SUPPLEMENTARY MATERIALS FOR NEURAL FIELDS AND NOISE-INDUCED PATTERNS IN NEURONS ON LARGE DISORDERED NETWORKS

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In the Supplementary Materials we prove some additional results, and include further numerical simulations. Recall the n-dimensional system with averaged connectivity,

$$
\begin{aligned}
dv^j_{\alpha,t} = \Big(-\sum_{\beta=1}^{q} L_{\alpha\beta} v^j_{\beta,t} \quad &+ n^{-1} \sum_{k=1}^{n} \sum_{\beta=1}^{q} \mathcal{K}_{\alpha\beta}(x^j_n, x^k_n) f_\beta(v^k_t) + I_{\alpha,t}(x^j_n) \Big) dt \\
&+ \sum_{\beta=1}^{q} G_{\alpha\beta,t}(x^j_n) dW^j_{\beta,t}, \\
v^i_\alpha = z^i_\alpha &
\end{aligned}
\tag{0.1}
$$

which shares with the original particle system identical Brownian motions and initial conditions. Our first result concerns the push-forward function Φ_T referred to in Section 6.

1. Transformation of the Uncoupled System to the Coupled System. Consider the empirical measure generated by (0.1), that is, the particle system with average coupling

$$\grave{\mu}^n_T = n^{-1} \sum_{j=1}^{n} \delta_{x^j_n, v^j}.$$

Our first main aim is to prove Lemma 6.2. We must thus find a continuous mapping $\Phi_T : X_T \to Y_T$, such that $\grave{\mu}^n_T = \Phi_T\big(\tilde{\mu}^n_T\big)$ identically, with $\tilde{\mu}^n_T$ given by

$$\tilde{\mu}^n_T = n^{-1} \sum_{j \in \mathbb{N}_n} \delta_{b^j} \quad b^j = \big(x^j_n, u^j_0, \{\tilde{W}^j_t : t \in [0,T]\}\big), \quad \tilde{W}^j_{\alpha,t} = \sum_{\beta=1}^{q} \int_0^t G_{\alpha\beta,s}(x^j_n) dW^j_{\beta,s} \tag{1.1}$$

This allows us to 'push-forward' the Large Deviations Principle for the uncoupled system (as noted in Section 6.1 of the main paper) to obtain a Large Deviations Principle for the coupled system. To the knowledge of these authors, the first scholar to apply this technique to the Large Deviations of interacting particle systems was Tanaka [3]. One of these authors has used this technique to determine the Large Deviations of a spatially-distributed network of interacting neurons in [2].

The mapping Φ_T is defined in two steps, using an intermediate mapping ψ_T that we are now going to discuss. The mapping ψ_T can be thought of as as transforming the characteristics of the noise empirical measure $\tilde{\mu}^n_T$ to the characteristics of $\grave{\mu}^n_T$.

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Define

$$\psi_T : Y_T \times D \times \mathbb{R}^q \times C\big([0,T],\mathbb{R}^q\big) \to D \times C\big([0,T],\mathbb{R}^q\big)$$
$$(\nu, x, z, w) \mapsto (x, v),$$

where $v \in C\big([0,T],\mathbb{R}^q\big)$ is defined to be such that for all $(\alpha, t) \in \mathbb{N}_q \times [0,T]$

$$v_{\alpha,t} = z_\alpha + \int_0^t \Big(-\sum_{\beta\in\mathbb{N}_q} L_{\alpha\beta} v_{\beta,s} + \sum_{\beta\in\mathbb{N}_q} \mathbb{E}^{(y,u)\sim\nu}\big[\mathcal{K}_{\alpha\beta}(x,y) f_\beta(u_s)\big] + I_\alpha(s,x) \Big) ds + w_{\alpha,t}.$$

LEMMA 1.1. *The transformation ψ_T is well-defined. Furthermore ψ_T is continuous.*

Proof. To prove well-definedness, fix $(\nu, x, z, w) \in Y_T \times D \times \mathbb{R}^q \times C([0,T],\mathbb{R}^q)$, and consider the map $\Lambda \colon C([0,T],\mathbb{R}^q) \to C([0,T],\mathbb{R}^q)$ defined by

$$\Lambda(r)_{\alpha,t} := z_\alpha + \int_0^t \Big(-\sum_{\beta\in\mathbb{N}_q} L_{\alpha\beta} r_{\beta,s} + \sum_{\beta\in\in\mathbb{N}_q} \int \mathcal{K}_{\alpha\beta}(x,y) f_\beta(u_s) \nu(dy,du) + I_\alpha(s,x) \Big) ds + w_{\alpha,t}.$$

We claim that Λ has a unique fixed point v. If this holds, then the fixed point v satisfies $\psi_T(\nu,x,z,w) = (x,v)$, which means ψ_T is a well-defined operator on $Y_T \times D \times \mathbb{R}^q \times C([0,T],\mathbb{R}^q)$ to $\mathbb{R}^q \times C([0,T],\mathbb{R}^q)$. To prove that Λ has a unique fixed point, we introduce the norm

$$\|v\|_\rho = \sup_{t\in[0,T]} e^{-\rho t} \|v_t\|_{\mathbb{R}^q},$$

which is equivalent to $\|\cdot\|_T$ on $C([0,T],\mathbb{R}^q)$ for any $\rho > 0$, because $e^{-\rho T}\|v\|_T \leq \|v\|_\rho \leq \|v\|_T$. We prove that Λ is a contraction on $(C([0,\tau],\mathbb{R}^q), \|\cdot\|_\rho)$, and hence on $(C([0,\tau],\mathbb{R}^q), \|\cdot\|_T)$, for any $\rho > \|L\|$ where the latter is the operator norm of L, seen as an operator on $\mathbb{R}^q$ to $\mathbb{R}^q$. For any $p, r \in (C([0,\tau],\mathbb{R}^q), \|\cdot\|_\rho)$, we estimate

