Uncertainty Quantification in Neurobiological Networks

Daniele Avitabile
Amsterdam Dynamics, VU Amsterdam
Inria MathNeuro Team, Sophia Antipolis

Brown University,
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Cite using the instructions on the Git repo https://tinyurl.com/yu7wcmba

www.danieleavitabile.com www.amsterdam-dynamics.nl @AvitabileD

RU-VU Math Neuro Research Group

VU VRIJE UNIVERSITEIT AMSTERDAM | Radboud University

- Numerical Analysis for spatially extended problems
- Uncertainty Quantification
- Inverse Problems and Data Assimilation
- Model Reduction

Francesca Cavallini

Svetlana Dubinkina

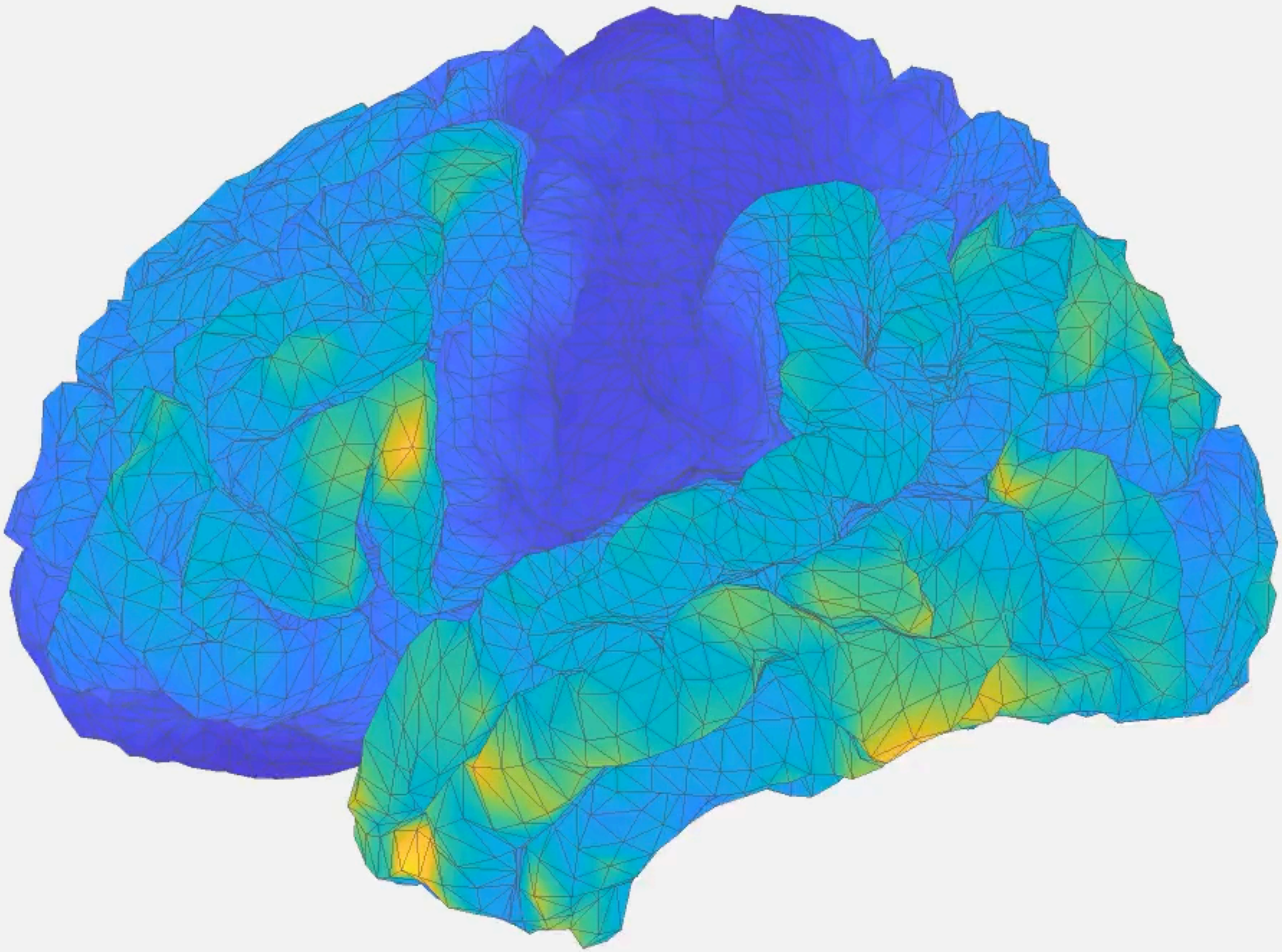
Gabriel Lord

Khadija Meddouni

mathematical-neuroscience.zulipchat.com

Voltage dynamics on realistic geometries

- Cortical domain from imaging data
- Synaptic connections from tractography data

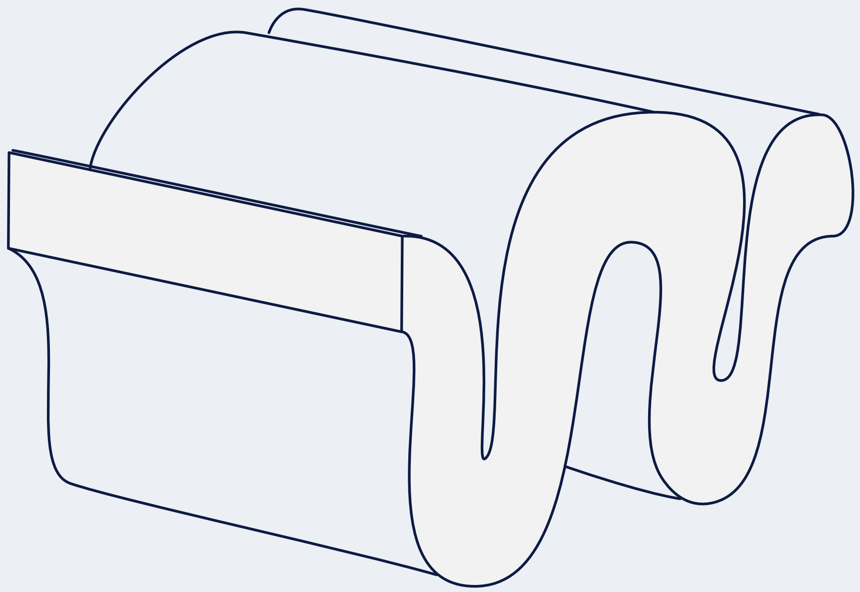


[S. Petros]

Neural field models

- Cortex: compact domain in $D \subset \mathbb{R}^3$

D



- Deterministic Neural Field

$$\partial_t u(x,t) = -u(x,t) + \int_D w(x,x') f(u(x',t)) dx' + g(x,t), \quad (x,t) \in D \times J$$

$$u(x,0) = v(x), \quad x \in D$$

Why neural fields?

- NFs are prototypical models of large-scale cortical activity
- Nonlocal, nonlinear, defined on continuum cortices
- The framework developed here applies to:
  - Connectomic models
  - Multi-population NFs
  - Hybrid neural mass-field models

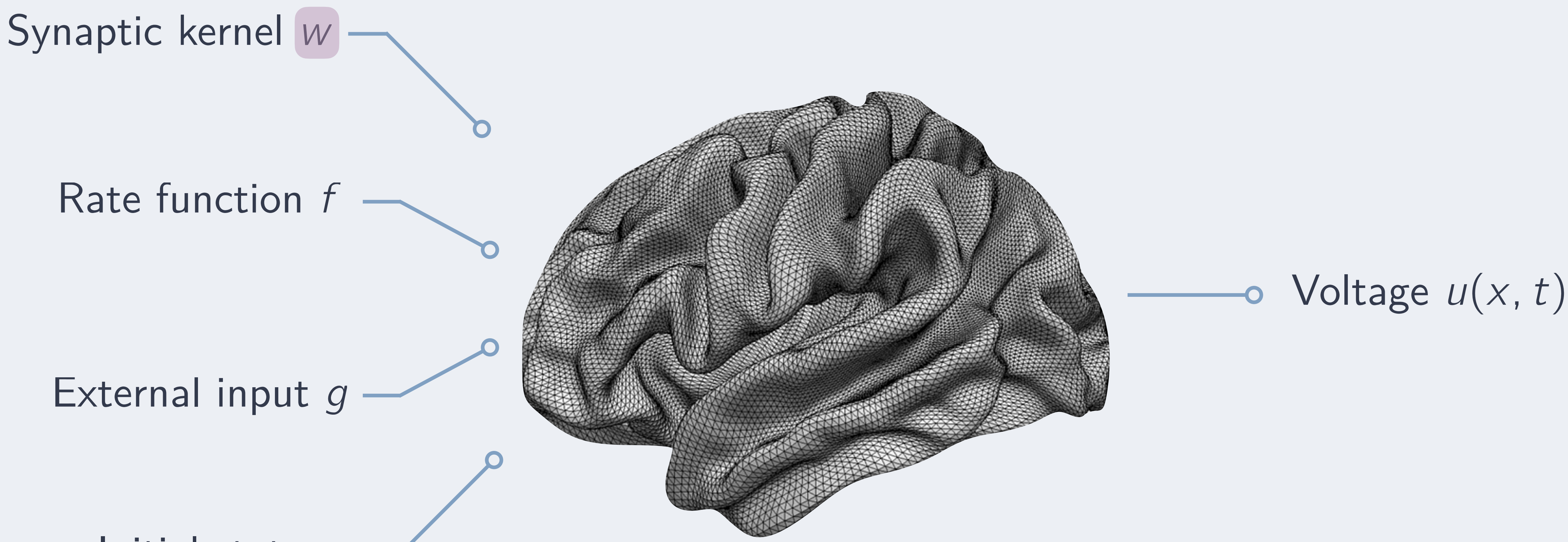
- Claim: it applies to a much-wider class of models, in particular 2nd generation NFs

$$\partial_t u_1(x,t) = N_1(u_1(x,t),\cdots,u_p(x,t)) + \sum_{i=1}^{p}\int_D w_{1j}(x,x')u_j(x',t)d\mu(x') + g_1(x,t),$$

$$\dots$$

$$\partial_t u_p(x,t) = N_p(u_1(x,t),\cdots,u_p(x,t)) + \sum_{i=1}^{p}\int_D w_{pj}(x,x')u_j(x',t))d\mu(x') + g_p(x,t),$$

Model prediction

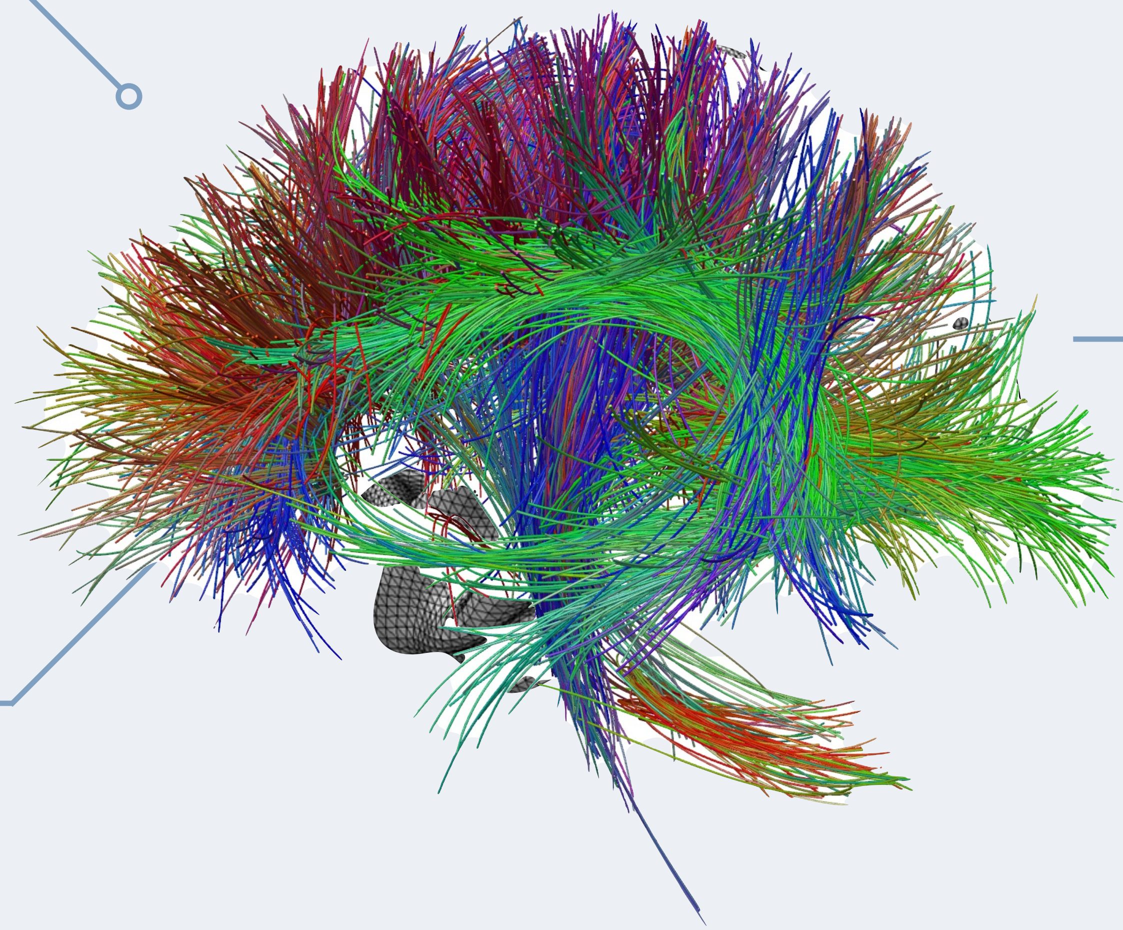


$$
\begin{aligned}
\partial_t u(x,t) &= -u(x,t) + \int_D \boxed{w(x,x')} f(u(x',t))\,dx' + g(x,t), && (x,t) \in D \times J \\
u(x,0) &= v(x), && x \in D
\end{aligned}
$$

[Amari] [Bressloff] [Coombes] [Ermentrout] [Faugeras et al.] [Faye] [van Gils] [Inglis] [Kilpatrick] [Laing] [Mc Laurin] [Meijer] [Potthast & beim Graben] [Wilson & Cowan]

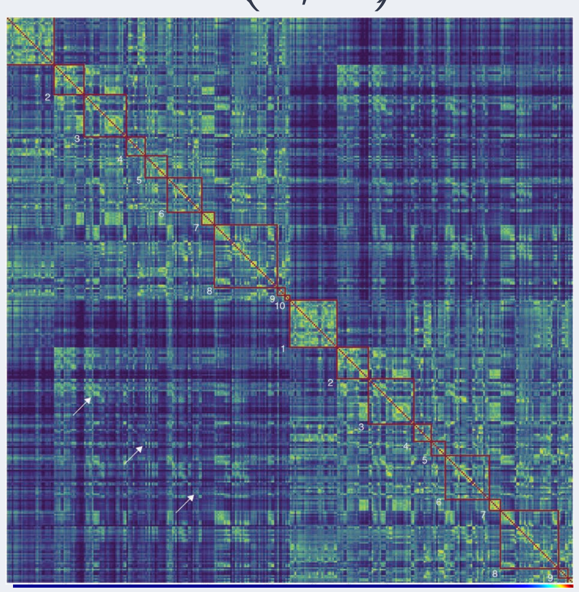
Deterministic or random data?

