Hydrodynamic Limits of the Boltzmann Equation: A Rigorous Derivation of the Navier-Stokes system

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Third Part

Rigorous convergence towards Navier-Stokes-Fourier system

Navier-Stokes scaling

Consider $\mathrm{St} = \varepsilon$ and the re-scaled Boltzmann equation

$$\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} \mathcal{Q}(F^\varepsilon, F^\varepsilon)$$

This corresponds to the scaling

$$F^\varepsilon(t, x, v) = F\left(\varepsilon^{-2} t, \varepsilon^{-1} x, v\right)$$

where

$$\partial_t F + v \cdot \nabla_x F = \mathcal{Q}(F, F)\,.$$

Ansatz

$$F^\varepsilon = \mathcal{M} + \varepsilon f^\varepsilon$$

where

$$\mathcal{M} = \mathcal{M}_{(1,0,1)}$$

is some steady Maxwellian state.

Ansatz

$$F^\varepsilon = \mathcal{M} + \varepsilon f^\varepsilon$$

where $\mathcal{M} = \mathcal{M}_{(1,0,1)}$ then

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \mathscr{L} f^\varepsilon + \frac{1}{\varepsilon} \mathcal{Q}(f^\varepsilon, f^\varepsilon) \tag{1.1}$$

where $\mathscr{L}$ is the linearized Boltzmann operator around some fixed $\mathcal{M}$

$$\mathscr{L} f = \mathcal{Q}(\mathcal{M}, f) + \mathcal{Q}(f, \mathcal{M})$$

Properties of the linearized operator

Natural space for $\mathscr{L}$ is the space $L^2(\mathbb{R}^3, \mathcal{M}^{-\frac{1}{2}}(v)\mathrm{d}v)$

Proposition

On the space $L^2_v(\mathcal{M}^{-\frac{1}{2}})$, the linearized operator, with domain

$$\mathscr{D}(\mathscr{L}) = \{f \in L^2_v(\mathcal{M}^{-\frac{1}{2}})\,;\, \Sigma(\cdot)f \in L^2_v(\mathcal{M}^{-\frac{1}{2}})\}$$

splits as

$$\mathscr{L}f(v) = \Sigma(v)f - \mathcal{K}f(v)$$

where

$$\Sigma(v) = \int_{\mathbb{R}^3\times\mathbb{S}^2} \mathcal{M}_\star B(|v - v_*|, \sigma)\mathrm{d}v_*\mathrm{d}\sigma$$

and

$$\mathcal{K}f(v) = \int_{\mathbb{R}^3\times\mathbb{S}^2} B(|v - v_*|, \sigma)\mathcal{M}\mathcal{M}_\star \left[\left(\frac{f}{\mathcal{M}}\right)' + \left(\frac{f}{\mathcal{M}}\right)'_\star - \left(\frac{f}{\mathcal{M}}\right)_\star\right] \mathrm{d}v_*\mathrm{d}\sigma.$$

Properties of the linearized operator

Proposition (Continued...)

It holds

1. *there is $\nu_\star > 0$ such that*

$$\nu_\star\,(1+|v|) \leqslant \Sigma(v) \leqslant \nu_\star^{-1}\,(1+|v|)\,, \qquad v \in \mathbb{R}^3\,.$$

2. *$\mathcal{K}$ is a compact operator on $L^2_v(\mathcal{M}^{-\frac{1}{2}})$.*
3. *$(-\mathscr{L}, \mathscr{D}(\mathscr{L}))$ is a self-adjoint nonnegative operator with*

$$\mathrm{Ker}\mathscr{L} = \mathrm{Span}\left\{\mathcal{M}, v_1\mathcal{M}, \ldots, v_d\mathcal{M}, |v|^2\mathcal{M}\right\}$$

Properties of the linearized operator

Corollary (Spectral gap)

There is $\lambda_\star > 0$ *such that*

$$\langle \mathscr{L} f, f \rangle \leqslant -\lambda \|f - \boldsymbol{\pi}_0 f\|^2_{L^2_v(\mathcal{M}^{-\frac{1}{2}})}, \qquad f \in \mathscr{D}(\mathscr{L})$$

where $\boldsymbol{\pi}_0$ *is the orthogonal projection over* $\mathrm{Ker}\mathscr{L}$*: if* $g = g(x,v) \in L^2_x L^2_v(\mathcal{M}^{-\frac{1}{2}})$ *then*

$$\boldsymbol{\pi}_0 g(x,v) = \left[\varrho_g(x) + \boldsymbol{u}_g(x) \cdot v + \frac{1}{2}\theta_g(x)\left(|v|^2 - d\right)\right]\mathcal{M}(v)$$

with

$$\varrho_g(x) = \int_{\mathbb{R}^d} g(x,v)\mathrm{d}v, \qquad \boldsymbol{u}_g(x) = \int_{\mathbb{R}^d} v g(x,v)\mathrm{d}v$$

and

$$\theta_g(x) = \frac{1}{d}\int_{\mathbb{R}^d} \left(|v|^2 - d\right) g(x,v)\mathrm{d}v.$$

Properties of the linearized operator

Corollary (Fredholm alternative)

On the space $L^2_v(\mathcal{M}^{-\frac{1}{2}})$, *one has*

$$\mathrm{Range}\mathscr{L} = (\mathrm{Ker}\mathscr{L})^{\perp} = \left\{ g \in L^2_v(\mathcal{M}^{-\frac{1}{2}})\ ;\ \int_{\mathbb{R}^3} g \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} \mathrm{d}v = 0 \right\}$$

and $\mathscr{L}_{|\mathrm{Ker}\mathscr{L}^{\perp}}$ *is invertible: for any* $f \in \mathrm{Im}\mathscr{L}$, *the equation*

$$\mathscr{L} g = f$$

has a unique solution $g \in \mathrm{Range}(\boldsymbol{Id} - \boldsymbol{\pi}_0)$.

Need for *a priori* estimates

Study of the *full linearized* operator

$$\mathcal{G}_\varepsilon h = \varepsilon^{-2}\mathscr{L} h - \varepsilon^{-1} v \cdot \nabla_x h \,.$$

in the Hilbert setting

$$\mathcal{H} := \mathbb{H}^m_x L^2_v(\mathcal{M}^{-\frac{1}{2}}), \qquad m > \frac{d}{2}$$

Hypocoercivity

There is a competition between the *coercive effect* of $\mathscr{L}$ and the *conservative effect* of transport $-v \cdot \nabla_x$ (even for $\varepsilon = 1$).

Recall that coercivity occurs only on $\mathrm{Range}(\mathbf{Id} - \boldsymbol{\pi}_0)$. We introduce here

$$\mathbf{P}_0(g) := \sum_{i=1}^{d+2} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} g \, \Psi_i \, \mathrm{d}x \, \mathrm{d}v \right) \Psi_i \, \mathcal{M}$$

and

$$\mathcal{H}_1 := \mathbb{H}^m_x L^2_v(\mathcal{M}^{-\frac{1}{2}} \langle \cdot \rangle^{\frac{1}{2}})$$

L^2-hypocoercivity

Hypocoercivity

- Understand the mixing between the transport operator which is skew-adjoint and conservative with the collision operator which is dissipative in velocity.
- Very convenient to study hydrodynamical problems because it heavily relies on the *micro–macro* decomposition of the solution

$$f^\varepsilon = \mathbf{P}_0 f^\varepsilon + (\mathbf{Id} - \mathbf{P}_0) f^\varepsilon$$

- Villani, 2005
- Hérau, 2006, Dolbeault, Mouhot, Schmeiser, 2009
- alternative path Guo, 2010 (for $\varepsilon = 1$).

