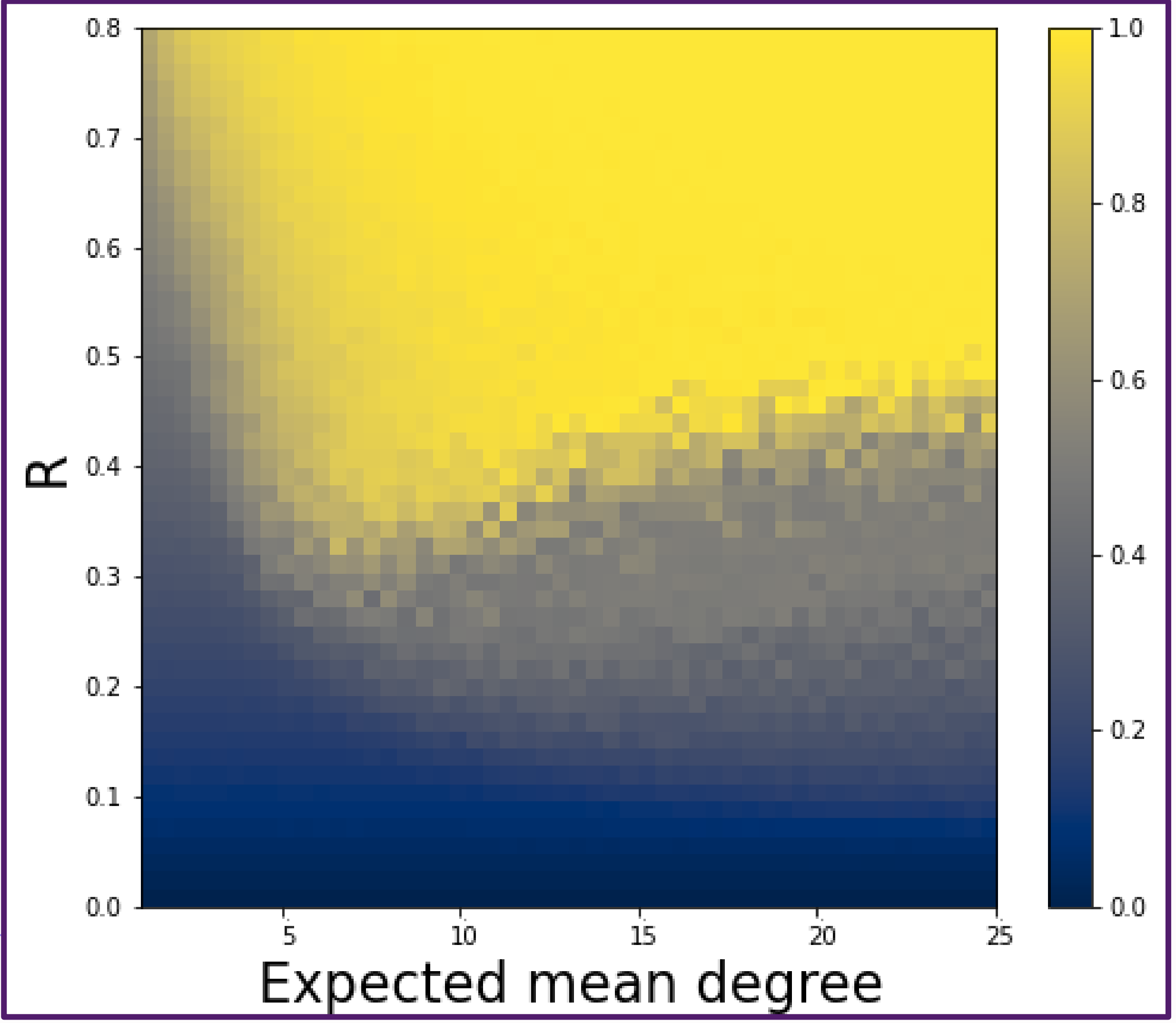
Here $w_{ij} \in [0,1]$ correspond to the **level of connectedness** of two agents, and k_i is the degree of agent i, representing **how well connected** it is.

In this case, it is possible to prove that if $\phi(\cdot)$ is always positive, then the system reaches consensus (similarly to the complete network case from earlier).

