Numerical and Analytical Methods for Spatially-Extended Neurobiological Networks

Part 2

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Cite using the instructions on the Git repo https://tinyurl.com/yu7wcmba

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Outline

- Yesterday

  $$\partial_t u(x,t) = -u(x,t) + \int_D w(x,y) f(u(y,t))\,dy$$

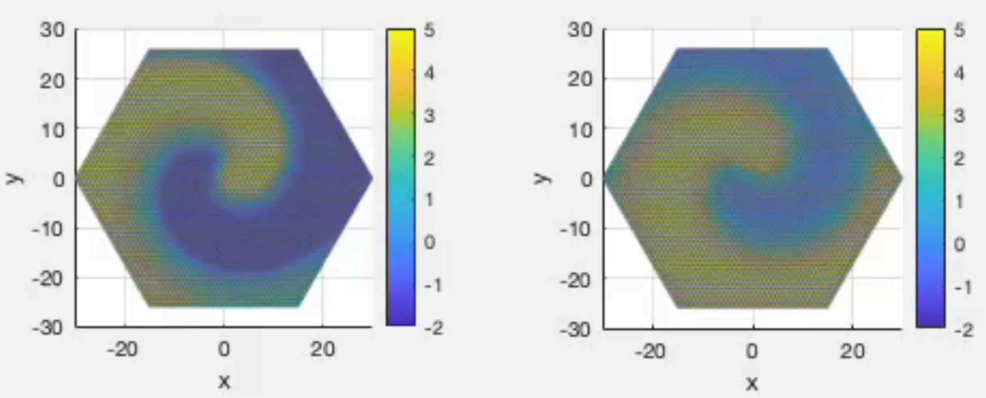
  Pattern formation through Turing bifurcation