L^2-hypocoercivity

Proposition

On the space $\mathcal{H}$, there exists a norm $|||\cdot|||_{\mathcal{H}}$ with associated inner product $\langle\langle \cdot, \cdot \rangle\rangle_{\mathcal{H}}$ equivalent to the standard norm $\| \cdot \|_{\mathcal{H}}$ for which there exist $\mathrm{a}_1 > 0$ and $\mathrm{a}_2 > 0$ such that

$$\langle\langle \mathcal{G}_\varepsilon h, h \rangle\rangle_{\mathcal{H}} \leqslant -\frac{\mathrm{a}_1}{\varepsilon^2} \|(\mathbf{Id} - \boldsymbol{\pi}_0)\, h\|^2_{\mathcal{H}_1} - \mathrm{a}_1 \|h\|^2_{\mathcal{H}_1} - \mathrm{a}_2 |||h|||^2_{\mathcal{H}} \tag{1.2}$$

holds true for any $h = h^\perp = (\mathbf{Id} - \mathbf{P}_0)h \in \mathscr{D}(\mathcal{G}_\varepsilon) \subset \mathcal{H}$.

Remark that the equivalent norm $|||\cdot|||_{\mathcal{H}}$ actually depends on ε but the "equivalence of norms" is uniform, there exists $C_{\mathcal{H}} > 0$ independent of $\varepsilon \in (0,1)$ such that

$$C_{\mathcal{H}} \|h\|_{\mathcal{H}} \leqslant |||h|||_{\mathcal{H}} \leqslant C_{\mathcal{H}}^{-1} \|h\|_{\mathcal{H}} \,, \qquad \forall h \in \mathcal{H} \,.$$

L^2-hypocoercivity

For general $f \in L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})$, the splitting $f = f^\perp + \boldsymbol{\pi}_0 f$, $f^\perp = (\mathbf{Id} - \boldsymbol{\pi}_0)f$ implies that

$$\|f\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} = \|f^\perp\|^2_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} + \|\varrho[f]\|^2_{L^2_x} + \|u[f]\|^2_{L^2_x} + \|\theta[f]\|^2_{L^2_x}\,.$$

where

$$\varrho[f] := \int_{\mathbb{R}^d} f(\cdot, v)\,\mathrm{d}v\,, \qquad u[f] = \int_{\mathbb{R}^d} v\, f(\cdot, v)\,\mathrm{d}v \in \mathbb{R}^d\,,$$

and $\theta[f]$ is defined through

$$\varrho[f] + \theta[f] = \frac{1}{d}\int_{\mathbb{R}^d} |v|^2 f(\cdot, v)\,\mathrm{d}v\,.$$

Notice that, for $f \in \mathscr{D}(\mathcal{G}_\varepsilon) \cap \mathrm{Range}(\mathbf{Id} - \mathbf{P}_0)$, one has $\mathbf{P}_0 f = \int_{\mathbb{T}^d} \boldsymbol{\pi}_0 f\,\mathrm{d}x = 0$ so that $(-\Delta_x)^{-1}\varrho[f], (-\Delta_x)^{-1} u_k[f]$ and $(-\Delta_x)^{-1}\theta[f]$ are well-defined and

$$\left\|(-\Delta_x)^{-1}\varrho[f]\right\|_{\mathbb{H}^2_x} \lesssim \|\varrho[f]\|_{L^2_x} \lesssim \|f\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}\,,$$

$$\left\|(-\Delta_x)^{-1} u_k[f]\right\|_{\mathbb{H}^2_x} \lesssim \|u_k[f]\|_{L^2_x} \lesssim \|f\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}$$

$$\text{and} \qquad \left\|(-\Delta_x)^{-1}\theta[f]\right\|_{\mathbb{H}^2_x} \lesssim \|\theta[f]\|_{L^2_x} \lesssim \|f\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}\,.$$

L^2-hypocoercivity

We define then

$$\psi_k[f](x) = \int_{\mathbb{R}^d} p_k(v) f(x,v) \mathrm{d}v, \qquad \boldsymbol{\Theta}_{k\ell}[f](x) := \int_{\mathbb{R}^d} p_{k\ell}(v) f(x,v) \mathcal{M}(v)\, \mathrm{d}v\,, \qquad x \in \mathbb{T}^d$$

for suitable (well-chosen second-order polynomial) functions $p_k, p_{k\ell}$ chosen so that

$$\boldsymbol{\Theta}_{k\ell}[f] = \boldsymbol{\Theta}_{k\ell}[f^\perp] \quad \text{if} \quad k \neq \ell\,, \quad \text{while} \quad \boldsymbol{\Theta}_{kk}[f] = \boldsymbol{\Theta}_{kk}[f^\perp] - \frac{d-1}{2}\theta[f]\,.$$

Since $\boldsymbol{b}_k$ and $p_{k\ell}$ are polynomial function, a simple use of Cauchy-Schwarz inequality shows that

$$\|\psi_k[f]\|_{L^2_x} + \|\boldsymbol{\Theta}_{k\ell}[f]\|_{L^2_x} \lesssim \|f\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}\,,$$

and

$$\|\boldsymbol{\Theta}_{k\ell}[f]\|_{L^2_x} \lesssim \|\theta[f]\|_{L^2_x} + \|f^\perp\|_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}\,.$$

L^2-hypocoercivity

On $\mathcal{H}$, one defines

Definition (Equivalent inner product)

If $\mathbf{P}_0 f = \mathbf{P}_0 g = 0$, then we define

$$
\begin{aligned}
\langle\!\langle f, g\rangle\!\rangle_H := \langle f, g\rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})} \\
&+ \varepsilon\eta_1 \sum_{k=1}^{d} \left(\langle \partial_{x_k} (-\Delta_x)^{-1} \theta[f], \psi_k[g]\rangle_{L^2_x} + \langle \partial_{x_k} (-\Delta_x)^{-1} \theta[g]\, , \psi_k[f]\rangle_{L^2_x}\right) \\
&+ \varepsilon\eta_2 \sum_{k,\ell=1}^{d} \left(\langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[f]\, , \boldsymbol{\Theta}_{k\ell}[g]\rangle_{L^2_x} + \langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[g]\, , \boldsymbol{\Theta}_{k\ell}[f]\rangle_{L^2_x}\right) \qquad (1.3) \\
&+ \varepsilon\eta_3 \sum_{k=1}^{d} \left(\langle \partial_{x_k} (-\Delta_x)^{-1} \varrho[f]\, , u_k[g]\rangle_{L^2_x} + \langle \partial_{x_k} (-\Delta_x)^{-1} \varrho[g]\, , u_k[f]\rangle_{L^2_x}\right)
\end{aligned}
$$

and set $\langle\!\langle f, g\rangle\!\rangle_H = \langle f, g\rangle_{L^2_{x,v}(\mathcal{M}^{-\frac{1}{2}})}$ *otherwise.*

L^2-hypocoercivity

Suitable choice of η_1, η_2, η_3 gives

$$\langle\!\langle\, \mathcal{G}_\varepsilon h, h \,\rangle\!\rangle_{\mathcal{H}} \leqslant -\frac{\mathrm{a}_1}{\varepsilon^2} \left\|(\mathbf{Id} - \boldsymbol{\pi}_0)\, h\right\|_{\mathcal{H}_1}^2 - \mathrm{a}_1 \|h\|_{\mathcal{H}_1}^2 - \mathrm{a}_2 |\!|\!| h |\!|\!|_{\mathcal{H}}^2$$

for $h = h^\perp$.

Exercise

The inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathcal{H}}$ *associated to the norm* $|\!|\!|\cdot|\!|\!|_{\mathcal{H}}$ *on* $\mathcal{H}$ *is such that for any* $g \in \mathcal{H}_1$

$$\langle\!\langle (\mathbf{Id} - \boldsymbol{\pi}_0)\, \mathcal{Q}(g, g), g \rangle\!\rangle_{\mathcal{H}} \lesssim \|g\|_{\mathcal{H}_1} \|g\|_{\mathcal{H}} \;\left\|(\mathbf{Id} - \boldsymbol{\pi}_0)\, g\right\|_{\mathcal{H}_1} \tag{1.4}$$

Hint: $\langle\!\langle (\mathbf{Id} - \boldsymbol{\pi}_0)\, \mathcal{Q}(g, g), g \rangle\!\rangle_{\mathcal{H}} = \langle \mathcal{Q}(g, g), (\mathbf{Id} - \boldsymbol{\pi}_0)\, g \rangle$.

