Fokker-Planck représentation of stochastic neural fields: derivation, analysis and application to grid cells Lecture 2 : existence of solutions

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Vrije Universiteit Amsterdam,
From Microscopic Dynamics
to Continuum Models

ÉCOLE CENTRALE LYON

One population model

Recap on the model

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial s}\left(\Big[\Phi_{\bar\rho}(\mathbf{x},t) - s\Big]\rho\right) + \sigma\frac{\partial^2 \rho}{\partial s^2},$$

where $\Phi_{\bar\rho}(\mathbf{x},t)$ is given by

$$\Phi_{\bar\rho}(\mathbf{x},t) = \Phi\left(W * \bar\rho\,(\mathbf{x},t) + B\right),$$

where $B > 0$ is constant and

$$\bar\rho(\mathbf{x},t) = \int_0^\infty s\rho(\mathbf{x},s,t)\,\mathrm{d}s, \qquad \Phi_{\bar\rho}(\mathbf{x},t)\rho(\mathbf{x},0,t) - \sigma\frac{\partial \rho}{\partial s}(\mathbf{x},0,t) = 0.$$

Simplest case

Let's take the simplest possible case : $\Phi \equiv 0$ and $\sigma = 1$. Then,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial s}\Big(- s\rho\Big) + \frac{\partial^2 \rho}{\partial s^2}, \qquad \frac{\partial \rho}{\partial s}(\mathbf{x}, 0, t) = 0.$$

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Then, use the change of variable $y = e^t s$ and $\tau = \frac{1}{2}(e^{2t} - 1)$. Denoting $\alpha(\tau) = (2\tau + 1)^{-\frac{1}{2}} = e^{-t}$, the function

$$q(x, y, \tau) = \alpha(\tau)\rho\Big(x, \underbrace{y\alpha(\tau)}_{=s}, \underbrace{-\log(\alpha(\tau))}_{=t}\Big),$$

is solution to

$$\frac{\partial q}{\partial \tau} = \frac{\partial^2 q}{\partial^2 y}, \qquad \frac{\partial q}{\partial y}(\mathbf{x}, 0, t) = 0.$$

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the function q is now solution at each (x, y, τ) of the problem

$$\frac{\partial q}{\partial \tau} = \frac{\partial^2 q}{\partial y^2} - \Psi(x,\tau)\frac{\partial q}{\partial y}, \qquad \Psi(x,\tau) = \Phi_{\bar{\rho}}(x, -\log(\alpha(\tau)))\alpha(\tau), \tag{1}$$

with $\bar{\rho} = \alpha\bar{q}$ and boundary condition

$$\Psi(x,\tau)q(x, 0, \tau) - \frac{\partial q}{\partial y}(x, 0, t) = 0.$$

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Let us absorb the drift in the time derivative with the variable

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$$\begin{array}{rcll} \dfrac{\partial u}{\partial \tau}(x,z,\tau) & = & \dfrac{\partial^2 u}{\partial z^2}(x,z,\tau), & z \in (\gamma(x,\tau),+\infty), \\ \gamma(x,\tau) & = & -\int_0^{\tau} \Psi(x,\eta) d\eta, & \\ \dfrac{\partial u}{\partial y}(x,\gamma(x,\tau),\tau) & = & \Psi(x,\tau) u(x,\gamma(x,\tau),\tau), & \\ \Psi(x,\tau) & = & \Phi(\alpha(\tau) W * [\bar{u}(x,\tau) - \gamma(x,\tau)] + B)\alpha(\tau), & \end{array}$$

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$$\begin{aligned}
\frac{\partial u}{\partial \tau}(x,z,\tau) &= \frac{\partial^2 u}{\partial z^2}(x,z,\tau), && z \in (\gamma(x,\tau), +\infty),\\
\gamma(x,\tau) &= -\textstyle\int_0^\tau \Psi(x,\eta) d\eta, &&\\
\frac{\partial u}{\partial y}(x,\gamma(x,\tau),\tau) &= \Psi(x,\tau) u(x,\gamma(x,\tau),\tau), &&\\
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This is **still not** an easy problem :(.

Did we make any progress ?

