

Numerical computation of coherent structures in spatially-extended systems

Daniele Avitabile

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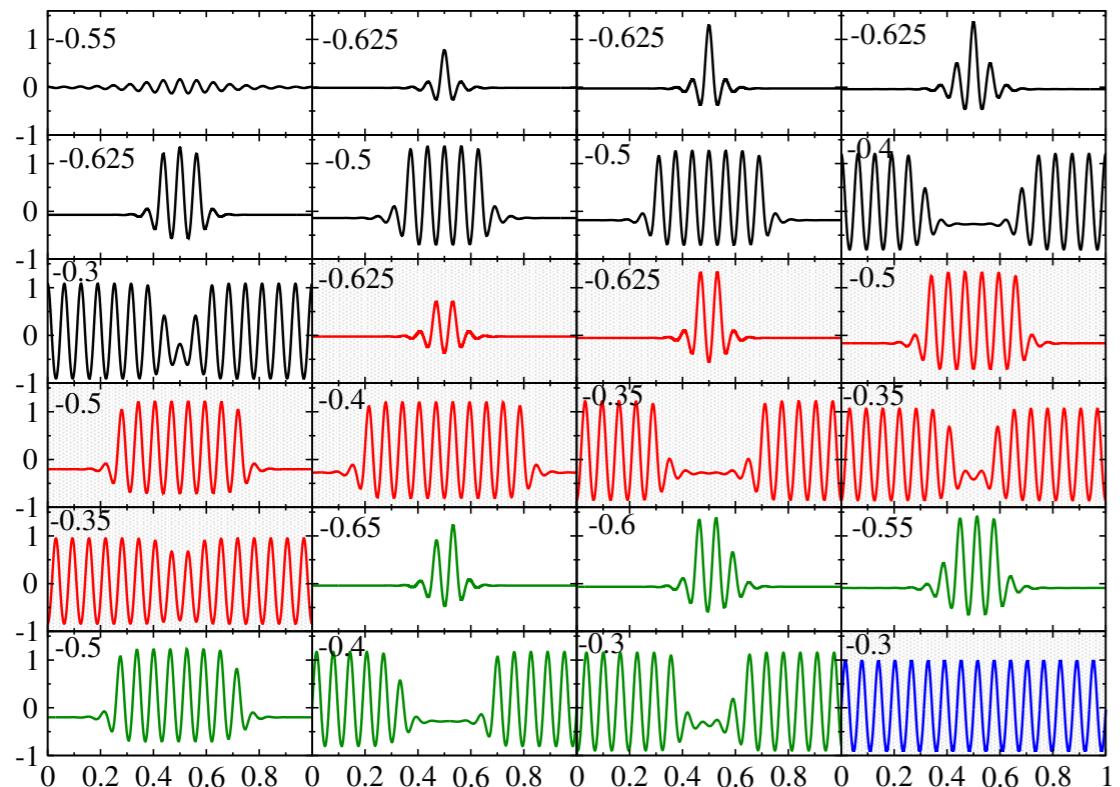
and

MathNeuro Team, Inria Sophia Antipolis

Example of coherent structures

Localised steady states

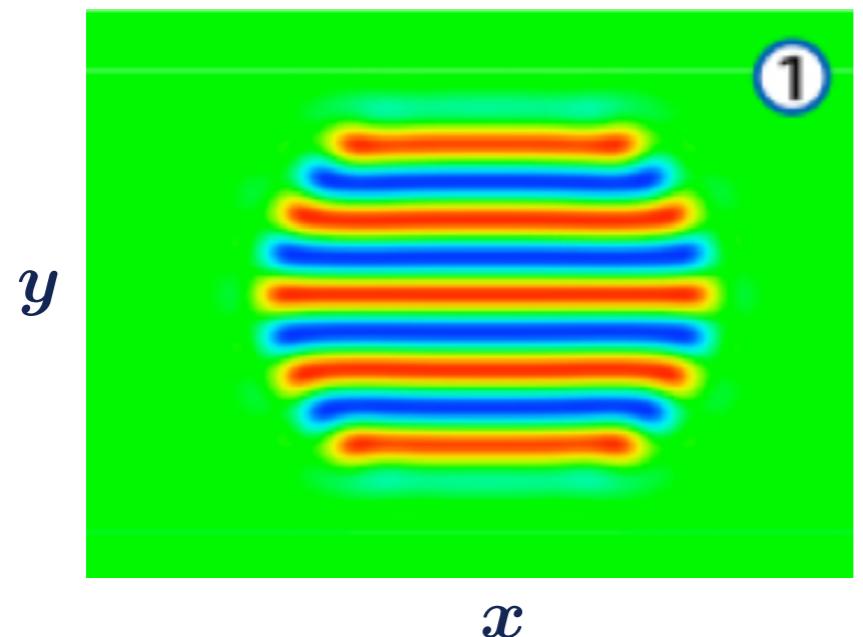
Steady states
(Swift-Hohenberg Eq.)



[Thiele, Archer, Robbins,
Gomez, Knobloch]

Localised steady states

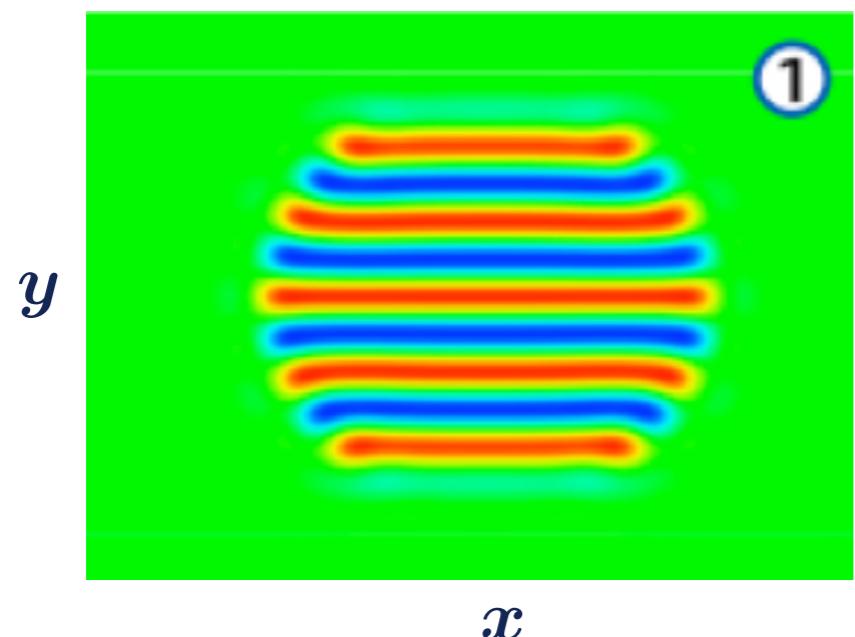
Localised steady states
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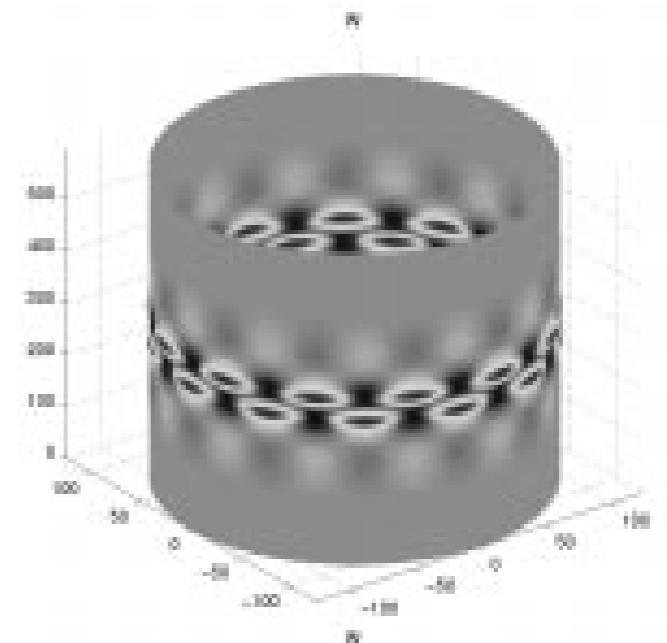
[A., Burke, Lloyd,
Sandstede, Knobloch]

Localised steady states

Localised steady states
(Swift-Hohenberg Eq.)



Localised steady state
(von Kármán Eq.)

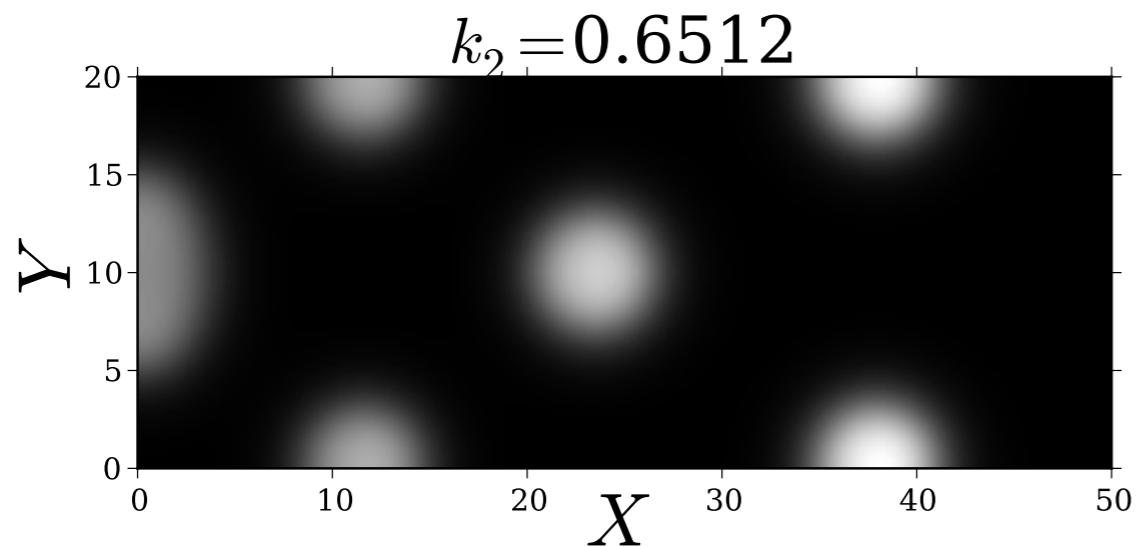


[Champneys, Hunt, Lord]

[A., Burke, Lloyd,
Sandstede, Knobloch]

Steady states

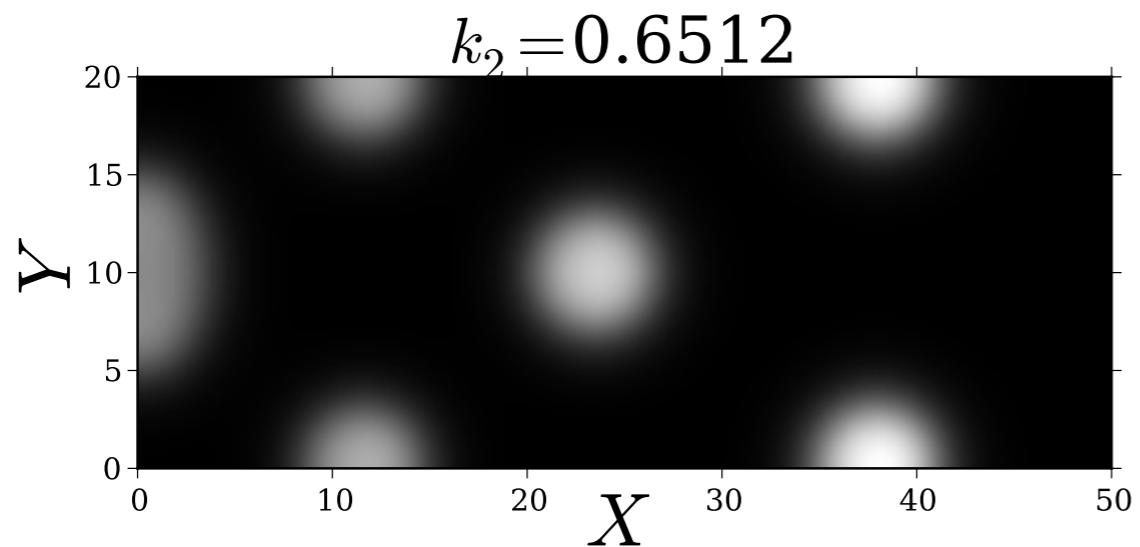
Stationary pattern
(Reaction-Diffusion-Advection system)



[Brena-Medina, A.,
Champneys, Ward]

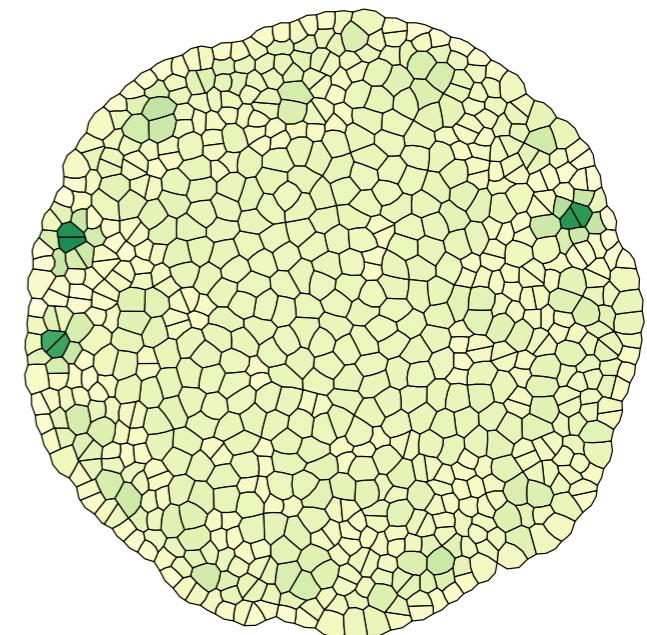
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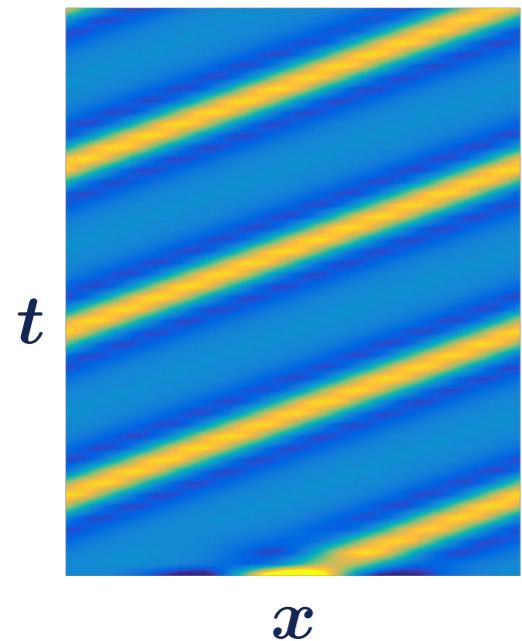
Localised auxin spots
(plant root section)



[Draelants, A., Vanroose]

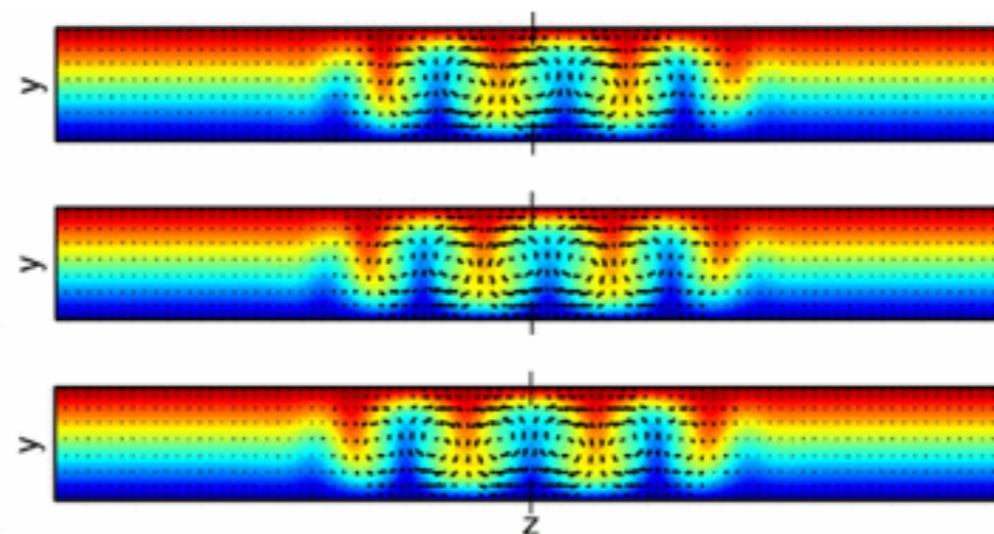
Travelling waves

Travelling wave
(Neural Markov-chain)



[A., Wedgwood]

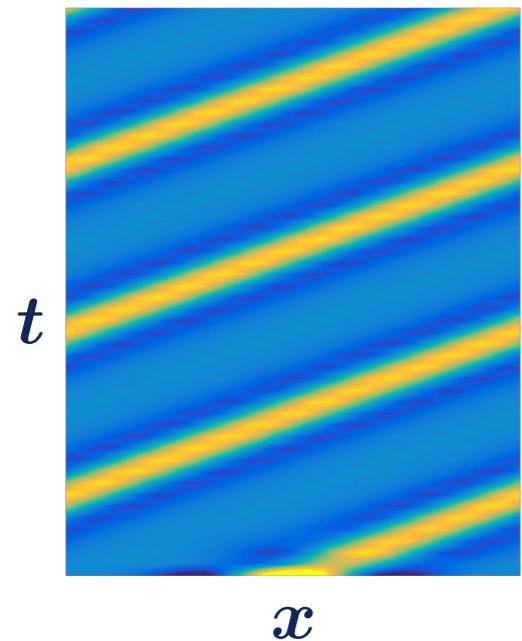
Travelling wave
(Navier-Stokes)



[Schneider, Gibson, Burke]