$$\begin{aligned}
\|\Lambda(r) - \Lambda(p)\|_\rho &\leq \sup_{t\in[0,T]} e^{-\rho t} \int_0^t e^{\rho s} \|e^{-\rho s} L(r_s - p_s)\|_{\mathbb{R}^q}\, ds \\
&\leq \|L\|\, \|r-p\|_\rho \sup_{t\in[0,T]} e^{-\rho t} \int_0^t e^{\rho s}\, ds \\
&= \|L\|\, \|r-p\|_\rho \sup_{t\in[0,T]} \frac{1-e^{-\rho t}}{\rho} \leq \frac{\|L\|}{\rho} \|r-p\|_\rho
\end{aligned}$$

which proves that Λ is a contraction on $(C([0,T],\mathbb{R}^q), \|\cdot\|_\rho)$, for any $\rho > \|L\|$. This proves Λ has a unique fixed point in $(C([0,T],\mathbb{R}^q), \|\cdot\|_T)$.

Notice that the function $(y,v) \to \mathcal{K}_{\alpha\beta}(x,y) f_\beta(v)$ is Lipschitz and bounded, for all $x \in D$. We thus find that there must exist a universal constant C_2 such that for all $\mu, \tilde{\mu} \in Y_t$, for any $x \in D$,

$$\Big| \int_0^t \sum_{\beta\in\mathbb{N}^q} \int \mathcal{K}_{\alpha\beta}(x,y) f_\beta(u_s) \mu(dy,du) - \int_0^t \sum_{\beta\in\mathbb{N}^q} \int \mathcal{K}_{\alpha\beta}(x,y) f_\beta(u_s) \tilde{\mu}(dy,du) \Big| \leq C_2 d_{Y_t}(\mu, \tilde{\mu}) \tag{1.2}$$

∎

Since the drift term in is uniformly Lipschitz in v, standard techniques imply that ψ_T is continuous.

We next define $\Phi_T : X_T \to Y_T$ to be such that

$$\Phi_T(\mu) := \mu \circ \psi_T(\Phi_T(\mu), \cdot, \cdot, \cdot)^{-1}. \tag{1.3}$$

In words, $\Phi_T(\mu)$ is the push-forward of μ by the characteristic map ψ_T. The transformation Φ_T is useful thanks to the following result.

LEMMA 1.2. *With unit probability,*

$$\grave{\mu}^n_T = \Phi_T\big(\tilde{\mu}^n_T\big), \tag{1.4}$$

Proof. This is almost immediate from the definitions. ◻

LEMMA 1.3. *For any $\mu \in X_T$ there exists a unique solution $\Phi_T(\mu) \in X_T$ to the fixed point identity (1.3). Furthermore the associated mapping $\mu \mapsto \Phi_T(\mu)$ is continuous over X_T.*

Proof. Fix $\mu \in X_T$. For any $\gamma \in Y_T$, define $\Gamma_{\mu,T}(\gamma) := \mu \circ \psi_T(\gamma, \cdot, \cdot, \cdot)^{-1} \in Y_T$. One easily checks that for any $\gamma, \tilde{\gamma} \in Y_T$, there is a constant $C > 0$ such that

$$d_{Y_t}\big(\Gamma_{\mu,t}(\gamma), \Gamma_{\mu,t}(\tilde{\gamma})\big) \leq Ct d_{Y_t}\big(\gamma, \tilde{\gamma}\big). \tag{1.5}$$

Hence for small enough t, $\Gamma_{\mu,t}$ is a contraction and there is a unique fixed point solution $\mu_{*,t}$ to (1.3).

Next, define the space $\tilde{Y}_t$ to consist of all measures $\nu \in Y_T$ such that the law of the variables upto time t is identical to $\mu_{*,t}$. For $\gamma, \tilde{\gamma} \in \tilde{Y}_t$ we find that for $s \geq t$,

$$d_{Y_s}\big(\Gamma_s(\gamma), \Gamma_s(\tilde{\gamma})\big) \leq C(s-t) d_{Y_s}\big(\gamma, \tilde{\gamma}\big). \tag{1.6}$$

Furthermore the constant C depends on T only. Hence Γ is a contraction for small enough $s - t$, and we obtain a unique fixed point in $\tilde{Y}_t$. Iterating this argument, we obtain a unique fixed point $\Phi_T(\mu)$ upto time T.

For the continuity, let $\mu, \tilde{\mu} \in X_T$, and let ξ_ε be a measure that is within $\varepsilon \ll 1$ of realizing the infimum in the definition of the Wasserstein metric. That is, we write ξ_ε to be the law of coupled random variables $(x, u_0, w), (\tilde{x}, \tilde{u}_0, \tilde{w}) \in D \times \mathbb{R}^q \times C([0,T], \mathbb{R}^q)$, and ξ_ε is such that

$$\mathbb{E}^{\xi_\varepsilon}\big[\|x - \tilde{x}\| + \|u_0 - \tilde{u}_0\| + \|w - \tilde{w}\|_T\big] \leq \varepsilon. \tag{1.7}$$

Write $\psi_T\big(\Phi_T(\mu), x, u_0, w\big) := (x, u_{[0,T]})$ and $\psi_T\big(\Phi_T(\tilde{\mu}), \tilde{x}, \tilde{u}_0, \tilde{w}\big) := (\tilde{x}, \tilde{u}_{[0,T]})$. Substituting definitions, and employing the Lipschitz property in (1.2), we find that there is a constant $C > 0$ (chosen independently of T) such that

$$\begin{aligned} \sup_{t \leq T} \sup_{1 \leq \alpha \leq q} \big|u_\alpha(t) - \tilde{u}_\alpha(t)\big| \leq{}& \sup_{1 \leq \alpha \leq q} \big|u_\alpha(0) - \tilde{u}_\alpha(0)\big| \\ &+ C \int_0^T \Big\{ \sup_{t \leq T} \sup_{1 \leq \alpha \leq q} \big|u_\alpha(t) - \tilde{u}_\alpha(t)\big| + d_{Y_t}\big(\Phi_t(\mu), \Phi_t(\tilde{\mu})\big) \\ &\qquad + \|x - \tilde{x}\| \Big\} dt + \sup_{t \leq T} \sup_{1 \leq \alpha \leq q} \big|w_\alpha(t) - \tilde{w}_\alpha(t)\big|. \end{aligned} \tag{1.8}$$