Consequences

Proposition (Strong decay of the linearized semigroup)

Let $(\mathcal{U}_\varepsilon(t))_{t\geqslant 0}$ be the strongly continuous semigroup in $\mathcal{H}$ generated by $\mathcal{G}_\varepsilon$, it holds

$$\|\mathcal{U}_\varepsilon(t)f\|_{\mathcal{H}} \lesssim \exp\left(-\frac{\mathrm{a}_1}{\varepsilon^2}\right)\|(\mathbf{Id}-\boldsymbol{\pi}_0)\,f\|^2_{\mathcal{H}_1}\,.$$

Remark (Crucial point for nonlinear analysis)

The nonlinear dynamics occurs on $\mathrm{Range}\,(\mathbf{Id}-\boldsymbol{\pi}_0)$ *since*

$$\boldsymbol{\pi}_0\mathcal{Q}(f,f)=0\,.$$

Consequences

Recall

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \mathscr{L} f^\varepsilon + \frac{1}{\varepsilon} \mathcal{Q}(f^\varepsilon, f^\varepsilon)$$

with

$$f_{\rm in}^\varepsilon \in \mathrm{Range}\left(\mathbf{Id} - \boldsymbol{\pi}_0\right).$$

This gives

$$\partial_t f^\varepsilon = \mathcal{G}_\varepsilon f^\varepsilon + \frac{1}{\varepsilon} \mathcal{Q}(f^\varepsilon, f^\varepsilon)$$

or in mild form

$$\begin{aligned} f^\varepsilon(t) &= \mathcal{U}_\varepsilon(t) f_{\rm in}^\varepsilon + \frac{1}{\varepsilon} \int_0^t \mathcal{U}_\varepsilon(t-s) \mathcal{Q}(f^\varepsilon(s), f^\varepsilon(s)) \mathrm{d}s \\ &= \mathcal{U}_\varepsilon(t) f_{\rm in}^\varepsilon + \frac{1}{\varepsilon} \int_0^t \mathcal{U}_\varepsilon(t-s) \left(\mathbf{Id} - \boldsymbol{\pi}_0\right) \mathcal{Q}(f^\varepsilon(s), f^\varepsilon(s)) \mathrm{d}s \end{aligned}$$

Energy estimate

Take the inner product (in $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathcal{H}}$) of the equation

$$\partial_t f^\varepsilon = \mathcal{G}_\varepsilon f^\varepsilon + \frac{1}{\varepsilon} (\boldsymbol{Id} - \boldsymbol{\pi}_0)\, \mathcal{Q}(f^\varepsilon, f^\varepsilon)$$

with f^ε, it gives

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |\!|\!| f^\varepsilon(t) |\!|\!|_{\mathcal{H}}^2 &= \langle\!\langle \mathcal{G}_\varepsilon f^\varepsilon, f^\varepsilon \rangle\!\rangle_{\mathcal{H}} + \frac{1}{\varepsilon} \langle\!\langle (\boldsymbol{Id} - \boldsymbol{\pi}_0)\, \mathcal{Q}(f^\varepsilon, f^\varepsilon), f^\varepsilon \rangle\!\rangle_{\mathcal{H}} \\
&\leqslant -\frac{\mathrm{a}_1}{\varepsilon^2} \, \| (\mathbf{Id} - \boldsymbol{\pi}_0)\, f^\varepsilon(t) \|_{\mathcal{H}_1}^2 - \mathrm{a}_2 \| f^\varepsilon(t) \|_{\mathcal{H}_1}^2 - \mathrm{a}_1 |\!|\!| f^\varepsilon(t) |\!|\!|_{\mathcal{H}}^2 \\
&\qquad + \frac{C}{\varepsilon} \| f^\varepsilon(t) \|_{\mathcal{H}_1} \| f^\varepsilon(t) \|_{\mathcal{H}} \, \| (\mathbf{Id} - \boldsymbol{\pi}_0)\, f^\varepsilon(t) \|_{\mathcal{H}_1}
\end{aligned}$$

Energy estimate

Young/Cauchy-Schwarz inequality in last term, for any $\eta > 0$, there is $C_\eta > 0$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||f^\varepsilon(t)|||^2_{\mathcal{H}} \leqslant -\mathrm{a}_1|||f^\varepsilon(t)|||^2_{\mathcal{H}} - \frac{1}{\varepsilon^2}(\mathrm{a}_1-\eta)\,\|(\mathbf{Id}-\boldsymbol{\pi}_0)\,f^\varepsilon(t)\|^2_{\mathcal{H}_1} \\ -\left(\mathrm{a}_2 - C_\eta\|f^\varepsilon(t)\|^2_{\mathcal{H}}\right)\|f^\varepsilon(t)\|^2_{\mathcal{H}_1} \\ \leqslant -\mathrm{a}_1|||f^\varepsilon(t)|||^2_{\mathcal{H}} - \frac{1}{\varepsilon^2}(\mathrm{a}_1-\eta)\,\|(\mathbf{Id}-\boldsymbol{\pi}_0)\,f^\varepsilon(t)\|^2_{\mathcal{H}_1} \\ -\left(\mathrm{a}_2 - \widetilde{C}_\eta|||f^\varepsilon(t)|||^2_{\mathcal{H}}\right)\|f^\varepsilon(t)\|^2_{\mathcal{H}_1}\end{aligned}$$

Choose $\eta \ll 1$ so that $\mathrm{a}_1 - \eta > \frac{1}{2}\mathrm{a}_1$. Then

$$|||f^\varepsilon(t)|||^2_{\mathcal{H}} < \frac{\mathrm{a}_2}{\widetilde{C}_\eta} \implies \frac{\mathrm{d}}{\mathrm{d}t}|||f^\varepsilon(t)|||^2_{\mathcal{H}} \leqslant 0.$$

Energy estimate

Theorem (Well-posedness and stability)

There is an explicit $\delta > 0$ *and* $C > 0$ *such that, if*

$$\|f_{\rm in}^\varepsilon\|_{\mathcal{H}} \leqslant \delta$$

then, for any $\varepsilon > 0$*, there is a unique solution* $f^\varepsilon(t) \in \mathcal{H}$ *to*

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \mathscr{L} f^\varepsilon + \frac{1}{\varepsilon} \mathcal{Q}(f^\varepsilon, f^\varepsilon)$$

with

$$\|f^\varepsilon(t)\|_{\mathcal{H}} \leqslant C \|f_{\rm in}^\varepsilon\|_{\mathcal{H}} \exp\left(-\frac{1}{2} \mathrm{a}_1 t\right) \qquad \forall t \geqslant 0,$$

$$\int_0^T \|f^\varepsilon(t)\|_{\mathcal{H}_1}^2 \, \mathrm{d}t \leqslant C \|f_{\rm in}^\varepsilon\|_{\mathcal{H}}^2, \qquad \forall T > 0,$$

and

$$\int_0^T \|(\mathbf{Id} - \boldsymbol{\pi}_0)\, f^\varepsilon(t)\|_{\mathcal{H}_1}^2 \leqslant C \varepsilon^2 \|f_{\rm in}^\varepsilon\|_{\mathcal{H}}^2, \qquad \forall T > 0.$$

Proof of the estimates is an Exercise based upon Gronwall argument (existence via suitable approximation procedure).

Enough to prove convergence

$$\|f^\varepsilon\|_{L^\infty_t(\mathcal{H})} + \|f^\varepsilon\|_{L^2_t(\mathcal{H}_1)} \lesssim \|f^\varepsilon_{\mathrm{in}}\|_{\mathcal{H}}$$

and

$$\|(\mathbf{Id} - \boldsymbol{\pi}_0)\, f^\varepsilon\|_{L^2_t(\mathcal{H}_1)} \lesssim \varepsilon,$$

and

$$\int_{t_1}^{t_2} \|(\mathbf{Id} - \boldsymbol{\pi}_0) f^\varepsilon(\tau)\|_{\mathcal{H}_1}\,\mathrm{d}\tau \lesssim \varepsilon\sqrt{t_2 - t_1}$$

for any $0 \leqslant t_1 \leqslant t_2 \leqslant T$.