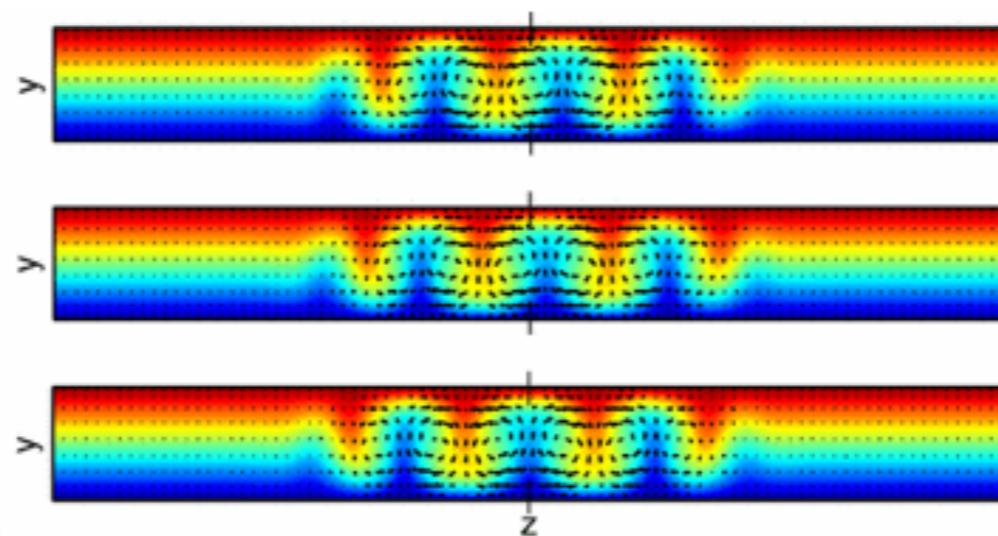
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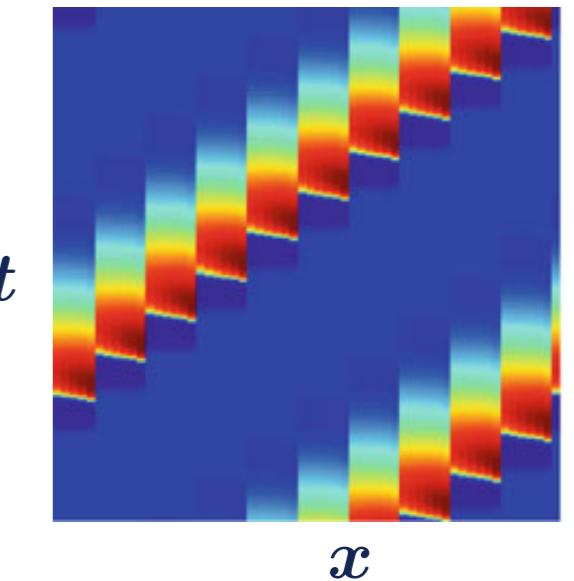
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Travelling wave
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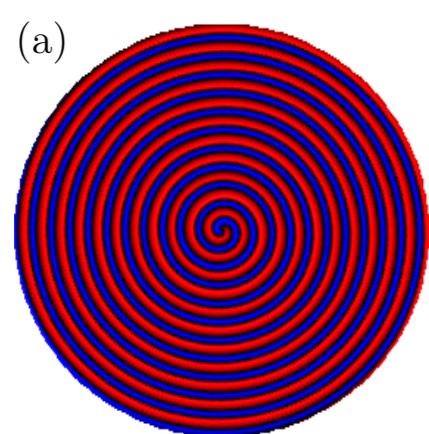
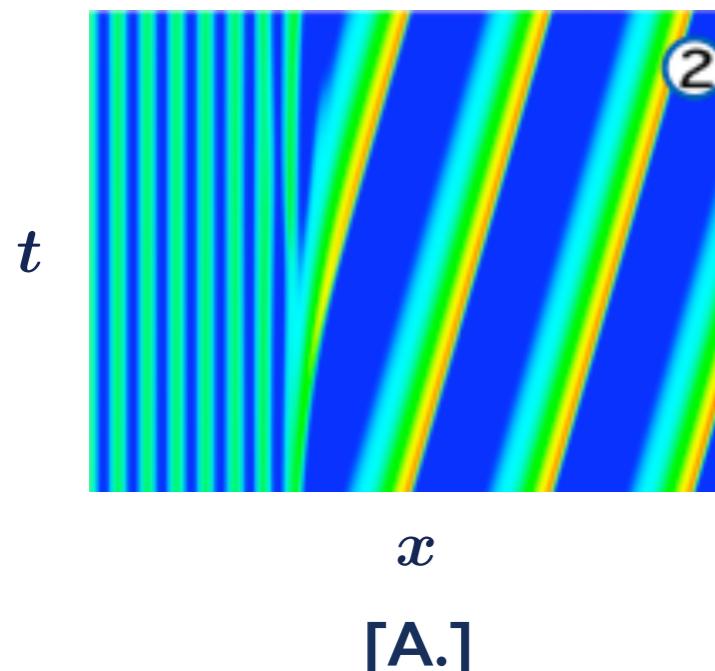
Lurching wave
(Neural field model)



[Wasyleenko, Cisternas,
Laing, Kevrekidis]

Other examples of coherent structures

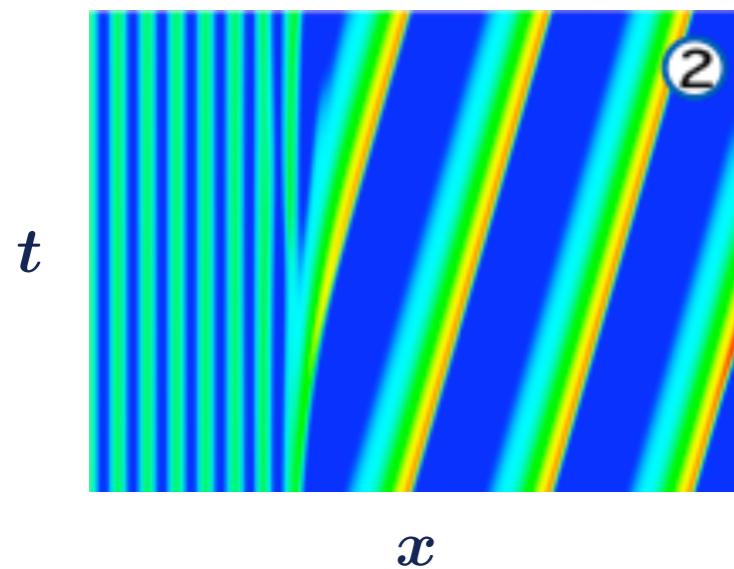
Defect/spiral waves
(Reaction-diffusion system)



[Wheeler, Barkley]

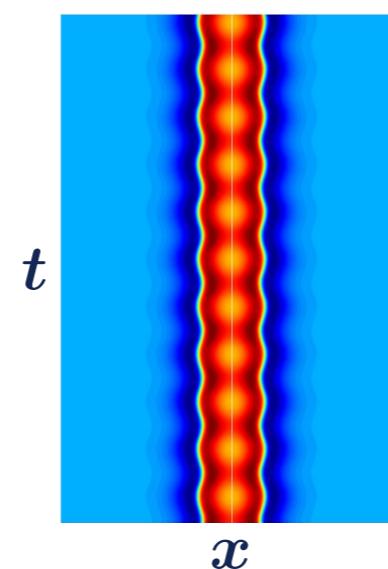
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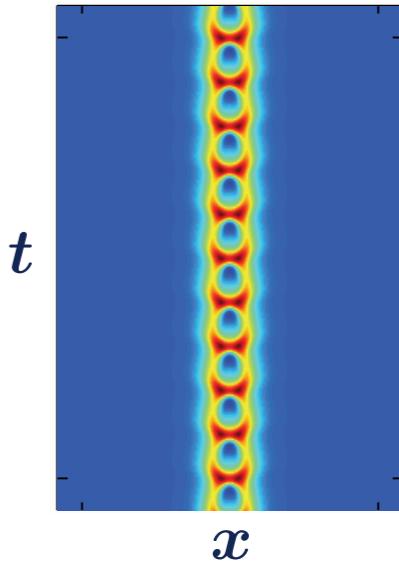
[A.]

Oscillon
(Neural field)

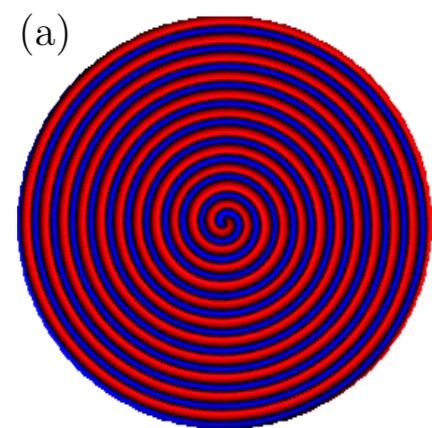


[Coombes, Schmidt, A.]

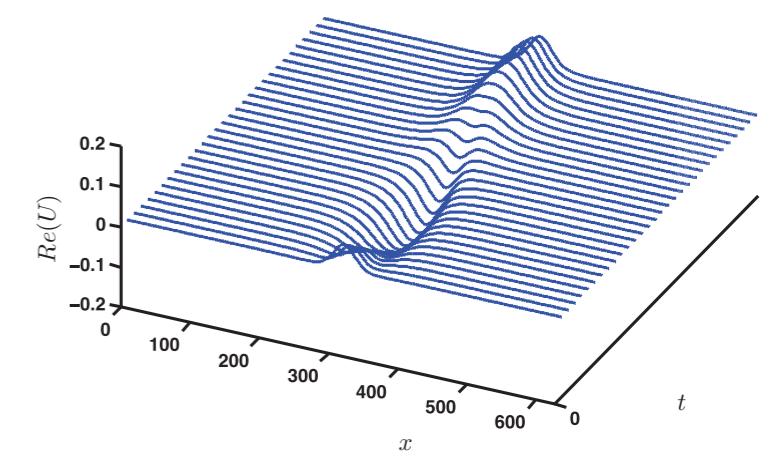
Oscillons
(Complex Ginzburg-Landau Eq.)



[Burke, Yochelis, Knobloch]



[Wheeler, Barkley]



[Alnahdi, Niesen, Rucklidge]

Problem setting

■ Coherent structures: solutions to nonlinear evolution equations with a particular spatial and temporal structure

■ Problem setting: compute special solutions to an Initial-Boundary-Value problem

Evolution equations $\partial_t u(x, t) = \mathcal{F}(u; p)(x, t), \quad (x, t) \in \Omega \times (0, T]$

Boundary conditions $\mathcal{B}(u; p)(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T]$

Initial conditions $u(x, 0) = u_0(x), \quad x \in \Omega \cup \partial\Omega$

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- Auto, MatCont, XPP, Coco, DDE-Biftool, Knut, etc.

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- Auto, MatCont, XPP, Coco, DDE-Biftool, Knut, etc.
- PDE2Path, LOCA, OOMPH
- DIY - version of the day [see [Codes](#)]

Time stepping

Example of time stepping

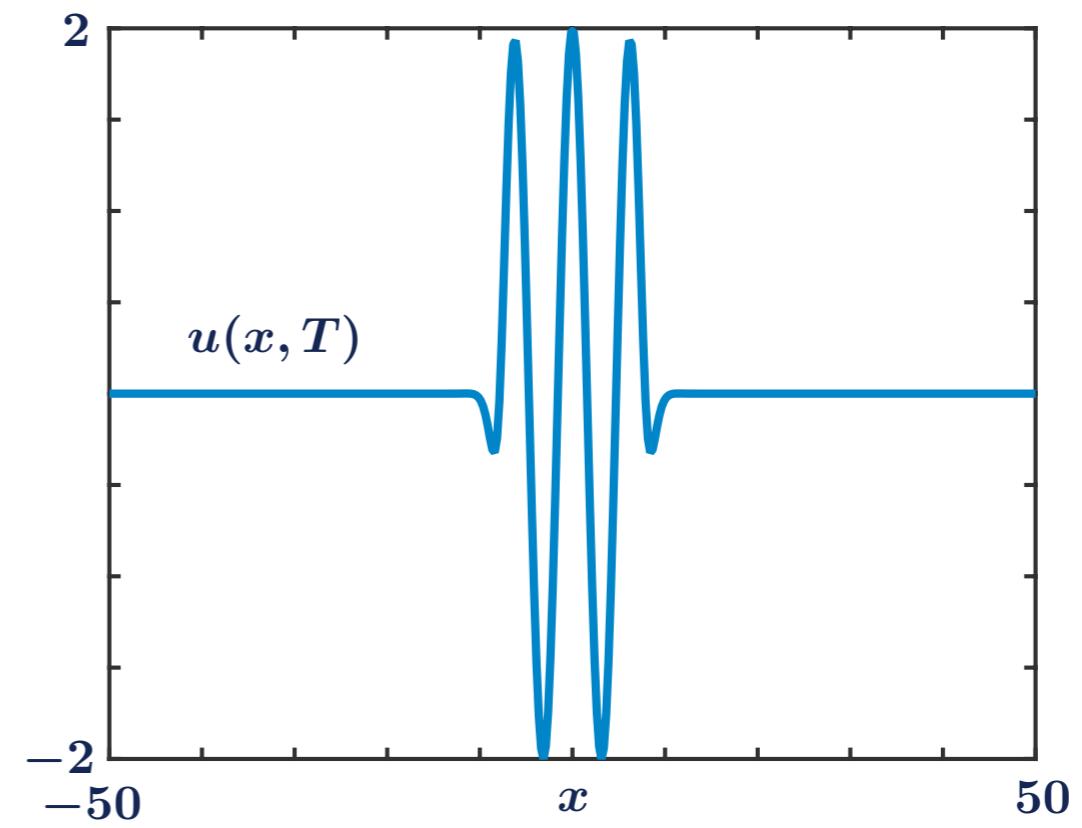
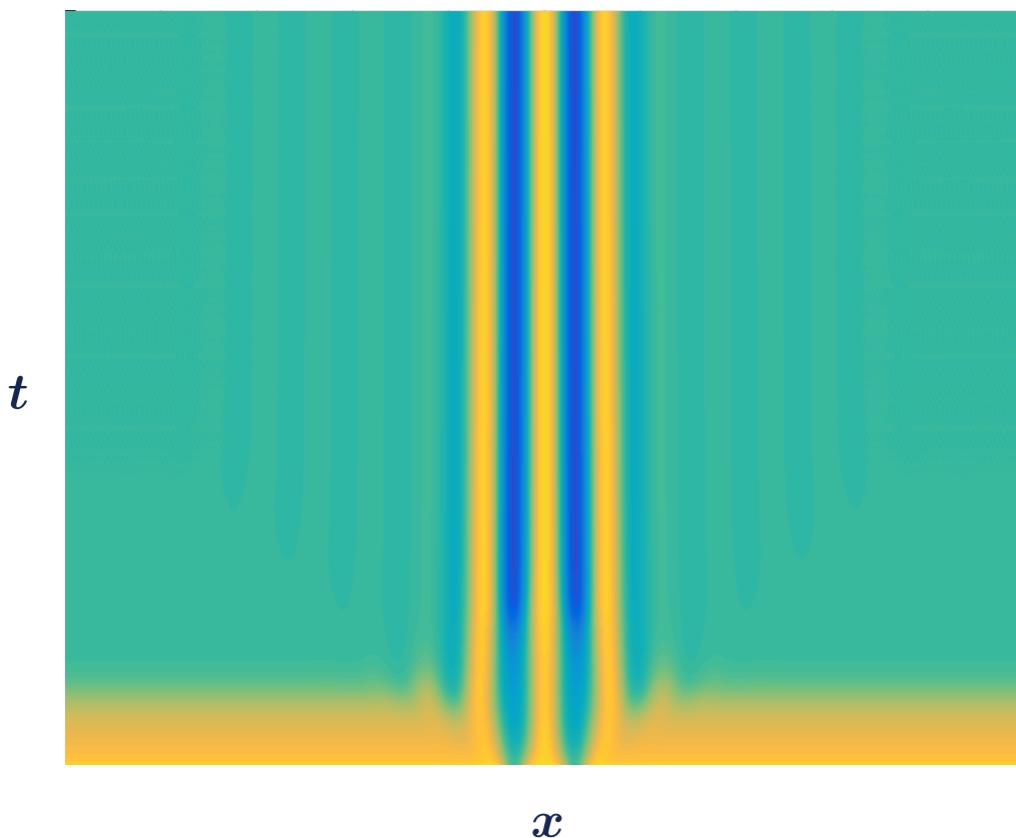
■ 1D spatial domain $\Omega = [-L, L] \in \mathbb{R}$

■ Swift-Hohenberg equation with cubic-quintic nonlinearity

$$\partial_t u = -(1 + \partial_x^2)^2 u - \mu u + \nu u^3 - u^5$$

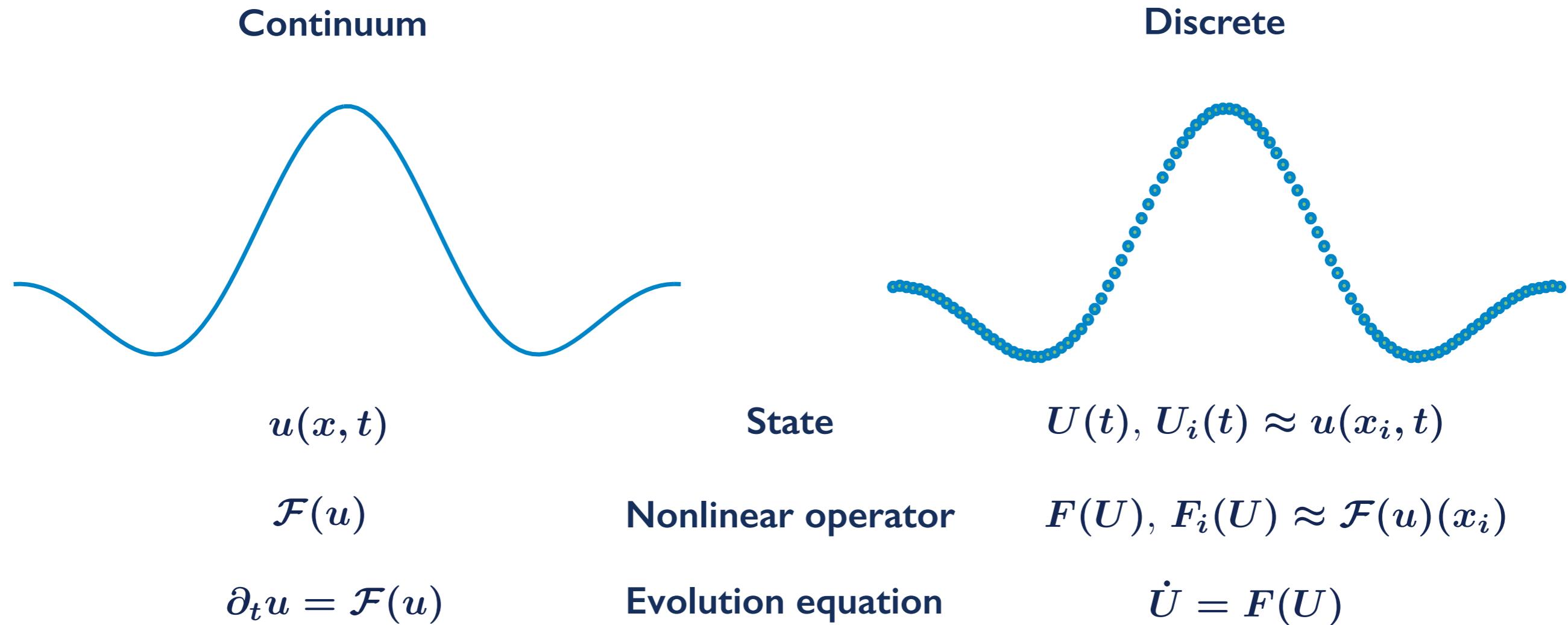
■ Periodic boundary conditions

■ Localised initial condition $u(x, 0) = 2 + \cos(x)[\tanh(x + 8) - \tanh(x - 8)]$



Method of lines

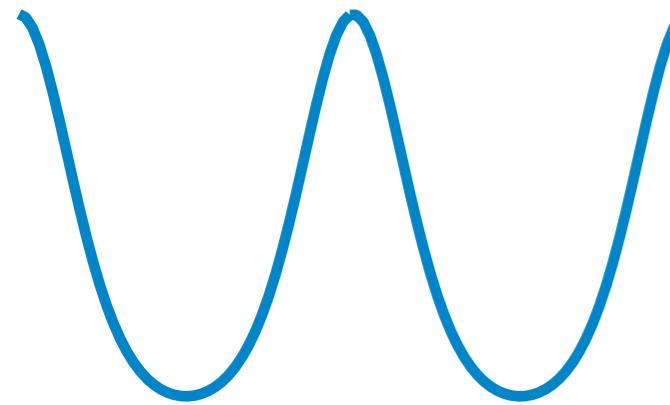
- A possible strategy is to use the method of lines
 - I. Discretise spatial coordinate in the PDE
 - 2. Time step the resulting ODE