- Today

  See various application of numerical methods for neural fields

  Develop a theory for Galerkin and collocation schemes

  Code this



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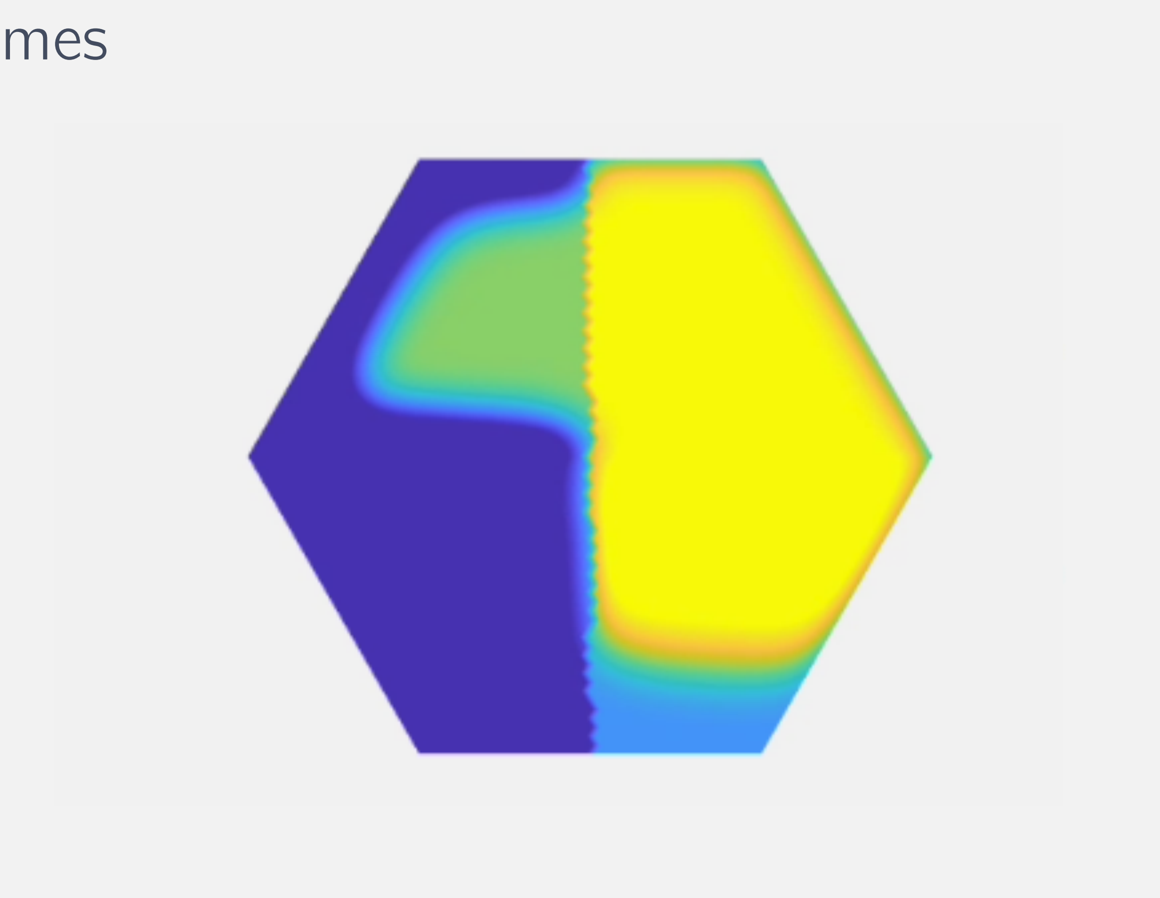
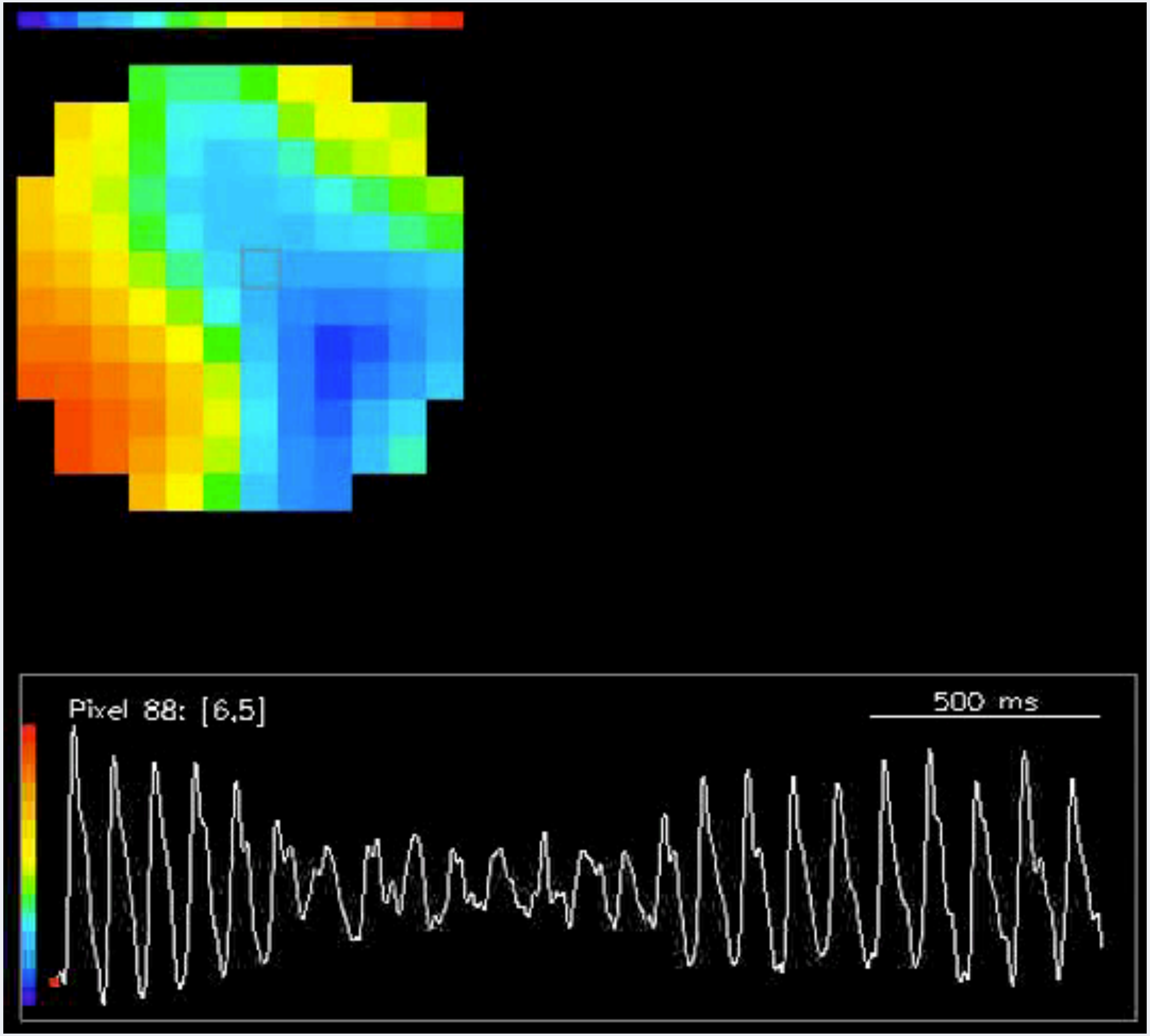
Code