Taking expectations of both sides with respect to ξ_ε, and writing

$$y_s = \mathbb{E}\big[\sup_{t \leq s} \|u_\alpha(t) - \tilde{u}_\alpha(t)\| + \|x - \tilde{x}\|\big], \tag{1.9}$$

we obtain that for all $t \geq 0$,

$$y_t \leq \varepsilon + C \int_0^t \left\{ y_s + d_{Y_s}\big(\Phi_s(\mu), \Phi_s(\tilde{\mu})\big) + d_{X_T}(\mu, \tilde{\mu}) + \varepsilon \right\} ds + d_{X_T}(\mu, \tilde{\mu}) \tag{1.10}$$

since

$$\mathbb{E}^{\xi_\varepsilon}\big[\sup_{t \leq T} \sup_{1 \leq \alpha \leq q} \big| w_\alpha(t) - \tilde{w}_\alpha(t) \big| \big] \leq \varepsilon + d_{X_T}(\mu, \tilde{\mu}).$$

Note that, by definition of the Wasserstein Metric,

$$d_{Y_s}\big(\Phi_s(\mu), \Phi_s(\tilde{\mu})\big) \leq y_s.$$

Taking $\varepsilon \to 0^+$, we thus find that

$$d_{Y_t}\big(\Phi_t(\mu), \Phi_t(\tilde{\mu})\big) \leq C \int_0^t \big(2 d_{Y_s}\big(\Phi_s(\mu), \Phi_s(\tilde{\mu})\big) + d_{X_T}(\mu, \tilde{\mu}) \big) ds + d_{X_T}(\mu, \tilde{\mu}).$$

An application of Gronwall's Inequality then implies that there exists a constant $\tilde{C}_T$ such that

$$d_{Y_T}\big(\Phi_T(\mu), \Phi_T(\tilde{\mu})\big) \leq \tilde{C}_T d_{X_T}(\mu, \tilde{\mu}). \tag{1.11}$$

Thus Φ_T is Lipschitz (and also continuous). ◻

2. Bounding the Original Particle System. Our main results only concern the convergence of empirical averages of bounded continuous functions. In fact it is possible to show that the empirical average of certain unbounded continuous functions also converges. This is particularly desirable for our Gaussian Application, because Gaussian distributions are most conveniently described in terms of their first and second moments. The main result of this section is the following Corollary to Theorem 3.9.

COROLLARY 2.1. *Let $h : \mathbb{R}^q \mapsto \mathbb{R}$ be continuous and such that*

$$\|h(z)\| \leq \mathrm{Const}\|\mathrm{z}\|^2.$$

Then for any $t \leq T$, any continuous function $g : D \mapsto \mathbb{R}$, it holds that

$$\lim_{n \mapsto \infty} \Big| n^{-1} \sum_{j \in \mathbb{N}_n} g(x_n^j) h(u_t^j) - \mathbb{E}^{(x,u) \sim \bar{\mu}_t}\big[g(x) h(u)\big] \Big| = 0. \tag{2.1}$$

Proof. For any $c > 0$, define $h_c : \mathbb{R}^q \mapsto \mathbb{R}$ to be such that

$$h_c(u) = h(cu/\|u\|) \text{ in the case that } \|u\| \geq c \tag{2.2}$$

and if $\|u\| < c$, define $h_c(u) = h(u)$. Notice that h_c is continuous and bounded, and also that

$$\lim_{c \to \infty} \sup_{t \leq T} \mathbb{E}^{(x,u) \sim \bar{\mu}_t}\big[g(x)\big(h(u) - h_c(u)\big)\big] = 0. \tag{2.3}$$

Since h_c is continuous and bounded, Theorem 1 implies that

$$\lim_{n \to \infty} \Big| n^{-1} \sum_{j \in I_n} g(x_n^j) h_c(u_t^j) - \mathbb{E}^{(x,u) \sim \bar{\mu}_t}\big[g(x) h_c(u)\big] \Big| = 0. \tag{2.4}$$

We therefore wish to show that

$$\lim_{c\to\infty}\lim_{n\to\infty}\left|n^{-1}\sum_{j\in I_n} g(x_n^j)\big(h(u_t^j)-h_c(u_t^j)\big)\right| = 0. \tag{2.5}$$

Indeed (2.3), (2.4) and (2.5) suffice for the Corollary, because they imply that

$$\lim_{n\to\infty}\left|n^{-1}\sum_{j\in I_n} g(x_n^j)h(u_t^j) - \mathbb{E}^{(x,u)\sim\bar{\mu}_t}\big[g(x)h(u)\big]\right| = \\ \lim_{c\to\infty}\lim_{n\to\infty}\left|n^{-1}\sum_{j\in I_n} g(x_n^j)\big(h(u_t^j)-h_c(u_t^j)\big) \right. \\ \left. - \mathbb{E}^{(x,u)\sim\bar{\mu}_t}\big[g(x)\big(h(u)-h_c(u)\big)\big]\right| = 0. \tag{2.6}$$

It only remains to prove (2.5), and in fact this is a consequence of Lemma 2.2 below.□

We next bound the tails of the second moment of the original particle system.