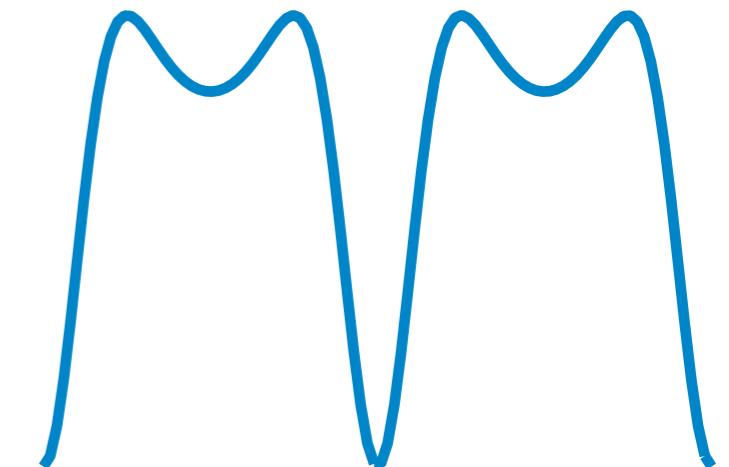
Approximation of derivatives [Trefethen]

Differential operator

$$\partial_{xx}: u(x) \mapsto v(x) = u''(x)$$



$$u \mapsto \partial_{xx} u$$



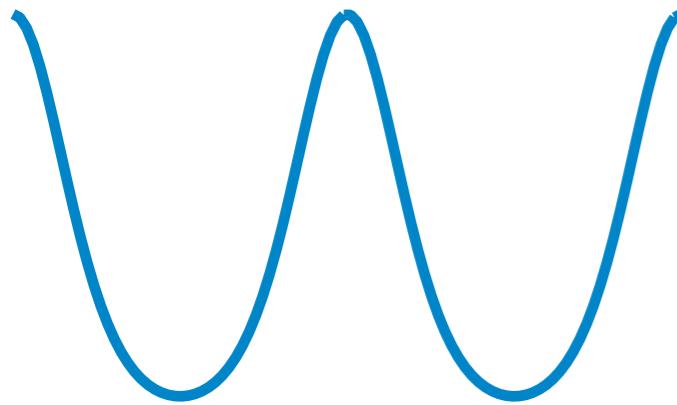
$$u(x) = \exp\left(\cos \frac{4\pi x}{L}\right)$$

$$\partial_{xx} u(x) = \frac{16\pi^2}{L^2} e^{\cos \frac{4\pi x}{L}} \left[\sin^2 \frac{4\pi x}{L} - \cos \frac{4\pi x}{L} \right]$$

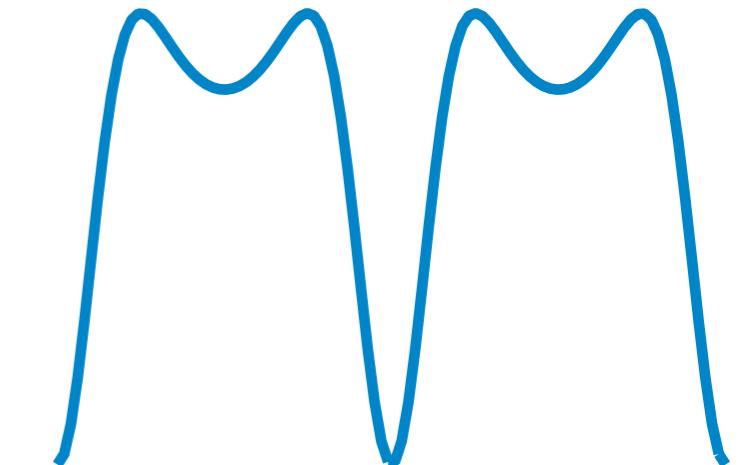
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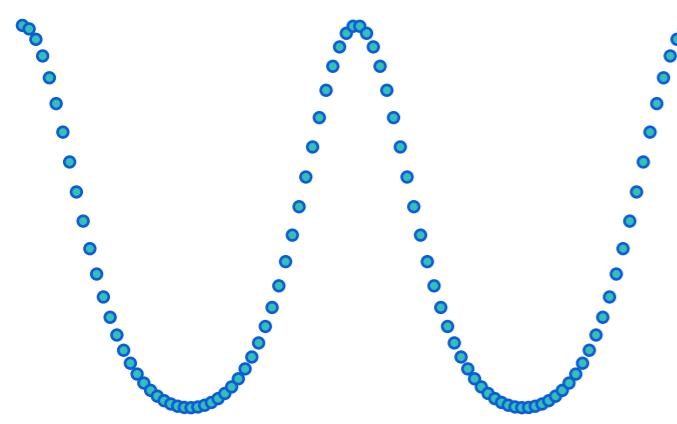


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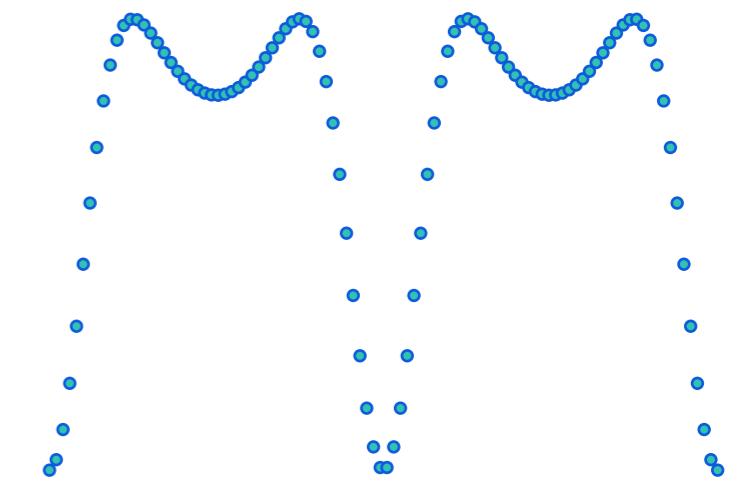
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Differentiation matrix

$$D_{xx}: U \mapsto V, \quad V_i = \frac{U_{i+1} - 2U_i + U_{i-1}}{h_x^2} \approx u''(x_i)$$



$$U \mapsto D_{xx} U$$



Discretising the Swift-Hohenberg operator

■ Swift-Hohenberg equation with cubic-quintic nonlinearity

$$\partial_t u = \mathcal{F}(u) \quad \mathcal{F}(u) = -(1 + \partial_x^2)^2 u - \mu u + \nu u^3 - u^5$$

■ Discretised Swift-Hohenberg equation with cubic-quintic nonlinearity

$$\dot{U} = F(U) \quad F(U) = -(I_n + D_{xx})^2 U - \mu U + \nu U^{\odot 3} - U^{\odot 5}$$

■ Note the Hadamard product/power symbol (component-wise product/power)

Product between vectors

$$(U \odot V)_i = U_i V_i$$

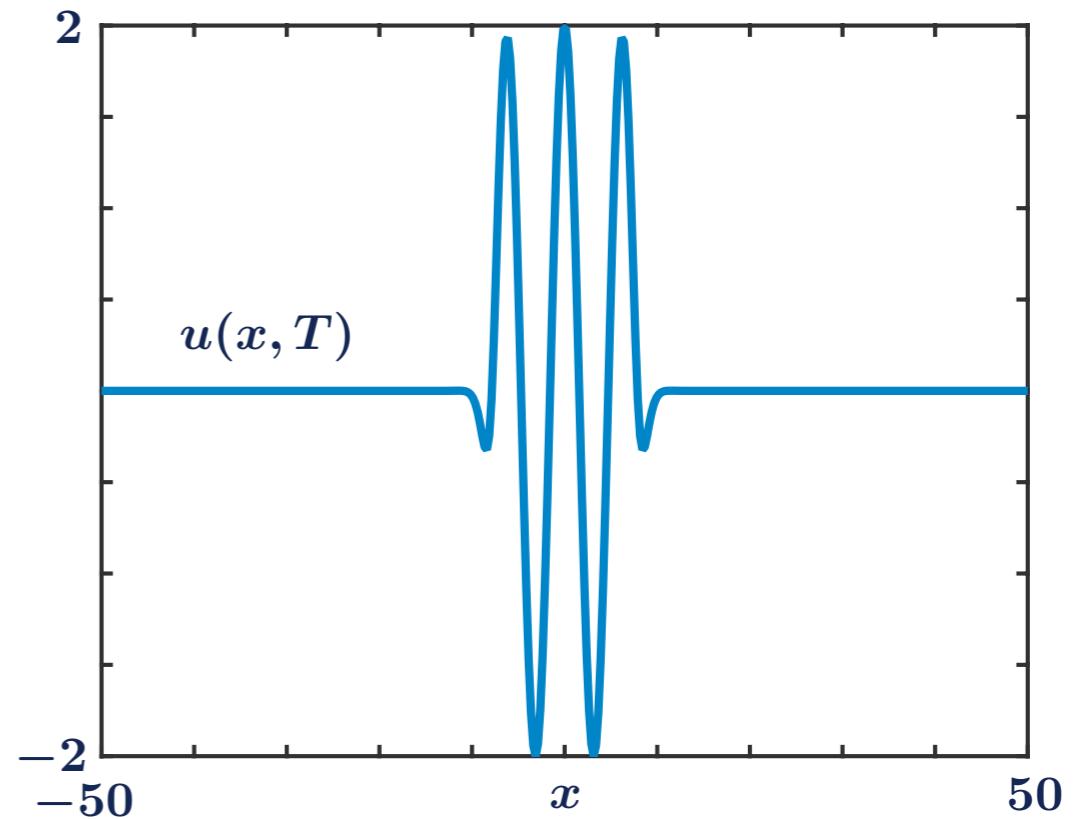
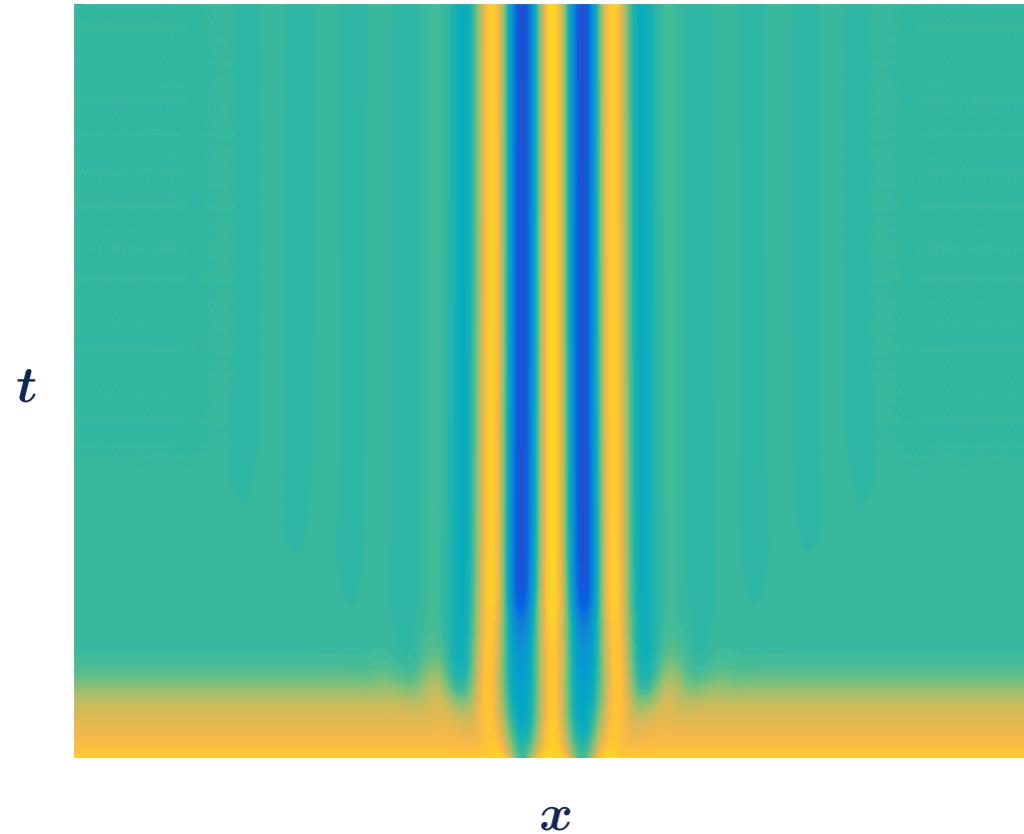
Power of a vector

$$(U^{\odot 3})_i = (U \odot U \odot U)_i = U_i^3$$

■ The similarity between $\mathcal{F}(u)$ and $F(U)$ is striking (and useful in practice)!

Comments on time stepping

See Codes



■ Discretised Swift-Hohenberg equation with cubic-quintic nonlinearity

$$\dot{U} = F(U) \quad F(U) = -(I_n + D_{xx})^2 U - \mu U + \nu U^{\odot 3} - U^{\odot 5}$$

- I used standard stiff ODE integrators, but other choices are available/advisable
1. Semi-implicit time-steppers
 2. Splitting methods
 3. Exponential time steppers
 4. Other problem-dependent time steppers (symplectic integrators, etc.)

Numerical continuation

Numerical continuation - introduction

- We want to study how steady states of $\partial_t u = \mathcal{F}(u; p)$ depend upon parameters
- In principle, we could change p and perform a new simulation. However
 1. There may be long transients before a new steady state is reached
 2. Only stable steady states are observable with time simulations
 3. For fixed p there may coexisting steady states, and which one is found depends on the initial condition
- Main idea behind numerical continuation:
 1. Start from (u_0, p_0) such that $\mathcal{F}(u_0, p_0) = 0$
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 4. Solutions are collected into branches with possibly different symmetry properties
 5. Can be applied to periodic orbits, travelling waves, spiral waves

Continuation of steady states in ODEs

See Codes

■ Evolution equation

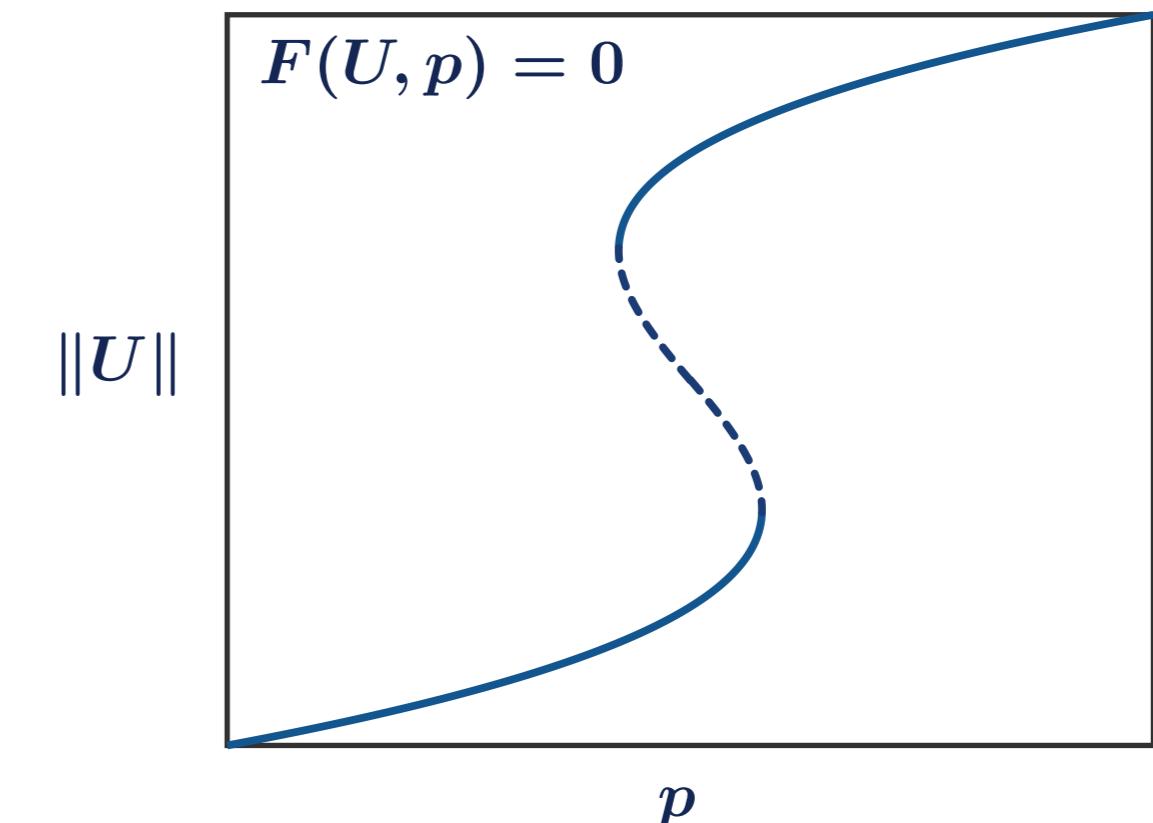
$$\dot{U} = F(U; p)$$

■ Branch of steady states

$$F(U(s), p(s)) = 0$$

■ Stability of a steady state (U_*, p_*)

$$D_U F(U_*, p_*) \tilde{U} = \lambda \tilde{U}$$



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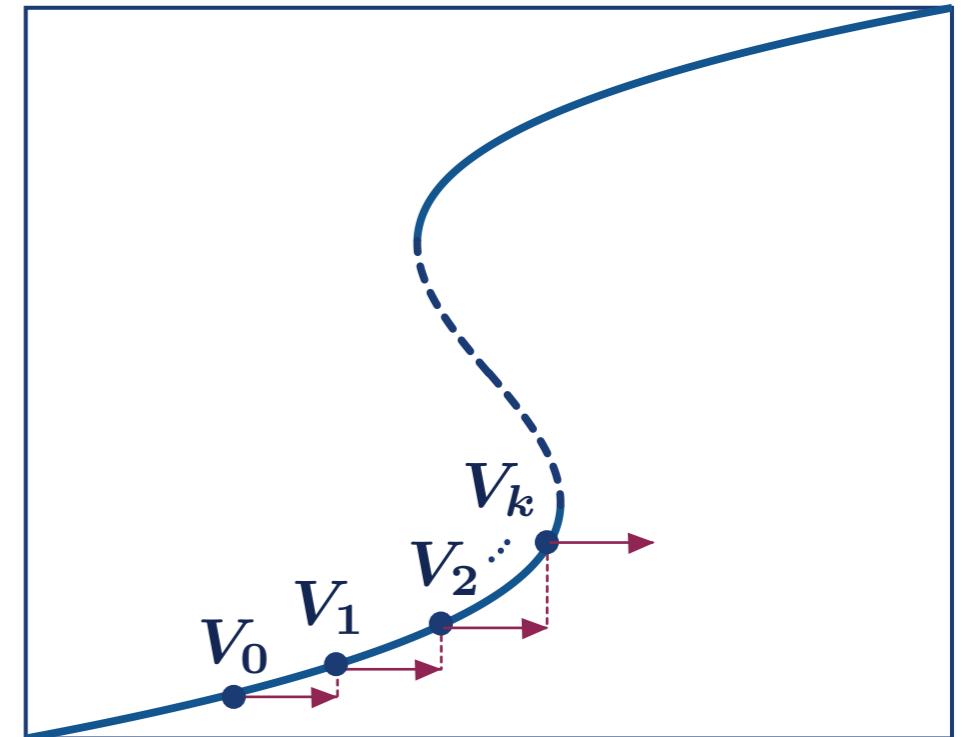
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■ Assume knowledge of $V_0 = (U_0, p_0) \in \mathbb{R}^{n+1}$ such that $F(U_0, p_0) = 0$

I. Fix h

2. For each $k > 0$ find (U, p) such that

$$\begin{array}{ll} F(U, p) = 0 & \text{or} \\ p - p^{k-1} - h = 0 & G(V; p^{k-1}, h) = 0 \end{array}$$