It is also possible to compare the dynamics of the system for different interaction functions...

... and for different (random) networks.


Network (nodes coloured by initial opinion)
Dynamics with interaction function ϕ_1
Opinion
ϕ_1
Dynamics with interaction function ϕ_2
ϕ_2
Time
Dynamics with interaction function ϕ_3
ϕ_3
Time


R
Expected mean degree

Adaptive networks

It makes sense to allow the network to **evolve in time**. We will do this by assuming that:

- Relationships (edge weights) form gradually over time, if the agents share similar beliefs.
- Existing relationships decay if individuals do not interact (i.e. not like-minded).

The model then becomes a coupled system for opinions and weights:

$$\frac{dx_i}{dt} = \frac{1}{k_i}\sum_{j=1}^{N} w_{ij}\phi(|x_j - x_i|)(x_j - x_i)$$

$$\frac{dw_{ij}}{dt} = \phi(|x_j - x_i|)f^+(W)_{ij} - \left(1 - \phi(|x_j - x_i|)\right)f^-(W)_{ij},$$

where f^+ and f^- describe how weights increase or decrease in time.

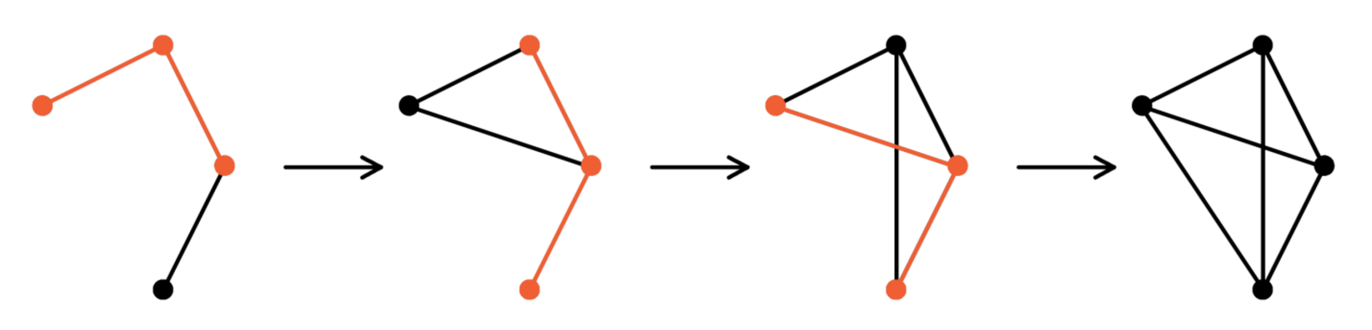
Examples of network dynamics

- Logistic weight dynamics:

$$f^+(W)_{ij} = w_{ij}\big(1 - w_{ij}\big) \quad \text{and } f^-(W)_{ij} = w_{ij}\big(1 - w_{ij}\big).$$

- Friend of a friend weight dynamics

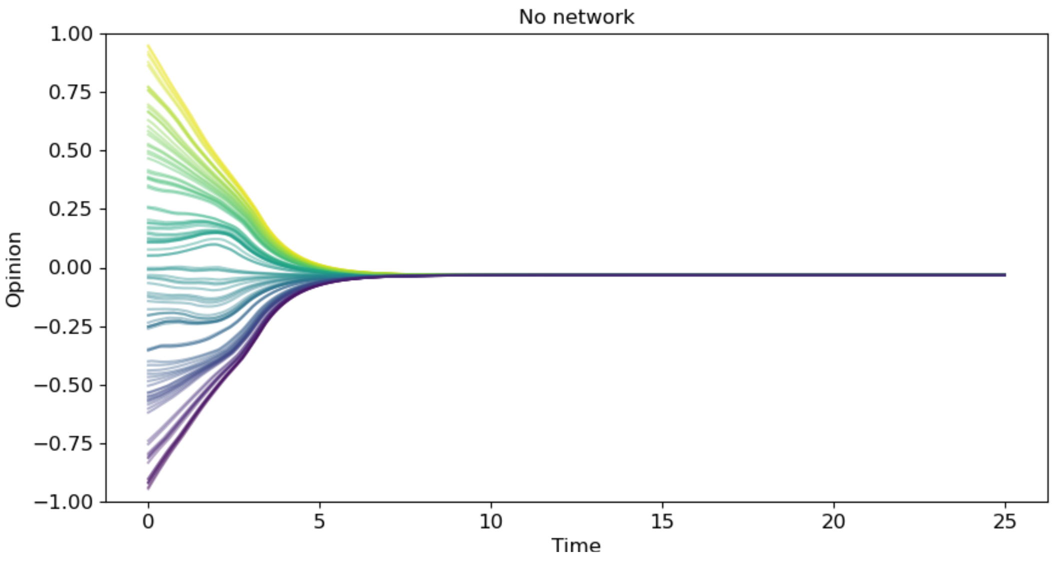
$$f^+(W)_{ij} = \left(w_{ij} + \tfrac{1}{N}(W^2)_{ij}\right)\big(1 - w_{ij}\big) \quad \text{and } \; f^-(W)_{ij} = w_{ij}.$$