Synaptic kernel w



Voltage $u(x, t)$

$w(x, x')$



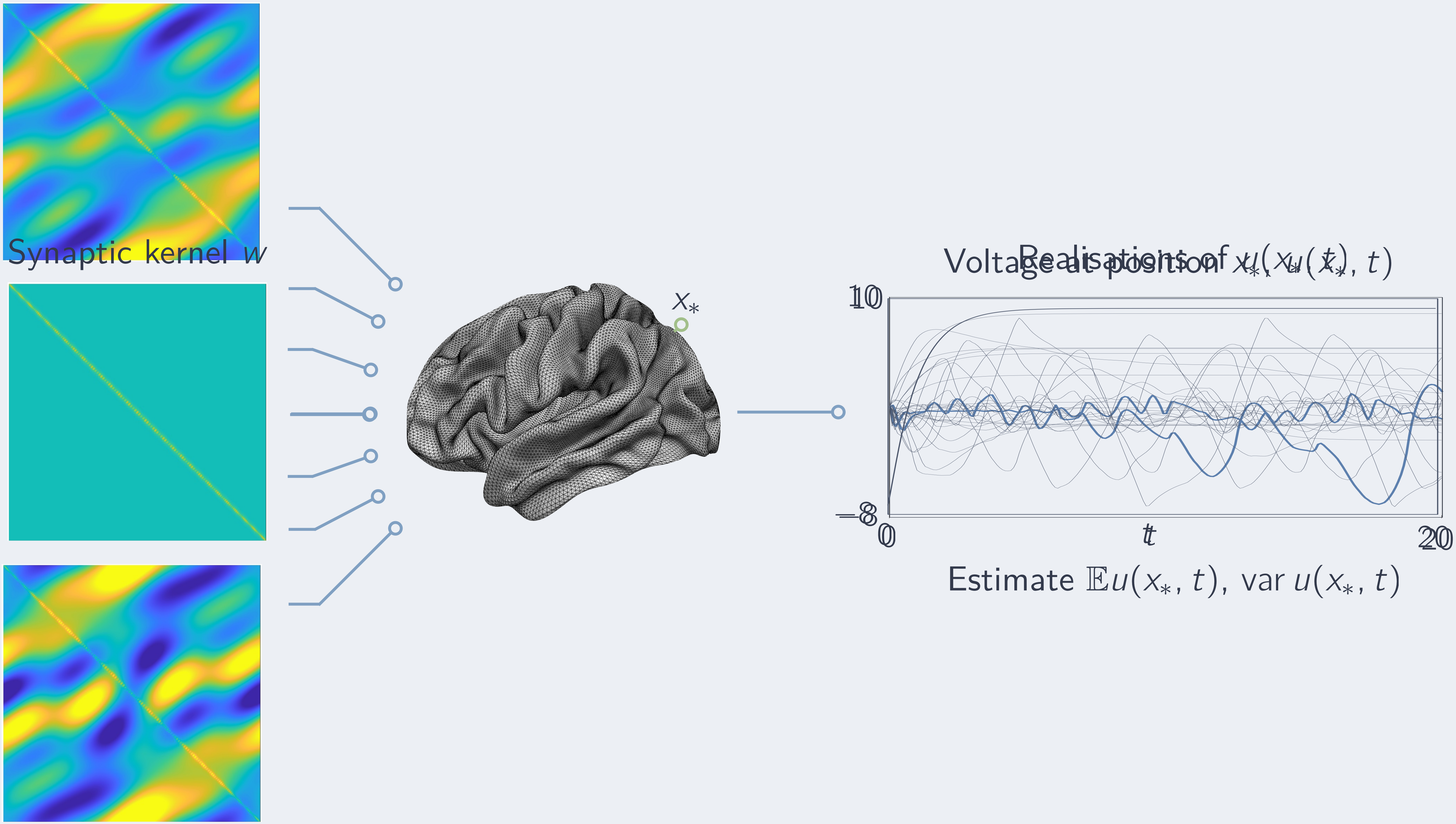
x

x'

[Rosen, Halgren]

"Overall, connection strengths are log-normally distributed"

Forward uncertainty quantification



Main objectives

- Wishlist for a numerical scheme
  - Spatiotemporal accuracy
  - Accuracy in estimating mean and variances
  - Scalable
  - Allows multiple noise data (kernel, firing rates, external inputs, initial states, etc.)
  - Provably accurate
- A tenative strategy
  1. Discretise integrals using a quadrature rule
  2. Perform Montecarlo simulations
  3. Compute averages and variances

New projection schemes

New stochastic collocation schemes

Collocation scheme (heuristic)

$$\partial_t u(x,t) = -u(x,t) + \int_D w(x,x')f(u(x',t))dx'$$

$\downarrow$ Evaluate at $x = x_i$

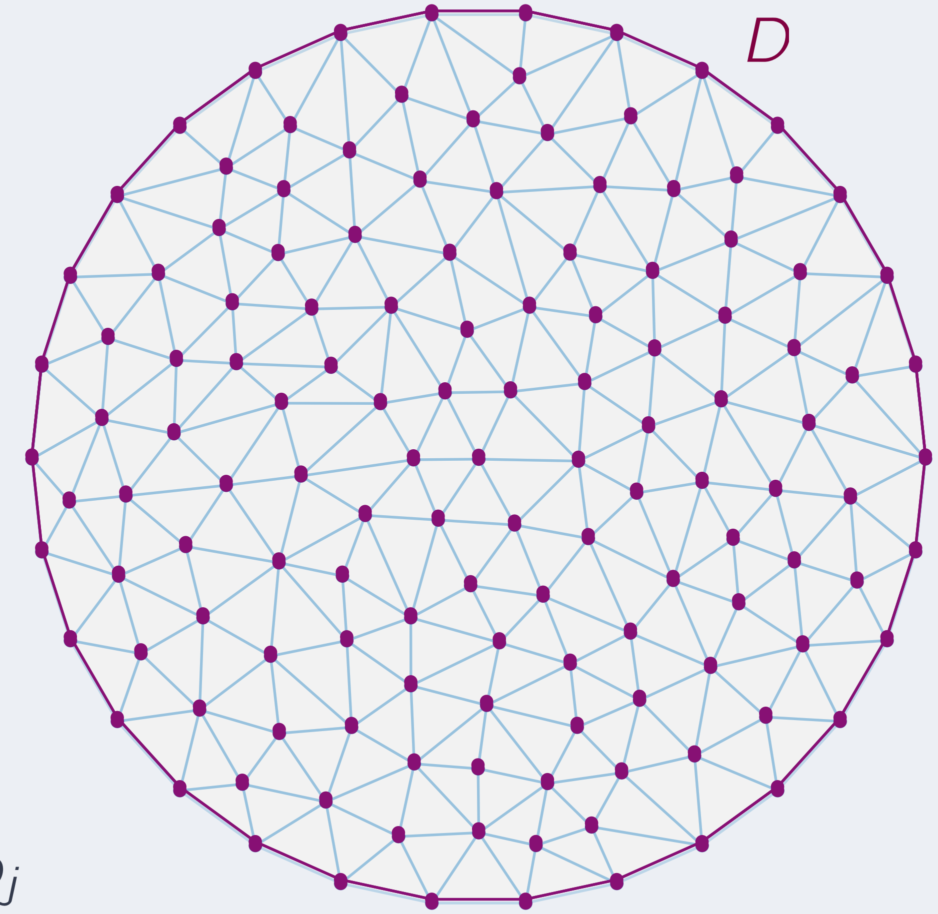
$$\partial_t u(x_i,t) = -u(x_i,t) + \int_D w(x_i,x')f(u(x',t))dx'$$

$\downarrow$ Approximate integral $\int_D h(x') \approx \sum_j h(x_j)\rho_j$

$$U_i'(t) = -U_i(t) + \sum_j w(x_i,x_j)f(U_j(t))\rho_j$$

$\downarrow$

Time step, Error Analysis

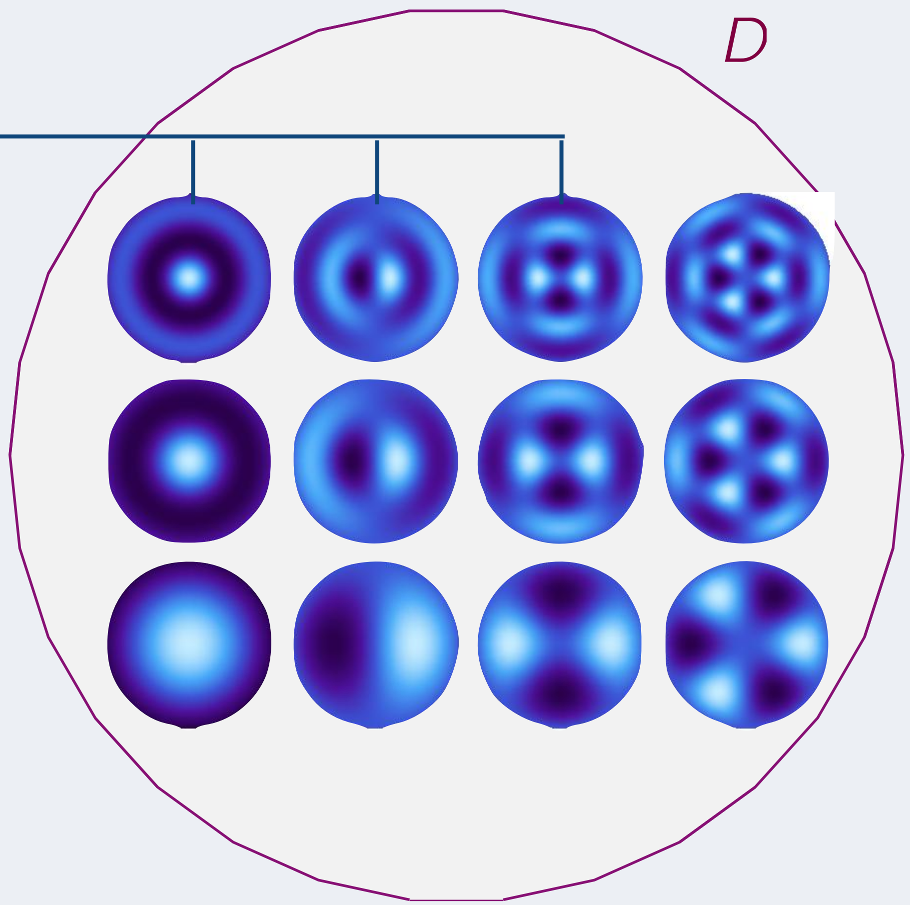


[Lima, Buckwar]

[Avitabile, Lima, Coombes]

Galerkin scheme (heuristic)

$$\partial_t u(x,t) = N(u)(x,t)$$



$$u(x,t) = a_1(t)\varphi_1(x) + a_2(t)\varphi_2(x) + a_3(t)\varphi_3(x) + \cdots$$

$\downarrow$ Truncate $u(x,t) \approx \sum_j^n a_j(t)\varphi_j(x)$

$$\sum_j \langle \varphi_i, \varphi_j \rangle a_j'(t) = \left\langle \varphi_i, N\Big(\sum_j a_j(t)\varphi_j\Big)\right\rangle$$

$\downarrow$ Approximate integrals $\langle \varphi_i, \varphi_j \rangle \approx \sum_k \varphi_i(x_k)\varphi_j(x_k)\rho_k$

$$\sum_j M_{ij} a_j'(t) = F(a_1(t), \dots, a_n(t))$$

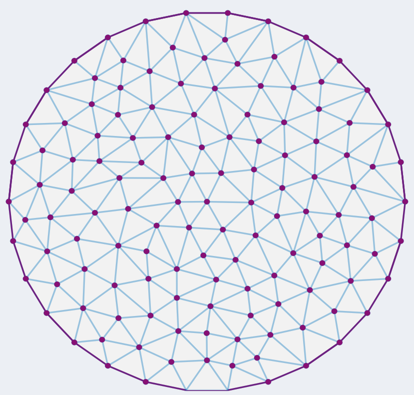
$\downarrow$

Time step, Error Analysis

Two separate schemes

$$\partial_t u(x,t) = -u(x,t) + \int_D w(x,x')f(u(x',t))dx' := N(u)(x,t)$$

“Approximate” $u,\ N(u)$



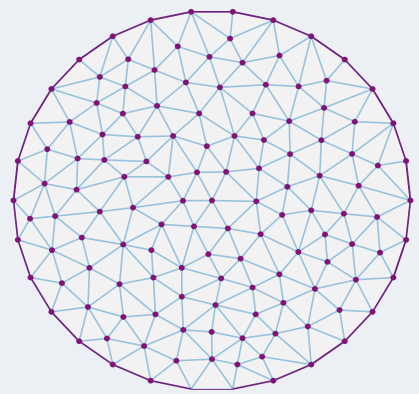
↓

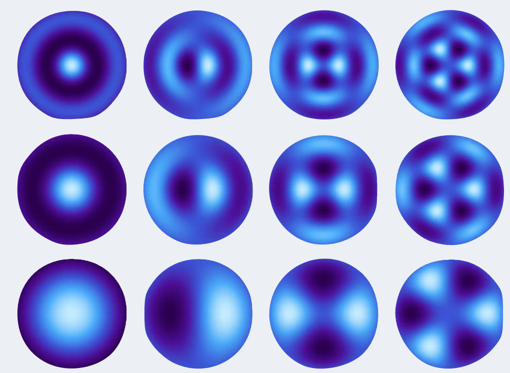
Quadrature

↓

Time step

“Approximate” $u,\ N(u)$





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Quadrature

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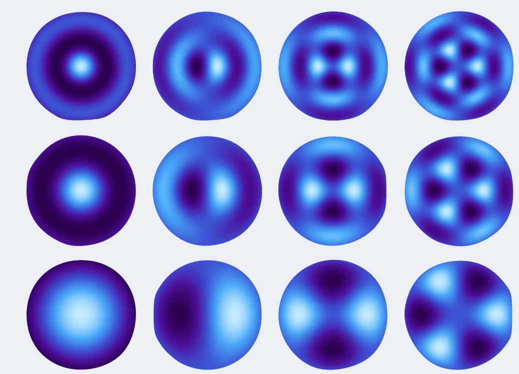
Time step

