Mean field limits for interacting particle systems, their inference, and applications

Part 2 – Interacting Particle Systems and their mean field limits

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From microscopic dynamics to continuum limits - ACDC Autumn School

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Outline

Interacting particle systems

From interacting particle systems to their mean-field limits

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Interacting particle systems

We saw earlier that interacting particle systems are appropriate to model a population of agents of the "same type". This means that everyone behaves in the same way and interacts at the same rate.

Example 1: Cell dynamics: one (or multiple) population(s) of cells interact in some experimental environment. The cell(s) can interact via attraction or repulsion, or other interesting dynamics.

Example 2: Birds flying in a group, with interactions via what is visible to them. Can exhibit very interesting behaviour, such as flocking, or milling.

Modelling approaches

There are several ways of modelling these types of systems. Today I will focus on (stochastic) **interacting particle systems**:

- Simple models for each particle (usually based on Newton's Laws or similar).
- Already saw an example earlier, now we will consider the general case.
- We will see that these are analytically and computationally hard to tackle.
- Since in common applications, we would have a very large number of particles, we can consider macroscopic limits: model the density of agents as the number of particles $N \to \infty$ using a **mean-field approach**.

The basic model

I will consider a class of first-order **weakly** interacting particle systems in one dimension:[1]

$$dX_t^i = V'(X_t^i)dt + \frac{1}{N}\sum_{j\neq i} K\left(X_t^i - X_t^j\right)dt + \sqrt{2\sigma}dW_t^i\,, \qquad X_0^i = x_0^i, \qquad i,j = 1,\ldots,N,$$

where:

- X_t^i denotes the position/state of particle i at time t
- $\mathrm{V}(x)$ is a **confining potential** (encodes the domain in some sense)
- $\mathrm{K}(x)$ is an **interaction potential** (encodes interactions) – in some cases we need that $\mathrm{K}(0) = 0$ and $\mathrm{K}'(0) = 0$.
- W_t^i are independent Brownian motions and σ is the strength of the noise (sometimes I will write β^{-1} which is more common in physics)
- x_0^i are initial conditions which can be deterministic or stochastic (independently distributed with some chosen law)
- The scaling $\frac{1}{N}$ is the mean-field scaling and it is critical for us, as it keeps the strength of interactions at order 1.

1 – this is for simplicity; similar results can be obtained in higher dimensions, and for second-order systems of the type $dX_t^i = V_t^i dt; dV_t^i = K\left(X_t^i - X_t^j\right)\,dt + \sqrt{2\sigma}dW_t^i$. Alternatively, one can also solve this SDE on a torus, and exclude the potential V, see Carrillo, Gvalani, Pavliotis and Schlichting, ARMA 2018.

Some examples

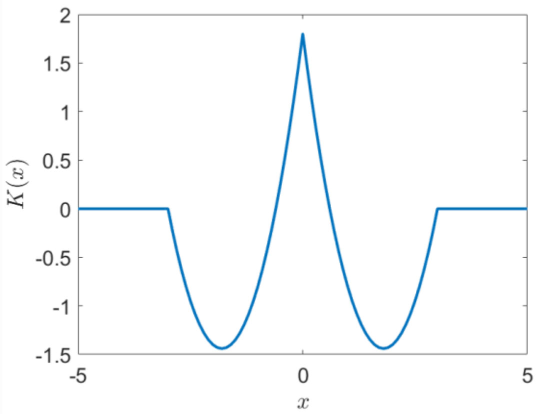
There are several examples of potentials, which depend on the application.

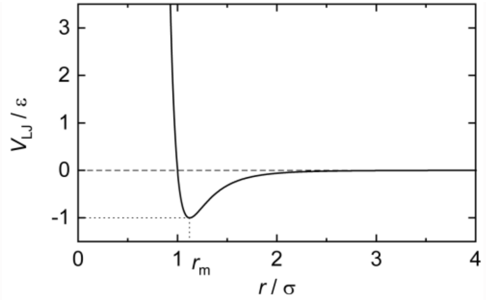
- Aggregation potentials (attraction/repulsion) for interactions (common for cell populations, animal populations, etc.).

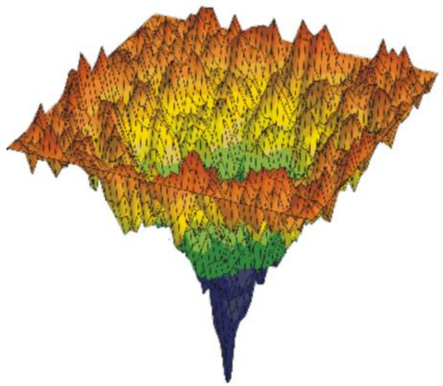
- Lennard-Jones interaction potentials (common in molecular dynamics simulations, density functional theory, chemistry for molecular interactions)

- Multi-well confining potentials (more well-known as energy landscapes, common in protein folding, cell evolution dynamics, etc.)

- For rational agents, the potentials can be a bit different, e.g., utility functions.







Empirical measure and N-particle distribution

To pass to the mean-field limit, it is important to define two measures:

- The **empirical measure**

$$\mu_N(t,x) = \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - X_t^i\right).$$

  - Contains all the information about the solution $(X_t^1, \dots, X_t^N)$.
  - Is a random probability measure[2]
  - The stochastic behaviour only vanishes as $N \to \infty$ $\implies$ important to quantify fluctuations if N remains finite[3] (we will discuss this later)

- The N **−particle or joint distribution**

$$F^N(t, x_1, \dots, x_N) = Law(X_t^1, \dots, X_t^N).$$

  - not experimentaly measurable, but
  - its marginals contain statistical information on the process

$$F_k^N(t, x_1, \dots, x_N) = \int_{\mathbb{R}^{n-k}} F^N(t, x_1, \dots, x_N) dx_{k+1} \cdots dx_N.$$

2 - In the deterministic case (no noise, $\sigma = 0$), this is a deterministic probability measure.
3 - See [J. Worsfold, T. Rogers, P. Milewski, SIAM J. Appl. Math (2023)]

N particle dynamics

It is useful to recall Itô's formula – this will allow us to write a PDE for the evolution of F^N.