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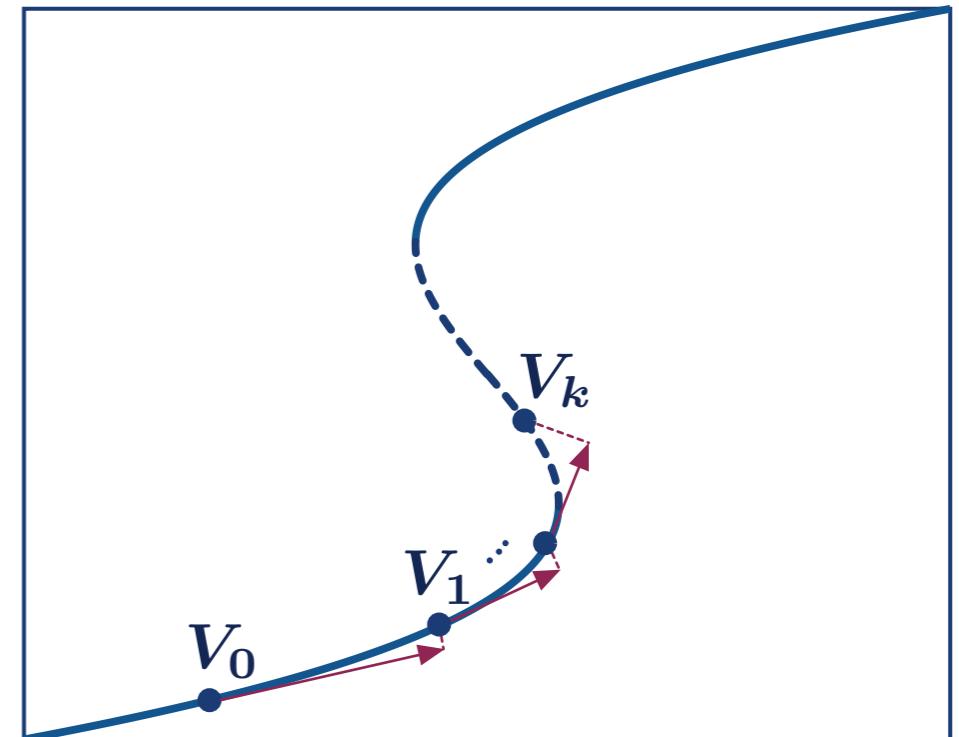
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■ Assume knowledge of $V_0 = (U_0, p_0) \in \mathbb{R}^{n+1}$ such that $F(U_0, p_0) = 0$

1. Fix h
2. For each $k > 0$
 - A. Predict along the tangent $\tilde{V}_k = V_k + hT_k$
 - B. Correct in the normal direction: find V such that

$$\begin{aligned} F(V) &= 0 \\ \langle V - \tilde{V}_k, hV_k \rangle &= 0 \end{aligned} \quad \text{or} \quad G(V; V_k, \tilde{V}_k, h) = 0$$

Newton's method

- At each continuation step, we solve the problem $G(V) = 0$
- Newton's method with initial guess V_0 : repeat until convergence
 1. Solve $D_V G(V_l) \tilde{V}_l = -G(V_l)$ for \tilde{V}_l
 2. Compute $V_{l+1} = V_l + \tilde{V}_l$
- At each continuation step we need to solve several linear problems
- In any continuation package, the functions $G(V)$, $D_V G(V)$ are constructed automatically from $F(U, p)$, $D_U F(U, p)$, $D_p F(U, p)$, which are provided by the user
- For PDEs, most computational time is spent in function evaluations and solutions of large linear problems

Application to the Swift-Hohenberg operator

Swift-Hohenberg operator and its Fréchet Derivative at u

$$\mathcal{F}(u) = -(1 + \partial_x^2)^2 u - \mu u + \nu u^3 - u^5$$

$$\partial_u \mathcal{F}(u) = -(1 + \partial_x^2)^2 - \mu + 3\nu u^2 - 5u^4$$

Application to the Swift-Hohenberg operator

■ Fix μ, ν . Find u such that

$$\begin{aligned} -(1 + \partial_x^2)^2 u - \mu u + \nu u^3 - u^5 &= 0 \\ u(L, t) - u(-L, t) &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} \mathcal{F}(u) &= 0 \\ \mathcal{B}(u) &= 0 \end{aligned}$$

Application to the Swift-Hohenberg operator

■ Fix μ, ν . Find u such that

$$\begin{aligned} -(1 + \partial_x^2)^2 u - \mu u + \nu u^3 - u^5 &= 0 & \mathcal{F}(u) = 0 \\ u(L, t) - u(-L, t) &= 0 & \text{or} \\ && \mathcal{B}(u) = 0 \end{aligned}$$

■ Discretised Swift-Hohenberg equation and its Jacobian at U

$$F(U) = -(I_n + D_{xx})^2 U - \mu U + \nu U^{\odot 3} - U^{\odot 5}$$

$$D_U F(U) = -(I_n + D_{xx})^2 - \mu I_n + \text{diag}(3\nu U^{\odot 2} - 5U^{\odot 4})$$

Application to the Swift-Hohenberg operator

- Fix μ, ν . Find u such that

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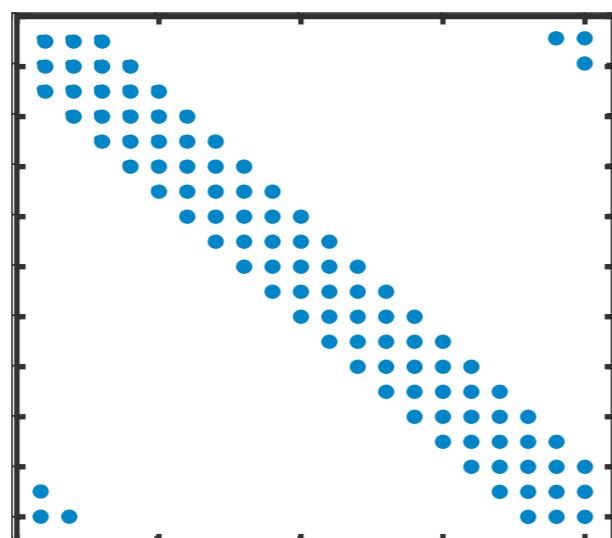
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- Sparsity pattern example $n = 20$

$$D_U F(U) =$$



$$-(I_n + D_{xx})^2$$

Fixed!

Application to the Swift-Hohenberg operator

- Fix μ, ν . Find u such that

$$-(1 + \partial_x^2)^2 u - \mu u + \nu u^3 - u^5 = 0 \quad \text{or} \quad \mathcal{F}(u) = 0$$
$$u(L, t) - u(-L, t) = 0 \quad \mathcal{B}(u) = 0$$

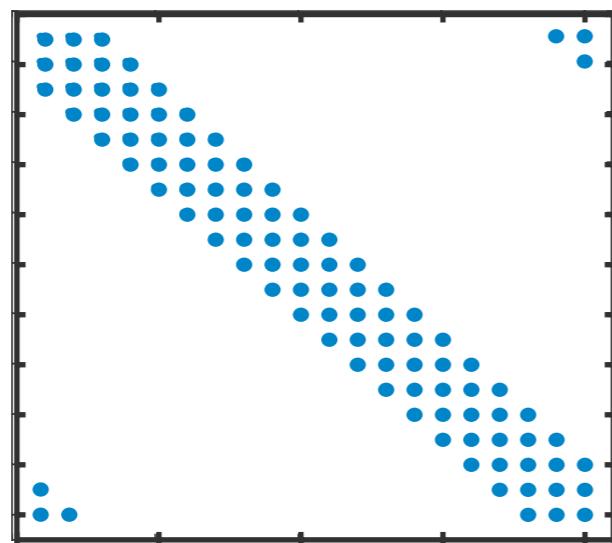
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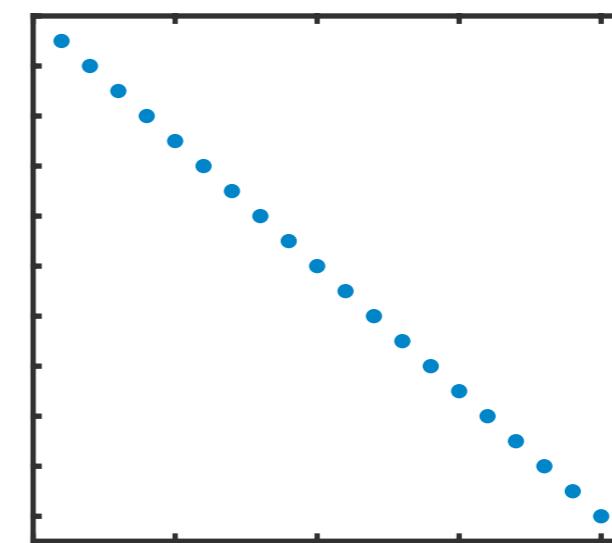
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Fixed!

+



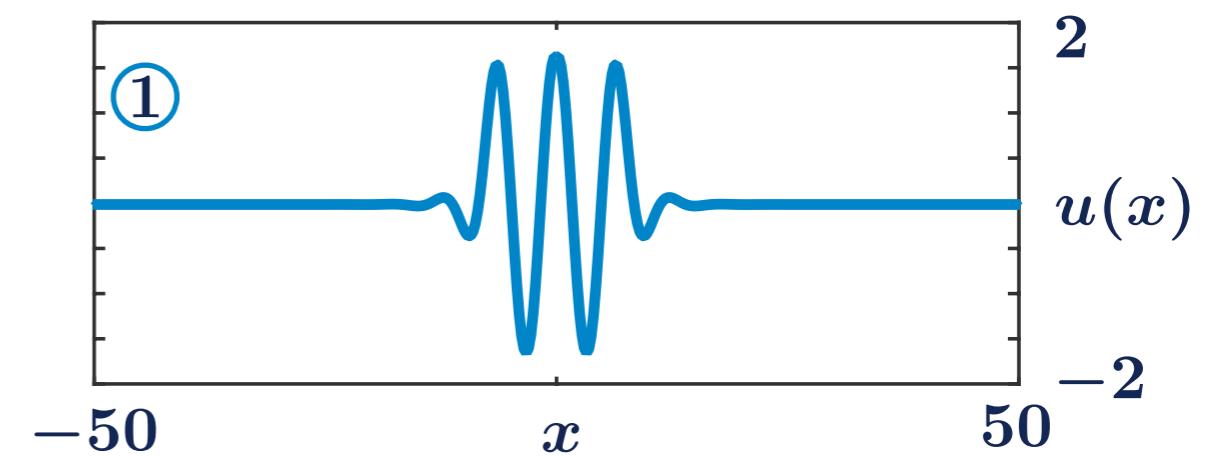
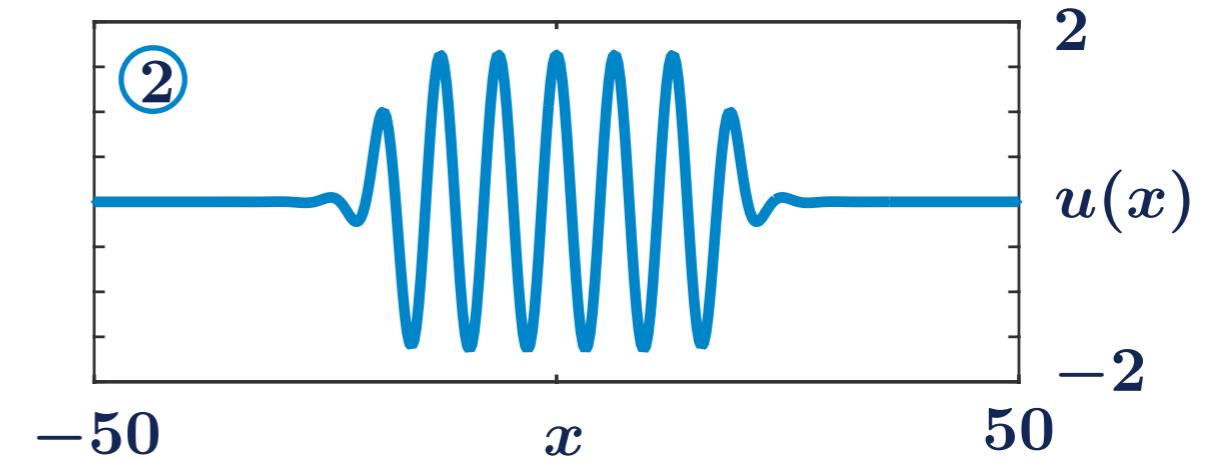
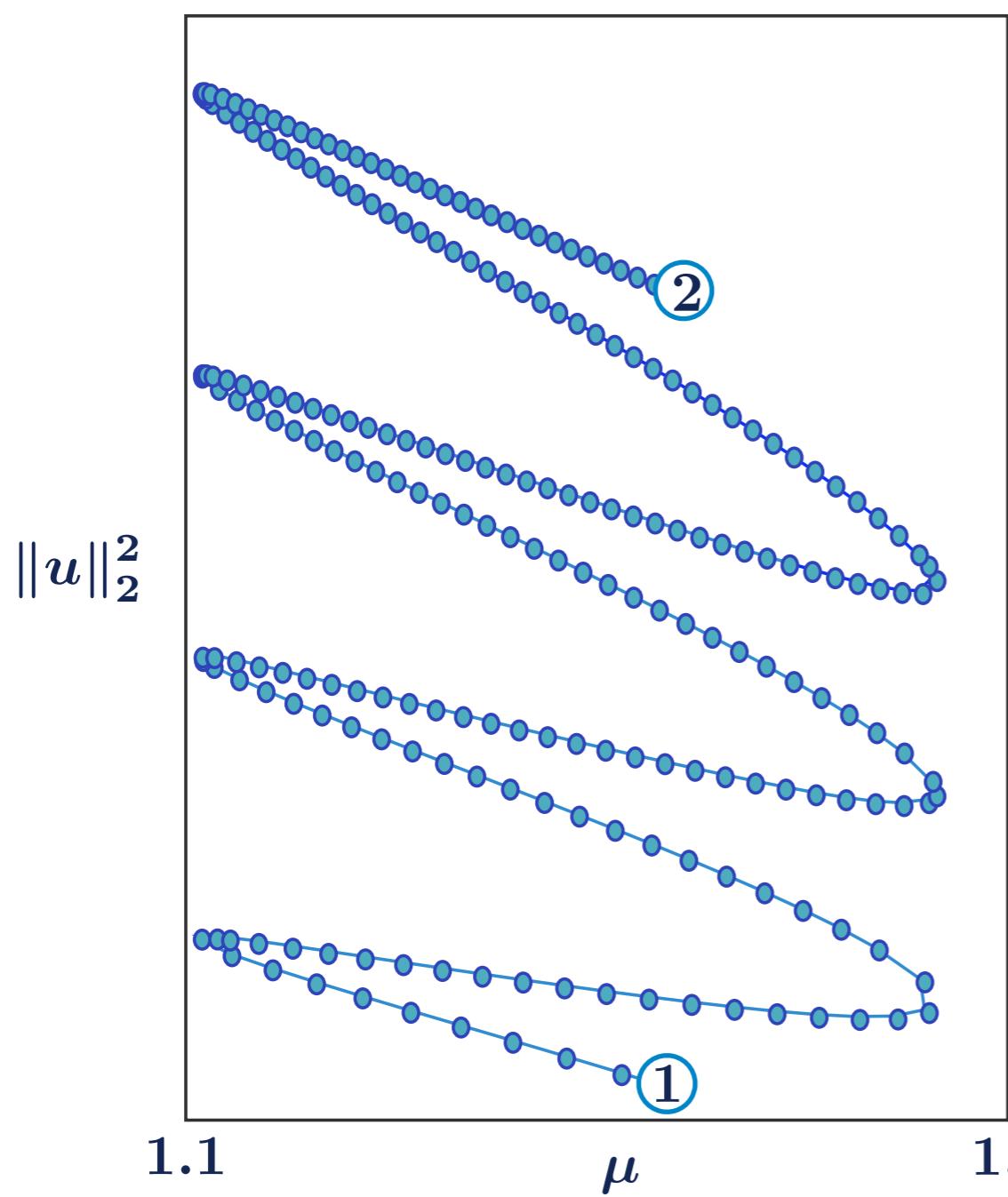
$$-\mu I_n + \text{diag}(3U^{\odot 2} - 5U^{\odot 4})$$

Variable

Example of numerical continuation

[See Codes](#)

$$\Omega = (-50, 50), \nu = 3, n = 400$$



Linear stability

Linear stability of steady states

■ Equilibrium u_* of the evolution equation

$$\partial_t u = \mathcal{F}(u), \quad \mathcal{F}(u_*) = 0$$

■ Study the evolution of small perturbations to u_*

$$u(x, t) = u_*(x) + \varepsilon \tilde{u}(x, t), \quad 0 < \varepsilon \ll 1, \quad \|\tilde{u}\| = \mathcal{O}(1)$$

■ At $\mathcal{O}(\varepsilon)$, the perturbations evolve according to

$$\partial_t \tilde{u} = \partial_u \mathcal{F}(u_*) \tilde{u} := \mathcal{L}(u_*) \tilde{u}$$

■ The ansatz $\tilde{u}(x, t) = e^{\lambda t} \psi(x)$ leads to the eigenvalue problem

$$\lambda \psi = \mathcal{L}(u_*) \psi$$

■ The state u_* is linearly stable if $\text{Spec} [\mathcal{L}(u_*)]$ is contained in the left half of the complex plane

Numerical calculations

See Codes

Continuum problem

$$\lambda\psi = \mathcal{L}(u_*)\psi$$

$$\mathcal{L}(u_*) = -(1 + \partial_x^2)^2 - \mu + 3\nu u_*^2 - 5u_*^4$$

Discrete problem

$$\lambda\Psi = L(U_*)\Psi$$

$$L(U) = -(I_n + D_{xx})^2 - \mu I_n + \text{diag}(3U^{\odot 2} - 5U^{\odot 4})$$

A function which computes L has already been used in the Newton iterations!