Recording of rat cortical slice



Generation of time-dependent patterns

- Compute steady state $u(x)$

$$0 = -u(x) + \int_D w(x,y) f(u(y), p)\, dy$$

- Eigenvalue problem

$$\lambda v(x) = -v(x) + \int_D w(x,y) f'(u(y), p) v(y)\, dy$$

- Solve nonlinear discretised problem

$$0 = -u + MF(u,p) \qquad 0 = N(u,p) \qquad u \in \mathbb{R}^n$$

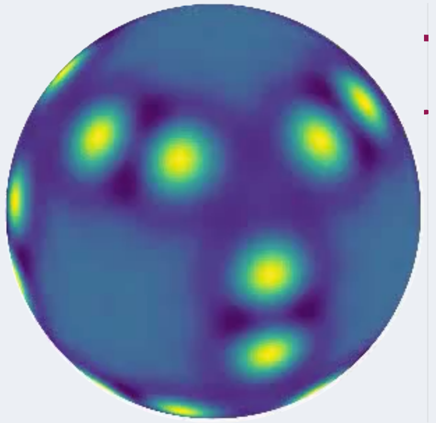
- Solve discretised eigenvalue problem

$$\lambda v = [-I + M \operatorname{diag} DF(u,p)]v \qquad \lambda v = D_u N(u,p) v \qquad D_u N(u,p) \in \mathbb{R}^{n \times n}$$

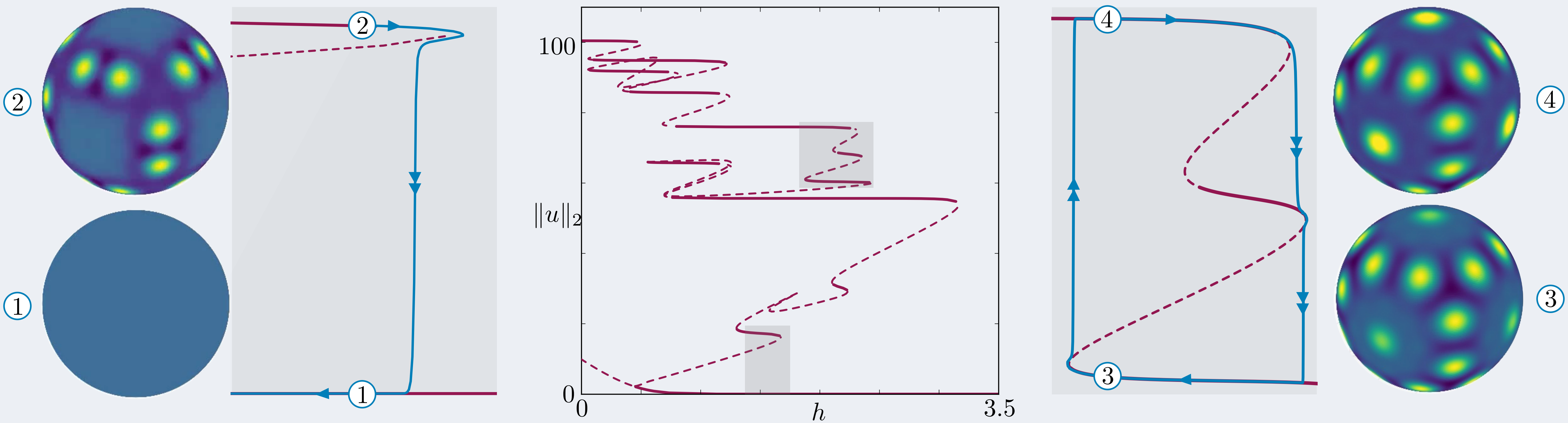
Example

Neural field on a sphere

$$\partial_t u(x,t) = -u(x,t) + \int_{\mathbb{S}^2} w(\langle x, y\rangle) f(u(y,t) - h)\, d\mu(y) \qquad x \in \mathbb{S}^2$$