Itô's formula (for our case)

Let $(X_t: t \geq 0)$ be a stochastic process which solves the SDE

$$dX_t = a(X_t, t)\, dt + \sqrt{2\sigma}\, dW_t.$$

Then, for a smooth function $f(x)$ we have

$$df(X_t) = a(X_t, t) f'(X_t)\, dt + \sigma f''(X_t)\, dt + \sqrt{2\sigma} f'(X_t)\, dW_t.$$

Applying Itô's formula to our initial system of SDEs and compute expectations with respect to the law of the process, F^N. This gives us the following partial differential equation:

$$\partial_t F^N(t, x_1, \dots, x_N) = -\sum_{i=1}^{N} \partial_{x_i} \left(V'(x_i) F^N + \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq i} K(x_i - x_j) F^N \right) + \sigma \sum_{i=1}^{N} \Delta_{x_i} F^N.$$

Note that when applying Itô's formula to $dX_t^i = V'(X_t^i)dt + \frac{1}{N}\sum_{j\neq i} K\left(X_t^i - X_t^j\right) dt + \sqrt{2\sigma} dW_t^i$, the last term vanishes because it is an Itô integral of a deterministic function.

The mean-field limit

To formally[4] pass to the limit, we use the mean field ansatz, i.e., we assume that

$$F^N(t, x_1, \dots, x_N) = \prod_{i=1}^{N} \rho(t, x_i) \quad \mathit{and} \quad F^N(0, x_1, \dots, x_N) = \prod_{i=0}^{N} \rho_0(x_i)\,.$$

Using this ansatz in the PDE for the evolution of the N-particle distribution, we can then integrate out the $N-1$ variables $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N$ and obtain a PDE for the evolution of $\rho(x_i)$:

$$\partial_t \rho(t, x_i) = -\partial_{x_i}\left(V'(x_i)\rho + \frac{N-1}{N}\rho \int_{\mathbb{R}} K(x_i - y)\, dy\right) + \sigma \partial_{x_i}^2 \rho.$$

Sending N to infinity, we obtain the **Fokker-Planck equation**

$$\partial_t \rho(t, x_i) = -\partial_{x_i}(V'(x_i)\rho + (K \star \rho)\rho) \ + \sigma \partial_{x_i}^2 \rho,$$

where denotes convolution.

4 - Passing rigorously to this limit can be done using martingale techniques or other classical stochastic analysis results, see [P.-E. Jabin and Z. Wang, Mean Field Limits for Stochastic Particle Systems, Active Particles Volume 1, (2017)] and references therein. This formal derivation follows Urbain Vaes' PhD Thesis, 2019.

Some relevant results

An alternative method consists of considering initial data $X_0^i = x_0^i$ i.i.d. with $Law(x_0^i) = f_0$, and constructing a particle system coupled to the original SDE:

$$d\bar{X}_t^i = V'(\bar{X}_t^i)\, dt + (K \star f_t)(\bar{X}_t^i)\, dt + \sqrt{2\sigma}\, dW_t^i, \qquad \bar{X}_0^i = x_0^i, \qquad i = 1, \dots, N.$$

where the W_t^i are the **same Brownian motions** as before, and f_t is the law of the $\bar{X}_t^i$.

This is known as the **McKean-Vlasov equation** (and is no longer an SDE because it depends on the law of the process).

One can check that f_t solves the Fokker–Planck equation on the previous slide, and show that the empirical measure μ_N converges in law to ρ, the solution to the Fokker–Planck equation.

Under appropriate conditions on K and V, it can be shown[5] that solutions to the McKean-Vlasov equation are **close to the solutions** of the original SDE, and use this to obtain bounds on the difference $|X_t^i - \bar{X}_t^i|$, as well as quantify large deviations.

5 - See [P.-E. Jabin and Z. Wang, Mean Field Limits for Stochastic Particle Systems, Active Particles Volume 1, (2017)]

Long time behaviour

Example of multi-well and multi-scale interacting potential

SNG, G.A. Pavliotis, *Journal of Nonlinear Science* 28, 905-941, 2018.
SNG, S. Kalliadasis, G.A. Pavliotis, P. Yatsyshin, *Physical Review E* 99, 032109, 2019
SNG, G.A. Pavliotis, U. Vaes, *Multiscale Modelling and Simulation*, 2020 (not discussed - 2nd order problem)

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A system of interacting particles

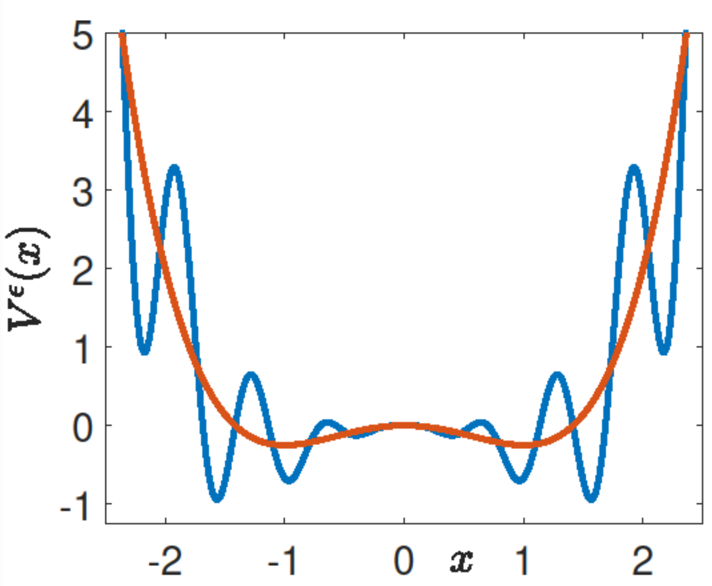
In some cases, we can use the mean-field limit to analyse the long time behaviour of these systems. We will focus on a particular example.

Consider a system of N weakly interacting particles given by

$$dX_t^i = \left(-V'\left(X_t^i\right) + \theta \left(X_t^i - \frac{1}{N} \sum_{j=1}^{N} X_j \right) \right) dt + \sqrt{2\sigma} dW_t^i.$$

Here, the particles interact via their mean, with strength θ, i.e., in a quadratic Currie-Weiss potential $K(x) = \frac{x^2}{2}$. We also consider multi-well confining potentials. For example:

- $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$
- $V_8(x) = hx^2(x^2 - 1)(x^2 - 4)(x^2 - 9)$
- $V_\epsilon(x) = V_0(x) + \frac{\delta x^2}{2} cos \left(\frac{x}{\epsilon}\right)$



$N \to \infty$ and the McKean-Vlasov equation

Using the previous arguments and using the Law of Large Numbers, we can formally study the mean field limit:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} X_j^t = \mathbb{E}[X_t],$$

where $\mathbb{E}[\cdot]$ is the expectation with respect to the one-particle distribution.

We pass to the limit $N \to \infty$ and obtain the McKean-Vlasov SDE for X_t

$$dX_t = \left(-V'(X_t) - \theta(X_t - \mathbb{E}[X_t])\right) dt + \sqrt{2\sigma} dW_t^i.$$

This SDE has a corresponding nonlinear Fokker-Planck equation:

$$\partial_t \rho(t, x_i) = \partial_x \left(V'(x)\rho + \theta\left(x - \int_{\mathbb{R}} x\, \rho(x,t)\, dx \right)\rho \right) + \sigma \partial_x^2 \rho,$$

Its steady states allow us to investigate the long-time behaviour of this system.