“Approximate” $u,\ N(u)$



↓

Quadrature

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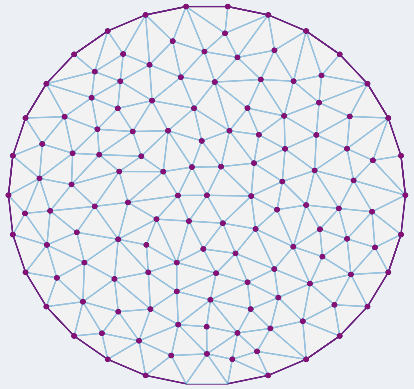
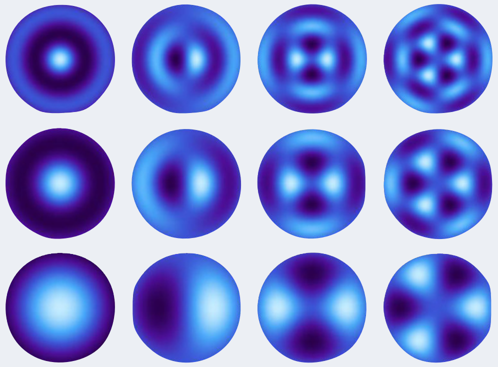
Time step

A unified error analysis

$$\partial_t u(x,t) = -u(x,t) + \int_D w(x,x')f(u(x',t))dx' := N(u)(x,t)$$

“Approximate” $u,\ N(u)$





Function spaces $\longmapsto$ Banach space error [Atkinson]

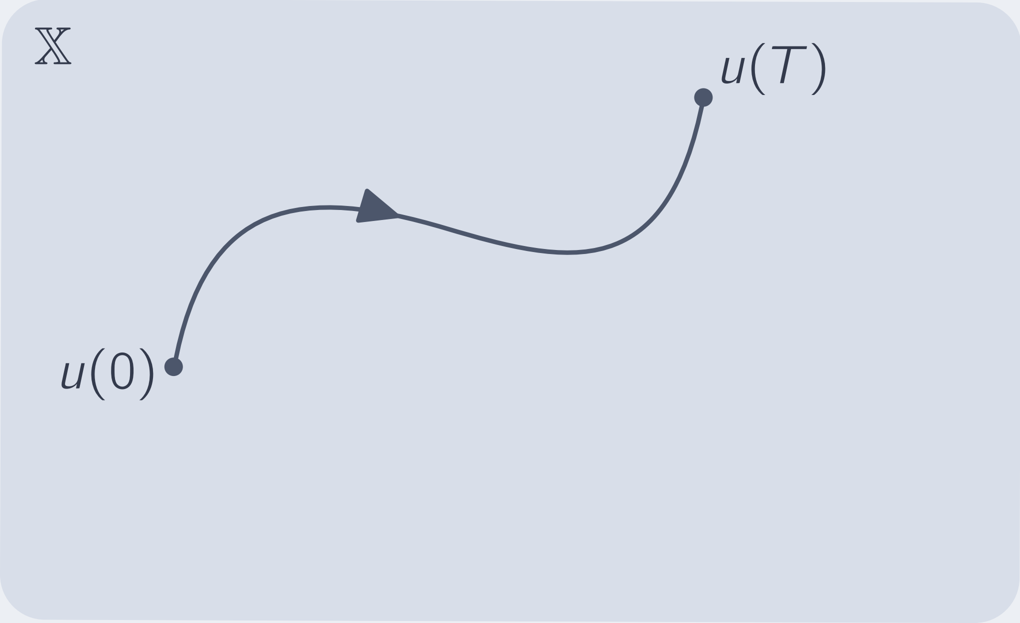
Discrete x — Quadrature — Quadrature

Discrete x, t — Time step — Time step

Discrete Error — Discrete Error

Cauchy problem for Neural Fields

- Banach spaces $\mathbb{X} = C(D)$, or $\mathbb{X} = L^2(D)$



- Classical integral and Nemitskii operators

$$W \colon v \mapsto \int_D w(\cdot, x')v(x')\,dx'$$

$$F(u)(x) = f(u(x))$$

- Nonlinear operator

$$N \colon J \times \mathbb{X} \to \mathbb{X}, \qquad (t, u) \mapsto -u + WF(u) + g(t)$$

Theorem (Existence of classical solutions) [Faugeras et al.] [Potthast et al.]

Under general conditions on w, f, g the initial value problems

$$u'(t) = N(t, u(t)), \qquad u(0) = v$$

admit solutions $u \in C^1(J, \mathbb{X})$, where $J = [0, T]$

Abstract approximation strategy

- Residual at a Neural field solution

$$R(u) = u' - N(\cdot\,, u) = 0, \qquad \text{on } J = [0, T]$$

- Approximation strategy
  - Select

$$\mathbb{X}_n = \mathrm{Span}\{\varphi_1, \dots, \varphi_n\}$$

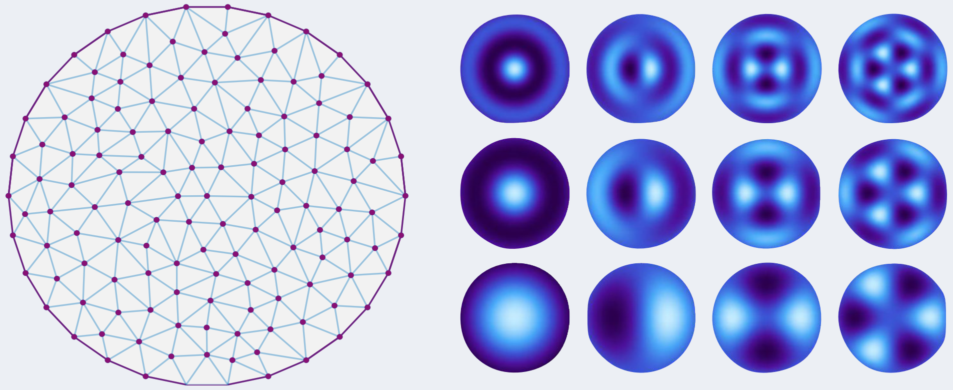
  - Set

$$u_n(t) = \sum_{j=1}^{n} U_j(t)\varphi_j \in \mathbb{X}_n$$

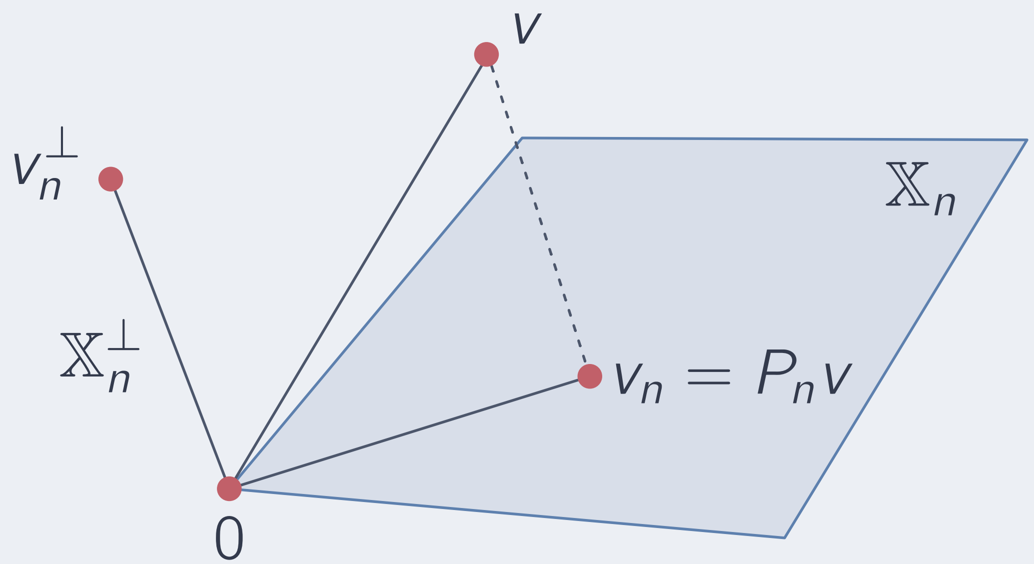
  - Make the residual "small"

$$R(u_n(t)) \approx 0, \qquad t \in J$$

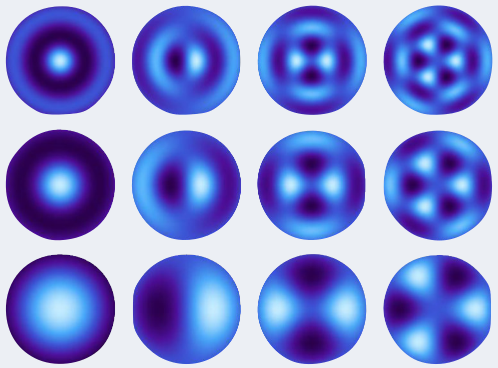
"Approximate" u, $N(u)$



Residual conditions - abstract schemes



Galerkin



$\mathbb{X} = L^2(D)$

Orthogonal projector

$$P_n \colon \mathbb{X} \to \mathbb{X}_n, \; v \mapsto \sum_j^n \langle v, \varphi_j \rangle \varphi_j$$

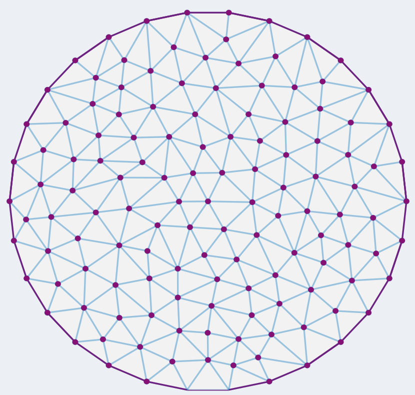
- Characterisation $\quad P_n v = 0 \iff \langle v, \varphi_j \rangle = 0 \quad \text{for all } j$

- Scheme

$$\begin{array}{ccc} P_n R(u_n) = 0 \text{ for all } x,\, t & \iff & u_n'(t) = P_n N(t, u_n(t)) \\ \Updownarrow & & \Updownarrow \\ \langle R(u_n), \varphi_j \rangle = 0 \text{ for all } j,\, t & \iff & \sum_j \langle \varphi_i, \varphi_j \rangle a_j'(t) = \Big\langle \varphi_i, N\big(\sum_j a_j(t)\varphi_j\big) \Big\rangle \end{array}$$

Residual conditions - abstract schemes

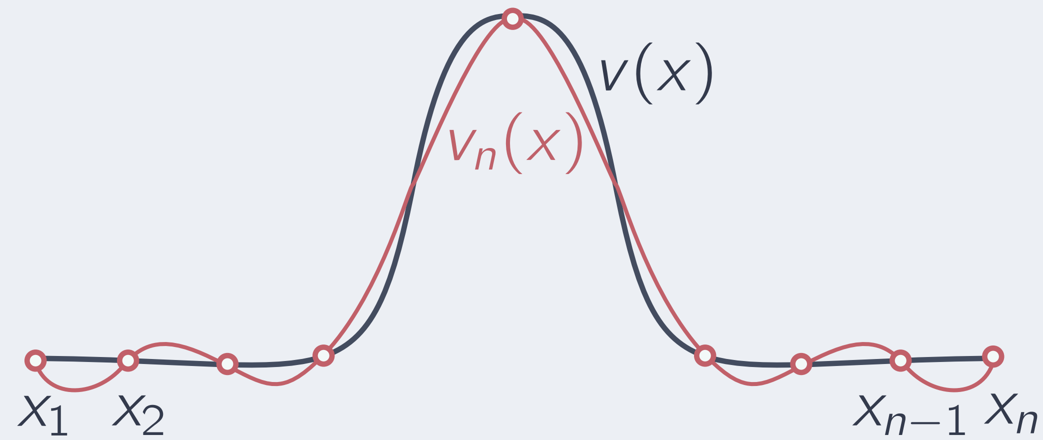
Collocation



$\mathbb{X} = C(D)$

Interpolating projector

$$P_n : \mathbb{X} \to \mathbb{X}_n, \; v \mapsto \sum_j^n v(x_j)\varphi_j$$



- Characterisation $\quad P_n v = 0 \iff v(x_j) = 0, \quad \text{for all } j$

- Scheme

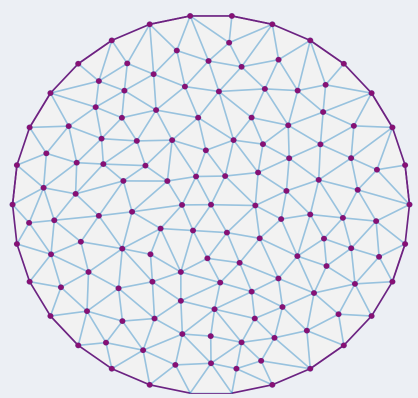
$$\begin{array}{ccc} P_n R(u_n) = 0 \text{ for all } x,\ t & \iff & u_n'(t) = P_n N(t, u_n(t)) \\ \Updownarrow & & \Updownarrow \\ R(u_n) = 0 \text{ for all } x_j,\ t & \iff & a_j'(t) = N\left(\sum_j a_j(t)\varphi_j\right) \end{array}$$