LEMMA 2.2. *For any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that*

$$\overline{\lim_{n\to\infty}}\, n^{-1}\log\mathbb{P}\left(n^{-1}\sum_{j\in I_n}\|u_t^j\|^2\chi\{\|u_t^j\|\geq c_\varepsilon\}\geq\varepsilon\right) < 0. \tag{2.7}$$

Proof. Write the matrix exponential as $Q_t = \exp\big(-tL\big)$. The solution of (**??**), written in its mild form, satisfies the identity

$$u_t^j = Q_t u_0^j + \int_0^t Q_{t-s}\Big(\frac{1}{n}\sum_{k=1}^n\frac{1}{\varphi_n}K^{jk}f(u_s^k)+I_s^j\Big)ds + \int_0^t Q_{t-s}G_s^j dW_s^j. \tag{2.8}$$

Taking the norm of both sides, squaring, and then using the inequality $(a_1+a_2+a_3+a_4)^2 \leq 4a_1^2+4a_2^2+4a_3^2+4a_4^2$, we find that

$$\|u_t^j\|^2 \leq 4\|Q_t u_0^j\|^2 + 4\left\|\int_0^t Q_{t-s}I_s^j ds\right\|^2 + \\ 4\left\|\int_0^t \frac{1}{n\varphi_n}\sum_{k=1}^n Q_{t-s}K^{jk}f(u_s^k)ds\right\|^2 + 4\left\|\int_0^t Q_{t-s}G_s^j dW_s^j\right\|^2. \tag{2.9}$$

Using Jensen's Inequality, and the fact that $|f|\leq f_{max}$, there is a constant such that

$$\left\|\int_0^t \frac{1}{n\varphi_n}\sum_{k=1}^n Q_{t-s}K^{jk}f(u_s^k)ds\right\|^2 \leq t\mathrm{Const}\int_0^{\mathrm{t}}\left(\frac{1}{\mathrm{n}\varphi_{\mathrm{n}}}\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathrm{K}^{\mathrm{jk}}\right)^2\mathrm{ds}. \tag{2.10}$$

Summing over j, and employing Hypothesis 3.5, we find that as long as c_ε is sufficiently large, it must hold that for all $t\leq T$,

$$4n^{-1}\sum_{j\in I_n}\left\|\int_0^t\frac{1}{n\varphi_n}\sum_{k=1}^n Q_{t-s}K^{jk}f(u_s^j)ds\right\|^2 \leq \frac{1}{4}c_\varepsilon. \tag{2.11}$$

Since it is assumed that the initial empirical measure converges, as long as c_ε is sufficiently large,

$$\overline{\lim_{n\to\infty}}\, n^{-1}\sum_{j\in I_n} 4\|Q_t u_0^j\|^2 \leq \frac{1}{4}c_\varepsilon. \tag{2.12}$$

Now the inputs are assumed to be uniformly bounded, i.e.

$$\sup_{j\in I_n}\sup_{s\leq T}\|I_s^j\| < \infty,$$

and hence as long as c_ε is sufficiently large, it must hold that

$$\overline{\lim_{n\to\infty}}\, n^{-1}\sum_{j\in I_n} 4\left\|\int_0^t Q_{t-s}I_s^j ds\right\|^2 \leq \frac{1}{4}c_\varepsilon. \tag{2.13}$$

For the stochastic integral, by Ito's Lemma,

$$\begin{aligned} n^{-1}\sum_{j\in I_n}\left\|\int_0^t Q_{t-s}G_s^j dW_s^j\right\|^2 = {} & n^{-1}\sum_{j\in I_n}\int_0^t \mathrm{tr}\big(\mathrm{Q_{t-s}G_s^j(G_s^j)^T Q_{t-s}^T}\big)\mathrm{ds} \\ & + 2n^{-1}\sum_{j\in I_n}\int_0^t (X_s^j)^T Q_{t-s}G_s^j dW_s^j \end{aligned} \tag{2.14}$$

where we have written

$$X_s^j = \int_0^t Q_{t-s}G_s^j dW_s^j \in \mathbb{R}^d.$$

Since $\sup_{j\in I_n}\sup_{s\leq T}\|G_s^j\| < \infty$, it holds that there is a constant such that for all $n\geq 1$ and all $t\leq T$,

$$n^{-1}\sum_{j\in I_n}\int_0^t \mathrm{tr}\big(\mathrm{Q_{t-s}G_s^j(G_s^j)^T Q_{t-s}^T}\big)\mathrm{ds} < \mathrm{Const}.$$

One can then show that for any $\delta > 0$,

$$\sup_{n\geq 1} n^{-1}\log\mathbb{P}\left(2n^{-1}\sum_{j\in I_n}\int_0^t (X_s^j)^T Q_{t-s}G_s^j dW_s^j \geq \delta\right) < 0. \tag{2.15}$$

We thus find that as long as c_ε is large enough

$$\overline{\lim_{n\to\infty}}\, n^{-1}\log\mathbb{P}\left(n^{-1}\sum_{j\in I_n} 4\left\|\int_0^t Q_{t-s}G_s^j dW_s^j\right\|^2 > \frac{1}{4}c_\varepsilon\right) < 0. \tag{2.16}$$

Combining the above results, we can thus conclude that

$$\overline{\lim_{n\to\infty}}\, n^{-1}\log\mathbb{P}\left(n^{-1}\sum_{j\in I_n}\|u_t^j\|^2\chi\{u_t^j\| \geq c_\varepsilon\} \geq \varepsilon\right) < 0. \tag{2.17}$$