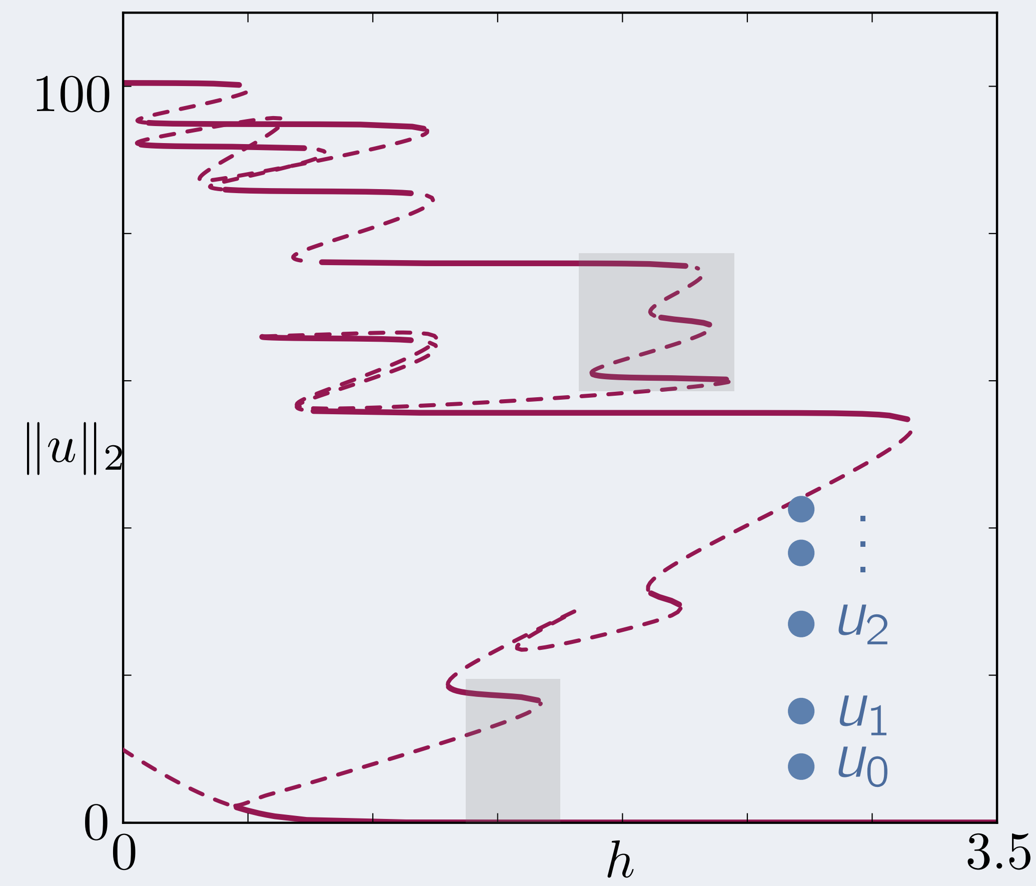
Numerical Bifurcation analysis



Newton's method and numerical continuation

Fix h, find u such that $N(u) = 0$

Sequence of approximations $\{u_k\}_k \quad u_k \to u$



Pseudocode

Guess u_0

For each $k \geq 0$

Solve $D_u N(u_k)x = -N(u_k)$

Set $u_{k+1} = u_k + x$

When convergence is attained

Compute eigenvalues of $D_u N(u_k)$

Solve $Ax = b$, with A large and dense

Large and dense eigenvalue problem

Use FFT on convolutional problems

- Neural field with convolutional structure, on a ring

$$0 = -u(x) + \int_{\mathbb{S}} w(x-y) f(u(y))\, dy, \qquad x \in \mathbb{S}$$

- Express RHS using Fourier Transform and its inverse

$$u \mapsto -u + \mathcal{F}^{-1}\Big[\mathcal{F}[w]\; \mathcal{F}[f(u)]\Big]$$

- Linear operator

$$v \mapsto -v + \mathcal{F}^{-1}\Big[\mathcal{F}[w]\; \mathcal{F}[f'(u)v]\Big]$$