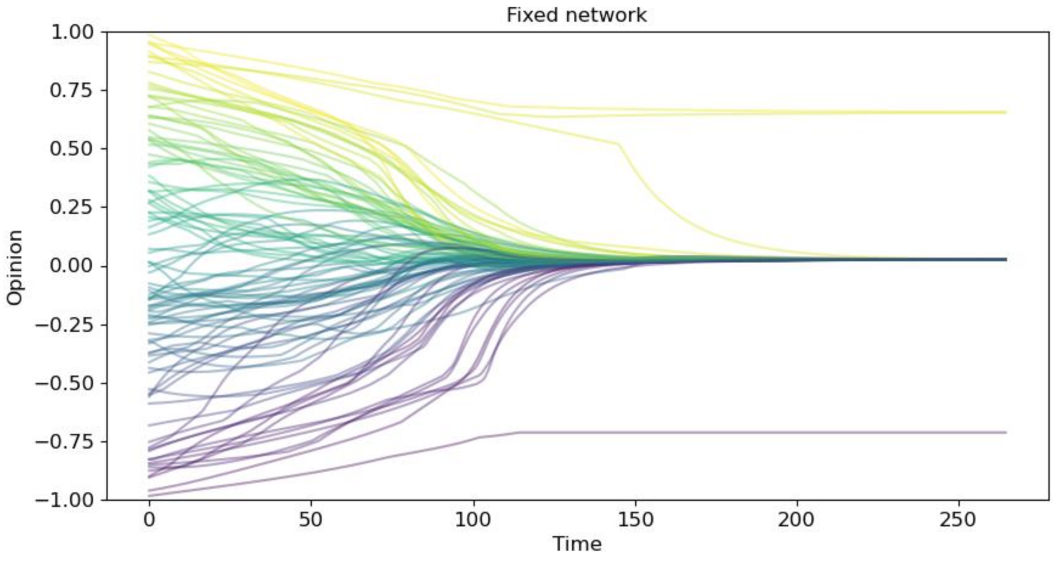


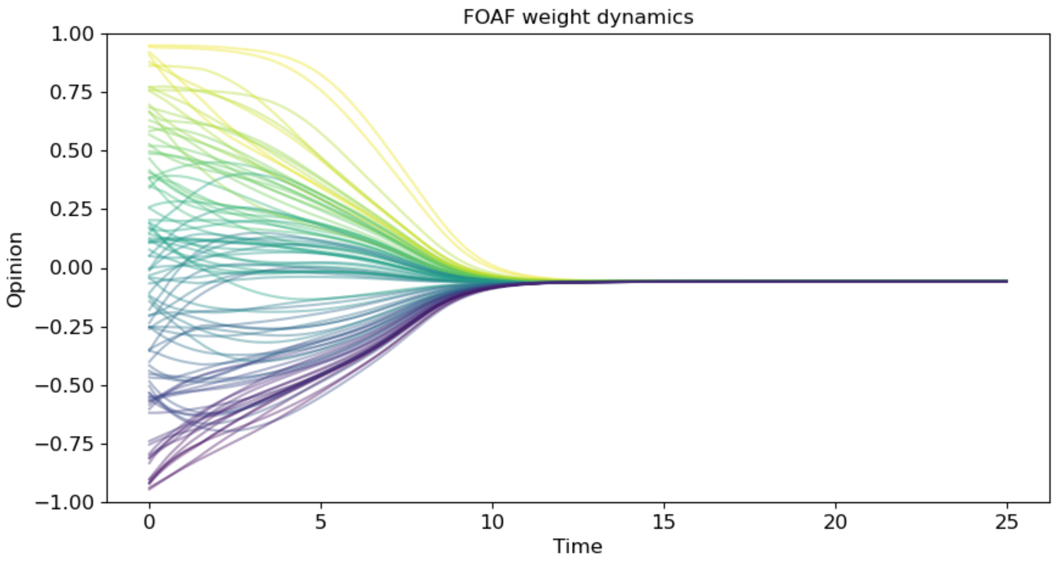
- Memory weight dynamics