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I. Standard factorisations for dense matrices (compute full spectrum)

Numerical calculations

See Codes

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1. Standard factorisations for dense matrices (compute full spectrum)
2. Arnoldi iterations for sparse matrices (compute a few eigenvalues)

Numerical calculations

See Codes

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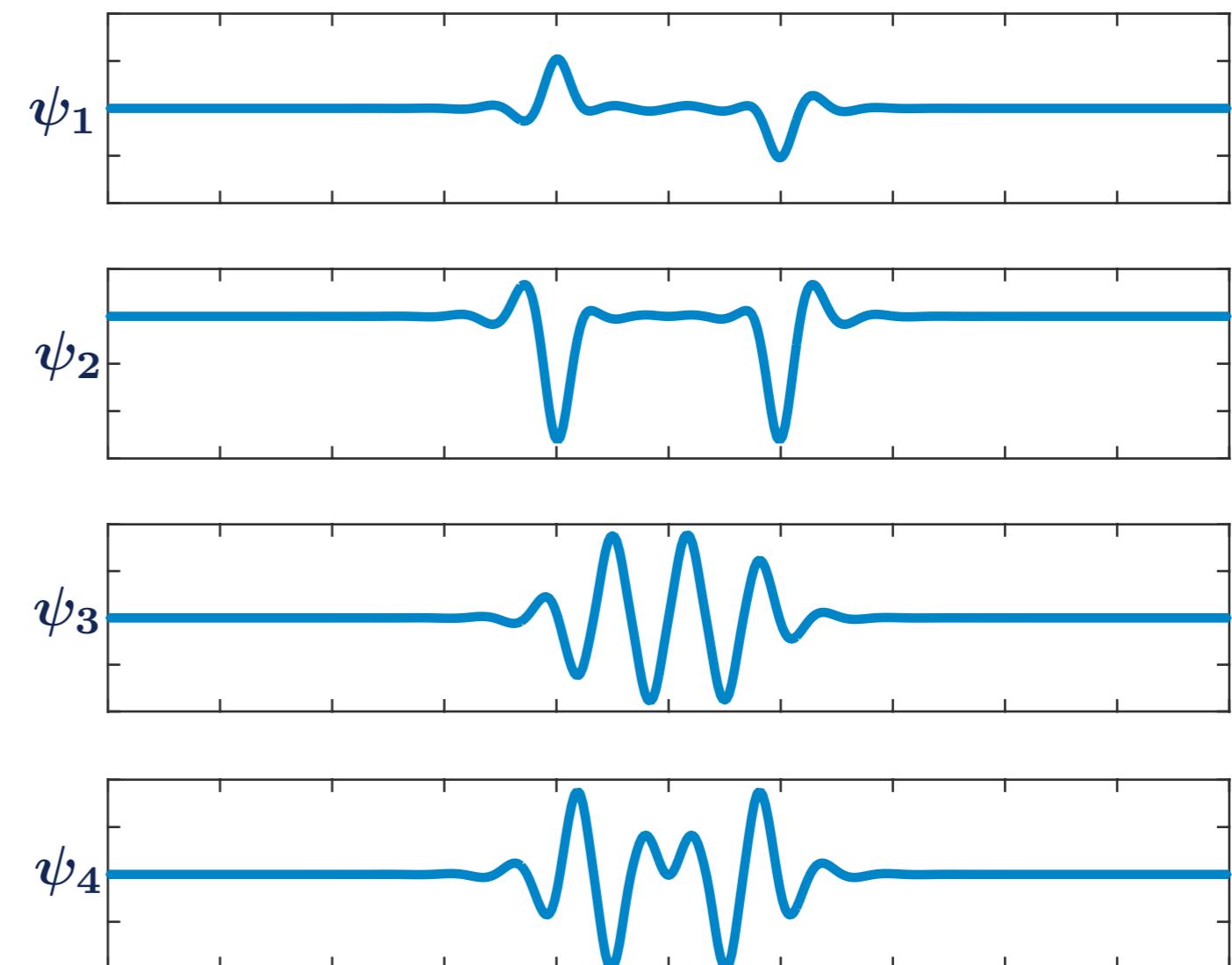
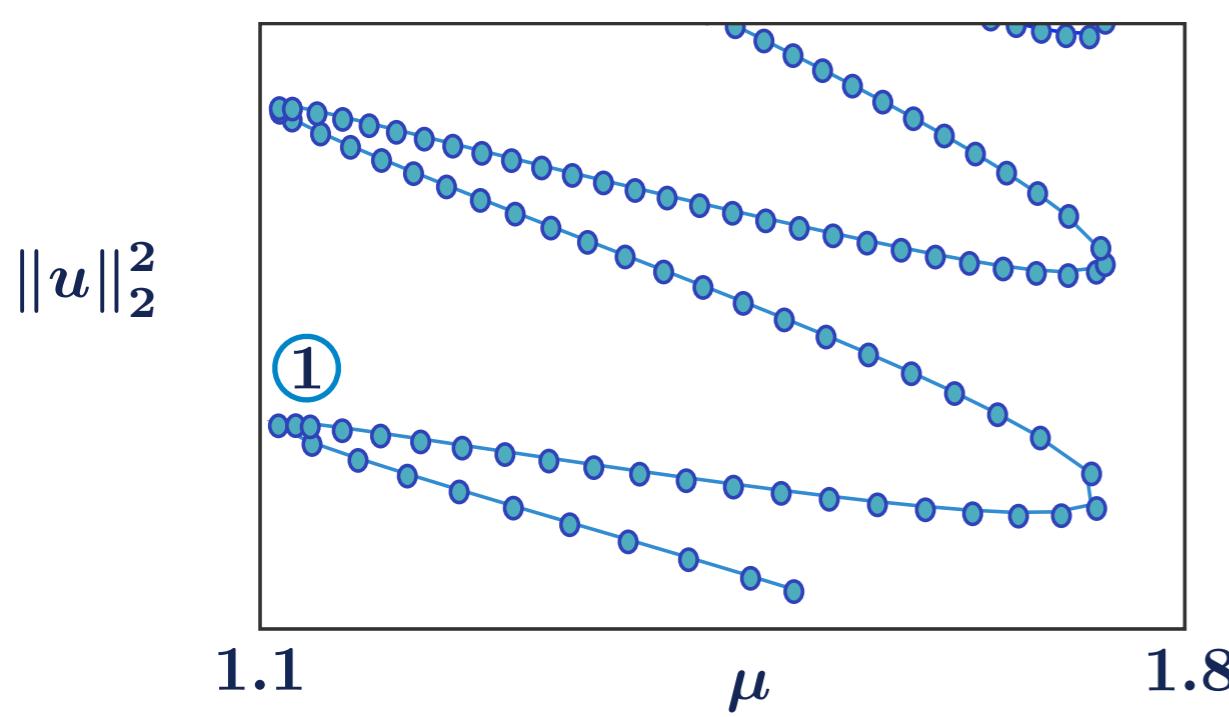
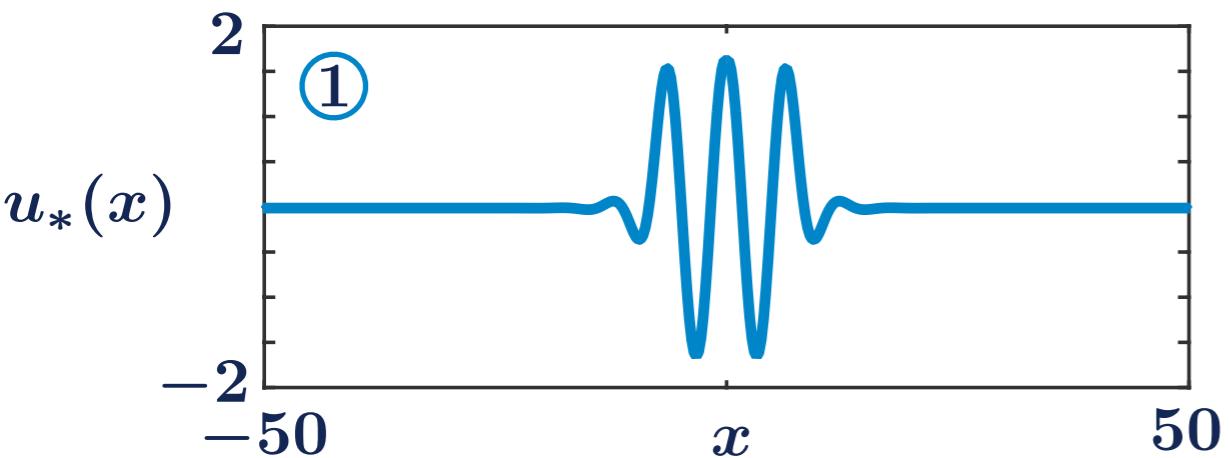
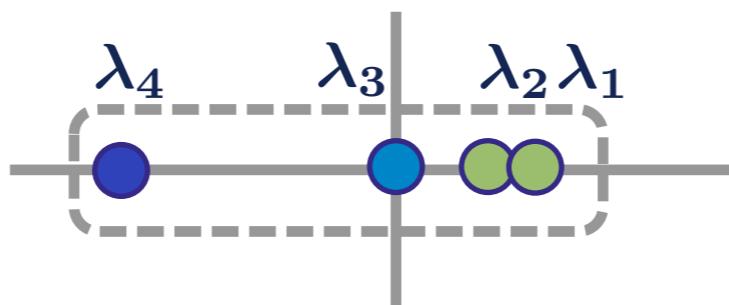
Numerical computation of eigenvalues

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Computations of eigenvalues

1. Can be done on-the-fly, during continuation
2. May be performed as a post-processing step

Results for a profile close to a saddle-node



Translation invariance

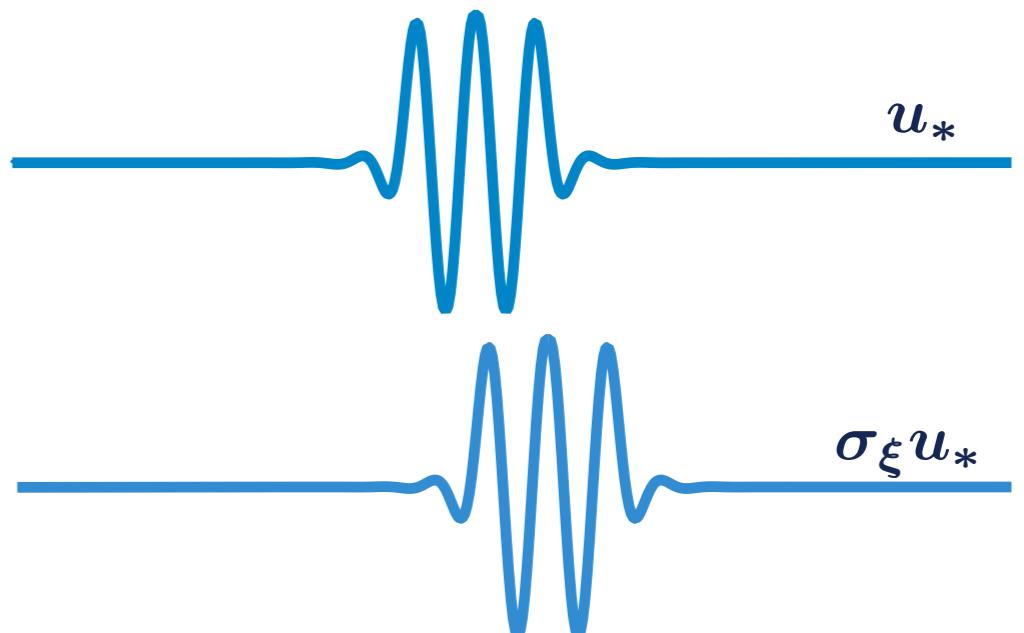
Translation invariance

- SH equation with periodic boundary conditions

$$\partial_t u = \mathcal{F}(u)$$

- Consider spatial translations via the action

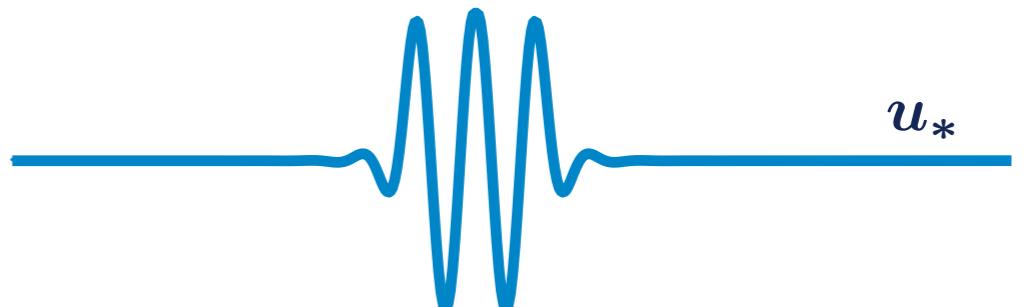
$$\sigma_\xi: v(\cdot) \mapsto v(\cdot + \xi), \quad \xi \in \mathbb{R}$$



Translation invariance

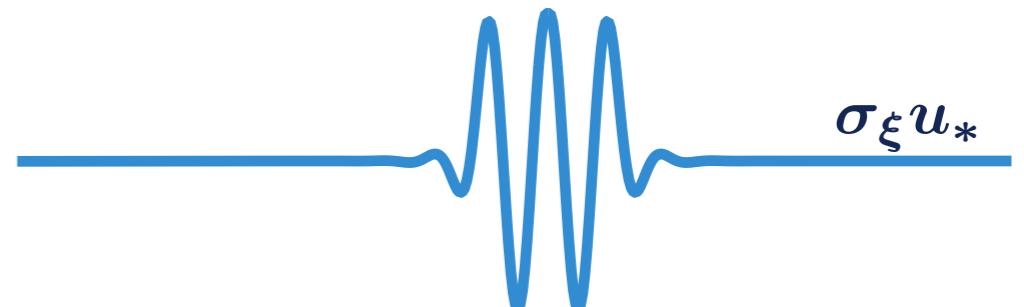
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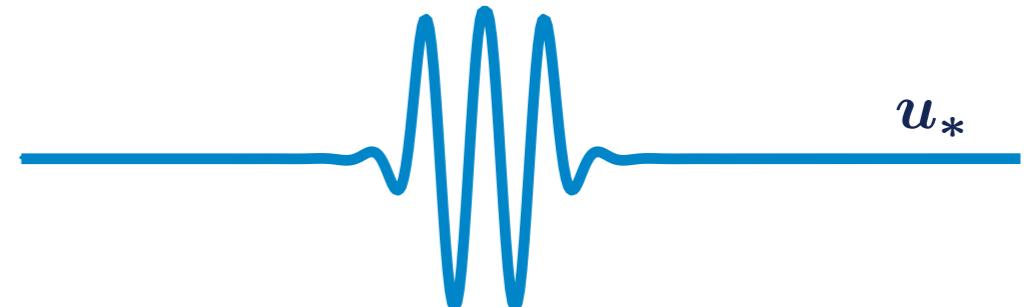
- The SH operator is equivariant under this action, that is, $\sigma_\xi \mathcal{F}(u) = \mathcal{F}(\sigma_\xi u)$

$$\begin{aligned}\sigma_\xi \mathcal{F}(u) &= \sigma_\xi [- (1 + \partial_x^2)^2 u - \mu u + \nu u^3 - u^5] \\ &= - (1 + \partial_x^2)^2 \sigma_\xi u - \mu \sigma_\xi u + \nu (\sigma_\xi u)^3 - (\sigma_\xi u)^5 \\ &= \mathcal{F}(\sigma_\xi u)\end{aligned}$$

Translation invariance

SH equation with periodic boundary conditions

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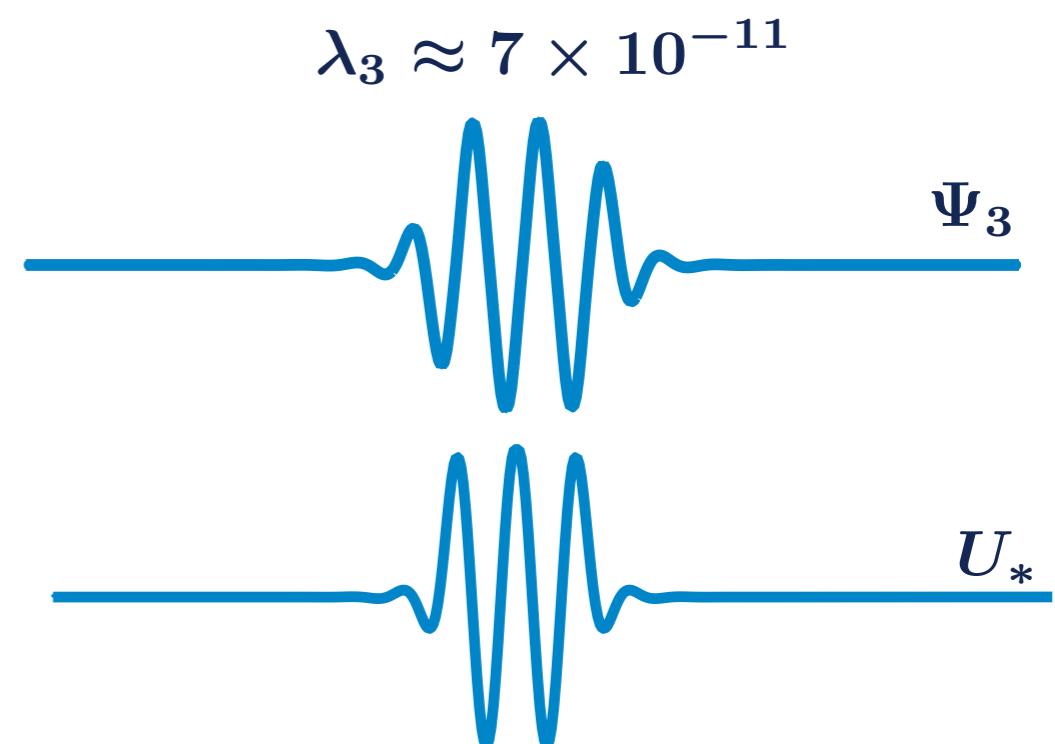
Consequences of translation invariance:

1. If $u_*(x)$ is a steady state, so is an arbitrary translation $u_*(x + \xi)$
2. The operator $\partial_u \mathcal{F}(u_*)$ has an eigenvalue 0 corresponding to eigenfunction $\partial_x u_*$

Numerical considerations

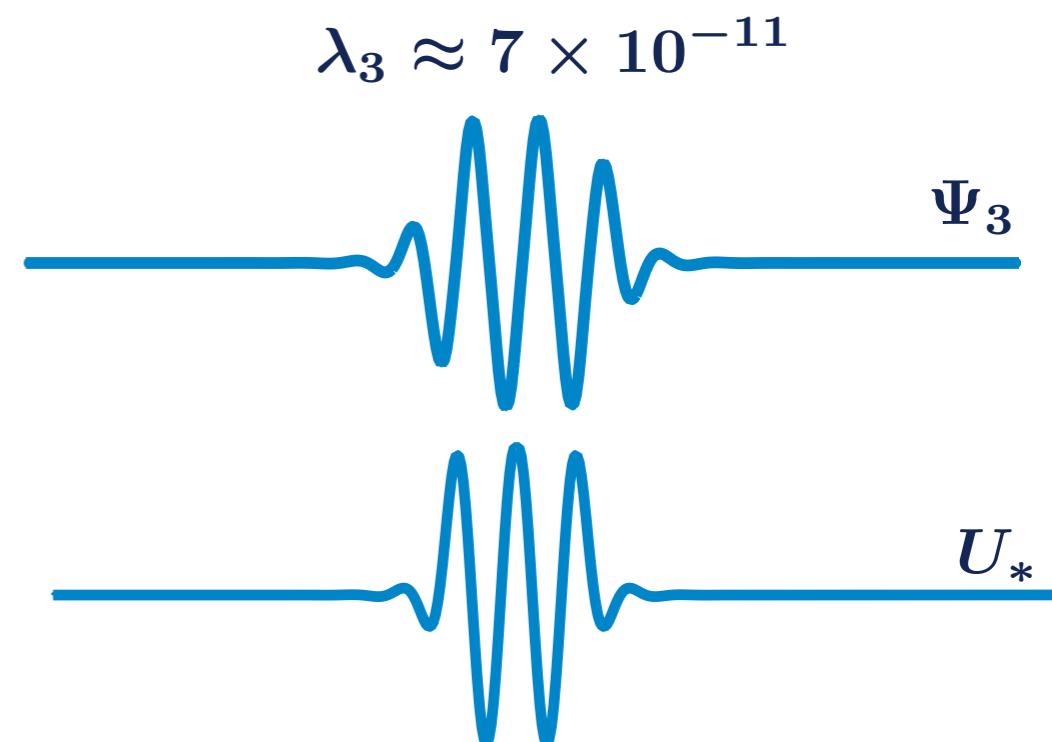
■ Our differentiation matrices impose periodicity

■ We expect an eigenvalue $\lambda \approx 0$ with eigenvector $\Psi \approx D_x U_*$



Numerical considerations

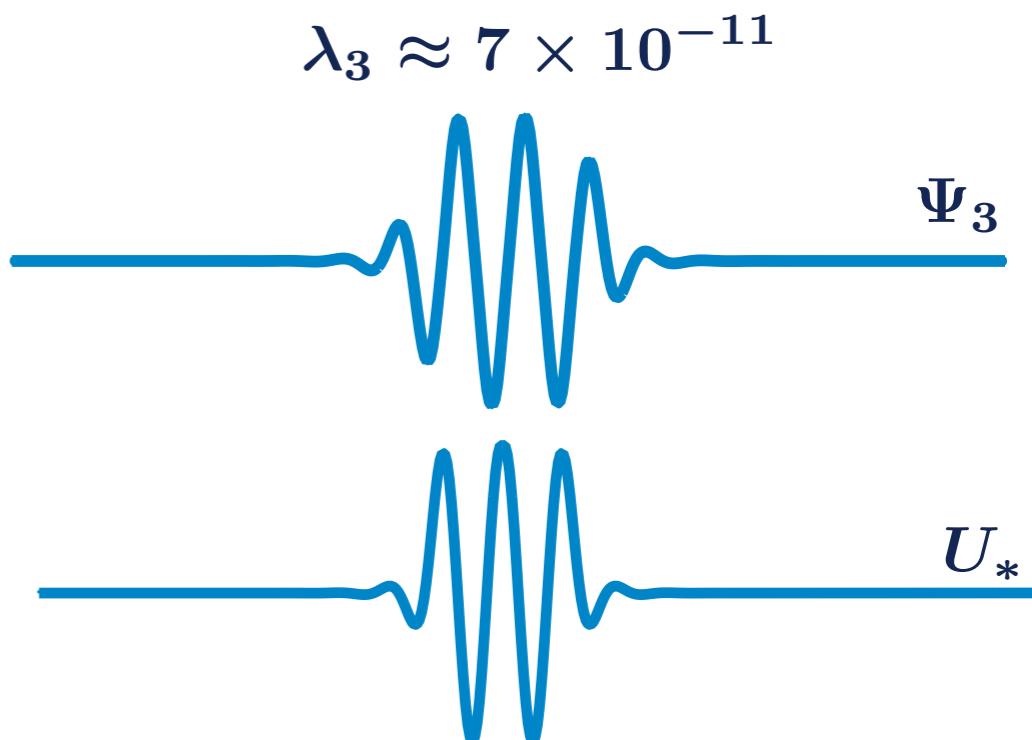
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- This means $D_U F(U_*)$ is nearly singular



Numerical considerations

- Our differentiation matrices impose periodicity
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- This means $D_U F(U_*)$ is nearly singular
- The solution U_* is not a regular zero of F

This is a disaster for Newton's iterates!