Multiple invariant measures

Invariant measures of the McKean-Vlasov SDE are steady states of the FP equation:

$$\partial_x \left(V'(x)\rho + \theta \left(x - \int_{\mathbb{R}} x\,\rho(x,t)\,dx \right)\rho + \beta^{-1}\partial_x \rho \right) = 0.$$

This admits a one-parameter family of solutions:

$$\rho_\infty(x;\theta,\beta,m) = \frac{e^{-\beta\left(V(x)+\theta\left(\frac{x^2}{2}-xm\right)\right)}}{Z(\theta,\beta;m)}, \qquad Z(\theta,\beta;m) = \int_{\mathbb{R}} e^{-\beta\left(V(x)+\theta\left(\frac{x^2}{2}-xm\right)\right)}\,dx,$$

subject to the constraint that they provide us with the correct formula for the first moment, known as the **self-consistency equation:**

$$m = \int_{\mathbb{R}} x\,\rho_\infty(x;\theta,\beta,m)\,dx =: R(m;\theta,\beta).$$

Solving this equation, we can find the invariant measures of the dynamics.

Critical temperature and other properties

For sufficiently small $\beta, m = 0$ is the only solution of the self-consistency equation. However, for **nonconvex** confining potentials, there exists a **critical temperature**, β_C, at which this is no longer true.[6]

To find β_C, we can differentiate the self-consistency equation at $m = 0$, and conclude that β_C is the solution of

$$Var\big(\rho_\infty(x;\theta,\beta,m=0)\big) = \int_{\mathbb{R}} x^2 \rho_\infty(x;\theta,\beta,m=0)\,dx = \frac{1}{\beta\theta}.$$

Another relevant feature is that the McKean-Vlasov equation is a **gradient flow**, with respect to the Wasserstein metric, for the **free energy functional**

$$\mathcal{F}[\rho] = \beta^{-1}\int_{\mathbb{R}} \rho\log\rho\,dx + \int_{\mathbb{R}} V(x)\rho\,dx + \frac{\theta}{2}\iint_{\mathbb{R}} \frac{(x-y)^2}{2}\rho(x)\rho(y)\,dx\,dy\,.$$

Steady states of the McKean-Vlasov equation are equilibrium points of the free energy functional.

6 - [Dawson, J. Stat.Phys 1983, Tamura,J. Fac.Sci. Univ. Tokyo 1984, Shiino, Phys. Rev. A 1987, Tugaut, Stochastics 2013]

Numerical results: Bistable potential[7]

The simplest example we can consider is the bistable potential,

$$V(x) = \frac{x^4}{4} - \frac{x^2}{2}.$$

For sufficiently large β (small σ), the self-consistency equation has two more solutions.

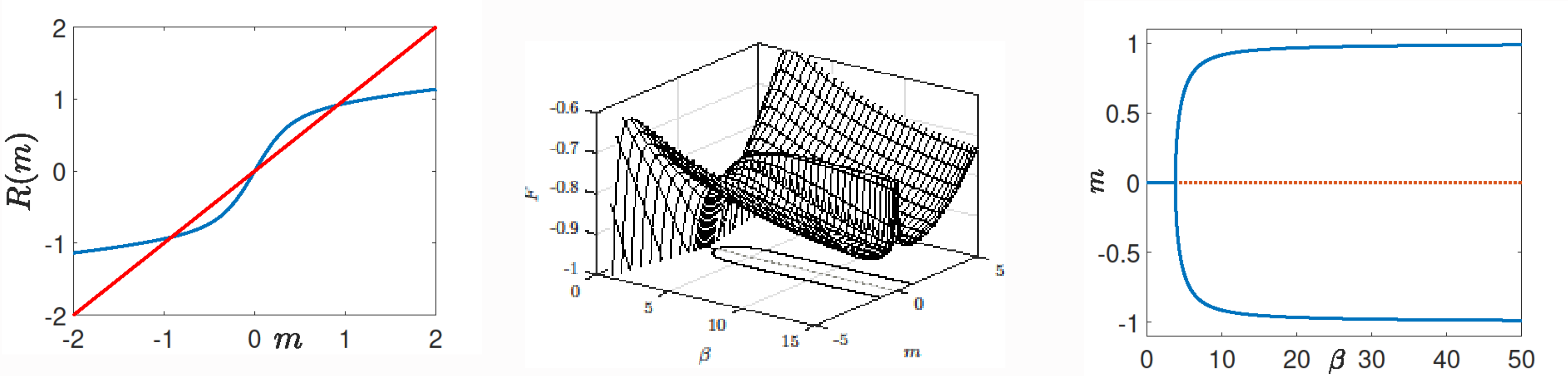


Figure: $R(m; 0.5; 10)$ against $y = x$ (left), free energy surface as a function of β and m (middle), and bifurcation diagram of m as a function of β for $\theta = 0.5$ (right).

7 – See, e.g. [Dawson, J. Stat.Phys 1983] for a detailed study of this case.

Other examples of potentials

We also considered the following confining potentials:

- Higher degree polynomial potentials such as

$$V_6(x) = h(x^6 - 5x^4 + 4x^2) = hx^2(x^2 - 1)(x^2 - 4),$$

$$V_8(x) = h(x^8 - 14x^6 + 49x^4 36x^2) = hx^2(x^2 - 1)(x^2 - 4)(x^2 - 9),$$

- Non-symmetric polynomial potentials by introducing a tilt

$$V(x) = \frac{1}{a_0}\left(\frac{x^4}{4} - \frac{x^2}{2}\right) + \kappa x,$$

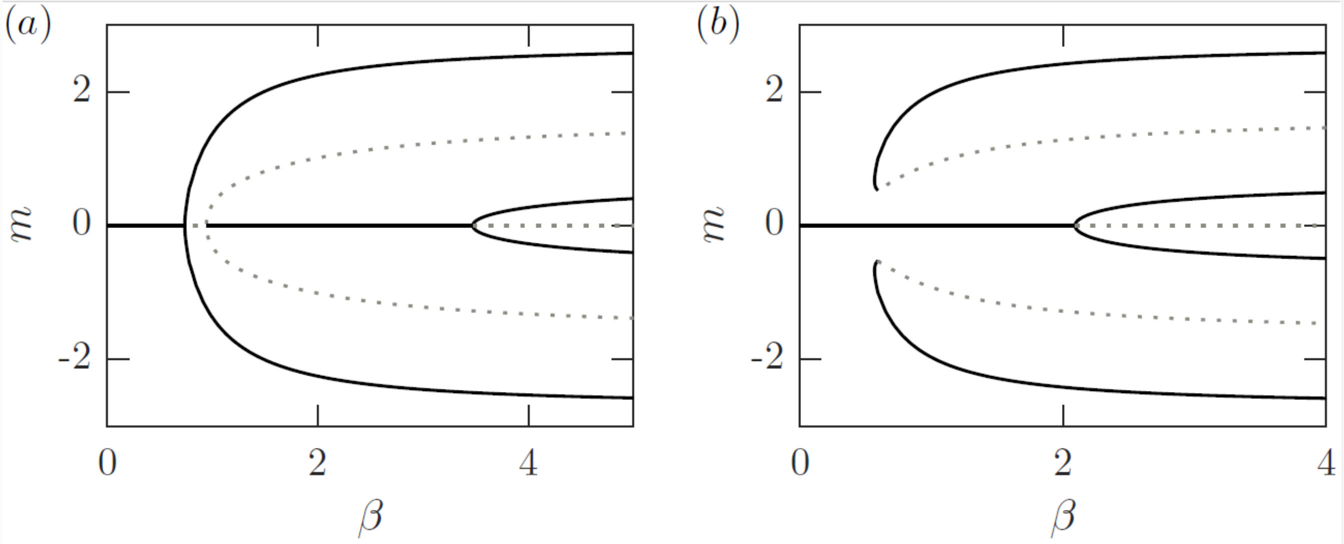
- Rational potentials with random locations of minima and barrier heights

$$V(x) = \frac{1}{\sum_{\ell=-n}^{n} \delta_\ell |x - c_\ell x_\ell|},$$

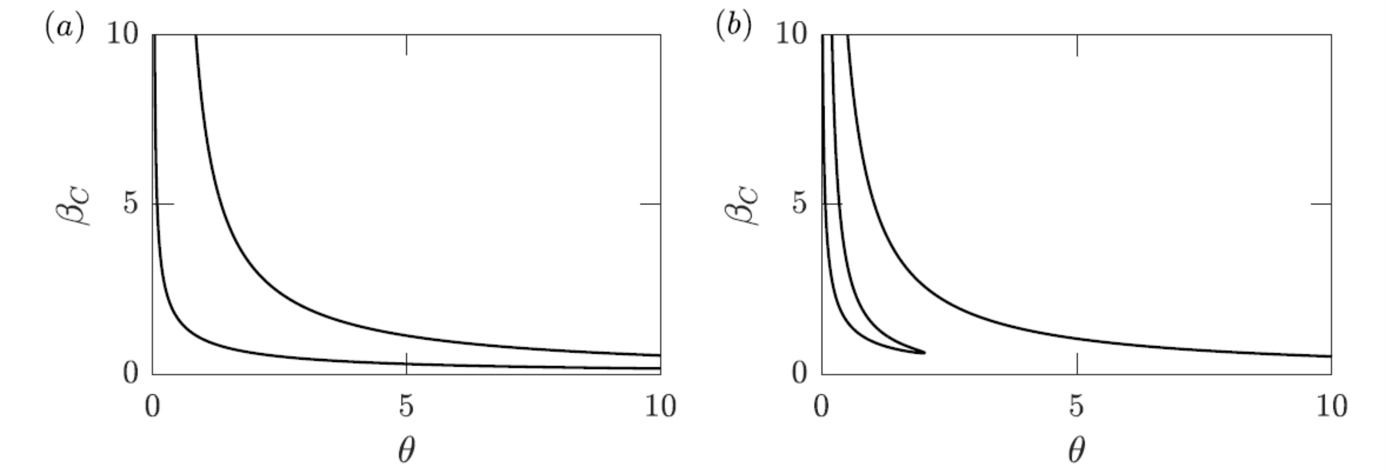
Where c_ℓ andδ_ℓ are random numbers drawn from a given distribution.

Symmetric multi-well potentials

Phase diagrams for the potential $V_8(x)$ for $h = 0.001$, and (left) $\theta = 1.5$, (right) $\theta = 2.5$.



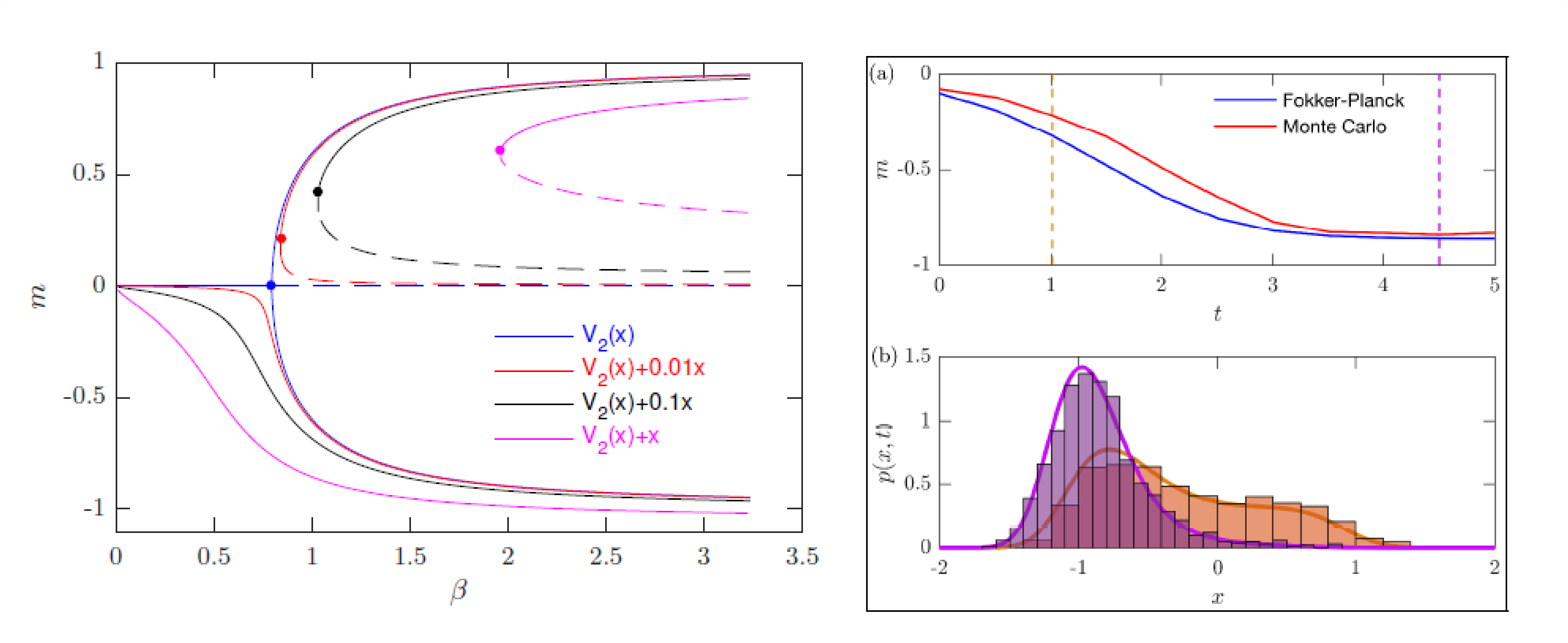
Critical temperature β_C as a function of θ for (left) $V_6(x)$ and (right) $V_8(x)$.