Schemes in operator form

- NF Equation $\quad u'(t) = N(t, u(t)), \qquad u(0) = u_0 \qquad$ ODE in $\mathbb{X}$

$$\downarrow$$

Collocation

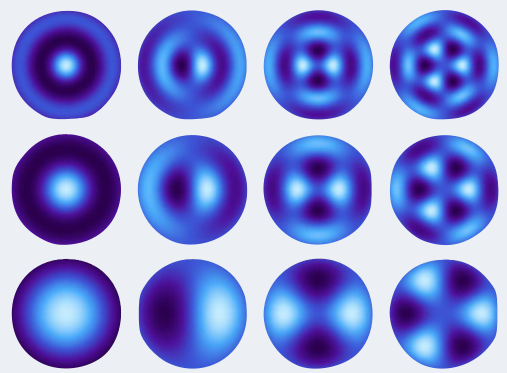


$\mathbb{X} = C(D)$

Interpolating projector

$$P_n \colon \mathbb{X} \to \mathbb{X}_n,\ v \mapsto \sum_j^n v(x_j)\varphi_j$$

Galerkin



$\mathbb{X} = L^2(D)$

Orthogonal projector

$$P_n \colon \mathbb{X} \to \mathbb{X}_n,\ v \mapsto \sum_j^n \langle v, \varphi_j\rangle\varphi_j$$

$$\downarrow$$

- NF Equation $\quad u_n'(t) = P_n N(t, u_n(t)), \qquad u_n(0) = P_n u_0 \qquad$ ODE in $\mathbb{X}_n$

Strong errors in Banach space

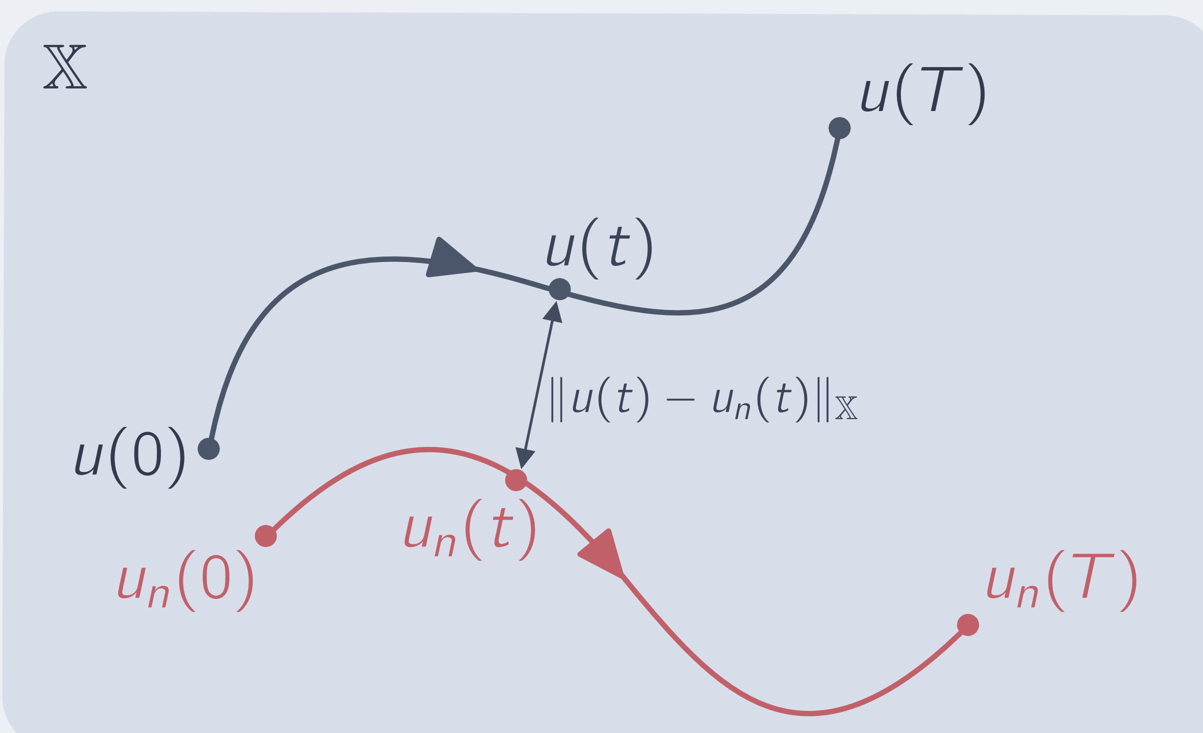
Theorem (Existence of classical solutions) Extends [Faugeras et al.]

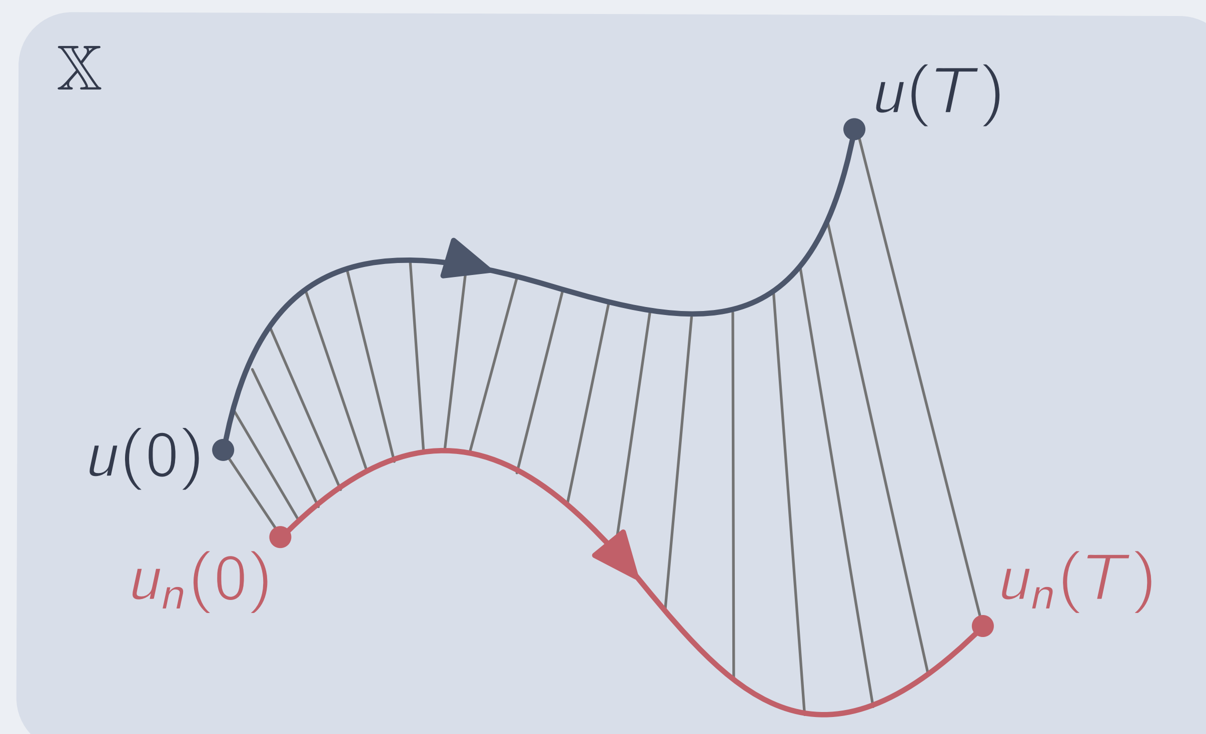
Under **general conditions** on w, f, g the initial value problems

$$\boldsymbol{u'(t)} = \boldsymbol{N}(\boldsymbol{t}, u(t)), \qquad \boldsymbol{u}(0) = v$$
$$u_n'(t) = P_n N(t, u_n(t)), \qquad u_n(0) = P_n v$$

admit solutions $\boldsymbol{u} \in C^1(J, \mathbb{X})$, $u_n \in C^1(J, \mathbb{X}_n)$, $J = [0, T]$

Proof: based on Banach-space version of Picard-Lindelöf theorem





Error $\qquad \|u - u_n\|_{\mathbb{X}} \qquad \|u - u_n\|_{C(J,\mathbb{X})} = \sup_{t \in J} \|u(t) - u_n(t)\|_{\mathbb{X}}$

Convergence result (deterministic)

Theorem (Convergence of projection schemes) Builds on [Atkinson]

If $P_n v \to v$ for all $v \in \mathbb{X}$, then $u_n \to u$ in $C(J, \mathbb{X})$, and

$$\frac{1}{M}\|u - P_n u\|_{C(J,\mathbb{X})} \leq \|u - u_n\|_{C(J,\mathbb{X})} \leq M\|u - P_n u\|_{C(J,\mathbb{X})}, \qquad M > 0$$

Even if $P_n v_* \not\to v_*$ for some $v_* \in \mathbb{X}$, it holds

$$\|u - u_n\|_{C(J,\mathbb{X})} \leq \alpha_n e^{\beta_n}$$

with $\{\beta_n\}$ bounded and $\alpha_n \to 0$ provided

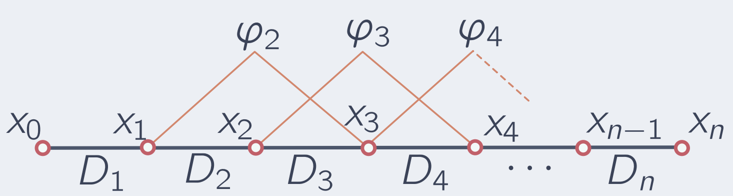
$$\|v - P_n v\|, \|W - P_n W\|, \|g - P_n g\|_{C(J,\mathbb{X})} \to 0$$

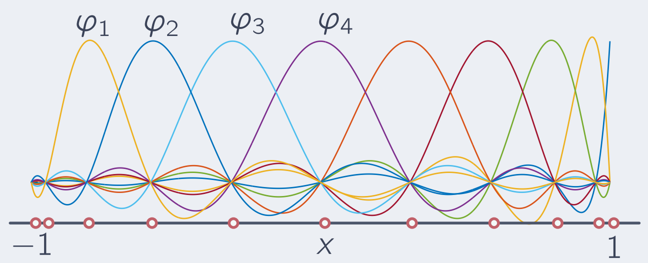
Remarks

The projector and the scheme "converge at the same speed"

The term $\|u - P_n u\|_{C(J,\mathbb{X})}$ can often be estimated using $\|v - P_n v\|_{\mathbb{X}}$, $v \in \mathbb{X}$

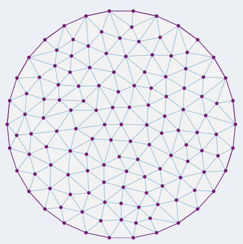
Scheme classification





Finite Elements
$D = \cup_j D_j$
φ_j locally supported

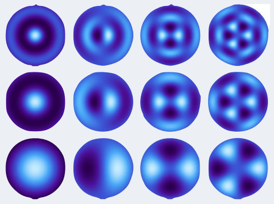
Spectral
φ_j supported in D



Collocation
Interpolatory P_n

Finite Elements Collocation

Spectral Collocation



Galerkin
Orthogonal P_n

Finite Elements Galerkin

Spectral Galerkin

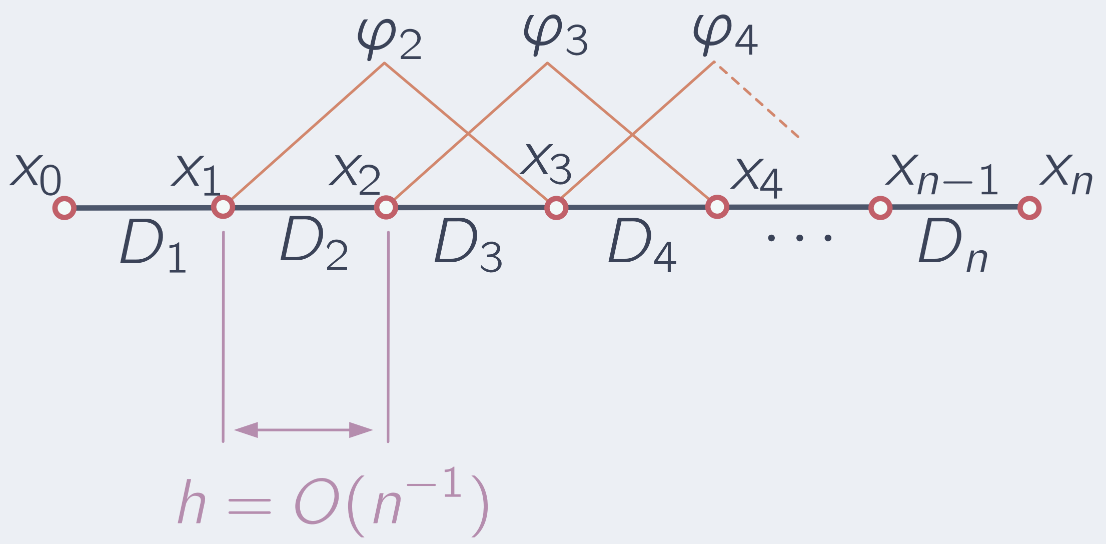
Finite Elements Collocation scheme

- Setup

  $\mathbb{X} = C(D)$, $\|\cdot\|_{\mathbb{X}} = \|\cdot\|_\infty$

  $D = [-1, 1] \subset \mathbb{R}$

  $\mathbb{X}_n = \mathrm{Span}\{\varphi_1, \dots, \varphi_n\}$



- Projection error bounds

$$\|v - P_n v\|_\infty \leq \begin{cases} \omega(v, h), & \text{if } v \in C(D) \\ \|v''\|_\infty h^2/8, & \text{if } v \in C^2(D) \end{cases}$$

Theorem (convergence of Finite Elements Collocation scheme)

It holds $\|u - u_n\|_{C(J, C(D))} \to 0$ as $n \to \infty$

If $u \in C(J, C^2(D))$,

$$\|u - u_n\|_{C(J,C(D))} \lesssim \max_{t \in J} \|u(t) - P_n u(t)\| = O(h^2) = O(n^{-2})$$

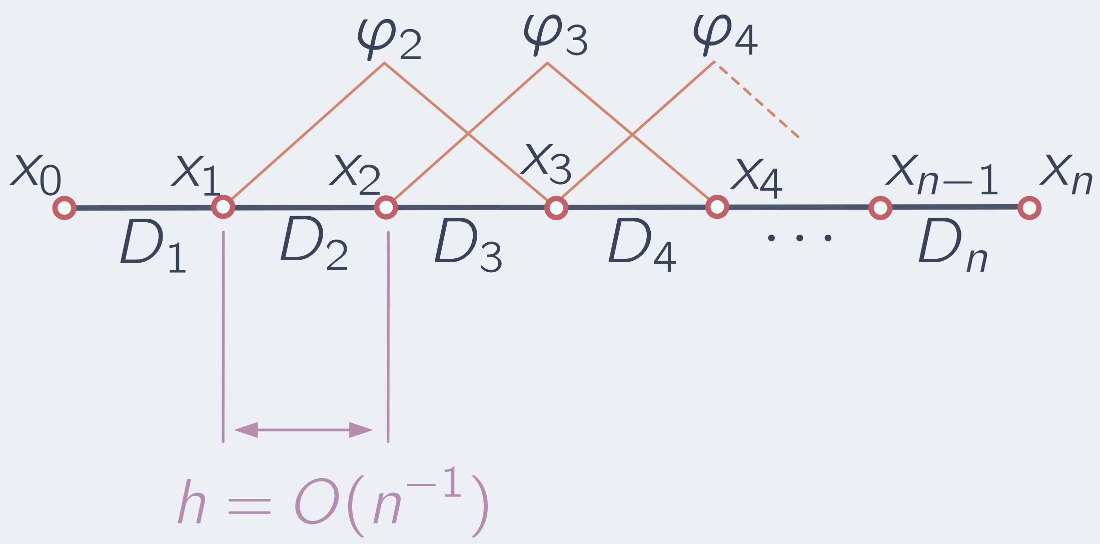
Finite Elements Collocation scheme

- Setup

  $\mathbb{X} = C(D)$, $\|\cdot\|_{\mathbb{X}} = \|\cdot\|_\infty$

  $D = [-1, 1] \subset \mathbb{R}$

  $\mathbb{X}_n = \mathrm{Span}\{\varphi_1, \dots, \varphi_n\}$



- Scheme with $O(n^{-2})$ accuracy

$$a_i'(t) = -a_i(t) + \sum_j \int_{D_j} w(x_i, y)\, f\left(\sum_k a_k(t)\varphi_k(y)\right) dy + g(x_i, t)$$

How to select quadrature?