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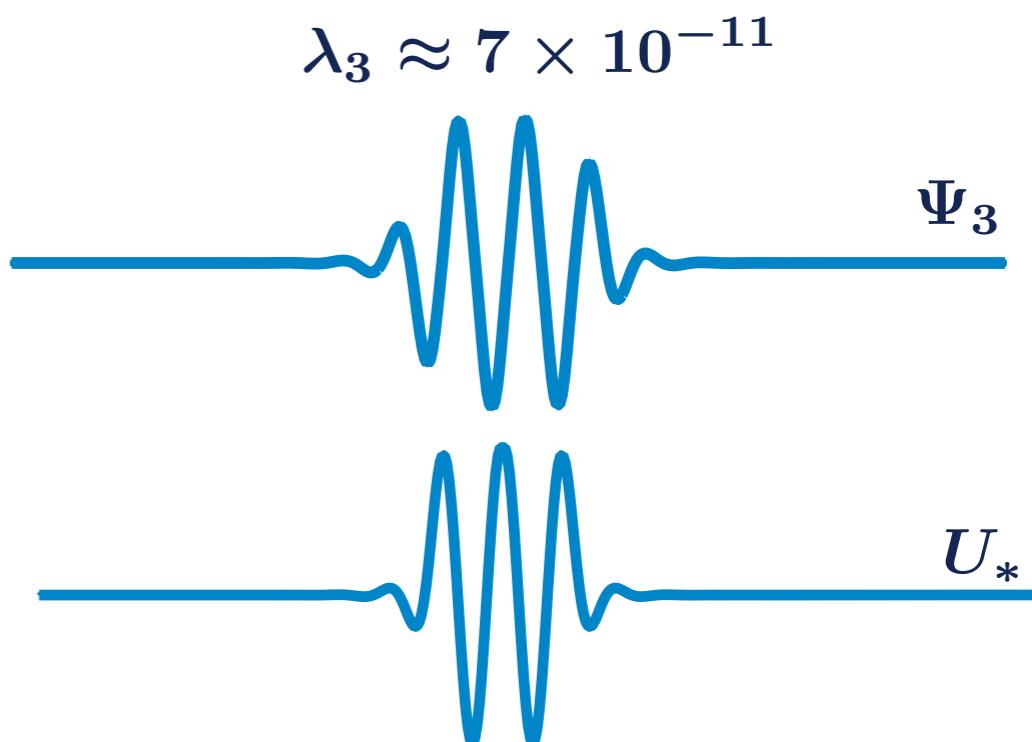
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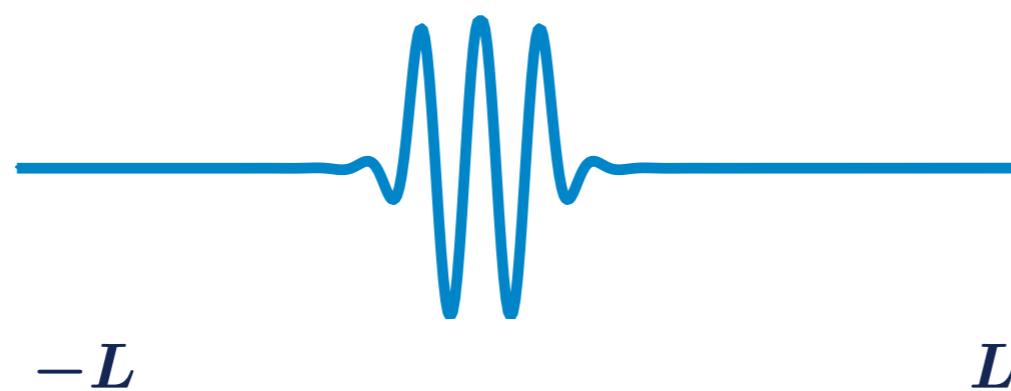
■ Possible remedies:

1. Remove invariance via boundary conditions
2. Regularise the problem via a phase condition



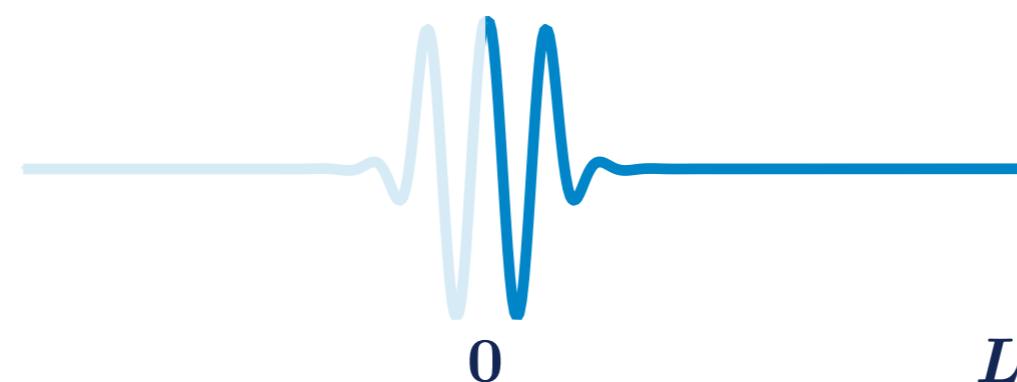
Regularisation via boundary conditions

See Codes



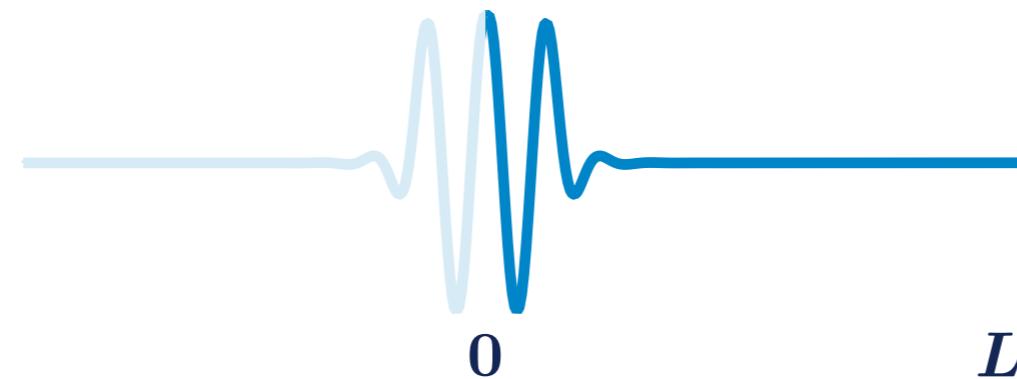
Regularisation via boundary conditions

See Codes



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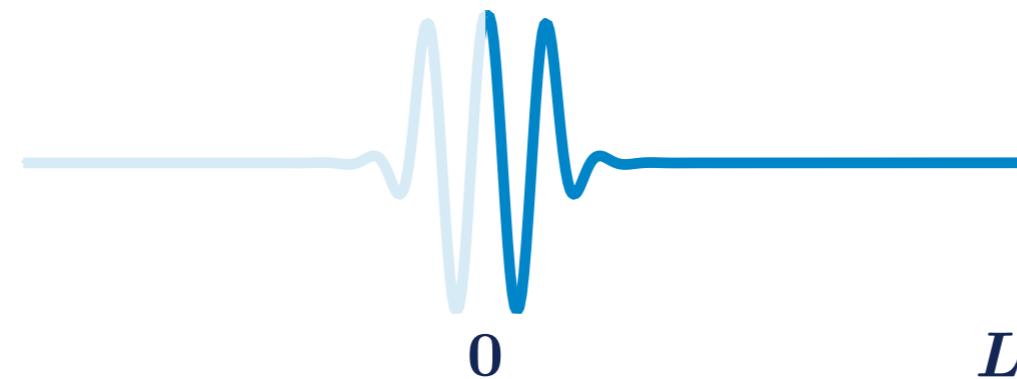
■ Pose the problem on half-domain with Neumann boundary conditions

$$-(1 + \partial_x^2)^2 u - \mu u + u^3 - u^5 = 0 \quad x \in (0, L)$$

$$\partial_x u = 0, \quad \partial_x^3 u = 0 \quad x = 0, L$$

Regularisation via boundary conditions

See Codes



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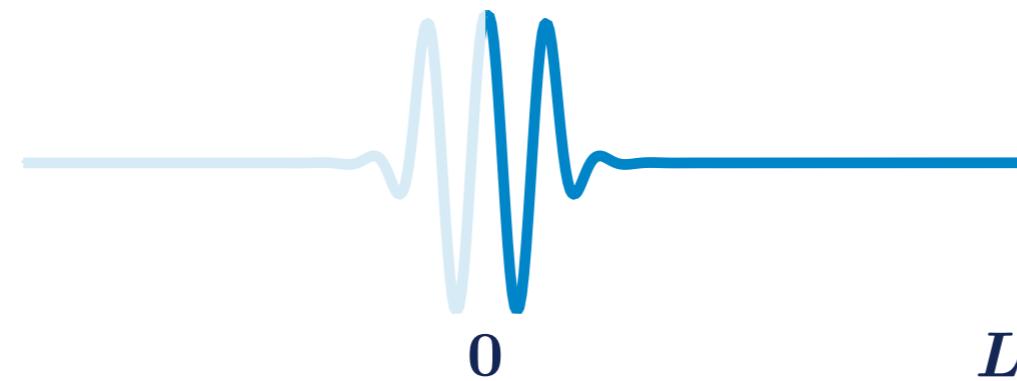
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■ Advantages:

1. Only a change in differentiation matrix is required
2. The same accuracy is obtained with half the number of grid points

Regularisation via boundary conditions

See Codes



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■ Advantages:

1. Only a change in differentiation matrix is required
2. The same accuracy is obtained with half the number of grid points

■ Disadvantages:

1. Only even solutions to the original BVP are computed
2. Stability calculations require some amendments

Regularisation via phase conditions

■ Solve for (u, c) the following BVP with periodic boundary conditions

$$-(1 + \partial_x^2)^2 u - \mu u + u^3 - u^5 + c \partial_x u = 0, \quad x \in [-L, L]$$

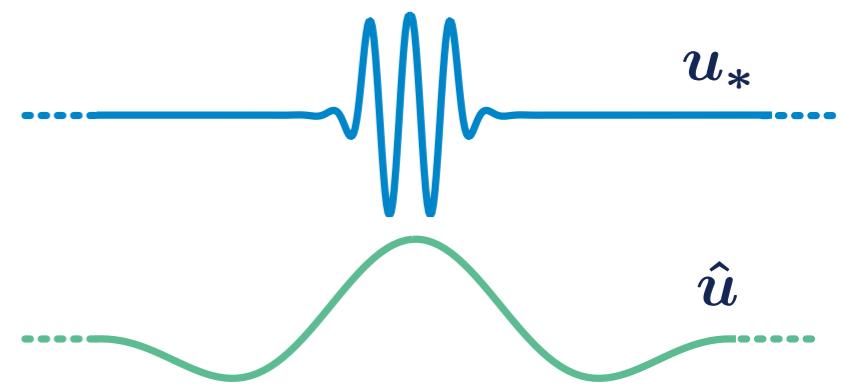
$$\mathcal{B}_{\text{per}}(u) = 0$$

$$\varphi(u, \hat{u}) = 0$$

■ Where

1. The original BVP has been perturbed by a term $c \partial_x u$
2. The solution \hat{u} is a reference/template solution
3. The last equation is a phase condition

$$\varphi(u, \hat{u}) = \int_{-L}^L \partial_x \hat{u}(x) (u(x) - \hat{u}(x)) dx$$

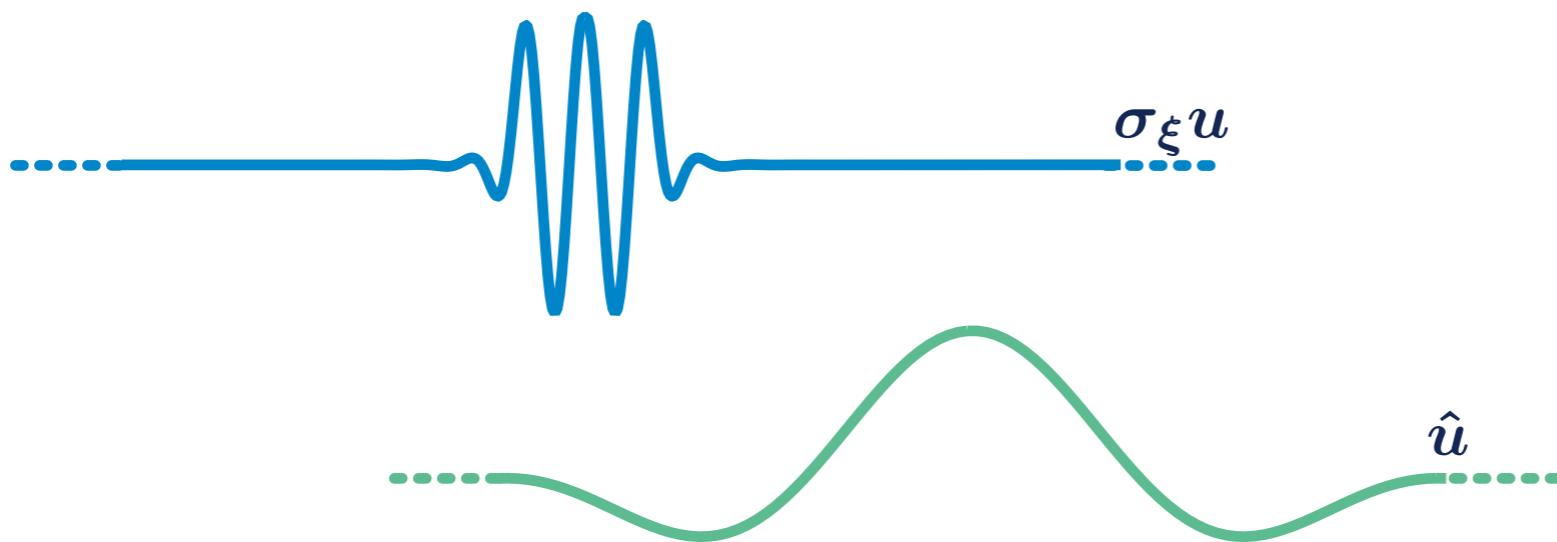


■ Things to note:

1. $(u_*, 0)$ is a regular solution to the BVP above (Bordering Lemma)
2. The phase condition tries to minimise the L_2 -distance to the template solution

Phase condition

[See Codes](#)

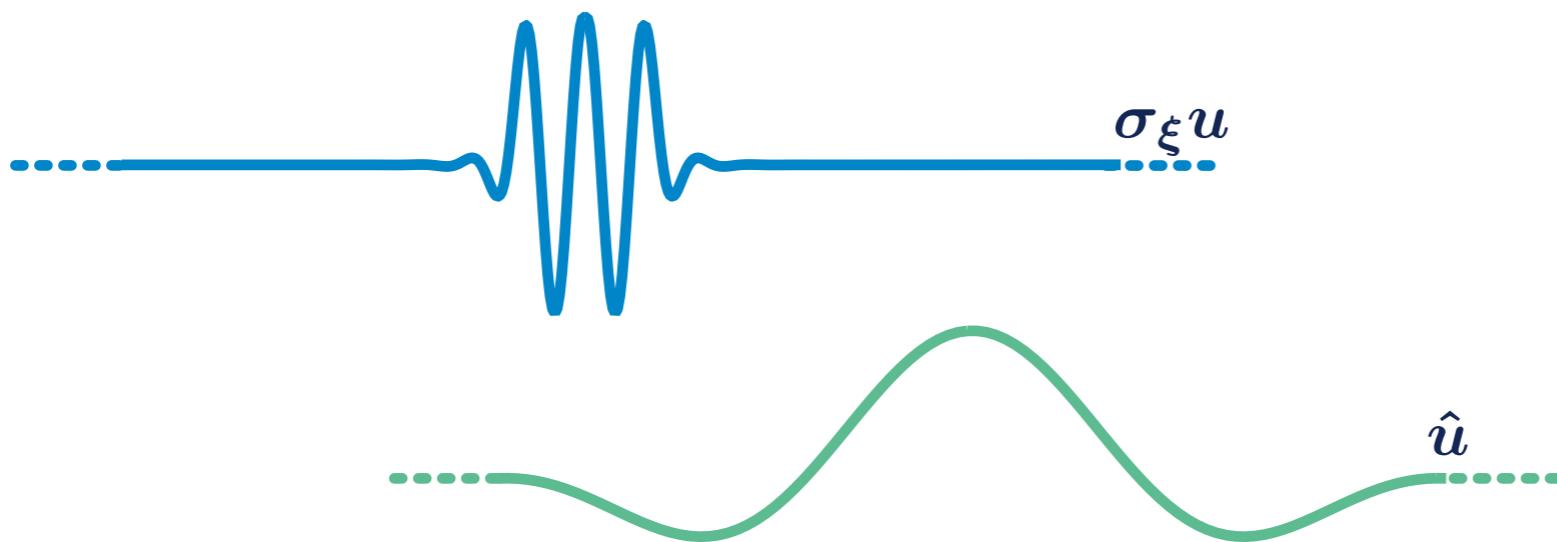


Distance between shifted solution and reference function

$$\mathcal{D}(u, \xi) = \int_{-L}^L |u(x - \xi) - \hat{u}(x)|^2 dx = \|\sigma_\xi u - \hat{u}\|_{L_2}^2$$