Remark

$\{(\mathbf{Id} - \boldsymbol{\pi}_0)\, f^\varepsilon\}_\varepsilon$ *converges* ***strongly*** *to* 0 *in* $L^2((0,T);\mathcal{H})$.

Enough to prove convergence

The formal arguments used to derive Navier-Stokes equation can now be made rigorous.

Corollary

Up to a subsequence, f^{ε} **converges weakly** *in* $L^2\left((0,T)\,;\mathcal{H}\right)$ *towards some* $f \in \mathrm{Ker}\mathscr{L}$*, i.e.*

$$f(t,x,v) = \left(\varrho(t,x) + u(t,x)\cdot v + \frac{1}{2}\theta(t,x)(|v|^2 - d\vartheta_1)\right)\mathcal{M}(v)$$

with

$$\varrho \in L^2\left((0,T)\,;\mathbb{H}^m_x(\mathbb{T}^d)\right)\,, \qquad \boldsymbol{u} \in L^2\left((0,T)\,;\left(\mathbb{H}^m_x(\mathbb{T}^d)\right)^d\right)\,,$$

$$\theta \in L^2\left((0,T)\,;\mathbb{H}^m_x(\mathbb{T}^d)\right)\,.$$

Enough to prove convergence

Lemma

Introduce for $(t,x) \in (0,T) \times \mathbb{T}^d$*:*

$$\boldsymbol{u}_\varepsilon(t,x) := \mathbb{P}\Big\langle v f^\varepsilon \Big\rangle \qquad \text{and} \qquad \boldsymbol{\theta}_\varepsilon(t,x) := \Big\langle \frac{1}{2}\Big(|v|^2 - (d+2)\Big) f^\varepsilon \Big\rangle .$$

Then, $\{\partial_t \boldsymbol{u}_\varepsilon\}_\varepsilon$ *and* $\{\partial_t \boldsymbol{\theta}_\varepsilon\}_\varepsilon$ *are bounded in* $L^1\left((0,T)\,;\mathbb{H}^m_x(\mathbb{T}^d)\right)$*. Consequently, up to the extraction of a subsequence,*

$$\int_0^T \|\boldsymbol{u}_\varepsilon(t) - \boldsymbol{u}(t)\|_{\mathbb{H}^{m-1}_x(\mathbb{T}^d)}\,\mathrm{d}t \xrightarrow[\varepsilon\to 0]{} 0$$

and

$$\int_0^T \|\boldsymbol{\theta}_\varepsilon(t,\cdot) - \boldsymbol{\theta}_0(t,\cdot)\|_{\mathbb{H}^{m-1}_x(\mathbb{T}^d)}\,\mathrm{d}t \xrightarrow[\varepsilon\to 0]{} 0$$

where

$$\boldsymbol{\theta}_0(t,x) := \Big\langle \frac{1}{2}(|v|^2 - (d+2)) f \Big\rangle = \frac{d}{2}\left(\varrho(t,x) + \theta(t,x)\right) - \frac{d+2}{2}\varrho(t,x) .$$

In other words, $\{\mathbb{P}u_\varepsilon\}_\varepsilon$ *(resp.* $\{\boldsymbol{\theta}_\varepsilon\}_\varepsilon$*) converges strongly to* $\boldsymbol{u} = \mathbb{P}\boldsymbol{u}$ *(resp.* $\boldsymbol{\theta}_0$*) in the space* $L^1\left((0,T)\,;\mathbb{H}^{m-1}_x(\mathbb{T}^d)\right)$*.*

The strong convergence of $\mathbb{P}u_\varepsilon$ towards $\boldsymbol{u}$ in $L^1\left((0,T);\mathbb{H}^{m-1}_x(\mathbb{T}^d)\right)$ and the weak convergence of u_ε implies that

$$\mathbb{P}\mathrm{Div}_x\left(u_\varepsilon\otimes u_\varepsilon-(\mathbf{Id}-\mathbb{P})u_\varepsilon\otimes(\mathbf{Id}-\mathbb{P})u_\varepsilon\right)\xrightarrow[\varepsilon\to 0]{}\mathbb{P}\mathrm{Div}_x\left(\boldsymbol{u}\otimes\boldsymbol{u}\right)\qquad\text{in}\qquad\mathscr{D}'_{t,x}\,.$$

To justify the convergence of $\mathbb{P}\mathrm{Div}_x\left\langle\boldsymbol{A}f^\varepsilon\right\rangle$ we only to prove that

$$\mathbb{P}\mathrm{Div}_x\left((\mathbf{Id}-\mathbb{P})u_\varepsilon\otimes(\mathbf{Id}-\mathbb{P})u_\varepsilon\right)\xrightarrow[\varepsilon\to 0]{}0\qquad\text{in}\qquad\mathscr{D}'_{t,x}\,.$$

and

$$\mathrm{div}_x\left(\beta_\varepsilon\left(\mathbf{Id}-\mathbb{P}\right)u_\varepsilon\right)\xrightarrow[\varepsilon\to 0]{}0\qquad\text{in}\qquad\mathscr{D}'_{t,x}\,.$$

where we set

$$\beta_\varepsilon:=\frac{1}{d}\left\langle|v|^2f^\varepsilon\right\rangle.$$

- The strong convergence of $\boldsymbol{\theta}_\varepsilon$ towards $\boldsymbol{\theta}_0$ in $L^1\left((0,T)\,;\mathbb{H}_x^{m-1}(\mathbb{T}^d)\right)$ together with the weak convergence of u_ε to $\boldsymbol{u}$ gives

$$\frac{2}{(d+2)}\mathrm{div}_x(u_\varepsilon \boldsymbol{\theta}_\varepsilon) \xrightarrow[\varepsilon\to 0]{} \frac{2}{(d+2)}\mathrm{div}_x(u\,\boldsymbol{\theta}_0) \qquad \text{in} \qquad \mathscr{D}'_{t,x}$$

- The strong convergence of $\mathbb{P}u_\varepsilon$ to $\boldsymbol{u}$ with the weak convergence of $\boldsymbol{\beta}_\varepsilon$ towards $\varrho+\theta=0$ gives

$$\mathrm{div}_x(\boldsymbol{\beta}_\varepsilon \mathbb{P}\boldsymbol{u}_\varepsilon) \xrightarrow[\varepsilon\to 0]{} \mathrm{div}_x\left(\boldsymbol{u}\,(\varrho+\theta)\right) = 0 \qquad \text{in} \qquad \mathscr{D}'_{t,x}$$

Consequence:

$$\mathrm{div}_x(\theta_\varepsilon u_\varepsilon) - \frac{2}{(d+2)}\mathrm{div}_x\left(\boldsymbol{\beta}_\varepsilon\left(\mathbf{Id}-\mathbb{P}\right)u_\varepsilon\right) \xrightarrow[\varepsilon\to 0]{} \boldsymbol{u}\cdot\nabla_x\theta \qquad \text{in} \qquad \mathscr{D}'_{t,x}\,.$$

Compensated compactness argument

Proposition (P. L. Lions and N. Masmoudi, 1999)

Let $c \neq 0$ and $T > 0$. Consider two families $\{\phi_\varepsilon\}_\varepsilon$ and $\{\psi_\varepsilon\}$ bounded in $L^\infty((0,T)\,;L^2_x(\mathbb{T}^d))$ and in $L^\infty((0,T)\,;\mathbb{H}^1_x(\mathbb{T}^d))$ respectively, such that

$$\begin{cases} \partial_t \nabla_x \psi_\varepsilon + \dfrac{c^2}{\varepsilon} \nabla_x \phi_\varepsilon = \dfrac{1}{\varepsilon} F_\varepsilon \\[2ex] \partial_t \phi_\varepsilon + \dfrac{1}{\varepsilon} \Delta_x \psi_\varepsilon = \dfrac{1}{\varepsilon} G_\varepsilon \end{cases}$$

where F_ε and G_ε converge strongly to 0 in $L^1((0,T)\,;\,L^2_x(\mathbb{T}^d))$. Then,

$$\mathbb{P}\mathrm{Div}_x\left(\nabla_x \psi_\varepsilon \otimes \nabla_x \psi_\varepsilon\right) \xrightarrow[\varepsilon \to 0]{} 0\,, \qquad \mathrm{div}_x\left(\phi_\varepsilon \nabla_x \psi_\varepsilon\right) \xrightarrow[\varepsilon \to 0]{} 0$$

in the sense of distributions on $(0,T) \times \mathbb{T}^d$.