- Choice of P_n dictates quadrature choice
- Here, we pick a quadrature that preserves $O(n^{-2})$ accuracy

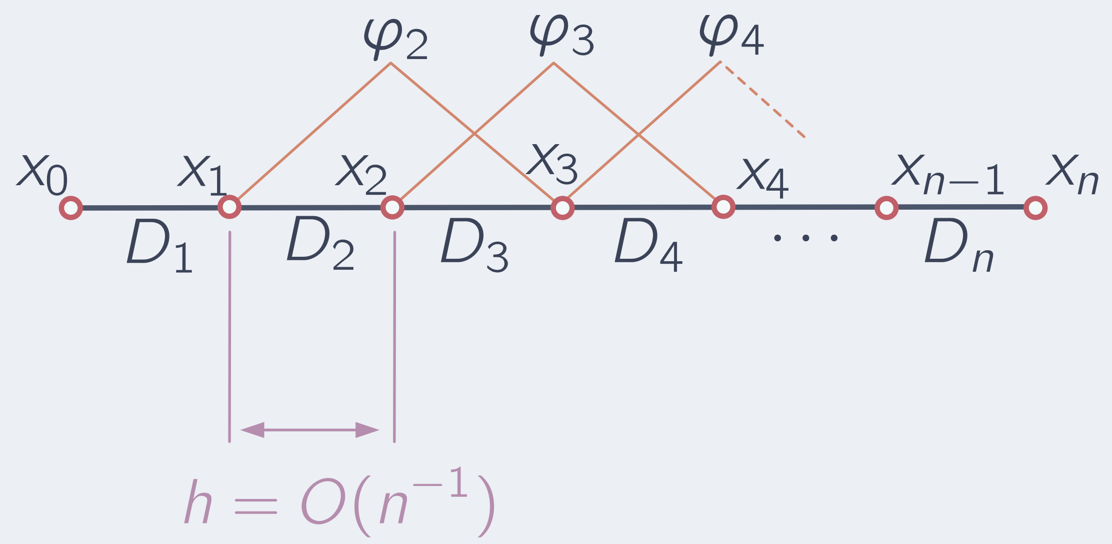
Galerkin Finite Elements

- Setup

  $\mathbb{X} = L^2(D)$, $\|\cdot\|_{\mathbb{X}} = \langle \cdot, \cdot \rangle_{L^2(D)}^{1/2}$

  $D = [-1, 1] \subset \mathbb{R}$

  $\mathbb{X}_n = \mathrm{Span}\{\varphi_1, \dots, \varphi_n\}$



Theorem (convergence of Galerkin Finite Elements scheme)

It holds $\|u - u_n\|_{C(J, L^2(D))} \to 0$ as $n \to \infty$

If $u \in C(J, C^2(D))$

$$\begin{aligned}
\|u - u_n\|_{C(J,L^2(D))} &= \max_{t \in J} \max_{z \in \mathbb{X}_n} \|u(t) - z\|_{L^2(D)} \\
&\le \max_{t \in J} \|u(t) - \mathcal{I}_n u(t)\|_{L^2(D)} \\
&\le |D|^{1/2} \max_{t \in J} \|u(t) - \mathcal{I}_n u(t)\|_{C(D)} = O(n^{-2})
\end{aligned}$$

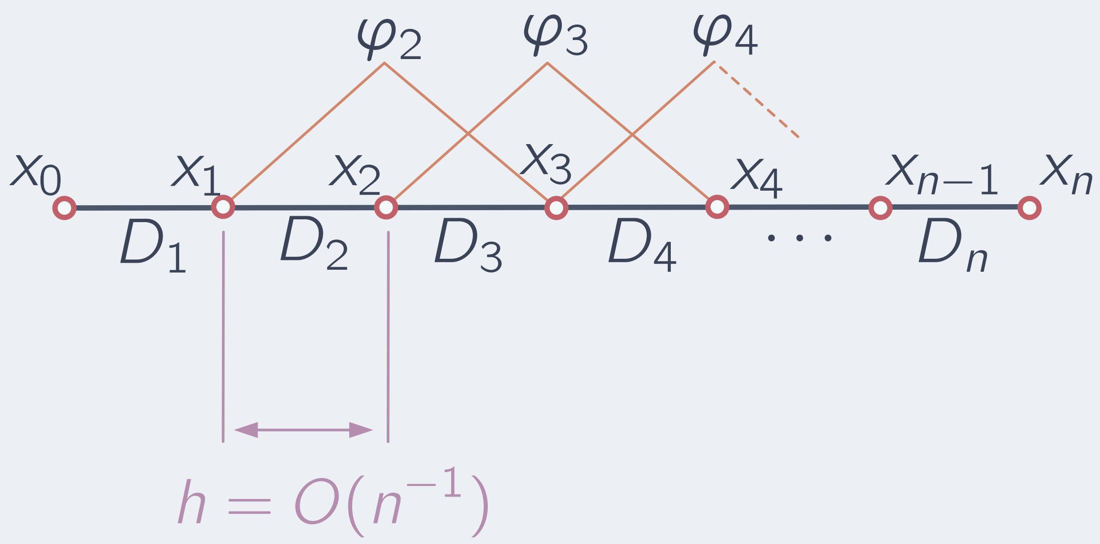
Galerkin Finite Elements Implementation

- Setup

  $\mathbb{X} = L^2(D)$, $\|\cdot\|_{\mathbb{X}} = \langle\cdot,\cdot\rangle_{L^2(D)}^{1/2}$

  $D = [-1, 1] \subset \mathbb{R}$

  $\mathbb{X}_n = \mathrm{Span}\{\varphi_1, \dots, \varphi_n\}$



- Scheme with $O(n^{-2})$ accuracy

$$\sum_j \langle \varphi_i, \varphi_j \rangle \, a_j'(t) = -\sum_j \langle \varphi_i, \varphi_j \rangle \, a_j(t) + \int_{\operatorname{supp}\varphi_i} \varphi_i(x) g(x,t) dx$$
$$+ \int_{\operatorname{supp}\varphi_i} \varphi_i(x) \int_D w(x,y) f\Big(\sum_j a_j(t)\varphi_j(y)\Big) dy\, dx$$

How to select quadrature?

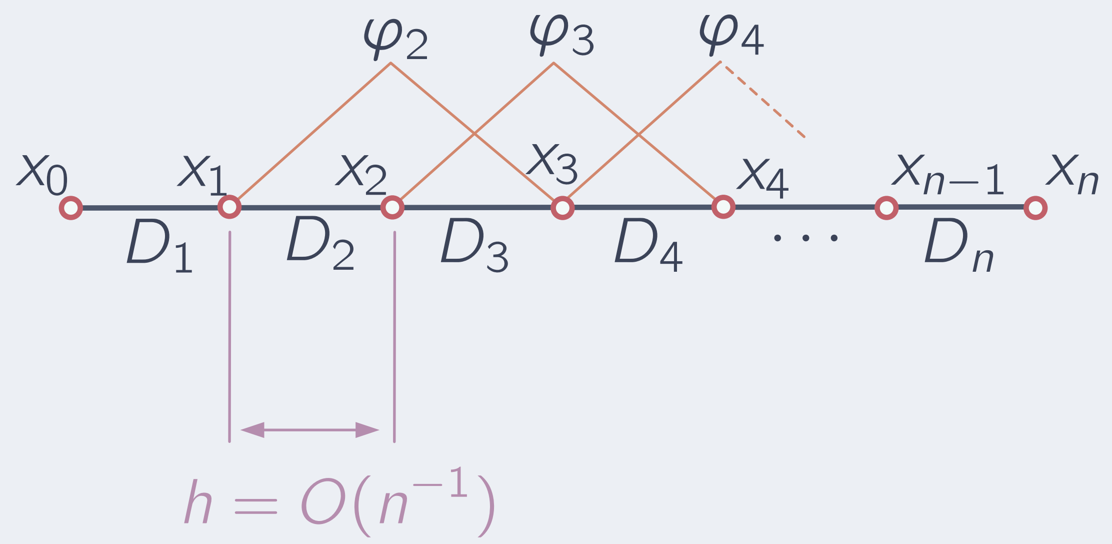
- Choice of P_n dictates quadrature choice
- Here, we pick a quadrature that preserves $O(n^{-2})$ accuracy

Finite Elements Galerkin scheme

- Setup

$\mathbb{X} = L^2(D) \qquad D = [-1, 1] \subset \mathbb{R}$

$\mathbb{X}_n = \mathrm{Span}\{\varphi_1, \ldots, \varphi_n\}$



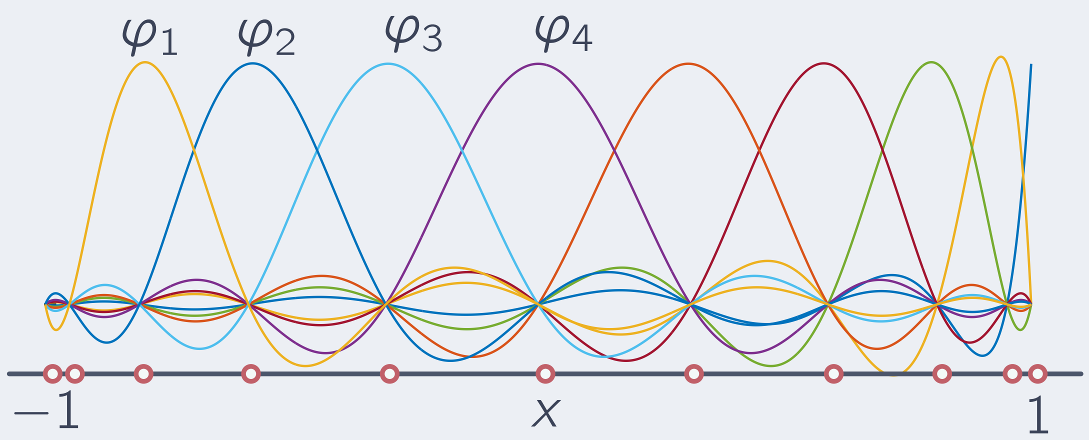
Theorem (convergence of Finite Elements Collocation scheme)

It holds $\|u - u_n\|_{C(J,C(D))} \to 0$ as $n \to \infty$

If $u \in C(J, C^2(D))$,

$$\|u - u_n\|_{C(J,C(D))} \lesssim \max_{t \in J} \|u(t) - P_n u(t)\| = O(h^2) = O(n^{-2})$$

Spectral Collocation scheme



- Setup

  $\mathbb{X} = C(D)$, $\|\cdot\|_{\mathbb{X}} = \|\cdot\|_\infty$ $\quad D = [-1, 1] \subset \mathbb{R}$

  $\mathbb{X}_n = \mathrm{Span}\{\varphi_1, \ldots, \varphi_n\}$

- Chebyshev interpolant (Lagrange polynomials, Chebyshev nodes)

$$u_n(x) = \sum_j u(x_j)\,\varphi_j(x) \qquad \varphi_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \qquad x_i = \cos\frac{i\pi}{n}$$

- In this case $P_n v_* \not\to v_*$ for some $v \in C(D)$

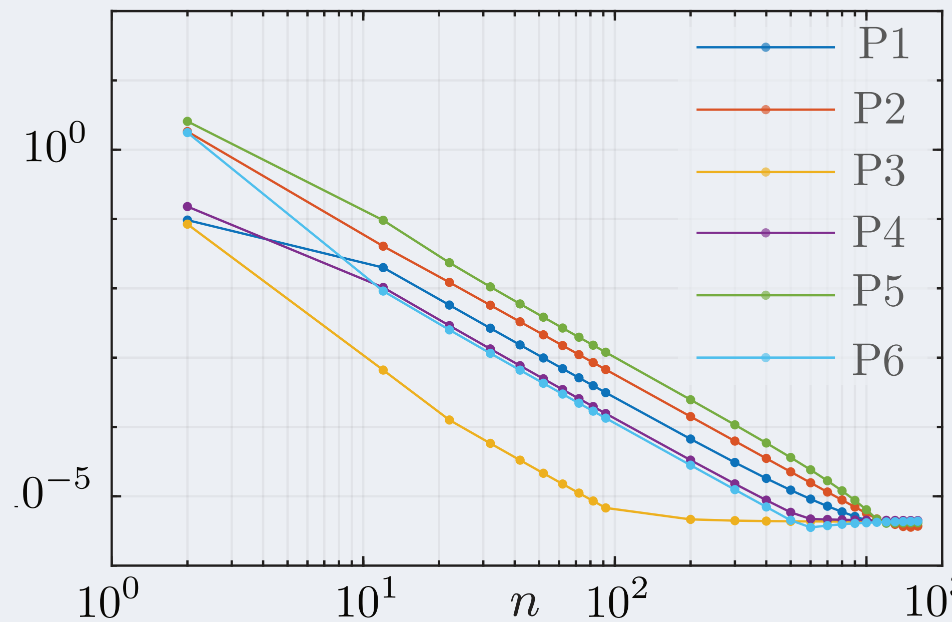
- Further conditions on the mth derivative are required