Phase condition

See Codes



Distance between shifted solution and reference function

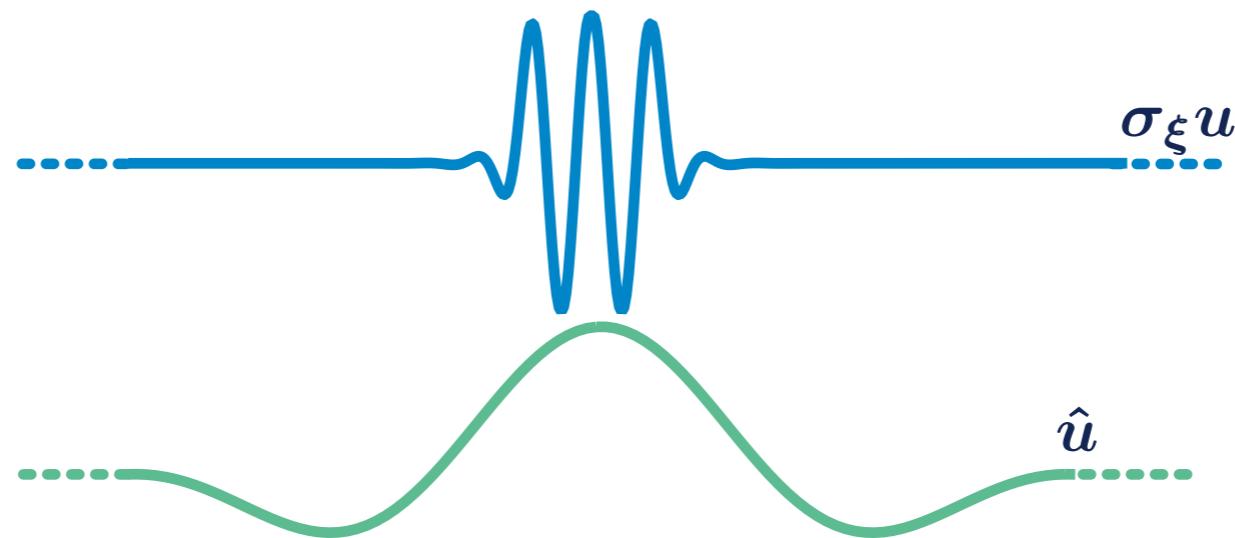
$$\mathcal{D}(u, \xi) = \int_{-L}^L |u(x - \xi) - \hat{u}(x)|^2 dx = \|\sigma_\xi u - \hat{u}\|_{L_2}^2$$

Differentiating w.r.t. ξ we obtain a necessary condition for a local minimum

$$\int_{-L}^L \partial_x \hat{u}(x) (u(x - \xi) - \hat{u}(x)) dx = 0$$

Phase condition

See Codes



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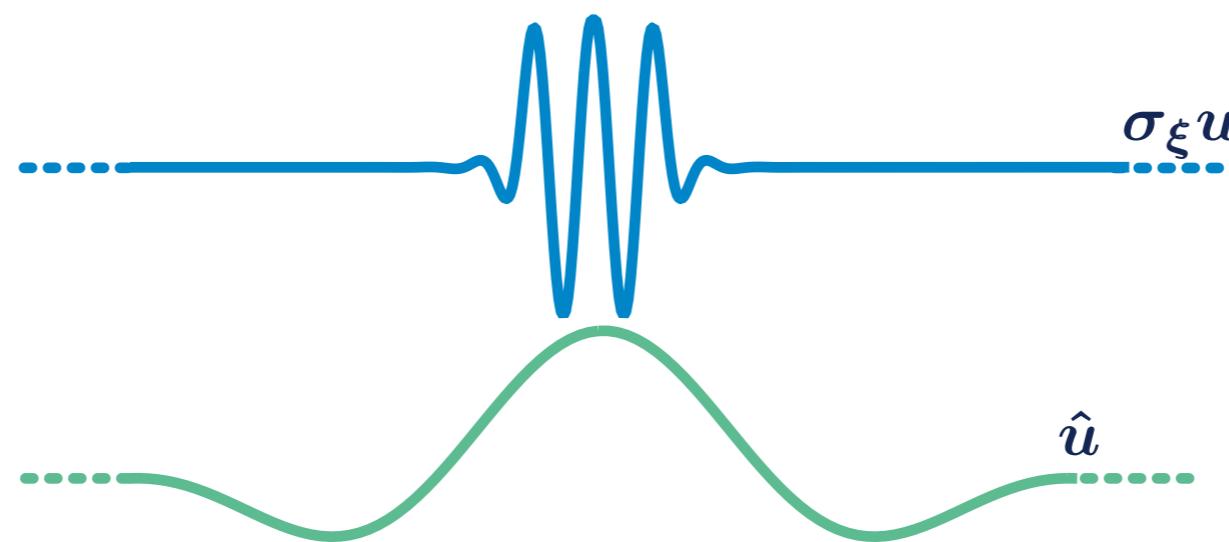
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$$\int_{-L}^L \partial_x \hat{u}(x) (u(x - \xi) - \hat{u}(x)) dx = 0$$

- The phase condition $\varphi(u, \hat{u}) = 0$ selects a solution for which the above equation holds at $\xi = 0$

Phase condition

See Codes



[Doedel]

[Beyn, Thümmler]

[Champneys, Sandstede]

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- Phase conditions (and borderings) can be constructed for more general symmetries

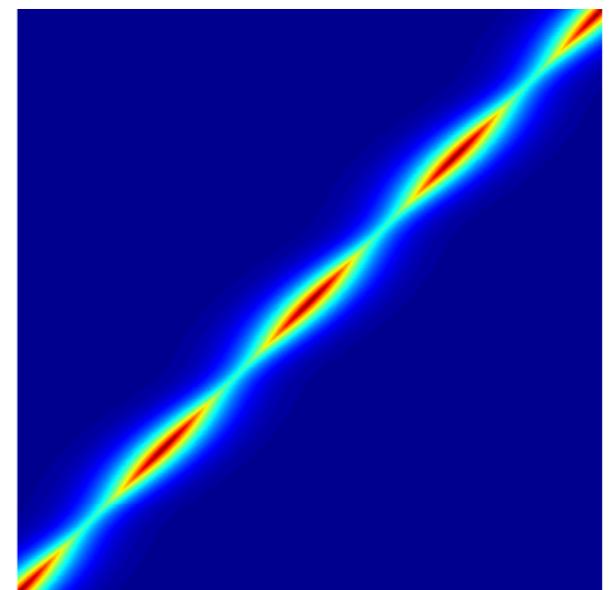
Neural fields: PDE and integral form

Integral Neural Field Model

■ Wilson-Cowan-Amari neural field equation

$$\partial_t u(x, t) = -u(x, t) + \int_{-\infty}^{\infty} W(x, y) f(u(y, t)) dy, \quad x \in \mathbb{R}$$

$W(x, y)$



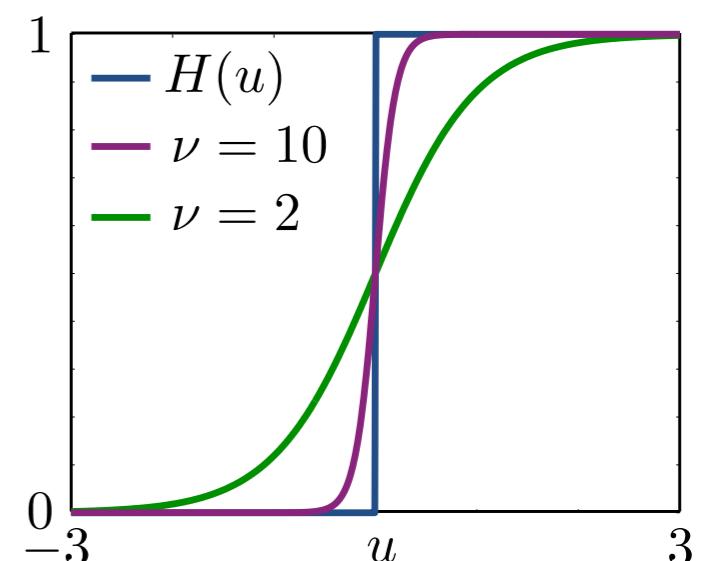
■ Heterogeneous connectivity function

$$\begin{aligned} W(x, y; a, \varepsilon) &= w(|x - y|) A(y; a, \varepsilon) \\ &= \frac{1}{2} e^{-|x-y|} (1 + a \cos(y/\varepsilon)), \end{aligned}$$

■ Firing rate function

$$f(u; h, \nu) = \frac{1}{1 + \exp(-\nu(u - h))}$$

[A. & Schmidt, Bressloff, Schmidt et al, Coombes & Laing]



PDE method [Laing]

■ Fourier transform

$$\partial_t \hat{u}(\xi, t) = -\hat{u}(\xi, t) + \hat{w}(\xi) (\widehat{Af(u)})(\xi, t),$$

where $\hat{w}(\xi) = P(\xi)/Q(\xi)$

■ Inverse Fourier transform

$$\mathcal{L}_Q [\partial_t u + u] = \mathcal{L}_P [A(x)f(u; h, \nu)]$$

■ For our choice of the kernel:

Evolution equation $(1 - \partial_x^2)\partial_t u = (\partial_x^2 - 1)u + A(x)f(u)$

Steady states $0 = (\partial_x^2 - 1)u + A(x)f(u)$

Stability $\lambda(1 - \partial_x^2)\psi = [\partial_x^2 - 1 + A(x)f'(u)]\psi$

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Stability $\lambda(1 - \partial_x^2)\psi = [\partial_x^2 - 1 + A(x)f'(u)]\psi$

■ Form the matrix $M = I_n - D_{xx}$

Time step using mass matrix $M\dot{U} = -MU + A \odot f(U)$

Steady states continuation $0 = -MU + A \odot f(U)$

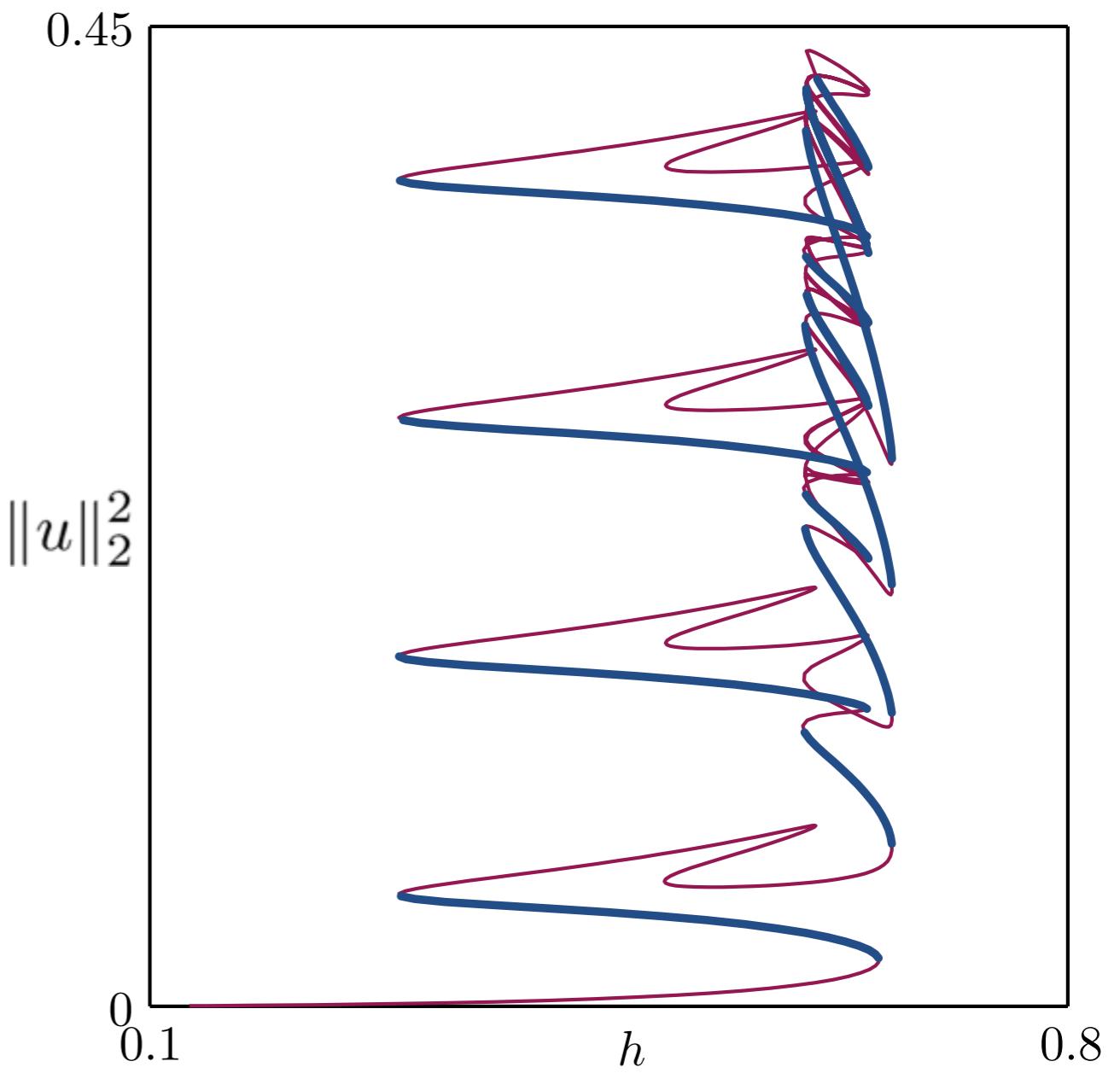
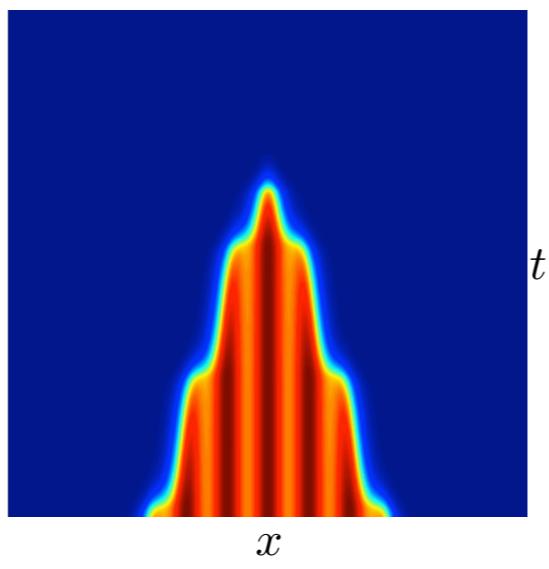
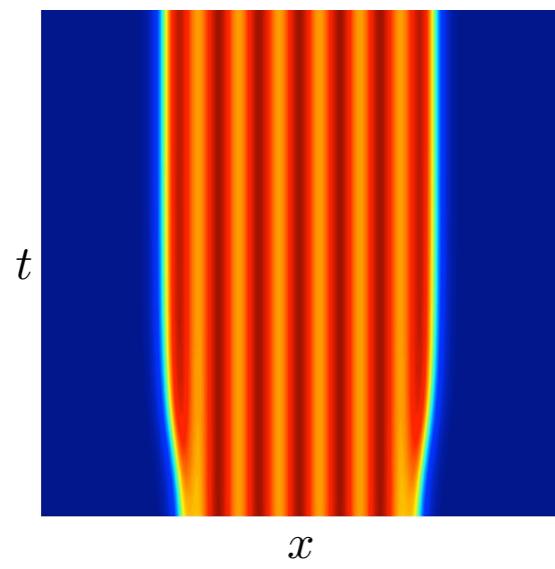
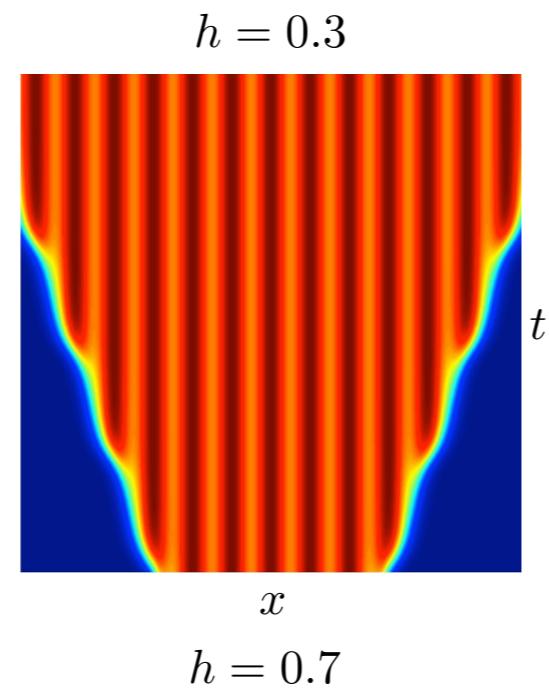
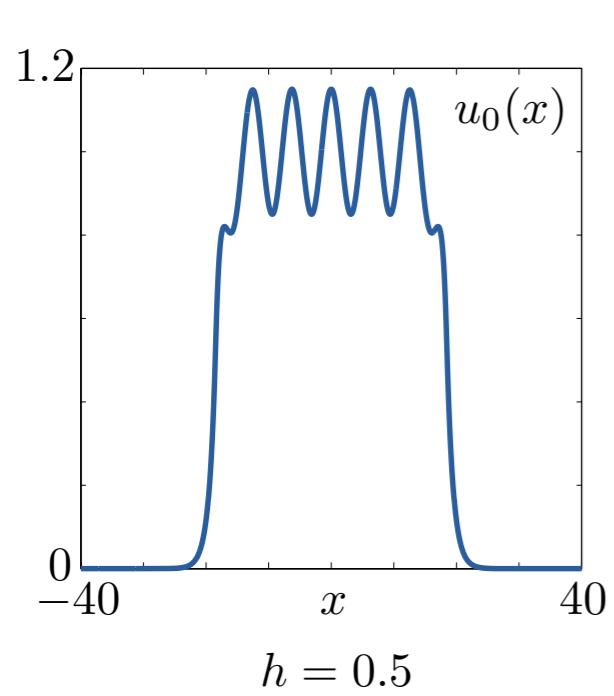
Steady states stability $\lambda M \Psi = [-M + \text{diag}(A \odot f'(U))]\Psi$

■ This fits into the framework used for the SH equation!

■ Read on for continuation of neural field models in integral form.