□

3. Proof that the Connectivity Assumptions are satisfied for a sparse Erdos-Renyi Random Digraph. We prove that the connectivity assumptions in Hypothesis 3.5 are satisfied in the case that the connections take on values in $\{-1, 0, 1\}$ and are sampled independently from a probability distribution. More precisely, we take φ_n to be a positive sequence that decreases to zero, and such that there exists a constant $l > 0$ such that

$$\sum_{n=1}^{\infty} \exp\big(-ln\varphi_n\big) < \infty. \tag{3.1}$$

It is also assumed that there exist continuous functions $p^+_{\alpha\beta}, p^-_{\alpha\beta} : D \times D \to \mathbb{R}^+$ such that

$$\mathbb{P}\big(K^{jk}_{\alpha\beta} = 1\big) = \varphi_n p^+_{\alpha\beta}(x^j_n, x^k_n) \tag{3.2}$$

$$\mathbb{P}\big(K^{jk}_{\alpha\beta} = -1\big) = \varphi_n p^-_{\alpha\beta}(x^j_n, x^k_n). \tag{3.3}$$

Furthermore it is assumed that $K^{jk}_{\alpha\beta}$ is probabilistically independent of $K^{ab}_{\gamma\delta}$ if either $a \neq j$, and /or $b \neq k$. This implies that the connectivity will not be symmetric (in general). We define the averaged connectivity to be such that

$$\mathcal{K}_{\alpha\beta}(x, y) = p^+_{\alpha\beta}(x, y) - p^-_{\alpha\beta}(x, y). \tag{3.4}$$

We start by proving the following Lemma.

Lemma 3.1. *For any $c \in \{-1, 1\}$,*

$$\overline{\lim_{n\to\infty}} (n\varphi_n)^{-1} \sup_{\alpha\in\mathbb{N}_q} \sup_{j\in\mathbb{N}_n} \sum_{k\in\mathbb{N}_n} \sum_{\beta\in\mathbb{N}_q} \chi\{K^{jk}_{\alpha\beta} = c\} < \infty. \tag{3.5}$$

Proof. Thanks to Chernoff's Inequality, for any $L > 0$, $c \in \{-1, 1\}$ and any $j \in \mathbb{N}_n$, $\alpha \in \mathbb{N}_q$, for any $a > 0$,

$$\mathbb{P}\bigg(\sum_{k\in\mathbb{N}_n} \sum_{\beta\in\mathbb{N}_q} \chi\{K^{jk}_{\alpha\beta} = c\} \geq Ln\varphi_n\bigg) \leq \mathbb{E}\bigg[\exp\bigg(a\sum_{k\in\mathbb{N}_n} \sum_{\beta\in\mathbb{N}_q} \chi\{K^{jk}_{\alpha\beta} = c\} - aLn\varphi_n\bigg)\bigg].$$

Our assumptions (3.2)-(3.3) dictate that there is a constant $C > 0$ such that for all $j, k \in \mathbb{N}_n$ and $\alpha \in \mathbb{N}_q$,

$$\mathbb{E}\bigg[\exp\bigg(a\sum_{\beta\in\mathbb{N}_q} \chi\{K^{jk}_{\alpha\beta} = c\}\bigg)\bigg] \leq 1 + \varphi_n C\big(\exp(qa) - 1\big). \tag{3.6}$$

We thus find that

$$\mathbb{E}\bigg[\exp\bigg(a\sum_{k\in\mathbb{N}_n} \sum_{\beta\in\mathbb{N}_q} \chi\{K^{jk}_{\alpha\beta} = c\} - aLn\varphi_n\bigg)\bigg] \leq \exp\bigg(n\varphi_n C\big(\exp(qa) - 1\big) - aLn\varphi_n\bigg). \tag{3.7}$$

Thus for large enough L,

$$\mathbb{P}\bigg(\sum_{k\in\mathbb{N}_n} \sum_{\beta\in\mathbb{N}_q} \chi\{K^{jk}_{\alpha\beta} = c\} \geq Ln\varphi_n\bigg) \leq \exp(-ln\varphi_n). \tag{3.8}$$

Thus, thanks to assumption (3.1),

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(\sum_{k\in\mathbb{N}_n} \sum_{\beta\in\mathbb{N}_q} \chi\{K^{jk}_{\alpha\beta} = c\} \geq Ln\varphi_n \Big) < \infty.$$

An application of the Borel-Cantelli Lemma then implies that

$$\overline{\lim_{n\to\infty}} (n\varphi_n)^{-1} \sup_{\alpha\in\mathbb{N}_q} \sup_{j\in\mathbb{N}_n} \sum_{k\in\mathbb{N}_n} \sum_{\beta\in\mathbb{N}_q} \chi\{K^{jk}_{\alpha\beta} = c\} \leq L. \tag{3.9}$$

□

We next obtain a bound on the operator norm of the connectivity matrix. For a constant $c > 0$, define the event

$$\mathcal{Q}_n = \Big\{ \sup_{w\in\mathbb{R}^{nq}:\|w\|=1} \sum_{j,k\in\mathbb{N}_n} \sum_{\alpha,\beta\in\mathbb{N}_q} \chi\big\{K^{jk}_{\alpha\beta} = 1\big\} w^j_\alpha w^k_\beta \leq cn\varphi_n \text{ and } \sup_{w\in\mathbb{R}^{nq}:\|w\|=1} \sum_{j,k\in\mathbb{N}_n} \sum_{\alpha,\beta\in\mathbb{N}_q} \chi\big\{K^{jk}_{\alpha\beta} = -1\big\} w^j_\alpha w^k_\beta \leq cn\varphi_n \Big\}. \tag{3.10}$$

We notice that if the event $\mathcal{Q}_n$ holds, then necessarily

$$\sup_{w\in\mathbb{R}^{nq}:\|w\|=1} \sum_{j,k\in\mathbb{N}_n} \sum_{\alpha,\beta\in\mathbb{N}_q} K^{jk}_{\alpha\beta} w^j_\alpha w^k_\beta \leq 2cn\varphi_n. \tag{3.11}$$

Lemma 3.2. *There exists a constant $c > 0$ such that, with unit probability there exists a random integer n_0 such that for all $n \geq n_0$, the event $\mathcal{Q}_n$ holds.*

Proof. Thanks to the Perron-Frobenius Thoerem, for $a \in \{-1, 1\}$,

$$\sup_{w\in\mathbb{R}^{nq}:\|w\|=1} \sum_{j,k\in\mathbb{N}_n} \sum_{\alpha,\beta\in\mathbb{N}_q} \chi\big\{K^{jk}_{\alpha\beta} = a\big\} w^j_\alpha w^k_\beta \leq \sup_{\alpha\in\mathbb{N}_q} \sup_{j\in\mathbb{N}_n} \sum_{k\in\mathbb{N}_n} \sum_{\beta\in\mathbb{N}_q} \chi\{K^{jk}_{\alpha\beta} = a\}. \tag{3.12}$$

The Lemma is thus a consequence of Lemma 3.1. □

The next lemma concerns the limits of the constants R^n of Hypothesis 3.5.