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial s}\left(\left[\Phi_{\bar{\rho}}(\mathbf{x}, t) - s\right]\rho\right) + \sigma \frac{\partial^2 \rho}{\partial s^2}, \tag{3}$$

$$\Phi_{\bar{\rho}}(\mathbf{x}, t) = \Phi\left(W * \bar{\rho}\,(\mathbf{x}, t) + B\right), \quad \bar{\rho}(\mathbf{x}, t) = \int_0^{\infty} s\rho(\mathbf{x}, s, t)\,\mathrm{d}s, \tag{4}$$

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Change variables to obtain

$$\begin{aligned}
\frac{\partial u}{\partial \tau}(x, z, \tau) &= \frac{\partial^2 u}{\partial z^2}(x, z, \tau), \qquad z \in (\gamma(x, \tau), +\infty), \\
\gamma(x, \tau) &= -\int_0^\tau \Psi(x, \eta) d\eta, \\
\frac{\partial u}{\partial y}(x, \gamma(x, \tau), \tau) &= \Psi(x, \tau) u(x, \gamma(x, \tau), \tau), \\
\Psi(x, \tau) &= \Phi(\alpha(\tau) W * [\bar{u}(x, \tau) - \gamma(x, \tau)] + B)\alpha(\tau),
\end{aligned}$$

Solve the easy part at least

$$\frac{\partial u}{\partial \tau}(x, z, \tau) = \frac{\partial^2 u}{\partial z^2}(x, z, \tau), \qquad z \in (\gamma(x, \tau), +\infty).$$

In order to obtain a Duhamel formula for u, we use the heat kernel

$$G(z, \tau, \xi, \eta) = \frac{1}{\sqrt{4\pi(\tau - \eta)}} e^{-\frac{(z-\xi)^2}{4(\tau-\eta)}}.$$

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This kernel satisfies the Green identity

$$\frac{\partial}{\partial \xi}\left(G\frac{\partial u}{\partial \xi} - u\frac{\partial G}{\partial \xi}\right) - \frac{\partial}{\partial \eta}(Gu) = 0.$$

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For each $x \in \mathbb{T}^d$, integrate on $(\gamma(x,\eta), +\infty)$ and then on $(0,\tau)$:

$$\int_0^\tau \int_{\gamma(x,\eta)}^{+\infty} \frac{\partial}{\partial \xi}\left(G\frac{\partial u}{\partial \xi}\right) d\xi d\eta - \int_0^\tau \int_{\gamma(x,\eta)}^{+\infty} \frac{\partial}{\partial \xi}\left(u\frac{\partial G}{\partial \xi}\right) d\xi d\eta$$
$$- \int_0^\tau \int_{\gamma(x,\eta)}^{+\infty} \frac{\partial}{\partial \eta}(Gu) d\xi d\eta = 0.$$

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$$\underbrace{\int_0^\tau \int_{\gamma(x,\eta)}^{+\infty} \frac{\partial}{\partial \xi}\left(G\frac{\partial u}{\partial \xi}\right) d\xi d\eta}_{I} - \underbrace{\int_0^\tau \int_{\gamma(x,\eta)}^{+\infty} \frac{\partial}{\partial \xi}\left(u\frac{\partial G}{\partial \xi}\right) d\xi d\eta}_{II}$$

$$- \underbrace{\int_0^\tau \int_{\gamma(x,\eta)}^{+\infty} \frac{\partial}{\partial \eta}(Gu) d\xi d\eta}_{III} = 0.$$

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$$\frac{\partial u}{\partial \tau}(x,z,\tau) \;=\; \frac{\partial^2 u}{\partial z^2}(x,z,\tau), \qquad z \in (\gamma(x,\tau), +\infty).$$

Computing I, II, III we get

$$u(x,z,\tau) = \int_0^{+\infty} G(z,\tau,\xi,0)u^0(x,\xi)d\xi \tag{6}$$

$$+ \int_0^{\tau} \frac{\partial G}{\partial \xi}(z,\tau,\gamma(x,\eta),\eta)u(x,\gamma(x,\eta),\eta)d\eta. \tag{7}$$

Time evolution problem

Define

$$v(x,\tau) = u(x,\gamma(x,\tau),\tau).$$

And use the heat Kernel to find a closed system

$$\begin{cases} v(x,\tau) & = & F_v[v,\gamma,\bar{u}](x,\tau) & \text{boundary value,} \\ \gamma(x,\tau) & = & F_\gamma[v,\gamma,\bar{u}](x,\tau) & \text{boundary position,} \\ \bar{u}(x,\tau) & = & F_{\bar{u}}[v,\gamma,\bar{u}](x,\tau) & \text{average.} \end{cases}$$

Then, painfully apply a Fixed point argument to the system.

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$\longrightarrow$ Global existence can be obtained with uniform bounds on v, γ and $\bar{u}$.

Thank you !