Breaking the symmetry - tilted bistable potential

We can study the effect of breaking the symmetry of these potentials by (1) **adding a tilt to a bistable potential:**

$$V(x) = \frac{1}{a_0}\left(\frac{x^4}{4} - \frac{x^2}{2}\right) + \kappa x.$$

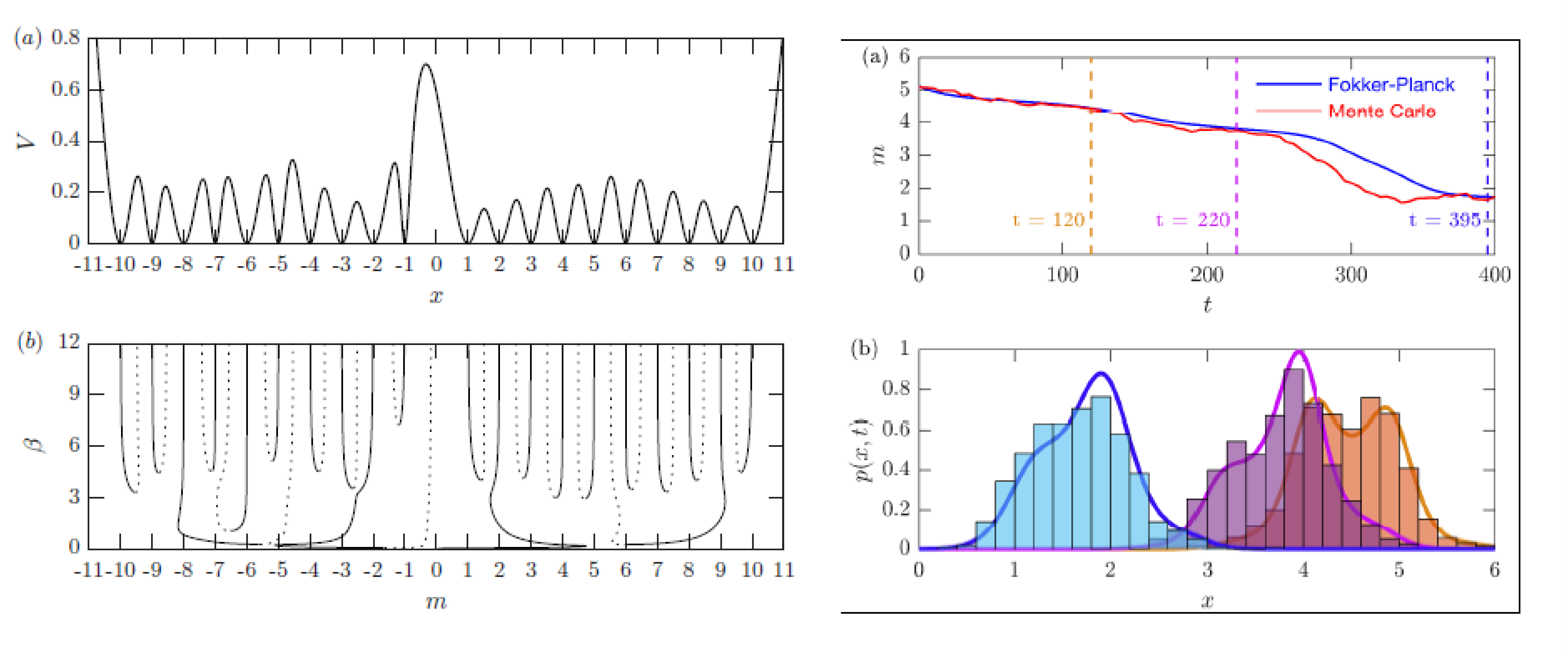


[20] Bifurcation diagrams of m as a function of β with $\theta = 2.5$, $a_0 = 0.25$, and $\kappa = 0, 0.01, 0.1, 1$ (left), and evolution of the density and first moment with $\kappa = 0.1$, $\beta = 1.5$ (right).

Breaking the symmetry – random potentials

We can study the effect of breaking the symmetry of these potentials by (2) **considering random potentials**:

$$V(x) = \frac{1}{\sum_{\ell=-n}^{n} \delta_\ell |x - c_\ell x_\ell|}.$$



21 Random potential and corresponding bifurcation diagrams of m as a function of β with $\theta = 1.5$ (left), and evolution of the density and first moment with $\beta \approx 2.66$ (right).

Multiscale potentials

Recall that in this case, we have $V_\epsilon(x) = V_0(x) + \frac{\delta x^2}{2} cos\ \left(\frac{x}{\epsilon}\right)$. We need to distinguish between small, but finite ϵ, and $\epsilon \to 0$.

It turns out it also matters the order with which we send $\epsilon \to 0$ and pass to the mean-field limit $N \to \infty$.

We can use homogenisation techniques to first obtain a homogenised SDE then obtain the mean field limit (first $\epsilon \to 0$ then $N \to \infty$), or first pass to the mean field limit and then send $\epsilon \to 0$.

We have shown[8] that if the oscillations are additive, then the two limits commute, and otherwise, we obtain different long-time behaviour. This can be seen from the self-consistency equation:

$\epsilon \to 0$ first

$$m = \int_{\mathbb{R}} \frac{x e^{-\beta\left(V_{eff}(x)+\psi(x)\right)}}{Z} dx\,,$$

where $\psi(x) = \beta^{-1} \log \int_0^{2\pi} e^{-\beta V_1(x,y)} dy\,.$

$N \to \infty$ first

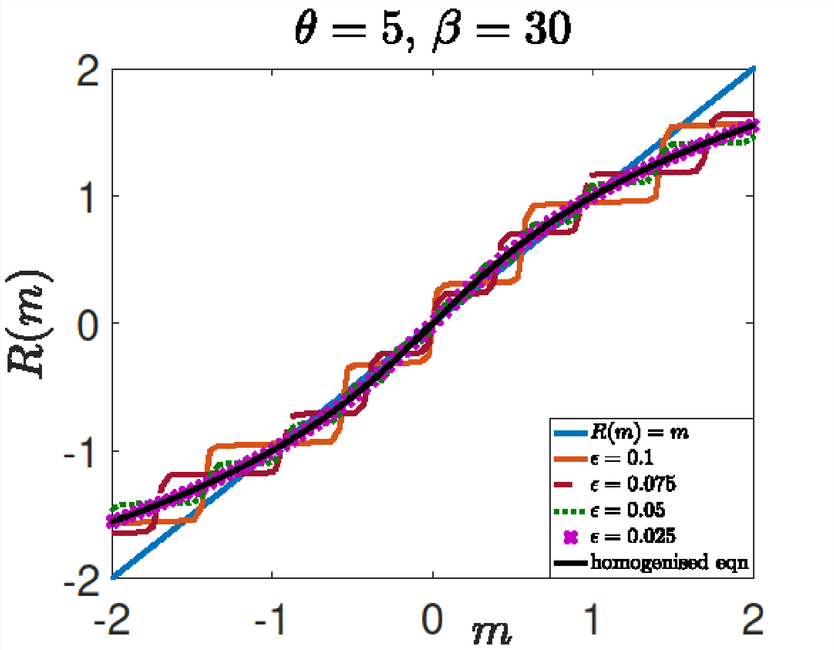
$$m^\epsilon = \int_0^L \int_{\mathbb{R}} \frac{x e^{-\beta\left(V_{eff}(x)+V_1(x,y)\right)}}{\bar{Z}} dx\, dy\,,$$

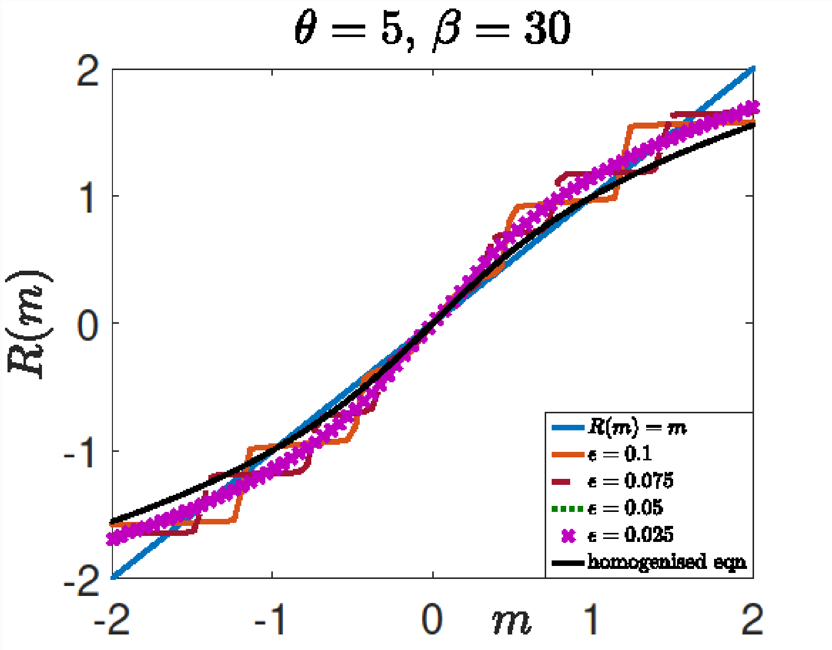
where $y = \frac{x}{\epsilon}$, and $V_1(x,y)$ is the multiscale term.

8 - see [SNG, G.A. Pavliotis, J. Nonlinear Sci 2018]

Numerical illustration of the nature of multi-scale potentials

From the previous expressions, we see that when $V_1(x, y) = V_1(y)$ **does not depend on x**, **the two limits commute**. However, if $V_1(x, y)$ **depends on x**, then ψ is not easy to compute and the limits **are not the same**.

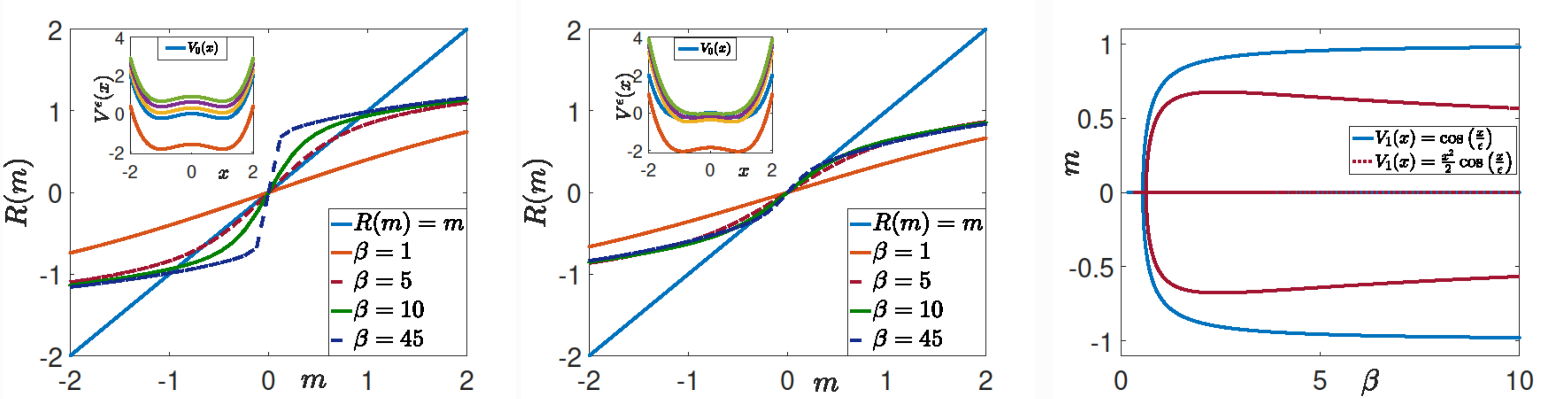




Plot of $R(m; \theta, \beta)$ and $R(m^\epsilon; \theta, \beta) = m$ for $\theta = 5$, $\beta = 30$, $\delta = 1$, and various values of ϵ for separable fluctuations (left) and multiplicative fluctuations (right).

Homogenised bistable potential

The bistable potential maintains its two extra solutions... But now the homogenised potential depends on β.



$R(m;\theta,\beta)$ compared to $y = x$ for $\theta = 0.5$, $\delta = 1$, and various values of β for the homogenised bistable potential with additive (left) and multiplicative (middle) fluctuations, and bifurcation diagram of m as a function of β for the additive (full line) and multiplicative (dashed line) fluctuations (right).

Finite ϵ: bistable potential, $\theta = 5, \delta = 1, \epsilon = 0.1$.