Lemma 3.3. *With unit probability,*

$$\lim_{n\to\infty} R^n = 0. \tag{3.13}$$

Proof. Define

$$R^n_+ = n^{-1} \sup_{\alpha\in\mathbb{N}_q} \sup_{y\in[-1,1]^{qn}} \sum_{j\in\mathbb{N}_n,\alpha\in\mathbb{N}_q} \Big(n^{-1} \sum_{k\in\mathbb{N}_n,\beta\in\mathbb{N}_q} \Big(\varphi_n^{-1} \chi\big\{K^{jk}_{\alpha\beta} = 1\big\} - p^+_{\alpha\beta}(x^j_n, x^k_n) \Big) y^k_\beta \Big)^2$$

$$R^n_- = n^{-1} \sup_{\alpha\in\mathbb{N}_q} \sup_{y\in[-1,1]^{qn}} \sum_{j\in\mathbb{N}_n,\alpha\in\mathbb{N}_q} \Big(n^{-1} \sum_{k\in\mathbb{N}_n,\beta\in\mathbb{N}_q} \Big(\varphi_n^{-1} \chi\big\{K^{jk}_{\alpha\beta} = -1\big\} - p^-_{\alpha\beta}(x^j_n, x^k_n) \Big) y^k_\beta \Big)^2$$

It suffices that we show that

$$\lim_{n\to\infty} R_-^n = 0 \tag{3.14}$$

$$\lim_{n\to\infty} R_+^n = 0. \tag{3.15}$$

Since the proofs are almost identical, we only prove (3.15). For $y \in [-1,1]^{qn}$, define

$$h_\alpha^j(y) = n^{-1}\Big|\sum_{k=1}^{n}\sum_{\beta=1}^{q}\big(\varphi_n^{-1}\chi\{K_{\alpha\beta}^{jk}=1\} - p_{\alpha\beta}^{+}(x_n^j,x_n^k)\big)y_\beta^k\Big|.$$

We need to show that

$$\lim_{n\to\infty}\Big\{n^{-1}\sup_{y\in[-1,1]^{qn}}\sup_{\alpha\in\mathbb{N}_q}\sum_{j\in I_n} h_\alpha^j(y)^2\Big\} = 0. \tag{3.16}$$

For a constant $C>0$, define also the event

$$\mathcal{U}_n = \Big\{\text{For } c=\pm 1,\ \sup_{\alpha\in\mathbb{N}_q}\sup_{j\in\mathbb{N}_n}\sum_{k\in\mathbb{N}_n}\sum_{\beta\in\mathbb{N}_q}\chi\{K_{\alpha\beta}^{jk}=c\} \le Cn\varphi_n\Big\}. \tag{3.17}$$

The constant $C>0$ is taken to be large enough that $\mathcal{U}_n$ always holds (for large enough n), which is possible thanks to Lemma 3.1.

For $m\in\mathbb{Z}^+$, we decompose

$$y_\alpha^j = y_\alpha^{(m),j} + \tilde{y}_\alpha^{(m),j} \tag{3.18}$$

where $\tilde{y}_\alpha^{(m),j}\in[0,m^{-1})$ and $y_\alpha^{(m),j} = am^{-1}$ for some integer a. Write

$$\mathcal{S}_m^n = \big\{y\in[-1,1]^{qn}\ :\ y_\alpha^j = a_\alpha^j/m \text{ for some integer } a_\alpha^j\big\}. \tag{3.19}$$

Notice that

$$\overline{\lim_{n\to\infty}}\, n^{-1}\log\big|\mathcal{S}_m^n\big| < \infty. \tag{3.20}$$

We write

$$\begin{aligned}\sum_{j\in\mathbb{N}_n}\big(h_\alpha^j(y)\big)^2 &= \sum_{j\in\mathbb{N}_n}\big\{h_\alpha^j(y^{(m)})^2 + h_\alpha^j(\tilde{y}^{(m)})^2 + 2h_\alpha^j(y^{(m)})h_\alpha^j(\tilde{y}^{(m)})\big\}\\ &\le \sum_{j\in\mathbb{N}_n}\big\{h_\alpha^j(y^{(m)})^2 + h_\alpha^j(\tilde{y}^{(m)})^2\big\}\\ &\quad + 2\Big\{\sum_{j\in\mathbb{N}_n} h_\alpha^j(y^{(m)})^2\Big\}^{1/2}\Big\{\sum_{j\in\mathbb{N}_n}\tilde{h}_\alpha^j(y^{(m)})^2\Big\}^{1/2},\end{aligned} \tag{3.21}$$

thanks to the Cauchy-Schwarz Inequality. Furthermore one verifies that (as long as the event $\mathcal{U}_n$ holds), for all $j\in\mathbb{N}_n$ and all $\alpha\in\mathbb{N}_q$,

$$\big|h_\alpha^j(\tilde{y}^{(m)})\big| \le Cm^{-1} + qm^{-1}\sup_{x,y\in\mathcal{D},\beta\in\mathbb{N}_q} p_{\alpha\beta}^{+}(x,y) := \tilde{C}m^{-1}, \tag{3.22}$$

and the RHS evidently goes to 0 uniformly as $m \to \infty$. In light of (3.21), in order that (3.16) holds it thus suffices that we show that

$$\lim_{m\to\infty}\lim_{n\to\infty} n^{-1} \sup_{\alpha\in\mathbb{N}_q}\sup_{y\in\mathcal{S}^n_m}\sum_{j\in\mathbb{N}_n} h^j_\alpha(y)^2 = 0. \tag{3.23}$$