  α-Hölder continuity gives $\qquad \|u - u_n\|_{C(J, C(D))} = O(n^{-(\alpha+m)} \log n)$

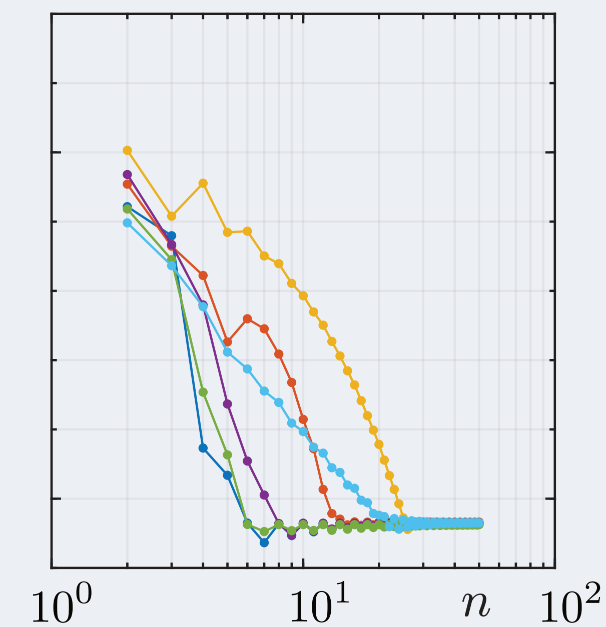
  bounded variation gives $\qquad \|u - u_n\|_{C(J, C(D))} = O(n^{-m})$

Convergence results

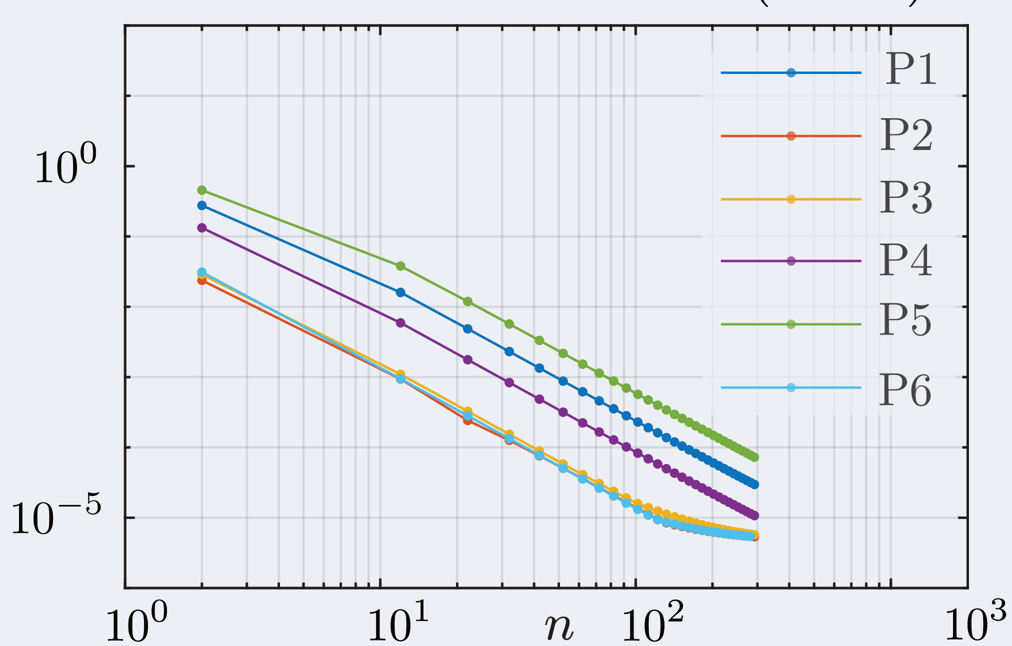
Finite Element Collocation
(Trapezium)



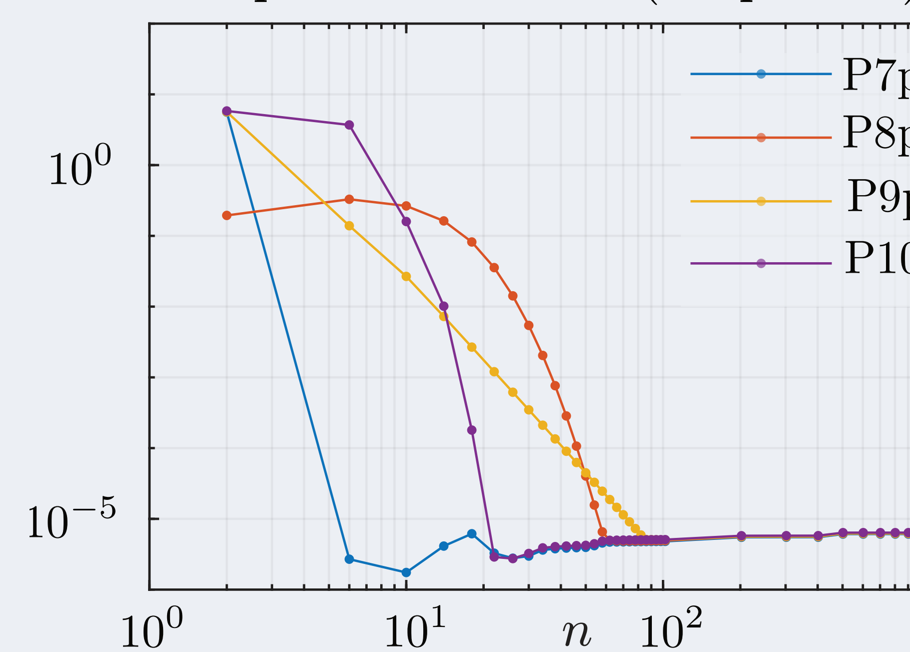
Spectral Collocation
(Clenshaw-Curtis)



Galerkin Finite Elements (Gauss)



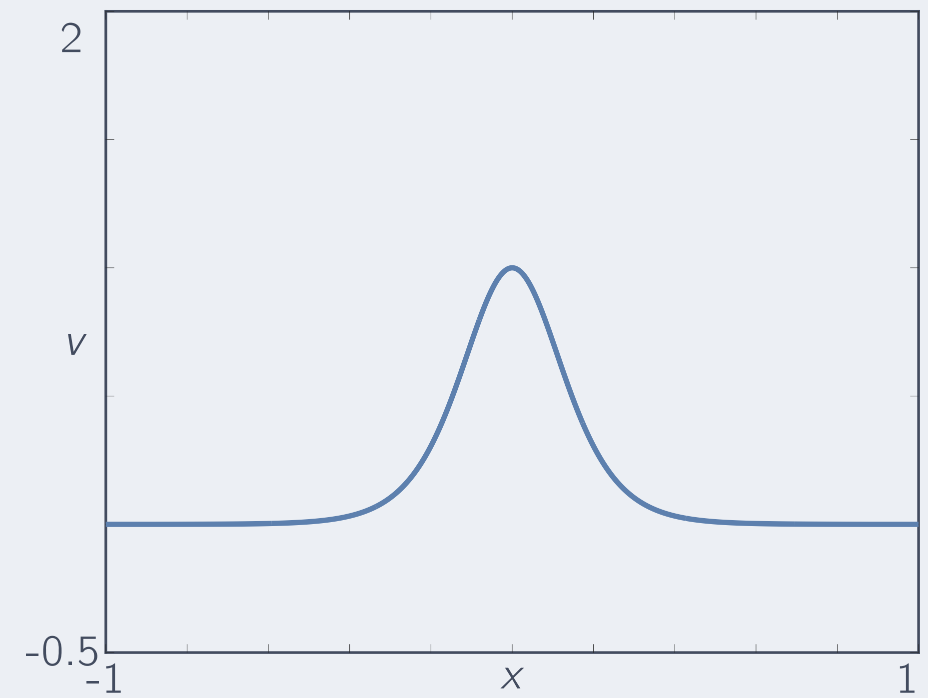
Spectral Galerkin (Trapezium)



Random inputs

- Deterministic initial condition

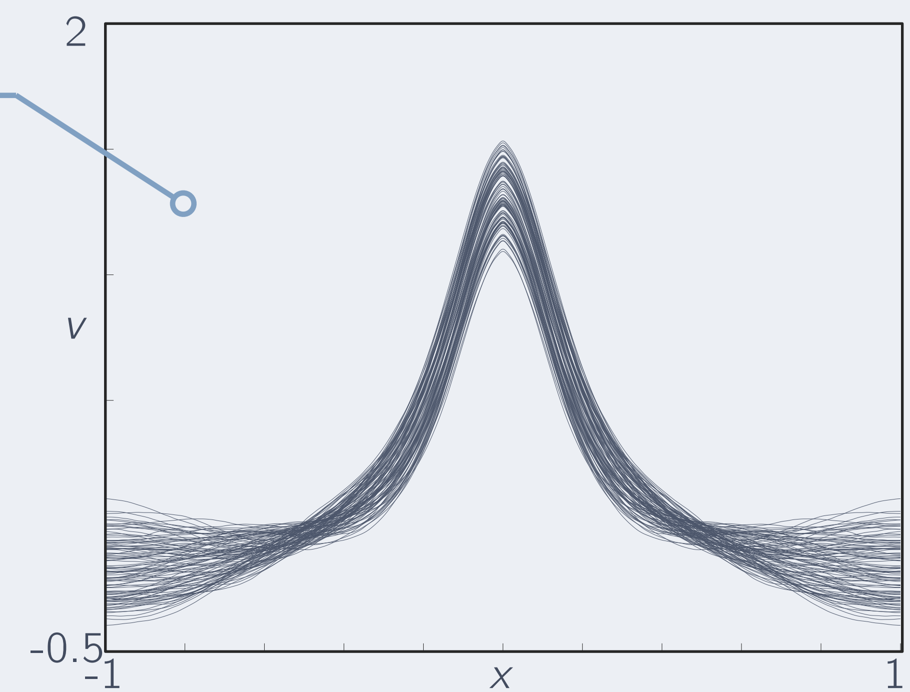
$$v(x), \qquad x \in D, \qquad v \in \mathbb{X} \in \{C(D), L^2(D)\}$$



- Initial condition as random field

$$v(x,\omega), \qquad x \in D, \omega \in \Omega \qquad v(\cdot,\omega) \in \mathbb{X}, \qquad \mathbb{P}\text{-a.s.}$$

100 realisations
$v \in L^2(\Omega, C(D))$



- Require $\quad v \in L^p(\Omega, \mathbb{X})$

$$\|v\|_{L^p(\Omega,\mathbb{X})} = \left(\int_\Omega \|v(\cdot,\omega)\|_{\mathbb{X}}^p \, d\mathbb{P}(\omega) \right)^{1/p} < \infty$$

Well-posedness of NFs with random data

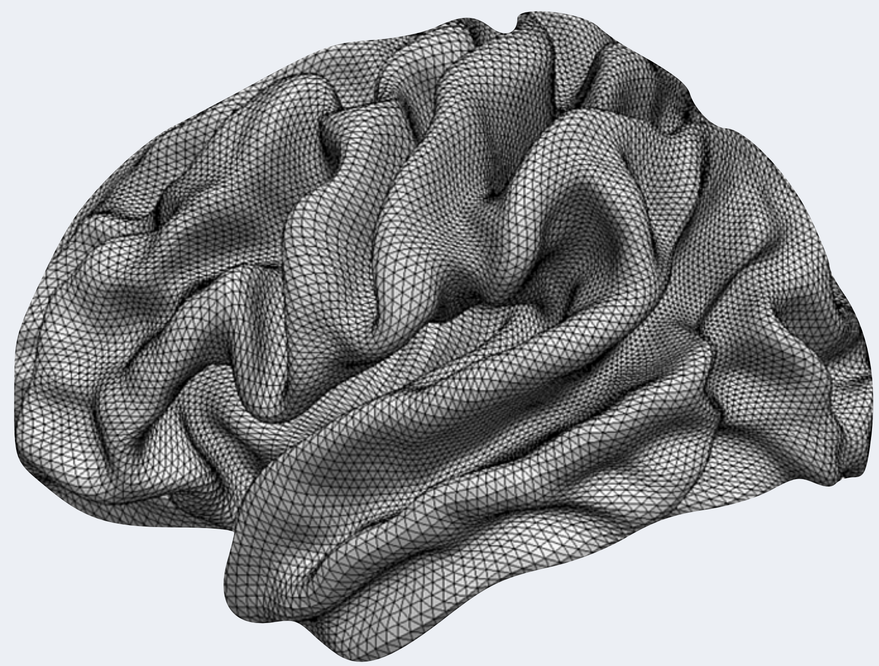
Initial state $v(x, t, \omega)$

$v \in L^p(\Omega, \mathbb{X})$

External input $g(x, t)$

Synaptic kernel $w(x, x')$

Rate function $f(u)$



Voltage $u(x, t, \omega)$

$u \in L^p(\Omega, C^1(J, \mathbb{X}))$

Theorem (well-posedness of NFs with random initial condition)

Under general assumptions on w, g, f, if $v \in L^p(\Omega, \mathbb{X})$, the problem

$$u'(t, \omega) = N(t, u(t, \omega)), \qquad u(0, \omega) = v(\omega)$$

admits a unique solution $u(\cdot, \omega)$ $\mathbb{P}$-almost surely, and $u \in L^p(\Omega, C^1(J, \mathbb{X}))$