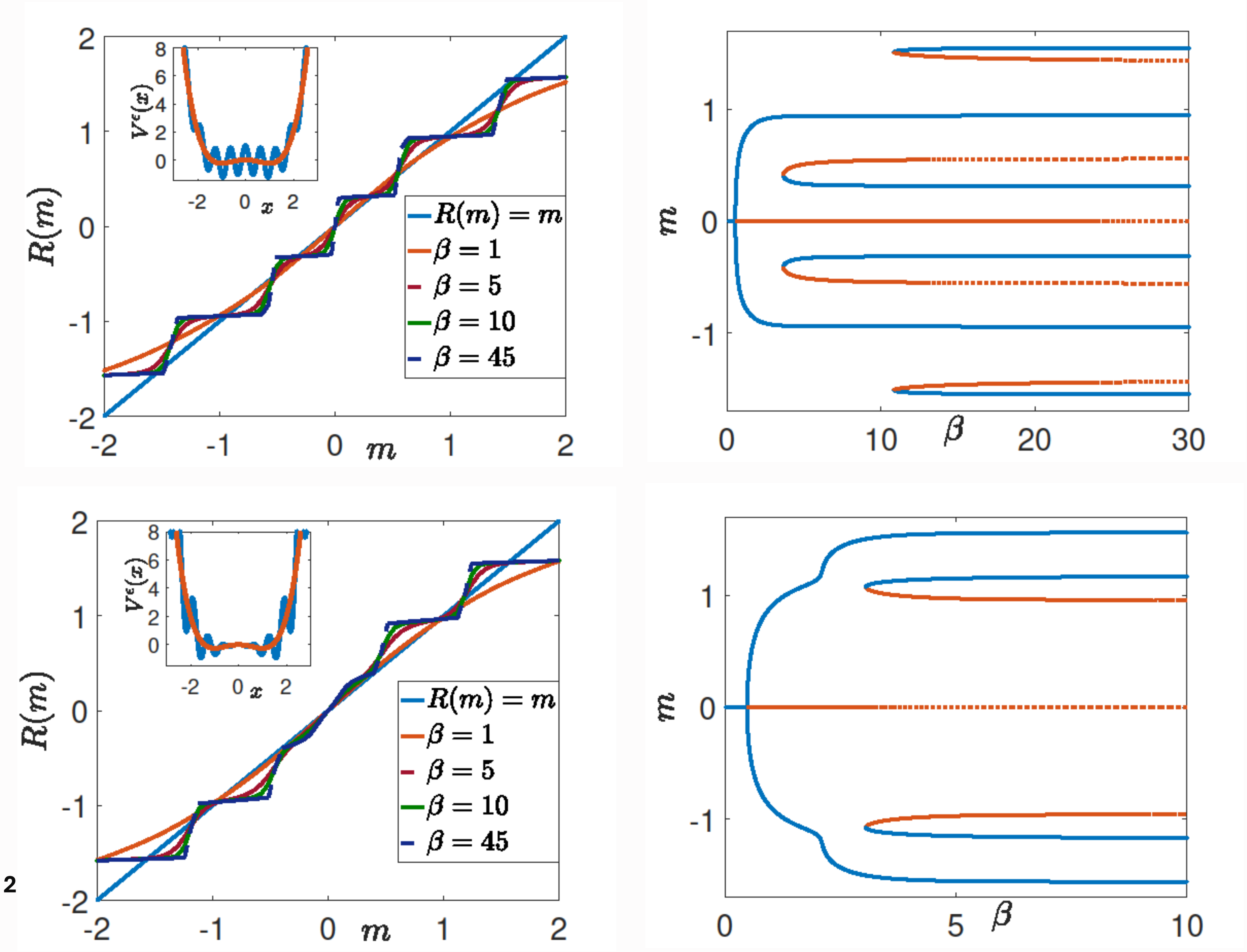


Figure: Behaviour of the bistable potential with multi-scale fluctuations for a fixed ϵ.

Top: Additive fluctuations $V_1\left(x, \frac{x}{\epsilon}\right) = \delta \cos\frac{x}{\epsilon}$

Bottom: Multiplicative fluctuations $V_1\left(x, \frac{x}{\epsilon}\right) = \delta x^2 \cos\frac{x}{\epsilon}$

Left: $R(m^\epsilon; \theta, \beta)$ for various values of β

Right: bifurcation diagram of m as a function of β. Full lines are stable steady states, and dashed lines are unstable ones.

Once we know more about the system…

There is a great interest in controlling interacting particle systems – knowing about all the possible steady states introduces the possibility of feedback control.

This is a recent area of research – see for example (very incomplete list):

- Using mean field limits to control particle systems [Bicego, Kalise, Pavliotis, Proceedings of the Royal Society A, 2025]
- Optimal control of the Fokker-Planck equation [Kalise, Moschen, Pavliotis, U. Vaes, IEEE Control Systems Letters, 2025]
- Model predictive control for these systems [Albi, Bicego, Herty, Huang, Kalise, Segala, Springer, book chapter, 2025]
- Linearisation based feedback control [Kalise, Moschen, Pavliotis, preprint 2025] based on periodic systems [Carrillo, Gvalani Pavliotis, Schlichting, ARMA 2019]

Other relevant work includes the case when the multi-scale dynamics happens in time, not in space. See, for example, [Goddard, Ottobre, Painter, Souttar, Proceedings of the Royal Society A, 2023].

Inference for SDEs

Parameter estimation – usual inverse problem setting

In several problems, one wants to estimate parameters present in our models (SDEs).

Consider an SDE that depends on a parameter,

$$dX_t = f(X_t; \theta)\, dt + dW_t,$$

where we assume we know the diffusion coefficient and $\sigma = 1$.

Intuitively, we want to find the best value of θ given an observation of a trajectory X_t. So we want to minimise

$$\phi(\theta; X_t) = \int_0^T \left|\dot{X}_t - f(X_t; \theta)\right|^2 \, dt.$$

However, X_t solves an SDE, so computing $\phi(\theta; X_t)$ is equivalent to integrating the square of the **derivative of a Brownian motion**!

Recall that...

The Brownian motion has unbounded variation - this means that it is not differentiable anywhere. In particular,

$$\mathbb{P}\left(\forall t > 0 : \limsup_{\Delta t \to 0} \left|\frac{B_{t+\Delta t} - B_t}{\Delta t}\right| = \infty\right) = 1.$$

For this reason, $\phi(\theta; X_t)$ is **almost surely infinite**, and we can't solve this inverse problem in the usual way!

A detour into likelihood based inference

If the problem we are modelling involves noise, we need to do something better. We can fix this by defining the **maximum likelihood estimator.** [8]

Assume we have a random variable X with probability distribution function $f(x;\theta)$, known up to *parameters* θ that we want to estimate from observations.