One observes that

$$\varepsilon\,\partial_t u_\varepsilon + \nabla_x \boldsymbol{\beta}_\varepsilon = -\mathrm{Div}_x \left\langle \boldsymbol{A}\, h_\varepsilon \right\rangle \tag{1.5}$$

whereas

$$\varepsilon \partial_t \boldsymbol{\beta}_\varepsilon + \mathrm{div}_x \left\langle \frac{1}{d} |v|^2 v\, f^\varepsilon \right\rangle = 0 \tag{1.6}$$

where we check easily that

$$\begin{aligned}
\mathrm{div}_x \left\langle \frac{1}{d} |v|^2 v\, f^\varepsilon \right\rangle &= \frac{2}{d} \mathrm{div}_x \left\langle \boldsymbol{b}\, f^\varepsilon \right\rangle + \frac{d+2}{d} \mathrm{div}_x u_\varepsilon \\
&= \frac{2}{d} \mathrm{div}_x \left\langle \boldsymbol{b}\, f^\varepsilon \right\rangle + \frac{d+2}{d} \mathrm{div}_x \left(\mathbf{Id} - \mathbb{P} \right) u_\varepsilon \,.
\end{aligned}$$

Write

$$(\mathbf{Id} - \mathbb{P})u_\varepsilon = \nabla_x \boldsymbol{U}_\varepsilon$$

with $\boldsymbol{U}_\varepsilon \in L^\infty\left((0,T)\,;\left(\mathbb{H}_x^{m-1}(\mathbb{T}^d)\right)^d\right)$. After applying $(\mathbf{Id} - \mathbb{P})$ to (1.5), we obtain that $\boldsymbol{U}_\varepsilon$ and β_ε satisfy

$$\begin{cases} \varepsilon\partial_t \nabla_x \boldsymbol{U}_\varepsilon + \nabla_x \beta_\varepsilon = \boldsymbol{F}_\varepsilon \\ \\ \varepsilon\partial_t \beta_\varepsilon + \frac{d+2}{d}\Delta_x \boldsymbol{U}_\varepsilon = \boldsymbol{G}_\varepsilon \end{cases} \tag{1.7}$$

with

$$\boldsymbol{F}_\varepsilon := -(\mathbf{Id} - \mathbb{P})\mathrm{Div}_x\left\langle \boldsymbol{A}\, f^\varepsilon \right\rangle, \qquad \boldsymbol{G}_\varepsilon := -\frac{2}{d}\mathrm{div}_x\left\langle \boldsymbol{b}\, f^\varepsilon \right\rangle.$$

are such that

$$\|\boldsymbol{F}_\varepsilon\|_{L^1((0,T)\,;\,\mathbb{H}_x^{m-1}(\mathbb{T}^d))} \lesssim \varepsilon \qquad \text{and} \qquad \|\boldsymbol{G}_\varepsilon\|_{L^1((0,T)\,;\,\mathbb{H}_x^{m-1}(\mathbb{T}^d))} \lesssim \varepsilon\,.$$

Both $\boldsymbol{F}_\varepsilon$ and $\boldsymbol{G}_\varepsilon$ converge *strongly* to 0 in $L^1((0,T)\,;L^2_x(\mathbb{T}^d))$ and

$$\boldsymbol{U}_\varepsilon \in L^\infty((0,T)\,;(\mathbb{H}^1_x(\mathbb{T}^d))^d)\,, \qquad \boldsymbol{\beta}_\varepsilon \in L^\infty((0,T)\,;L^2_x(\mathbb{T}^d))\,.$$

Then, from the *compensated compactness* argument, one deduces that

$$\mathbb{P}\mathrm{Div}_x\left((\mathbf{Id}-\mathbb{P})u_\varepsilon \otimes (\mathbf{Id}-\mathbb{P})u_\varepsilon\right) \xrightarrow[\varepsilon\to 0]{} 0 \qquad \text{in} \qquad \mathscr{D}'_{t,x}$$

and

$$\mathrm{div}_x\left(\boldsymbol{\beta}_\varepsilon\,(\mathbf{Id}-\mathbb{P})\,u_\varepsilon\right) \xrightarrow[\varepsilon\to 0]{} 0 \qquad \text{in} \qquad \mathscr{D}'_{t,x}\,.$$

Final result

Set

$$\mathscr{W}_\ell := \left(\mathbb{H}^\ell_x\left(\mathbb{T}^d\right)\right)^{d+2}, \quad \ell \in \mathbb{N}.$$

Theorem

For $m > \frac{d}{2}$, we suppose that there exists $(\varrho_0, u_0, \theta_0) \in \mathscr{W}_m$ such that

$$\|\boldsymbol{\pi}_0 f^\varepsilon_{\mathrm{in}} - f_0\|_{\mathcal{H}} \xrightarrow[\varepsilon \to 0]{} 0\,,$$

where

$$f_0(x,v) := \left(\varrho_0(x) + u_0(x)\cdot v + \frac{1}{2}\theta_0(x)(|v|^2 - d)\right)\mathcal{M}(v)\,.$$

Then, for any $T > 0$, the family of solutions

$$\{f^\varepsilon\}_\varepsilon \text{ converges weakly in } \quad L^2((0,T)\,,\mathbb{H}^m_x(L^2_v(\mathcal{M}^{-1}\mathrm{d}v))$$

to a limit f....

Final result

Theorem (Continued...)

The limit f is such that

$$f(t,x,v) = \left(\varrho(t,x) + \boldsymbol{u}(t,x)\cdot v + \frac{1}{2}\theta(t,x)(|v|^2 - d)\right)\mathcal{M}(v)\,,$$

where

$$(\varrho, \boldsymbol{u}, \theta) \in \mathcal{C}\left([0,T]\,;\mathscr{W}_{m-1}\right) \cap L^2\left((0,T)\,;\mathscr{W}_m\right)$$

is solution to the following incompressible Navier-Stokes-Fourier system

$$\begin{cases} \partial_t \boldsymbol{u} - \nu\,\Delta_x \boldsymbol{u} + \boldsymbol{u}\cdot\nabla_x\,\boldsymbol{u} + \nabla_x p = 0\,, \\ \partial_t\,\theta - \gamma\,\Delta_x\theta + \boldsymbol{u}\cdot\nabla_x\theta = 0\,, \\ \mathrm{div}_x\boldsymbol{u} = 0\,, \qquad \varrho + \theta = 0\,, \end{cases}$$

subject to initial conditions $(\varrho_{\rm in}, u_{\rm in}, \theta_{\rm in})$ defined by

$$u_{\rm in} := \mathbb{P}u_0\,, \quad \theta_{\rm in} := \frac{d}{d+2}\theta_0 - \frac{2}{(d+2)}\varrho_0\,, \quad \varrho_{\rm in} := -\theta_{\rm in}\,,$$

Improvement: going beyond the $L^2_v(\mathcal{M}^{-1})$ framework

Question

Is it possible to prove the convergence for initial data lying in some L^1_v-based space ?

We work with initial datum

$$f^\varepsilon_{\rm in} \in \mathcal{E} := L^1_v \mathbb{H}^m_x(\varpi_q), \qquad \varpi_q(v) = \left(1 + |v|^2\right)^{\frac{q}{2}}, \qquad \ell > \frac{d}{2}$$

and look for solutions with same kind of *a priori* estimates in $\mathcal{E}$ (uniformly with respect to ε).

First issue

Properties of the (spatially homogeneous) linearized operator $\mathscr{L}$ in the space $L^1_v(\varpi_q(v)\mathrm{d}v)$

Mouhot, 2005, Gualdani, Mischler, Mouhot, 2013 – Enlargement/factorization techniques.