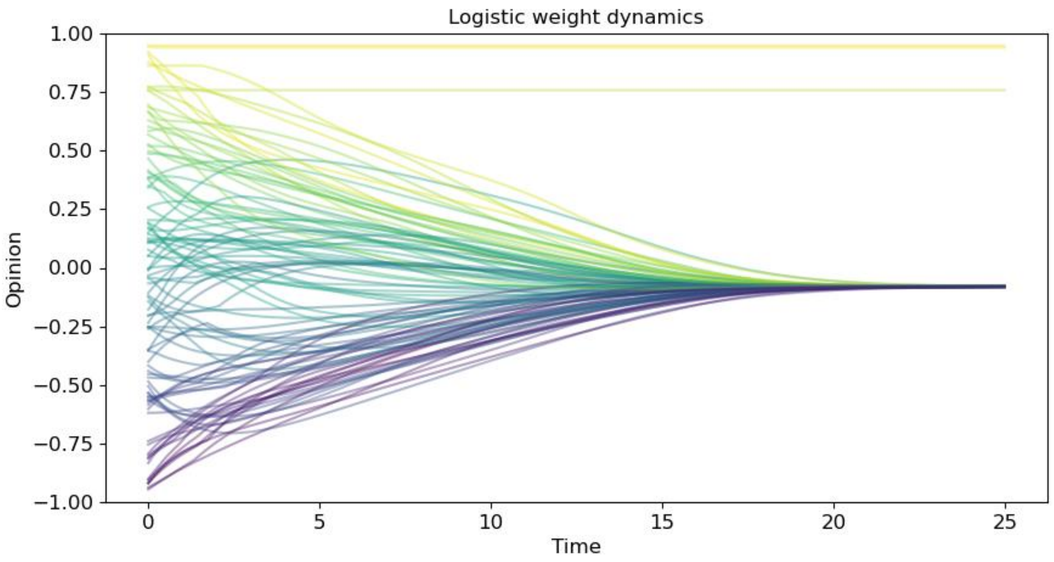
$$f^+(W)_{ij} = \big(1 - w_{ij}\big) \quad \text{and } f^-(W)_{ij} = w_{ij}.$$