Numerical computations

See Codes

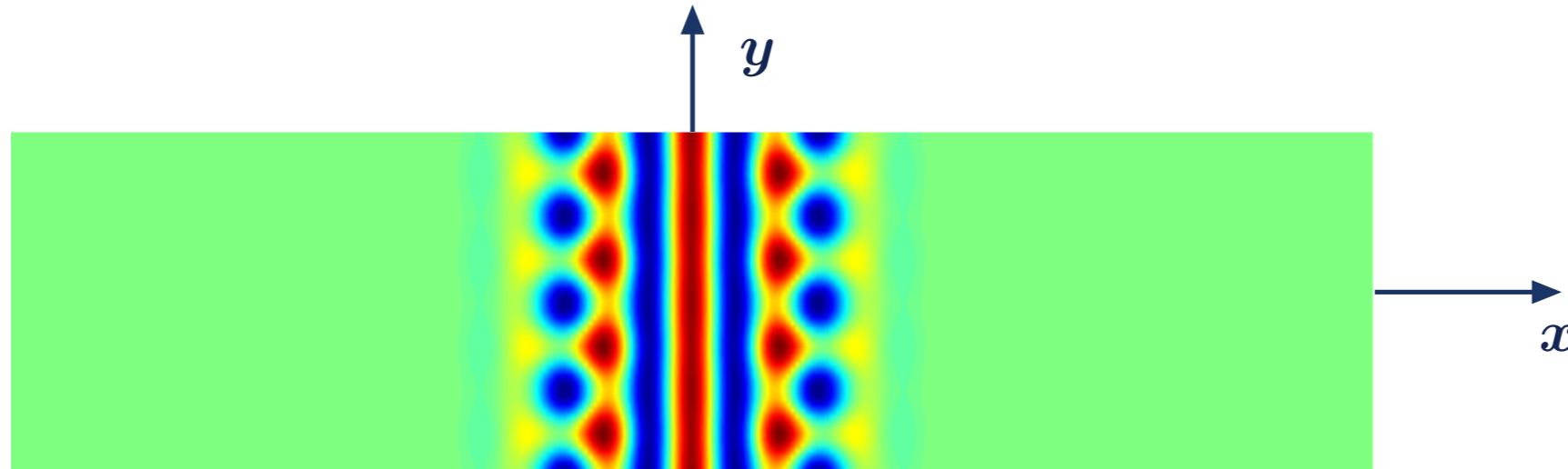


[A. & Schmidt]

From 1D to 2D

Swift-Hohenberg on a plane

See Codes



- Swift-Hohenberg equation posed on a 2D domain

$$\partial u_t = -(1 + \Delta)^2 u - \mu u + \nu u^3 - u^5, \quad (x, y) \in \mathbb{R} \times \mathbb{R}/2L_y\mathbb{Z}$$

- We would like to obtain a discrete form of this type

$$\dot{U} = -(I + L)^2 U - \mu U + \nu U^{\odot 3} - U^{\odot 5}$$

- The matrix L is the differentiation matrix for the Laplacian

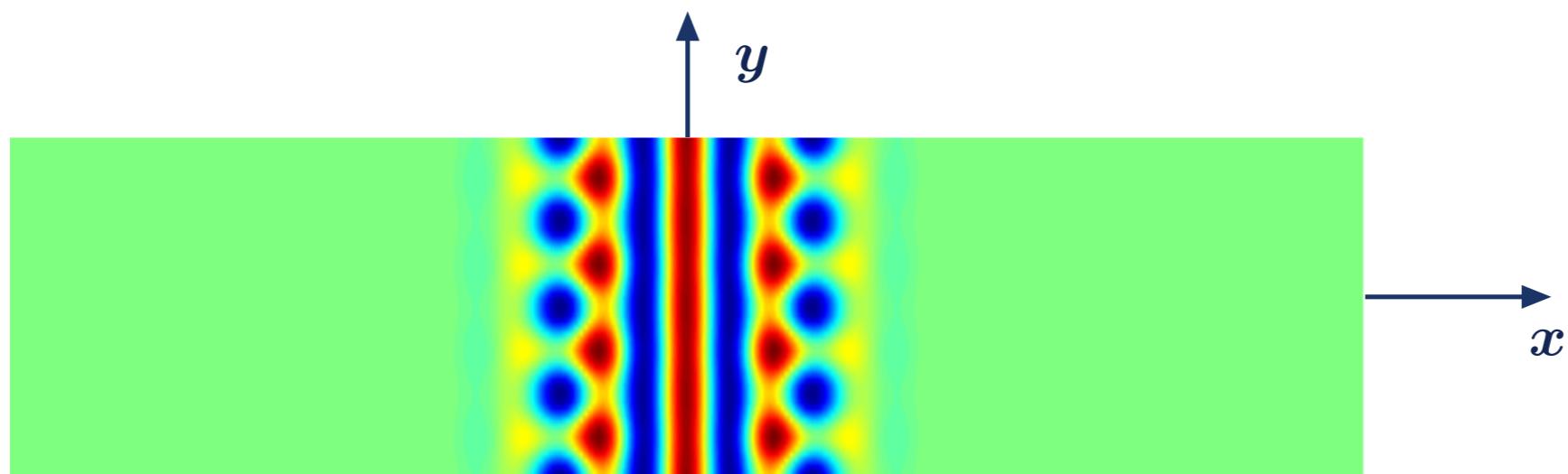
1. It can be computed elegantly for tensor grids
2. Different discretisations (or discretisation methods!) are allowed for x and y

Building the Laplacian for differentiation matrices

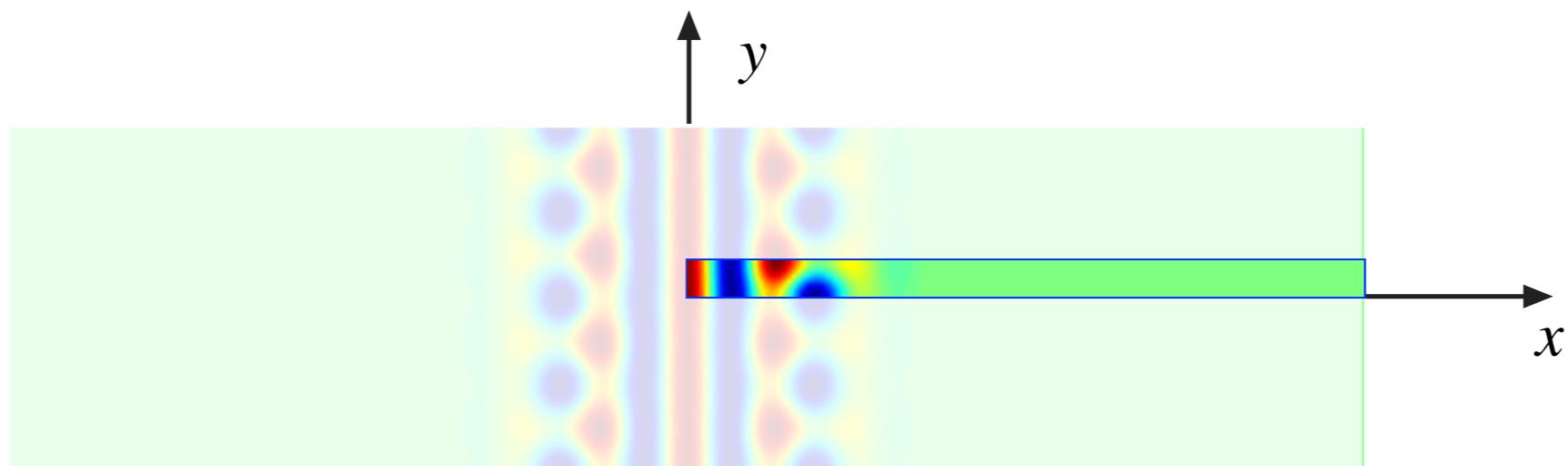


- Consider a grid of $N_x \times N_y$ nodes, and pose $u(x_i, y_j) \approx U_{i,j}$
- If the component of the unknown vector $\mathbf{U} \in \mathbb{R}^{N_x N_y \times 1}$ are in lexicographic order, the discrete Laplacian can be computed using the Kronecker product \otimes
$$L = D_{xx} \otimes I_y + I_x \otimes D_{yy}$$
- Where $D_{xx}, I_x \in \mathbb{R}^{N_x \times N_x}$ and $D_{yy}, I_y \in \mathbb{R}^{N_y \times N_y}$ are calculated independently
- The framework seen in 1D is easy to adapt to the 2D case

Notes on 2D calculations

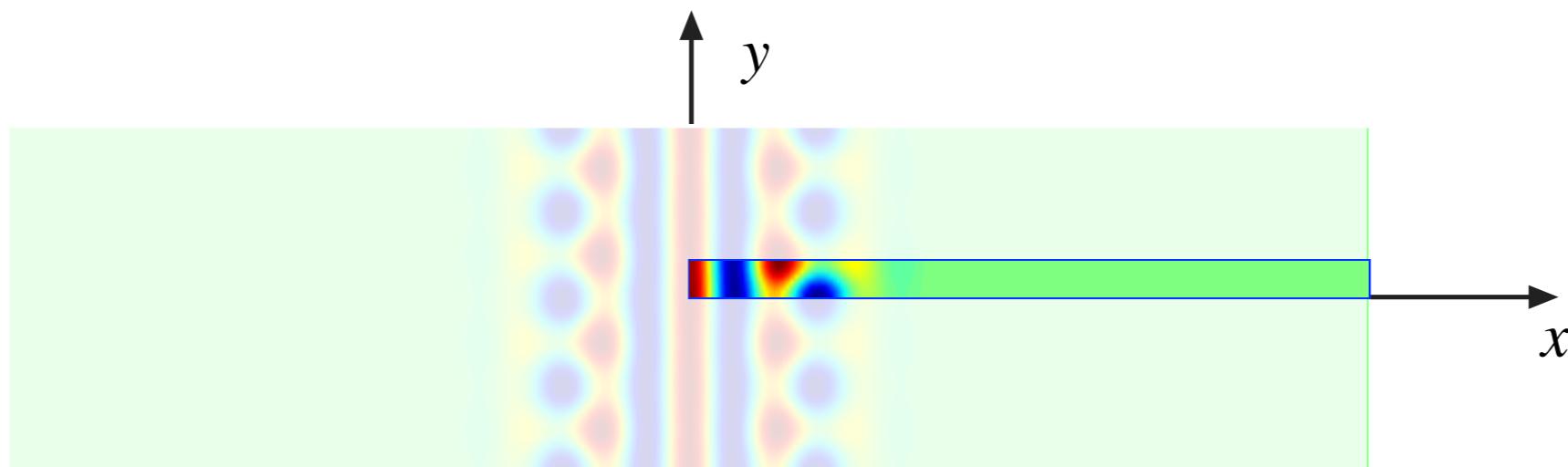


Notes on 2D calculations

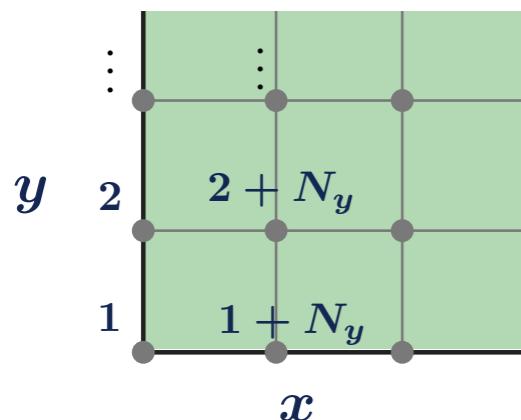


- Use symmetries whenever possible

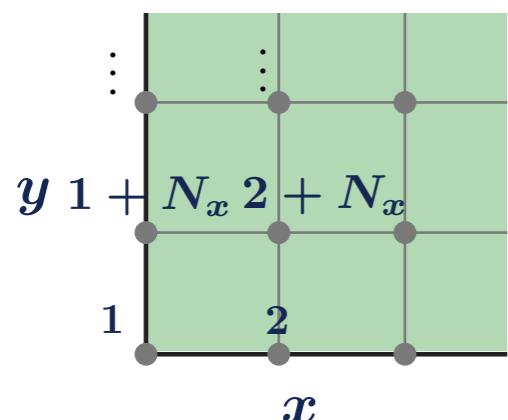
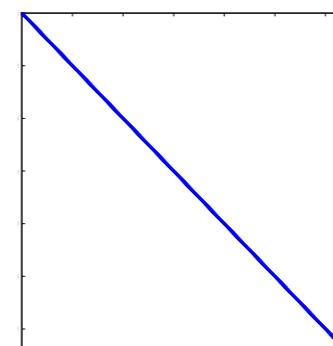
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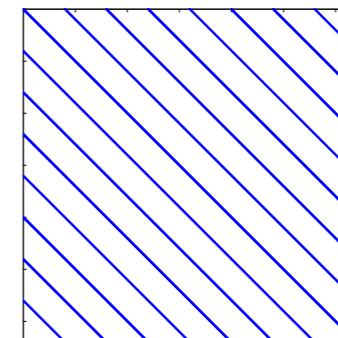
- Use symmetries whenever possible
- Ordering may affect the sparsity pattern/matrix bandwidth



$$L = D_{xx} \otimes I_y + I_x \otimes D_{yy}$$



$$L = D_{yy} \otimes I_x + I_y \otimes D_{xx}$$



Notes on 2D calculations

■ Linear algebra routines could be a bottleneck. Sometimes we need to

1. Use of efficient solvers for bordered systems
2. Use of Matrix-Free methods (Krylov methods)
3. Use preconditioners for iterative linear systems

■ Finite Element methods may be more efficient:

1. Allow a greater flexibility in allocating nodes
2. Mesh adaptation is possible
3. The tensor-product trick will not work in general

Spatio-temporal structures

Coherent structures

■ Coherent structures: solutions to nonlinear evolution equations with a special spatio-temporal structure

1. Travelling waves and solitons
2. Spiral waves in 2D
3. Breathers
4. Oscillons

■ Strategy to compute coherent structures

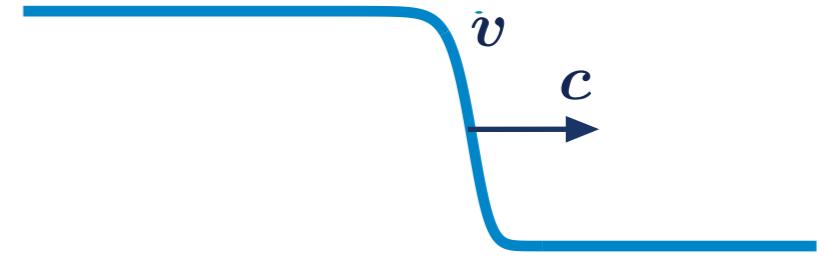
1. Prescribe a defining system of equations (a well-posed BVP, often symmetries require phase conditions)
2. Setup an eigenvalue problem for linear stability
3. Use the methods seen in the past sections

Travelling wave continuation

- We derive the defining problem for TW of

$$\partial_t u = \mathcal{F}(u)$$

$$0 = \mathcal{B}(u)$$



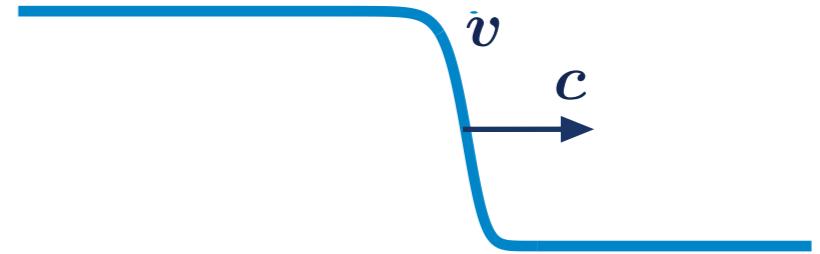
- Need to determine the *profile* v and the *wave speed* c

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- Travelling waves are special solutions to the evolution equation above
$$u(x, t) = v(x - ct)$$

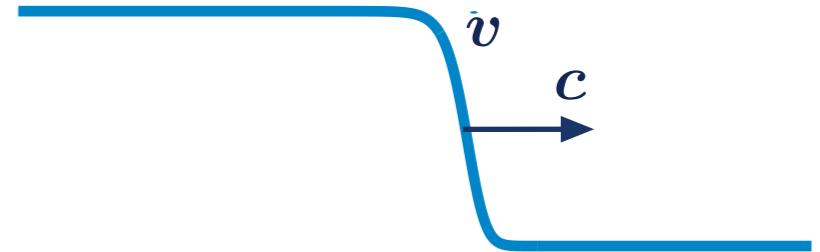
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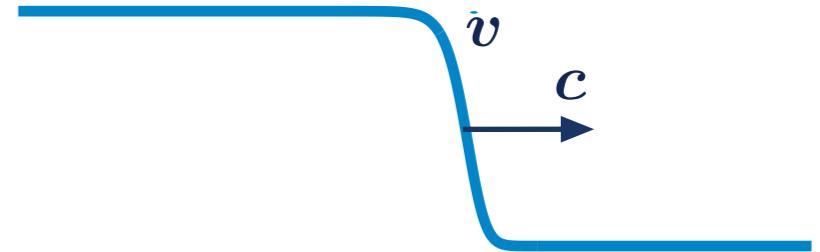
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- This is not a valid defining system!

1. We need one extra equation to determine c
2. The linearised operator has a 0 eigenvalue, with eigenfunction $\partial_z v$

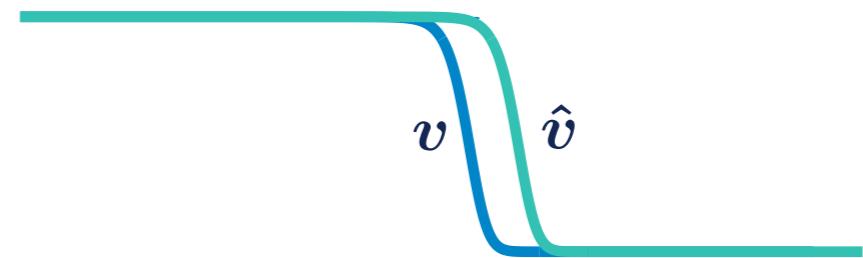
Travelling waves - defining system

■ Defining system: find (v, c) such that

$$c\partial_z v + \mathcal{F}(v) = 0$$

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$$\varphi(v, \hat{v}) = 0$$



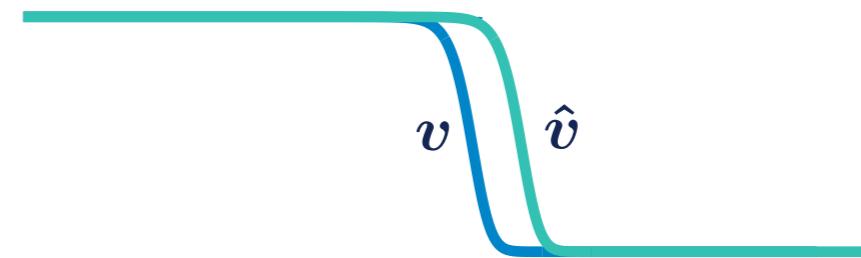
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■ Stability of a travelling wave (v_*, c_*)

$$u(x, t) = v_*(x - ct) + \varepsilon \tilde{v}(x - ct, t), \quad 0 < \varepsilon \ll 1, \quad \|\tilde{v}\| = \mathcal{O}(1)$$

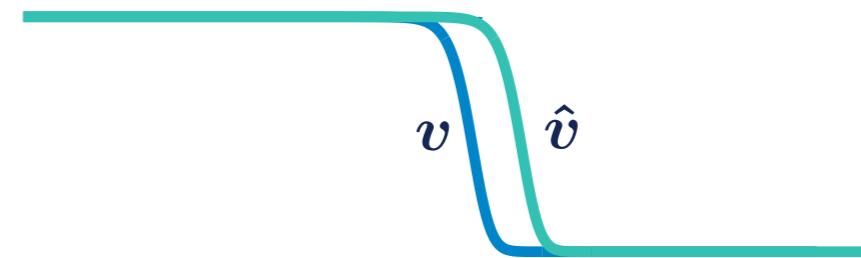
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■ At $\mathcal{O}(\varepsilon)$, the perturbations evolve according to

$$\partial_t \tilde{v} = [c_* \partial_z + \partial_u \mathcal{F}(v_*)] \tilde{v} := \mathcal{L}(v_*, c_*) \tilde{v}$$

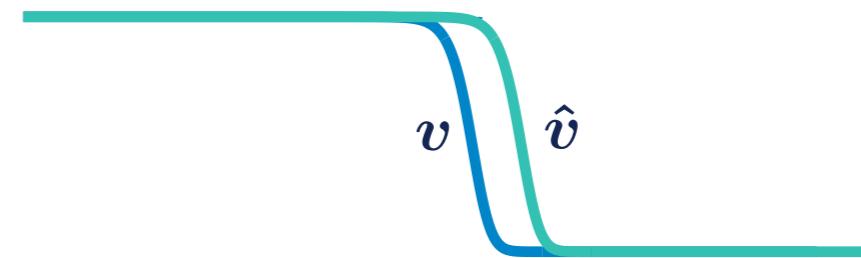
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■ The travelling wave is linearly stable if $\text{Spec} [\mathcal{L}(v_*, c_*)] \setminus \{0\}$ is contained in the left half of the complex plane

Conclusions

■ Coherent structures can be analysed following structured steps

1. Prescribe a well-defined BVP
2. Discretise the BVP using finite differences, spectral methods, FEM
3. Use standard numerical continuation techniques for path-following
4. Prescribe an eigenvalue problem
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- The steps above can also wrap around a numerical time stepper

$$\dot{U} = F(U; p), \quad \Phi_T: U(t) \mapsto U(t + T)$$

[Keller] [Lust] [Tuckermann, Barkley] [Cvitanovich et al.] [Sanchez-Umbria]

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