Enlargement/factorization

Lemma

For any $\delta > 0$, *there is a decomposition of* $\mathscr{L}$ *as*

$$\mathscr{L} f = \mathcal{A}^{(\delta)} f + \mathcal{B}^{(\delta)} f, \qquad f \in \mathscr{D}(\mathscr{L}) \subset L^1_v(\varpi_q(v)\mathrm{d}v)$$

where $\mathcal{A}^{(\delta)}$ *is a regularizing operator while* $\mathcal{B}^{(\delta)}$ *is dissipative. Precisely, for any* $k \in \mathbb{N}$ *and* $\delta > 0$,

$$\mathcal{A}^{(\delta)} f \in \mathbb{W}^{k,2}_v(\mathbb{R}^d) \text{ with compact support for any } f \in L^1_v(\varpi_1)$$

and there is $\nu_q > 0$ *such that*

$$\mathcal{B}^{(\delta)} + \nu_q \text{ is dissipative in } L^1_v(\varpi_q).$$

Consequences for the fully linearized operator

On the space $\mathcal{E} = \mathbb{H}^\ell_x L^1_v(\varpi_q)$, it holds

$$\begin{aligned} \mathcal{G}_\varepsilon f &:= \varepsilon^{-2} \mathscr{L} f - \varepsilon^{-1} v \cdot \nabla_x f \\ &= \mathcal{A}^{(\delta)}_\varepsilon f + \mathcal{B}^{(\delta)}_\varepsilon f \end{aligned}$$

with

$$\mathcal{A}^{(\delta)}_\varepsilon := \varepsilon^{-2} \mathcal{A}^{(\delta)}, \qquad \mathcal{B}^{(\delta)}_\varepsilon := \varepsilon^{-2} \mathcal{B}^{(\delta)} - \varepsilon^{-1} v \cdot \nabla_x .$$

Enlargement/factorization

Proposition

For any $\ell \geqslant 0$ and $q > 2$, there exist $\delta_q^\dagger > 0$ and $\nu_q > 0$ such that for any $\varepsilon \in (0,1]$,

$$\mathcal{B}_\varepsilon^{(\delta)} + \varepsilon^{-2}\nu_q \quad \textit{is dissipative in } \mathbb{H}_x^\ell L_v^1(\varpi_q)$$

for any $\delta \in (0, \delta_q^\dagger)$. In particular, for $\ell = 0$,

$$\int_{\mathbb{R}^d} \|h(\cdot,v)\|_{L_x^2}^{-1} \left(\int_{\mathbb{T}^d} \mathcal{B}_\varepsilon^{(\delta)}(h)(x,v)h(x,v)\,\mathrm{d}x \right) \varpi_q(v)\,\mathrm{d}v \leqslant -\varepsilon^{-2}\nu_q \|h\|_{L_x^2 L_v^1 \varpi_{q+1})}\,.$$

We will now work with fixed δ, q and simply writes $\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon$ and $\mu_0 = \nu_q$.

How to use this to the hydrodynamic limit?

Main idea

Split the solution $f^\varepsilon \in \mathcal{E}$ to

$$\partial_t f^\varepsilon = \mathcal{G}^\varepsilon f - \varepsilon^{-1} \mathcal{Q}(f^\varepsilon, f^\varepsilon)$$

into two parts

$$f^\varepsilon = f_0^\varepsilon + f_1^\varepsilon$$

one lying in $\mathcal{E}$ and one in the smaller space $\mathcal{H}$:

$$f_0^\varepsilon \in \mathcal{E}, \qquad f_1^\varepsilon \in \mathcal{H}\,.$$

Look for the evolution of both parts.

Remark

$\mathcal{A}_\varepsilon \in \mathscr{B}(\mathcal{E}, \mathcal{H})$ and on the following continuous embeddings:

$$\mathcal{H} \hookrightarrow \mathcal{E}_1 \hookrightarrow \mathcal{E}$$

with $\mathcal{E}_1 = \mathscr{D}(\mathscr{L}) \cap \mathcal{E} = \mathbb{H}^m_x L^1_v(\varpi_{q+1})$. No regularizing effect of $\mathcal{A}^\varepsilon$ in the x-variable, so same number of derivatives in $\mathcal{E}, \mathcal{H}$.

Splitting of the equation

Coupling system

$$
\left\{
\begin{aligned}
\partial_t f_0^\varepsilon &= \mathcal{B}_\varepsilon f_0^\varepsilon \\
&\quad + \varepsilon^{-1}\Big[\mathcal{Q}(f_0^\varepsilon, f_0^\varepsilon) + \mathcal{Q}(f_0^\varepsilon, f_1^\varepsilon) + \mathcal{Q}(f_1^\varepsilon, f_0^\varepsilon) + \underbrace{\pi_0 \mathcal{Q}(f_1^\varepsilon, f_1^\varepsilon)}_{=0}\Big], \\
f_0^\varepsilon(0,x,v) &= f_{\rm in}^\varepsilon(x,v) \in \mathcal{E}
\end{aligned}
\right.
\tag{1.8}
$$

and

$$
\left\{
\begin{aligned}
\partial_t f_1^\varepsilon &= \mathcal{G}_\varepsilon f_1^\varepsilon + \varepsilon^{-1}\left(\mathbf{Id} - \pi_0\right)\mathcal{Q}(f_1^\varepsilon, f_1^\varepsilon) + \mathcal{A}_\varepsilon f_0^\varepsilon\,, \\
f_1^\varepsilon(0,x,v) &= 0 \in \mathcal{H}\,.
\end{aligned}
\right.
\tag{1.9}
$$

If $f_0^\varepsilon, f_1^\varepsilon$ solve these coupled equations, then $f^\varepsilon = f_0^\varepsilon + f_1^\varepsilon \in \mathcal{E}$ is such that

$$
\partial_t f^\varepsilon = \mathcal{G}_\varepsilon f^\varepsilon + \varepsilon^{-1}\mathcal{Q}(f^\varepsilon, f^\varepsilon), \qquad f^\varepsilon(0,x,v) = f_{\rm in}^\varepsilon(x,v).
$$

Splitting of the equation

Main ideas:

- The equation for f_1^ε can be solved as in the previous case with the L^2-hypocoercivity of $\mathcal{G}_\varepsilon$ and the regularizing effect of $\mathcal{A}_\varepsilon$ (take care that $\mathcal{A}_\varepsilon$ is stiff

$$\|\mathcal{A}_\varepsilon f_0^\varepsilon\|_{\mathcal{H}} \lesssim \varepsilon^{-2}\|f_0^\varepsilon\|_{\mathcal{E}_1}$$

- The equation for f_0^ε is more involved and truly in $\mathcal{E}$. But $\mathcal{B}_\varepsilon$ is strongly dissipative on $\mathcal{E}$.

Assume that $f_0^\varepsilon \in \mathcal{E}$, $f_1^\varepsilon \in \mathcal{H}$ are solutions to (1.8)-(1.9) and that there exists $\Delta_0 \leqslant 1$ such that

$$\sup_{t \geqslant 0} \left(\|f_0^\varepsilon(t)\|_{\mathcal{E}} + \|f_1^\varepsilon(t)\|_{\mathcal{H}}\right) \leqslant \Delta_0 \,. \tag{1.10}$$

Prove *a priori* estimates for $f_0^\varepsilon, f_1^\varepsilon$.