Multiple random inputs

Initial state $v(x, t, \omega_v)$

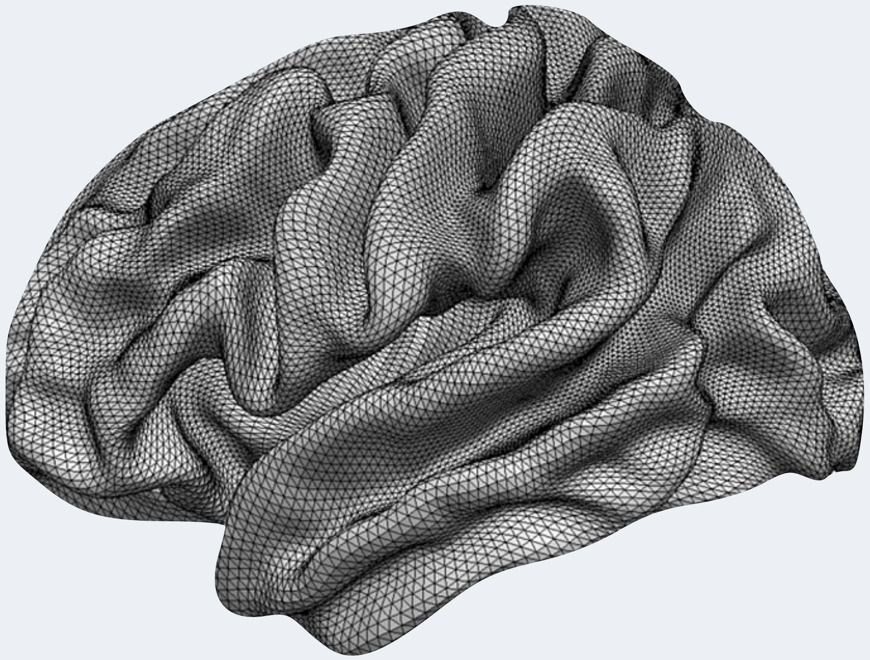
$v \in L^p(\Omega_v, \mathbb{X})$

External input $g(x, t, \omega_g)$

$g \in L^p(\Omega_g, C^0(J, \mathbb{X}))$

Synaptic kernel $w(x, x')$

Rate function $f(u)$



Voltage $u(x, t, \omega)$

$u \in L^p(\Omega, C^1(J, \mathbb{X}))$

Theorem (well-posedness of NFs with random data)

Assume general hypotheses on w, f, and

$$v \in L^p(\Omega_v, \mathbb{X}) \qquad g \in L^p(\Omega_g, C^0(J, \mathbb{X})) \qquad v \text{ and } g \text{ independent}$$

- NF with random data has a unique solution $u(\,\cdot\,, \omega)$ for almost all $\omega \in \Omega_v \times \Omega_g$,
- $u \in L^p(\Omega_v \times \Omega_g, C^1(J, \mathbb{X}))$
- $u_n \in L^p(\Omega_v \times \Omega_g, C^1(J, \mathbb{X}_n))$

Finite-dimensional noise assumption

Assumption (m-dimensional noise)

$$v(\cdot, \omega) = \tilde{V}(\cdot, Y_1(\omega), \dots, Y_m(\omega)), \qquad \text{deterministic } \tilde{V}(\cdot, y_1, \dots, y_m)$$

Example: truncated KL expansion of $h \in L^2(\Omega, L^2(D))$

$$h(x, \omega) = \mathbb{E}h(x, \cdot) + \sum_{j=1}^{\infty} \sqrt{\nu_j}\psi_j(x)Y_j(\omega), \qquad Y_j \sim U(-1, 1), \text{ i.i.d.}$$

$$v(x, \omega) = \mathbb{E}h(x, \cdot) + \sum_{j=1}^{m} \sqrt{\nu_j}\psi_j(x)Y_j(\omega), \qquad Y_j \sim U(-1, 1), \text{ i.i.d.}$$

Nonlinear KL expansions are also possible, unbounded Y are also possible

Lemma

Finite-dimensional random data $\Rightarrow$ finite-dimensional solutions u, u_n

Numerical Scheme for forward UQ

Assume m-dimensional noise with

$$(Y_1, \ldots, Y_m) \sim \rho = \rho_1 \cdots \rho_m \qquad \text{on } \Gamma_1 \times \cdots \times \Gamma_m \qquad \rho \text{ sub-Gaussian}$$

The key idea is to combine

- A spatial (interpolatory or orthogonal) projector

$$P_n v = \sum_{i=1}^{n} v_i \varphi_i(x) \qquad v_i = \begin{cases} v(x_i) & \text{if } \mathbb{X} = C(D) \\ \langle v, \varphi_i \rangle & \text{if } \mathbb{X} = L^2(D) \end{cases}$$

- An interpolatory projector

$$\mathcal{I}_q h = \sum_{j_1=1}^{q_1+1} \cdots \sum_{j_m=1}^{q_m+1} h(y_{j_1}, \ldots, y_{j_m})(l_{j_1} \otimes \cdots \otimes l_{j_m}) := \sum_{j=1}^{q} h(y_j)\psi_j(y)$$

Numerical Scheme for forward UQ

Assume m-dimensional noise with

$$(Y_1, \ldots, Y_m) \sim \rho = \rho_1 \cdots \rho_m \qquad \text{on } \Gamma_1 \times \cdots \times \Gamma_m \qquad \rho \text{ sub-Gaussian}$$

$$u'(y_j) = N(t, u(y_j), y_j) \qquad \rho dy\text{-a.e.}$$

$$\downarrow P_n$$

$$u_n'(y_j) = P_n N(t, u_n(y_j), y_j) \qquad \rho dy\text{-a.e.}$$

q realisations

$$\downarrow \mathcal{I}_q$$

$$u_{nq}(y) = (\mathcal{I}_q u_n)(y) = \sum_{j=1}^{q} u_n(y_j) \psi_j(y)$$

Error splitting: $u - u_{nq} = (u - u_n) + (u_n - u_{nq})$

P_n error $\qquad$ $\mathcal{I}_q$ error

[Babuska, Nobile, Tempone] [Zhang, Gunzburger]

Error analysis

adapting [Babuska, Nobile, Tempone]

Assume regularity of the random data (example on initial condition)

$$\frac{\|P_n \partial_{y_j}^k v(y)\|_{\mathbb{X}}}{1 + \|v(y)\|_{\mathbb{X}}} \le C k! \gamma_j^k \qquad \text{for all } j,\ k\ n, \text{ and } y$$

Theorem (Analyticity of NF solution)

Fix $n \in \mathbb{N}$. For any j the mapping $y_j \mapsto u_n(x, t, (y_j, \cdot))$ admits an analytic extension in a filled-in Bernstein ellipse $B_j \subset \mathbb{C}$, indepedent of n.

Theorem (Error splitting)

$$\begin{aligned} \|u(T) - u_{nq}(T)\|_{\mathbb{X}\otimes L^2_\rho(\Gamma)} &\le \|u(T) - u_n(T)\|_{\mathbb{X}\otimes L^2_\rho(\Gamma)} + \|u_n(T) - u_{nq}(T)\|_{\mathbb{X}\otimes L^2_\rho(\Gamma)} \\ &\lesssim n^{-k} + \sum_{j=1}^{m} e^{-r_j q_j} \end{aligned}$$

Proof sketch

In this proof sketch

- Neural Field problem is linear
- The initial condition v, is a 1-dimensional random variable
- All other input data are deterministic

- Introduce weighted function spaces [Babuska, Nobile, Tempone]

$$C^0_\sigma(\Gamma, \mathbb{X}) = \left\{ v : \Gamma \to \mathbb{X}, \quad v \in C^0(\Gamma, \mathbb{X}), \quad \max_{y \in \Gamma} \|\sigma(y) v(y)\|_{\mathbb{X}} < \infty \right\}$$

- Weight

$$\sigma(y_i) = \begin{cases} 1 & \text{if } \Gamma \text{ is bounded,} \\ e^{-\eta|y|} & \text{if } \Gamma_i \text{ is unbounded,} \end{cases} \qquad \eta \in \mathbb{R}_{\geq 0},$$

- Prove

$$v \in C^0_\sigma(\Gamma, \mathbb{X}) \hookrightarrow L^p_\rho(\Gamma, \mathbb{X}) \qquad \Rightarrow \qquad u_n \in C^0_\sigma(\Gamma, C^1(J, \mathbb{X})) \hookrightarrow L^p_\rho(\Gamma, C^1(J, \mathbb{X}))$$

Proof sketch

- Evolution problem for derivatives

$$\partial_t(\partial_y^k u_n) = A_n(\partial_y^k u_n), \quad t \in [0, T], \qquad\qquad A_n = -\mathbf{Id}_{\mathbb{X}} + P_n W$$
$$\partial_y^k u_n(0) = P_n \partial_y^k v$$

- Operator A_n is bounded, and generates a uniformly continuous semigroup

$$\|e^{tA_n}\| \leq e^{t\|A_n\|} \leq e^{t\beta} \qquad \text{for all } n$$

- Bound derivatives, using regularity of the data

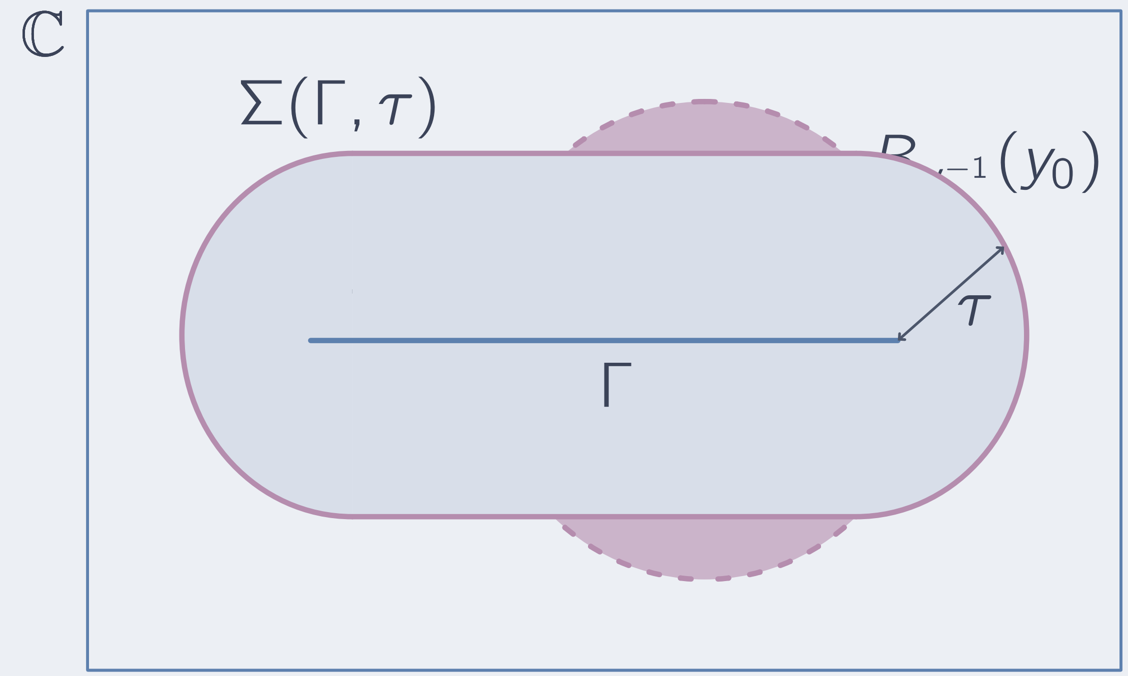
$$\begin{aligned}\left\|\partial_y^k u_n(t, y)\right\|_{\mathbb{X}} &\leq e^{t\|A_n\|} \left\|P_n \partial_y^k v\,(y)\right\|_{\mathbb{X}} \\ &\lesssim e^{t\beta}\,(1 + \|v\,(y)\|_{\mathbb{X}})\, k!\gamma^k \;\; =: D\big(t, \|v(y)\|_{\mathbb{X}}\big)\, k!\gamma^k\end{aligned}$$

- Fix t, and prove that $y \mapsto u_n(t, y)$ can be analytically extended

Proof sketch

- Fix T, define

$$\tilde{u}_n : \mathbb{C} \to \mathbb{X} \quad z \mapsto u_n(T, z)$$



- Fix $y_0 \in \Gamma$, consider the power series

$$\tilde{u}_n(z) := \sum_{k=0}^{\infty} \frac{(z - y_0)^k}{k!} \partial_y^k u_n\,(T, y_0)$$

- Series converges in $B_{\gamma^{-1}}(y_0)$ because

$$\|\partial_y^k u_n(T, y_0)\|_{\mathbb{X}} \lesssim D\big(T, \|v(y_0)\|_{\mathbb{X}}\big)\ k!\gamma^k$$

$$\sum_{k=0}^{p} \frac{(z - y_0)^k}{k!} \partial_y^k u_n\,(T, y_0) \lesssim \sum_{k=0}^{p} \left(\frac{z - y_0}{1/\gamma}\right)^k D(T, \|v(y_0)\|_{\mathbb{X}})$$

- Then $\tilde{u}_n$ is analytic in $\Sigma(\Gamma, \tau) \quad \tau \in (0, 1/\gamma)$

- This implies $\quad \|u_n(T) - u_{nq}(T)\|_{\mathbb{X} \otimes L^2_\rho(\Gamma)} \lesssim e^{-rq^\theta}$