Now for any $\delta > 0$, using a union-of-events bound,

$$\begin{aligned}\mathbb{P}\big(\sup_{y\in\mathcal{S}^n_m}\sum_{j\in\mathbb{N}_n} h^j_\alpha(y)^2 \geq n\delta, \mathcal{U}_n\big) \leq & \big|\mathcal{S}^n_m\big| \sup_{y\in\mathcal{S}^n_m}\mathbb{P}\Big(\mathcal{U}_n, \sum_{j\in\mathbb{N}_n} h^j_\alpha(y)^2 \geq n\delta\Big) \\ =&(m+1)^n \sup_{y\in\mathcal{S}^n_m}\mathbb{P}\Big(\mathcal{U}_n, \sum_{j\in\mathbb{N}_n} h^j_\alpha(y)^2 \geq n\delta\Big).\end{aligned} \tag{3.24}$$

Thanks to the Borel-Cantelli Lemma, in order that (3.23) holds it suffices that we show that for arbitrary $\delta > 0$,

$$\overline{\lim_{n\to\infty}}\, n^{-1}\log \sup_{y\in\mathcal{S}^n_m}\mathbb{P}\big(\mathcal{U}_n, \sum_{j\in\mathbb{N}_n} h^j_\alpha(y)^2 \geq n\delta\big) = -\infty, \tag{3.25}$$

since (3.24) and (3.25) imply that

$$\sum_{n=1}^{\infty}\mathbb{P}\Big(\sup_{y\in\mathcal{S}^n_m}\sum_{j\in\mathbb{N}_n} h^j_\alpha(y)^2 \geq n\delta, \mathcal{U}_n\Big) < \infty. \tag{3.26}$$

To this end, define

$$g^j_\varepsilon(y) = \chi\big\{\sup_{\alpha\in\mathbb{N}_q} h^j_\alpha(y) \geq \varepsilon\big\}. \tag{3.27}$$

Notice that if the event $\mathcal{U}_n$ holds, then

$$\big|h^j_\alpha(y)\big| \leq C + \sup_{\beta\in\mathbb{N}_q}\sup_{x,z\in D} p^+_{\alpha\beta}(x,z) := \bar{C}.$$

This means that (as long as the event $\mathcal{U}_n$ holds), then for all $y \in [-1,1]^{nq}$,

$$n^{-1}\sum_{j\in\mathbb{N}_n} h^j_\alpha(y)^2 \leq \bar{C}^2 n^{-1}\sum_{j\in\mathbb{N}_n} g^j_\varepsilon(y) + \varepsilon^2. \tag{3.28}$$

In order that (3.25) holds, it thus suffices that we show that for arbitrary $\varepsilon > 0$,

$$\lim_{n\to\infty} n^{-1}\log\sup_{y\in\mathcal{S}^n_m}\mathbb{P}\big(n^{-1}\sum_{j\in\mathbb{N}_n} g^j_\varepsilon(y) \geq \varepsilon\big) = -\infty. \tag{3.29}$$

To this end, by Chernoff's Inequality, for a constant $a > 0$,

$$\begin{aligned}\mathbb{P}\big(n^{-1}\sum_{j\in\mathbb{N}_n} g^j_\varepsilon(y) \geq \varepsilon\big) \leq & \mathbb{E}\Big[\exp\Big(a\sum_{j\in\mathbb{N}_n} g^j_\varepsilon(y) - a\varepsilon n\Big)\Big] \\ =& \exp\big(-a\varepsilon n\big)\prod_{j\in\mathbb{N}_n}\Big(1 + \mathbb{P}\big(g^j_\varepsilon(y) = 1\big)\big(\exp(a) - 1\big)\Big) \\ \leq & \exp\Big(np_n\big(\exp(a) - 1\big) - a\varepsilon n\Big)\end{aligned} \tag{3.30}$$