Energy method for f_0^ε

Using the dissipativity of $\mathcal{B}_\varepsilon$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f_0^\varepsilon(t)\|_{\mathcal{E}} \leqslant -\frac{\mu_0}{\varepsilon^2}\|f_0^\varepsilon(t)\|_{\mathcal{E}_1} + \frac{1}{\varepsilon}\Big(\|\mathcal{Q}(f_0^\varepsilon(t), f_0^\varepsilon(t))\|_{\mathcal{E}} + \|\mathcal{Q}(f_0^\varepsilon(t), f_1^\varepsilon(t))\|_{\mathcal{E}} + \|\mathcal{Q}(f_1^\varepsilon(t), f_0^\varepsilon(t))\|_{\mathcal{E}}\Big)$$

Using classical estimates for $\mathcal{Q}$: there exists $C > 0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|f_0^\varepsilon(t)\|_{\mathcal{E}} &\leqslant -\frac{\mu_0}{\varepsilon^2}\|f_0^\varepsilon(t)\|_{\mathcal{E}_1} + \frac{C}{\varepsilon}\Big(\|f_0^\varepsilon(t)\|_{\mathcal{E}} + \|f_1^\varepsilon(t)\|_{\mathcal{E}_1}\Big)\|f_0^\varepsilon(t)\|_{\mathcal{E}_1} \\ &\leqslant -\frac{1}{\varepsilon^2}\left(\mu_0 - \varepsilon C\Big(\|f_0^\varepsilon(t)\|_{\mathcal{E}} + \|f_1^\varepsilon(t)\|_{\mathcal{E}_1}\Big)\right)\|f_0^\varepsilon(t)\|_{\mathcal{E}_1} \\ &\leqslant -\frac{1}{\varepsilon^2}\left(\mu_0 - 2\varepsilon C\Delta_0\right)\|f_0^\varepsilon(t)\|_{\mathcal{E}_1},\end{aligned}$$

where we used (1.10). Thus, for ε small enough

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f_0^\varepsilon(t)\|_{\mathcal{E}} \leqslant -\frac{\mu_0}{2\varepsilon^2}\|f_0^\varepsilon(t)\|_{\mathcal{E}_1},$$

Strong decay of f_0^ε

$$\|f_0^\varepsilon(t)\|_{\mathcal{E}} \leqslant \|f_{\rm in}^\varepsilon\|_{\mathcal{E}} \exp\left(-\frac{\mu_0}{2\varepsilon^2}t\right).$$

Plug this in the equation for f_1^ε. Micro-macro decomposition

$$f_1^\varepsilon = \mathbf{P}_0 f_1^\varepsilon + (\mathbf{Id} - \mathbf{P}_0)\, f_1^\varepsilon\,.$$

Study of the macroscopic part

$$\|\mathbf{P}_0 f_1^\varepsilon\|_{\mathcal{H}} \lesssim \|f_0^\varepsilon\|_{\mathcal{E}}$$

since $\mathbf{P}_0 f^\varepsilon = 0$ and $\mathbf{P}_0 \in \mathscr{B}(\mathcal{E}, \mathcal{H})$.

Enough to study $h_1^\varepsilon := (\mathbf{Id} - \mathbf{P}_0)\, f_1^\varepsilon = \mathbf{P}_0^\perp f_1^\varepsilon$

$$\partial_t h_1^\varepsilon = \mathbf{P}_0^\perp \mathcal{G}_\varepsilon f_1^\varepsilon + \varepsilon^{-1}\mathbf{P}_0^\perp \mathcal{Q}(f_1^\varepsilon, f_1^\varepsilon) + \mathbf{P}_0^\perp \mathcal{A}_\varepsilon f_0^\varepsilon$$

with $\mathbf{P}_0^\perp \mathcal{G}_\varepsilon f_1^\varepsilon = \mathcal{G}_\varepsilon\left(\mathbf{P}_0^\perp f_1^\varepsilon\right) = \mathcal{G}_\varepsilon h_1^\varepsilon$.

Energy Method used before for the study in $\mathcal{H}$ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||h_1^\varepsilon(t)|||_{\mathcal{H}}^2 \leqslant -\frac{\mathrm{a}_1}{\varepsilon^2}\|\left(\mathbf{Id}-\boldsymbol{\pi}_0\right)h_1^\varepsilon(t)\|_{\mathcal{H}_1}^2 - \mathrm{a}_2\|h_1^\varepsilon(t)\|_{\mathcal{H}_1}^2 - \mathrm{a}_1|||h_1^\varepsilon(t)|||_{\mathcal{H}}^2$$
$$+\,\varepsilon^{-1}\langle\!\langle(\mathbf{Id}-\boldsymbol{\pi}_0)\,\mathcal{Q}(f_1^\varepsilon(t),f_1^\varepsilon(t)),h_1^\varepsilon(t)\rangle\!\rangle_{\mathcal{H}}$$
$$+\,\frac{C}{\varepsilon^2}\|h_1^\varepsilon(t)\|_{\mathcal{H}}\,\|f_0^\varepsilon(t)\|_{\mathcal{E}}$$

Estimate on $\mathcal{Q}$

$$\varepsilon^{-1}\langle\!\langle(\mathbf{Id}-\boldsymbol{\pi}_0)\,\mathcal{Q}(f_1^\varepsilon(t),f_1^\varepsilon(t)),h_1^\varepsilon(t)\rangle\!\rangle_{\mathcal{H}} \leqslant C\varepsilon^{-1}\|f_1^\varepsilon(t)\|_{\mathcal{H}_1}\|f_1^\varepsilon(t)\|_{\mathcal{H}}\,\|\left(\mathbf{Id}-\boldsymbol{\pi}_0\right)h_1^\varepsilon(t)\|_{\mathcal{H}_1}$$
$$+\,C\|f_1^\varepsilon(t)\|_{\mathcal{H}}^2\,\|\boldsymbol{\pi}_0 h_1^\varepsilon(t)\|_{\mathcal{H}}$$
$$\leqslant \frac{\eta}{\varepsilon^2}\,\|(\mathbf{Id}-\boldsymbol{\pi}_0)\,h_1^\varepsilon(t)\|_{\mathcal{H}_1}^2 + \frac{C^2}{4\eta}\|f_1^\varepsilon(t)\|_{\mathcal{H}}^2\|f_1^\varepsilon(t)\|_{\mathcal{H}_1}^2 + C\|f_1^\varepsilon(t)\|_{\mathcal{H}}^3\,, \qquad \eta>0$$

thanks to Young's inequality.

Choosing $\eta \leqslant \mathrm{a}_1$, one sees that there is some positive constant $c_0 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}|||h_1^\varepsilon(t)|||_{\mathcal{H}}^2 \leqslant -2\mathrm{a}_1\|h_1^\varepsilon(t)\|_{\mathcal{H}_1}^2 + c_0\|f_1^\varepsilon(t)\|_{\mathcal{H}}^2\left(\|f_1^\varepsilon(t)\|_{\mathcal{H}_1}^2 + \|f_1^\varepsilon(t)\|_{\mathcal{H}}\right) + \frac{c_0}{\varepsilon^2}\|f_1^\varepsilon(t)\|_{\mathcal{H}}\,\|f_0^\varepsilon(t)\|_{\mathcal{E}}$$

where

$$c_0\|f_1^\varepsilon(t)\|_{\mathcal{H}}^2\left(\|f_1^\varepsilon(t)\|_{\mathcal{H}_1}^2 + \|f_1^\varepsilon(t)\|_{\mathcal{H}}\right) \leqslant c_1\Delta_0^2\|h_1^\varepsilon(t)\|_{\mathcal{H}_1}^2 + c_1\Delta_0\|f_1^\varepsilon(t)\|_{\mathcal{H}}^2\,.$$

Key estimate

For $\Delta_0 > 0$ small enough so that $\widetilde{\mu}_1 := 2\mathrm{a}_1 - c_1\Delta_0^2 > 0$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}|||h_1^\varepsilon(t)|||_{\mathcal{H}}^2 \leqslant -\widetilde{\mu}_1\,\|h_1^\varepsilon(t)\|_{\mathcal{H}_1}^2 + c_1\Delta_0\|f_1^\varepsilon(t)\|_{\mathcal{H}}^2 + \frac{c_0}{\varepsilon^2}\|f_1^\varepsilon(t)\|_{\mathcal{H}}\,\|f_0^\varepsilon(t)\|_{\mathcal{E}}\,, \qquad \forall\, t \geqslant 0\,.$$

Consequently, for $\mu_1 > 0$ (related to $\widetilde{\mu}_1$ and the equivalent constant relating $|||\cdot|||$ to $\|\cdot\|_{\mathcal{H}}$)

$$\|h_1^\varepsilon(t)\|_{\mathcal{H}}^2 \lesssim \Delta_0\int_0^t e^{-\mu_1(t-s)}\|f_1^\varepsilon(s)\|_{\mathcal{H}}^2\,\mathrm{d}s + \frac{1}{\varepsilon^2}\int_0^t e^{-\mu_1(t-s)}\|f_1^\varepsilon(s)\|_{\mathcal{H}}\,\|f_0^\varepsilon(s)\|_{\mathcal{E}}\,\mathrm{d}s \tag{1.11}$$