Examples of network dynamics

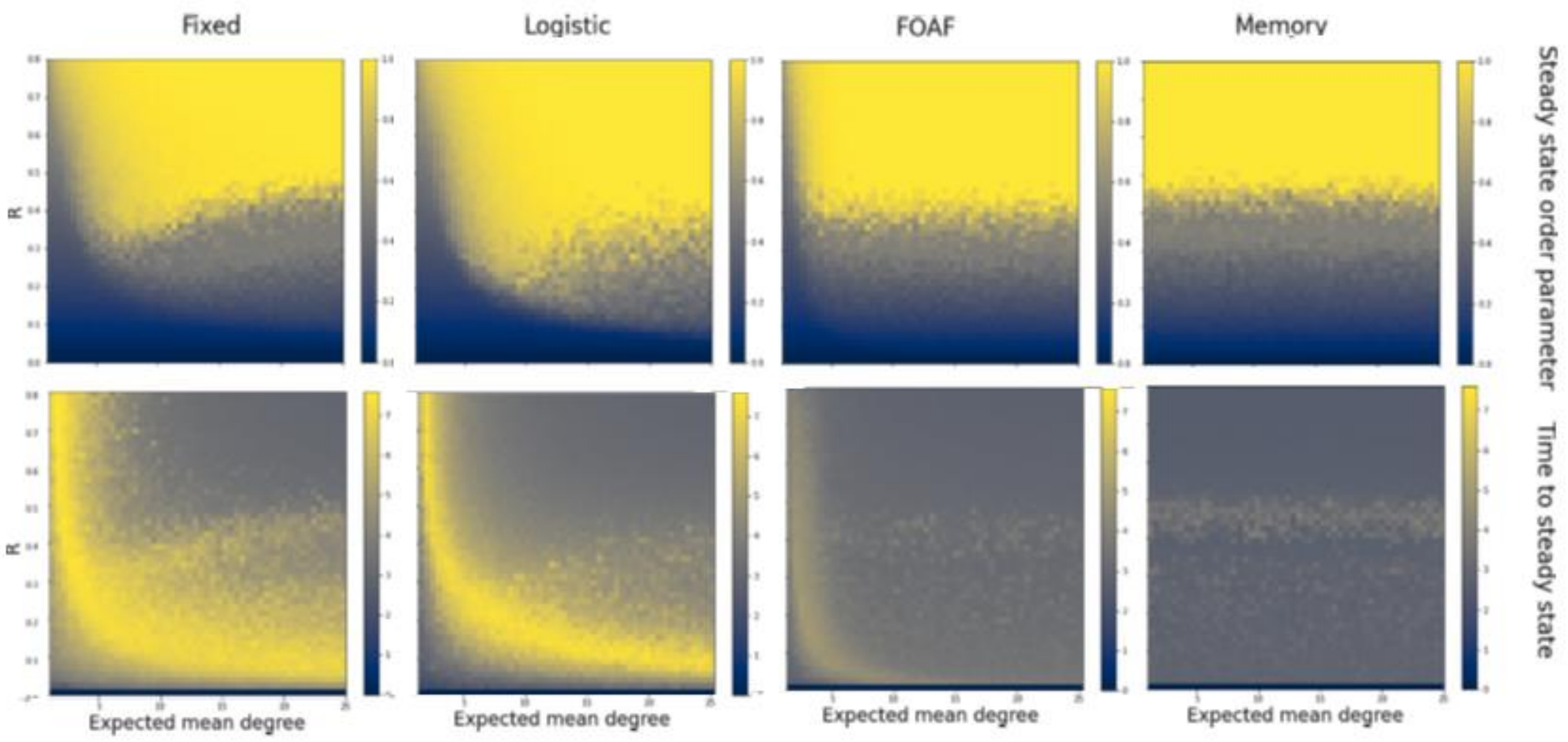








Influence of adaptive networks in the order parameter



We can also show that if the network evolves slowly, when compared to the opinions, then the system behaves as if the network is static. Conversely, if the network evolution is fast, we can analyse the “fast network” limit.

Discussion

Other things we can do and future work in this area

Discussion

There is a lot that can be done, both from the ABM and IPS points of view, and several topics we did not exmplore. Some examples include...

- Mean-field limits for IPS and analyse the mean field limit. Can lead to control, inference, etc. (after the break)
- When the ABM is also discrete in the state-space (e.g. voting yes/no in a referendum, being infected or susceptible to a disease, ...) we can obtain the mean-field limit directly from the ABM. This usually results in an ODE or SDE for the proportion of the population in a certain state (the SIR model you might be familiar with can be obtained this way!)
- Use a kinetic interpretation instead, leading to Boltzmann-type equations, from each we can obtain the mean-field limit (Sasha's work)
- Many other things (more to come in the next two sessoins)

After the break:
Mean-field limits for large populations, and inference for interacting particle systems.

Thank you for your attention!

After the break: Mean-field limits for large populations, and inference for interacting particle systems.

Susana.gomes@warwick.ac.uk

UNIVERSITY OF WARWICK

Enabling Quantification of EQUIP Uncertainty for Inverse Problems

LEVERHULME TRUST