Numerical experiment

- Neural field on 1D cortex, with random input g

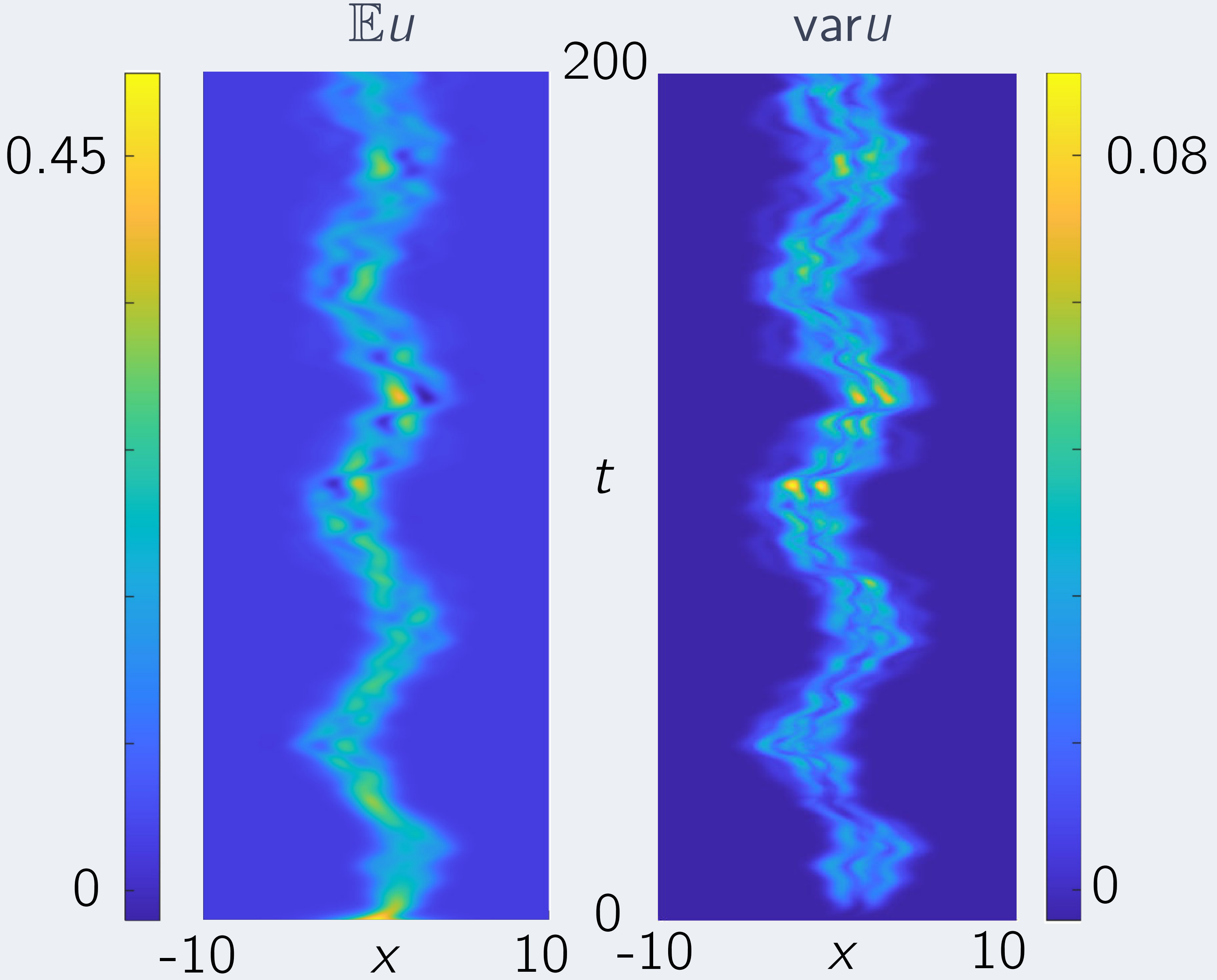
$$g(x, t, Y) = Ae^{-(x-c(t,Y))^2}$$
$$c(t, Y) = \sum_{j=1}^{3} Y_j \sin(Y_{2j-1} t)$$
$$Y_j \sim U(a_j, b_j) \quad \text{mut. independent}$$

- Parameter for this study

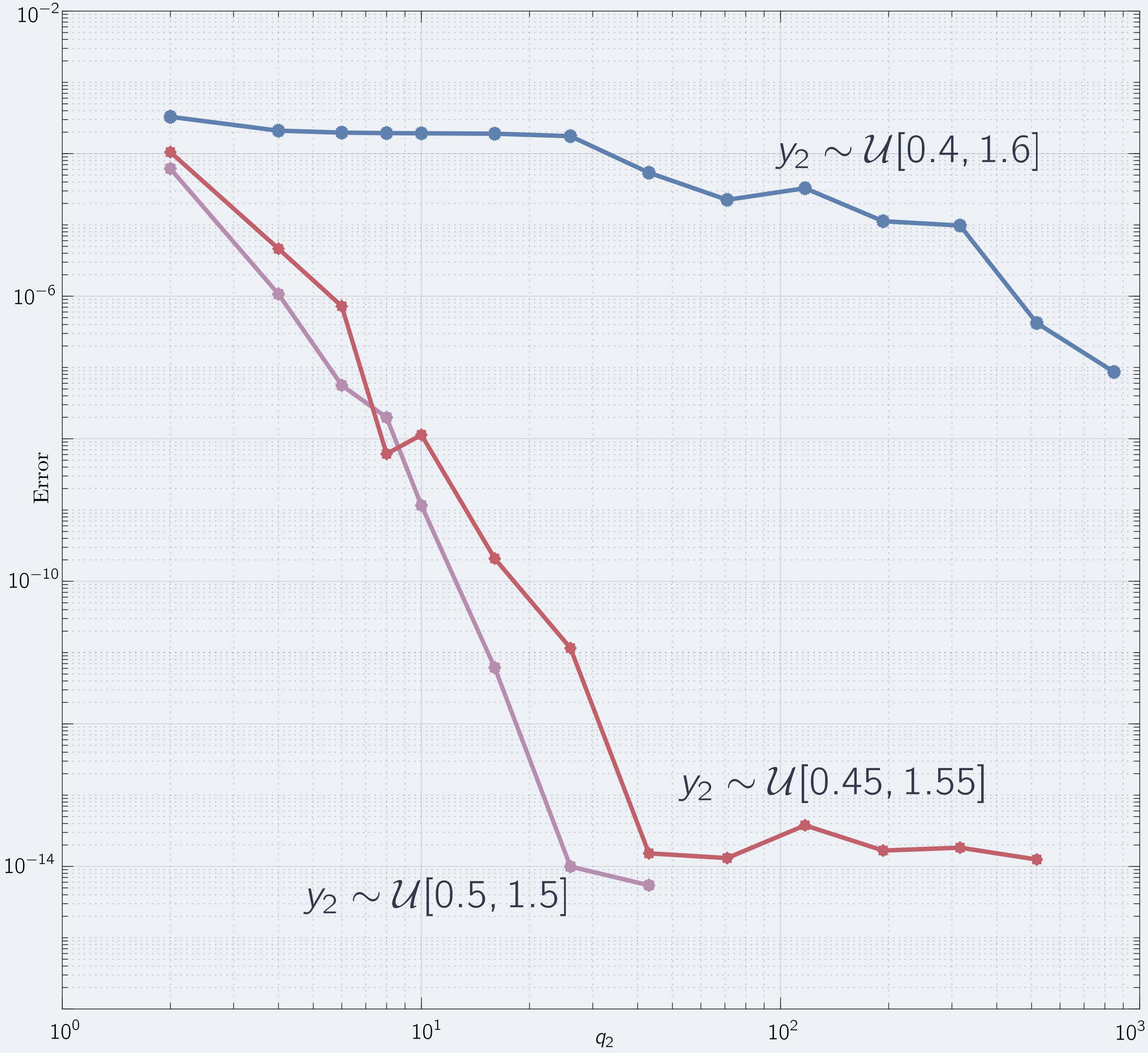
m = 6 random variables

n = 100 spatial nodes in D

q = 85 nodes in Γ



Convergence properties 4-paremeters



$y_i \sim \mathcal{U}[a_i, b_i]$

Neural field with noise

- Neural fields as SDE on $\mathbb{X} = H^m(D)$

$$du(t) = \left[Au(t) + KF(u(t))\right]dt + \sigma(u(t))dW(t)$$

- W is an $\mathbb{X}$-valued Wiener Q-Wiener process $\sigma(u)$ models additive or multiplicative noise

Theorem (Convergence of projection schemes)

The solution u_n of the P_n-projected stochastic neural field problem satisfies

$$\mathbb{E}\|u - u_n\|_{C^0(J,\mathbb{X})} \leq e^{\beta_n T}\mathbb{E}\|u - P_n u\|_{C^0(J,\mathbb{X})} \qquad \beta_n \propto \|P_n K\|$$

- Exponential scheme (several others are being considered)

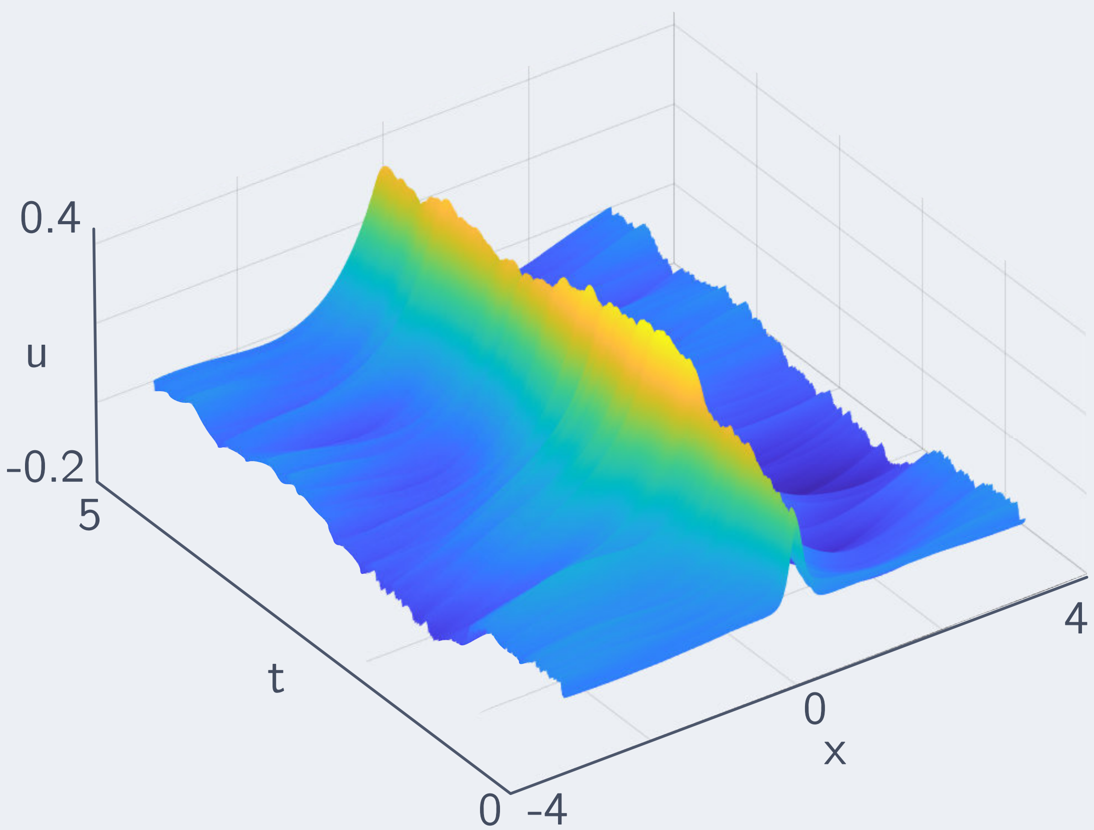
$$U_{k+1} = e^{-\Delta t}U_k + \Delta t e^{-\Delta t} KF(U_k) + \sigma(U_k)\Delta W_k$$

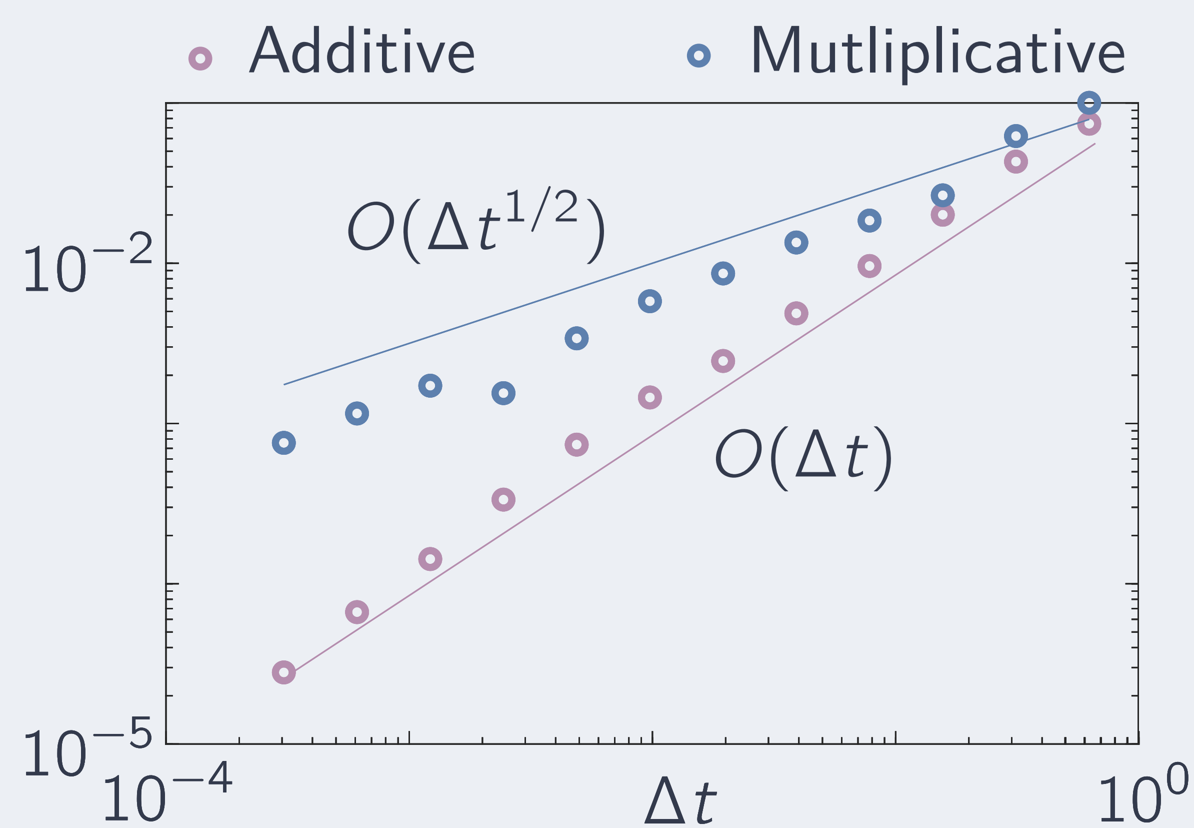
Convergence result

- Neural fields as SDE on $\mathbb{X} = H^m(D)$

$$du(t) = \big[Au(t) + KF(u(t))\big]dt + \sigma(u(t))dW(t) \qquad \sigma(u) = \begin{cases} \mathrm{Id}_{\mathbb{X}} & \text{(Additive)} \\ u & \text{(Multiplicative)} \end{cases}$$

- Estimates for $\mathbb{E} \sup_{t\in[0,T]} \|u(t) - u_*(t)\|^2_{L^2(D)}$





Conclusions

- Generic theory for NF with random data
  - Arbitrary cortices
  - Concurrent random inputs
  - Well-posedness of continuous and semi-discrete problems
  - Weak/Strong convergence theory for problem realisations
  - Analyticity/Stochastic collocation for linear NFs
- Extension to other methods for problems with random data
  - Multi-level MC methods
  - Stochastic FE methods
- UQ on Transcranial Magnetic Stimulation, with clinical data

Papers

- D. Avitabile, *Projection Methods for Neural Fields Equations*, (2023). To appear on SIAM Journal on Numerical Analysis

- D. Avitabile, F. Cavallini, S. Dubinkina, G. Lord, *Stochastic collocation for Neural Field Equations*, (2023). In preparation

Papers and codes available at www.danieleavitabile.com