Suppose that we have J independent observations $\{x_j\}_{j=1}^{J}$ of X.

We define the likelihood function

$$L\left(\{x_j\}_{j=1}^{J};\theta\right) = \prod_{j=1}^{J} f(x_j;\theta).$$

The maximum likelihood estimator (MLE) is

$$\hat{\theta} = \arg\max L\left(\{x_j\}_{j=1}^{J};\theta\right).$$

Example: $X \sim \mathcal{N}(\boldsymbol{\mu},\boldsymbol{\sigma}^2)$

The parameters are the mean and variance, $\theta = (\mu,\sigma)$. We have

$$f(x;\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

The likelihood function is

$$L\left(\{x_j\}_{j=1}^{J};\theta\right) = \frac{e^{-\sum_{j=1}^{J}\frac{(x_j-\mu)^2}{2\sigma^2}}}{(2\pi\sigma)^{\frac{J}{2}}}.$$

We maximise w.r.t. μ and σ and

$$\hat{\mu} = \frac{1}{J}\sum_{j-1}^{J} x_j, \qquad \hat{\sigma} = \frac{1}{J}\sum_{j=1}^{J}(x_j-\hat{\mu})^2.$$

8 – see [Casella and Berger, Statistical inference, 2002.]

Maximum likelihood inference for SDEs

In our case, the observations are a *series of discrete observations of a stochastic process*: $\{X_{i\Delta t}\}_{i=1}^{M}$, which solve an SDE:

$$dX_t = f(X_t;\theta)\,dt + dW_t \;\leftrightarrow X_{(\mathrm{i+1})\Delta\mathrm{t}} = X_{i\Delta t} + \; f(X_{i\Delta t};\theta)\,\Delta t + \Delta W_{i\Delta t}.$$

Using the fact that $\Delta W_{i\Delta t} \sim \mathcal{N}(0,\Delta t)$, we can see that

$$\mathbb{P}\left(X_{(i+1)\Delta t}\middle|X_{i\Delta t}\right) \sim \mathcal{N}(f(X_{i\Delta t};\theta)\Delta \mathrm{t},\Delta t),$$

and therefore, writing $f_i = f(X_{i\Delta t};\theta)$, we can write the law of the process X_t:

$$p_X^M = \frac{1}{(2\pi\Delta t)^M}\,exp\left(-\sum_{i=0}^{M}\left(\frac{(\Delta X_i)^2}{2\Delta t} + \frac{(b_i)^2\Delta t}{2} - b_i\Delta X_i\right)\right).$$

Similarly, the distribution function for Brownian motion is

$$p_W^M = \prod_{i=0}^{M-1}\frac{1}{\sqrt{2\pi\Delta t}}\,exp\left(-\frac{1}{2\Delta t}(\Delta W_i)^2\right) = \frac{1}{(2\pi\Delta t)^M}\,exp\left(-\frac{1}{2\Delta t}\sum_{i=0}^{M-1}(\Delta W_i)^2\right).$$

MLE for SDEs (continued)

Now we can calculate the ratio of the laws of the two processes, evaluated at the path $\{X_{i\Delta t}\}_{i=1}^{M}$:

$$\frac{p_X^M}{p_W^M} = \exp\left(-\frac{1}{2}\sum_{i=0}^{M-1}(b_i)^2\Delta t + \sum_{i=0}^{M-1} b_i \Delta X_i\right).$$

Taking the limit $M \to \infty$, we get the likelihood function

$$L(X_t;\theta) = \frac{d\mathbb{P}^X}{d\mathbb{P}^W} = \exp\left(\int_0^T f(X_s;\theta)dX_s - \frac{1}{2}\int_0^T \big(f(X_s\,;\theta)\big)^2 ds\right).$$

Rigorously, this can be done using **Girsanov's theorem**: one can show that the law of X_t, $\mathbb{P}^X$, is absolutely continuous with respect to the law of the Brownian motion, $\mathbb{P}^W$, [9] and therefore the **likelihood function** is defined by the Radon-Nikodym derivative of $\mathbb{P}^X$ w.r.t. $\mathbb{P}^W$ which is given by the expression above.

The maximum likelihood estimator given the observed path is then given by

$$\hat{\theta} = \arg\max_{\theta\in\Theta} L\big(\{X_t\}_{t\in[0,T]};\theta\big).$$

9 – See [Sørensen, International Statistical Review, 2004] or [Pavliotis, Stochastic processes and applications, 2014] for a justification, or [Liptser and Shiryaev, Statistics of random processes: I. General theory. Vol. 1., 2001] for a proof.

Example

Consider the stationary Ornstein-Uhlenbeck process

$$dX_t = -\alpha X_t \, dt + dW_t, \qquad X_0 \sim \mathcal{N}\left(0, \frac{\alpha}{2}\right).$$

The log-likelihood function is

$$\log L(\{X_t\}_{t\in[0,T]}; \alpha) = -\alpha \int_0^T X_t \, dX_t - \frac{\alpha^2}{2} \int_0^T X_t^2 \, dt \, .$$

From this, the Maximum Likelihood estimator is

$$\hat{\alpha} = -\frac{\int_0^T X_t \, dX_t}{\int_0^T X_t^2 \, dt}.$$

To evaluate this estimator, we use a trajectory: given a set of discrete equidistant observations $\{X_{i\Delta t}\}_{i=1}^{M}$, we have, for $X_j = X_{j\Delta t}$ and $\Delta X_j = X_{j+1} - X_j$,

$$\hat{\alpha} = -\frac{\sum_{j=0}^{M-1} X_j \Delta X_j}{\sum_{j=0}^{M-1} |X_j|^2 \Delta t}.$$

One can show that this Maximum Likelihood estimator becomes asymptotically unbiased in the **large sample limit** $M \to \infty$, for Δt fixed.

Discussion

This morning, I hopefully convinced you that systems of interacting particles are ubiquitous in applications such as physics, biology, chemistry, life and social sciences.

We saw that …

- Their behaviour can be characterised by McKean–Vlasov or Fokker–Planck equations.
- The latter can be used to, e.g. explore long time behaviour and phase transitions of solutions.
  - we saw examples of multi-well and multiscale potentials exhibiting phase transitions
  - depending on the parameters, we also observe topology changes in the bifurcation diagrams.
- We also explored inference (parameter estimation) for SDEs (not dependent on law of the process)
- Other (non-discussed) applications include **particle swarm optimisation** or more recently **consensus-based optimisation (CBO).**

After lunch:
Some case studies including pedestrian dynamics, cell population dynamics, and opinion dynamics (if there is time).

Thank you for your attention!

After lunch: Some case studies including pedestrian dynamics, cell population dynamics, and opinion dynamics (if there is time).

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