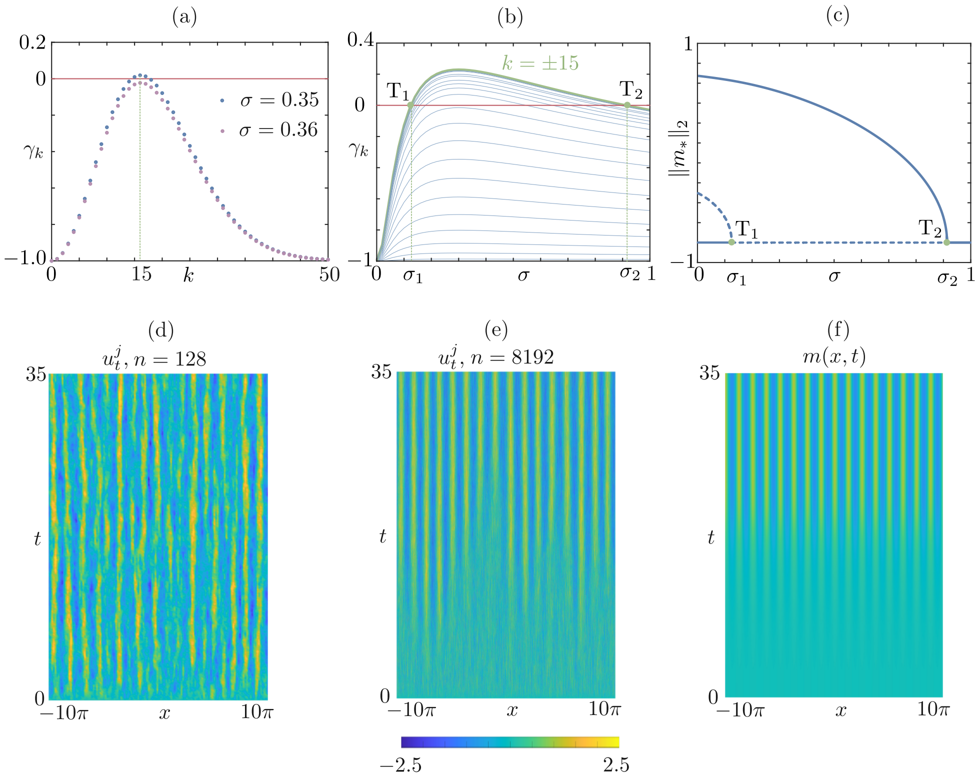

FIG. 3.1. *Noise-induced Turing-like bifurcation for the particle and mean-field model with one population ($q = 1$), posed on a ring of width $2l$, with synaptic kernel (3.32), neuronal firing rate (3.33), and mean-field firing rate (3.34). (a): values $\{\gamma_k\}_k$ (defined as in the main text of the manuscript), show that a Turing-like bifurcation of the homogeneous steady state is located between $\sigma = 0.35$ and $\sigma = 0.36$, with critical wavenumber $k_c = 15$. (b) Curves $\gamma_k(\sigma)$ for $k = \{0, \pm 1, \ldots, \pm 20\}$ show that the branch of homogeneous steady states is stable when there is no noise ($\sigma = 0$) and undergoes two noise-induced Turing-like bifurcations at $\sigma = \sigma_{1,2}$, both for $k_c = 15$. (c): numerical bifurcation analysis for spatially-extended equilibria of the mean-field shows that the first bifurcation is subcritical, while the second one is supercritical (dashed lines represent unstable branches). Numerical simulations of the particle system with kernel matrix for $n = 128$ (e), $n = 8192$ (f), and the mean-field for $\sigma = 0.58 \in (\sigma_1, \sigma_2)$ confirm that the bifurcation structure in (c). Parameters: $L = 1$, $l = 10\pi$, $I(t, x) \equiv 0$, $G(x, t) \equiv \sigma$, $B = 1.5$, $C = 7$, $\alpha = 10$, $\theta = 0.4$, $m_0(x) = 0.3\cos(k_c \pi x/l)$.*

where

$$p_n = \sup_{y \in \mathcal{S}_m^n} \sup_{j \in \mathbb{N}_n} \mathbb{P}\big(g_\varepsilon^j(y) = 1\big).$$

We define $a = -\log p_n$, and it remains for us to prove that

$$\lim_{n \to \infty} p_n = 0. \tag{3.31}$$

In fact (3.31) follows almost immediately from the Hoeffding Inequality [1]. ∎

3.1. Example of noise-induced Turing-like bifurcation. We give a further example of Turing-like bifurcation, in a model with a kernel that does not support localised solutions, as shown in the main text. We consider a network with $q = 1$ on a ring of width $2l$, $D = \mathbb{R}/2l\mathbb{Z}$ with distance dependent kernel $\mathcal{K}(x, y) = A(x - y)$, where

$$A(x) = \frac{C}{\sqrt{\pi}} e^{-x^2} - \frac{C}{B\sqrt{\pi}} e^{-(x/B)^2}, \qquad B \in \mathbb{R}_{>1}, \qquad C \in \mathbb{R}_{>0}, \tag{3.32}$$

linear coupling $L = 1$, forcing $I(t,x) \equiv 0$, $G(t,x) \equiv \sigma$, and with firing rate function

$$f(u) = \Phi(\alpha(u-\theta)), \qquad \Phi(u) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{u}{\sqrt{2}}\right)\right], \qquad \alpha \in \mathbb{R}_{>0}, \quad \theta \in \mathbb{R}, \tag{3.33}$$

which results in a mean-field firing rate of the form [4]

$$F(m,v) = \Phi\left(\alpha \frac{m-\theta}{\sqrt{1+\alpha^2 v}}\right). \tag{3.34}$$

The synaptic kernel A is locally excitatory and laterally inhibitory, and has been selected here as it is *balanced*, in the sense that its integral over $\mathbb{R}$ is null. If $D = \mathbb{R}$, it can be shown that $m_* = 0$ is a homogeneous equilibrium for any value of σ. We consider the problem on a ring of width $2l$, hence

$$\int_{-l}^{l} A(x)\, dx = C(\operatorname{erf} l - \operatorname{erf}(l/B)),$$

which, for large l is approximately null. We have computed a branch of σ-dependent homogeneous steady states using Newton's scheme, and the computed values of m_* do not appreciably differ from 0 in the selected parameter range.

In Figure 3.1(a) we show $\{\gamma_k\}_{k\in\mathbb{N}_{50}}$ for two values of σ, which provides evidence of a bifurcation at $\sigma_c \in (0.35, 0.36)$ with wavenumber $k_c = 15$. In Figure 3.1(b) we computed the curves $\gamma_k(\sigma)$ for $k = \{0, \pm 1, \dots, \pm 20\}$, from which we deduce that the homogeneous steady state is stable in the noiseless case $\sigma = 0$, and it undergoes two Turing-like bifurcations induced by noise.

We employed numerical bifurcation analysis tools to study the bifurcation structure of steady states to of the mean-field equation (see Figure 3.1(c)). We see that the primary bifurcation T_1 is subcritical, and a branch of stable spatially-periodic steady states emerge. In contrast, the secondary bifurcation, T_2, is supercritical and gives rise to stable spatially-periodic equilibria.

We carried out time simulations of the particle system with varying numbers of neurons, and of the mean-field equation to confirm the predictions of the numerical bifurcation analysis in Figure 3.1(d)–(f).

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