Exercise

Strong decay of $f_0^\varepsilon(t)$ implies

$$\frac{1}{\varepsilon^2}\int_0^t e^{-\mu_1(t-s)}\|f_1^\varepsilon(s)\|_{\mathcal{H}}\,\|f_0^\varepsilon(s)\|_{\mathcal{E}}\,\mathrm{d}s$$

$$\leqslant \|f_{\rm in}^\varepsilon\|_{\mathcal{E}}\,e^{-\mu_1\,t}\varepsilon^{-2}\int_0^t e^{-\frac{\mu_0}{2\varepsilon^2}s+\mu_1 s}\|f_1^\varepsilon(s)\|_{\mathcal{H}}\,\mathrm{d}s$$

$$\leqslant \|f_{\rm in}^\varepsilon\|_{\mathcal{E}}\,e^{-\mu_1\,t}\Delta_0\varepsilon^{-2}\int_0^t e^{-\frac{\mu_0}{4\varepsilon^2}s}\mathrm{d}s \leqslant \Delta_0\|f_{\rm in}^\varepsilon\|_{\mathcal{E}}\,e^{-\mu_1\,t}$$

for $\varepsilon > 0$ small enough. Thus

$$\|(\mathbf{Id}-\mathbf{P}_0)f_1^\varepsilon(t)\|_{\mathcal{H}}^2 \lesssim \Delta_0\|f_{\rm in}^\varepsilon\|_{\mathcal{E}}\,e^{-\mu_1 t} + \Delta_0\int_0^t e^{-\mu_1(t-s)}\|f_1^\varepsilon(s)\|_{\mathcal{H}}^2\,\mathrm{d}s\,.$$

Gronwall argument

Final estimates

$$\|f_0^\varepsilon(t)\|_{\mathcal{E}}^2 \leqslant \|f_{\rm in}^\varepsilon\|_{\mathcal{E}}^2 \exp\left(-\frac{\mu_0}{\varepsilon^2}\,t\right), \qquad t \geqslant 0,$$

and, for any $\lambda_1 < \mu_1$, one can choose Δ_0 small enough so that

$$\|f_1^\varepsilon(t)\|_{\mathcal{H}}^2 \leqslant C\left(\|f_{\rm in}^\varepsilon\|_{\mathcal{E}}^2 + \|f_{\rm in}^\varepsilon\|_{\mathcal{E}}\right)\exp\left(-\lambda_1 t\right).$$

Theorem

There exists $\delta > 0$ small enough such that, if

$$\|f_{\rm in}^\varepsilon\|_{\mathcal{E}} \leqslant \delta$$

then, the coupled system of equations (1.8)–(1.9) *admits unique solutions $f_0^\varepsilon(t), f_1^\varepsilon(t)$ with moreover*

$$\|f_0^\varepsilon\|_{L^\infty((0,T)\,;\,\mathcal{E})} \lesssim 1 \quad \textit{and} \quad \|f_0^\varepsilon\|_{L^1((0,T)\,;\,\mathcal{E}_1)} \lesssim \varepsilon^2$$

as well as

$$\|f_1^\varepsilon\|_{L^\infty((0,T)\,;\,\mathcal{H})} \lesssim 1 \quad \textit{and} \quad \|f_1^\varepsilon\|_{L^2((0,T)\,;\,\mathcal{H}_1)} \lesssim 1$$

Compactness and convergence

Theorem

There exists $f = \boldsymbol{\pi}_0(f) \in L^2\left((0,T);\mathcal{H}\right)$ *such that up to extraction of a subsequence, one has*

$$\begin{cases} \{f_0^\varepsilon\}_\varepsilon \text{ converges to } 0 \text{ strongly in } L^1((0,T);\mathcal{E}_1)\,, \\ \{f_1^\varepsilon\}_\varepsilon \text{ converges to } f \text{ weakly in } L^2\left((0,T);\mathcal{H}\right)\,. \end{cases}$$

In particular, there exist

$$\varrho \in L^2\left((0,T);\mathbb{H}_x^m(\mathbb{T}^d)\right)\,, \qquad \boldsymbol{u} \in L^2\left((0,T);\left(\mathbb{H}_x^m(\mathbb{T}^d)\right)^d\right)\,,$$
$$\theta \in L^2\left((0,T);\mathbb{H}_x^m(\mathbb{T}^d)\right)\,,$$

such that

$$f(t,x,v) = \left(\varrho(t,x) + \boldsymbol{u}(t,x)\cdot v + \frac{1}{2}\theta(t,x)(|v|^2 - d)\right)\mathcal{M}(v)\,.$$

With this, we recover the Navier-Stokes-Fourier system for $\varrho, \boldsymbol{u}, \theta$ as in the previous case.

This method in L^1_v space has been used for *inelastic Boltzmann equation* ALONSO, L., TRISTANI 2023-2025 and is particularly robust to handle also the presence of *source term S_ε*.

- Allow to treat source with $\|S_\varepsilon\| \simeq \infty$ as $\varepsilon \to 0$ provided $\boldsymbol{\pi}_0 S_\varepsilon = O(1)$.
- $L^1_v(\varpi_q)$ is a natural space for models having equilibrium state with heavier tails than Maxwellian.
- For inelastic model, need to take into account other kind of behaviours (self-similar change of variables induces additional drift term).

Alternative approach

Spectral method

- Mild formulation of both BE and Navier-Stokes equation with semigroup theory.
- Sharp description of the spectrum of the linearized Boltzmann operator $\mathcal{G}_\varepsilon$ according to Fourier modes.
- Decomposition of the semigroup in a dominant *fluid part* and *kinetic* and *oscillatory* parts.
- Fixed point argument exploiting the limit equation for ε small enough.
- Sharp results as far as regularity is concerned, quantitative convergence in strong form.

Method introduced by Bardos, Ukai, 1991, improved in Gallagher, Tristani, 2020 and further extended by Gervais, 2023. Unified version for general kinetic models with spectral gap Gervais, L. 2023.

Optimal results as far as regularity is concerned in recent contribution Carrapatoso, Gallagher, Tristani, 2025.

Alternative approach

Theorem (Carrapatoso, Gallagher, Tristani, 2025)

Let $\frac{3}{2} < m \leqslant 2$ be given. Consider $(\varrho_{\mathrm{in}}, \boldsymbol{u}_{\mathrm{in}}, \theta_{\mathrm{in}}) \in \mathbb{H}_x^{\frac{1}{2}}(\mathbb{T}^3)$ that are mean-free with $\nabla_x \cdot \boldsymbol{u}_{\mathrm{in}} = 0$ and $\varrho_{\mathrm{in}} + \theta_{\mathrm{in}} = 0$. Let

$$(\varrho, \boldsymbol{u}, \theta) \in L^\infty((0,T)\,;\,\mathbb{H}_x^{\frac{1}{2}}(\mathbb{T}^3)) \cap L^2((0,T)\,;\,\mathbb{W}_x^{\frac{3}{2},2}(\mathbb{T}^3))$$

be the unique solution to Navier-Stokes-Fourier system associated with the initial data $(\varrho_{\mathrm{in}}, \boldsymbol{u}_{\mathrm{in}}, \theta_{\mathrm{in}})$ for some $T > 0$. For some suitable choice of the initial datum $f_{\mathrm{in}}^\varepsilon$. Then, there is $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a unique solution

$$f^\varepsilon \in L^\infty((0,T)\,;\,\mathbb{H}_x^m(L_v^2(\mathcal{M}^{-\frac{1}{2}}))$$

to the Boltzmann equation and it converges strongly in $L_t^\infty \mathbb{W}_x^{\frac{3}{2},2}(L_v^2(\mathcal{M}^{-\frac{1}{2}}))$ towards

$$f(t,x,v) = \Big(\varrho(t,x) + \boldsymbol{u}(t,x)\cdot v + \frac{1}{2}\theta(t,x)(|v|^2 - d)\Big)\,\mathcal{M}(v)\,,$$

as $\varepsilon \to 0$.