- Main conclusions:

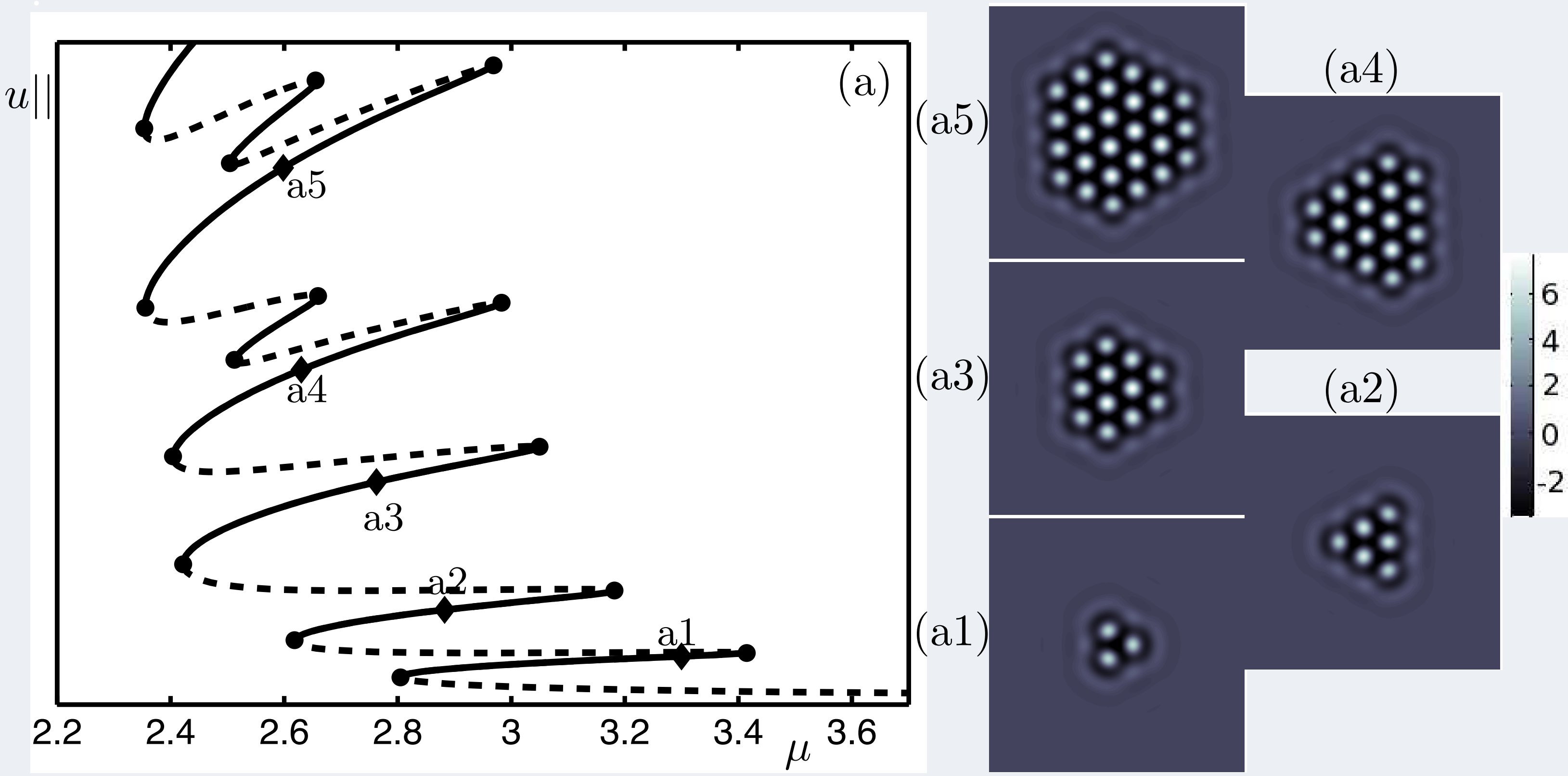
$u \mapsto N(u)$ evaluates in $O(n \log n)$ operations

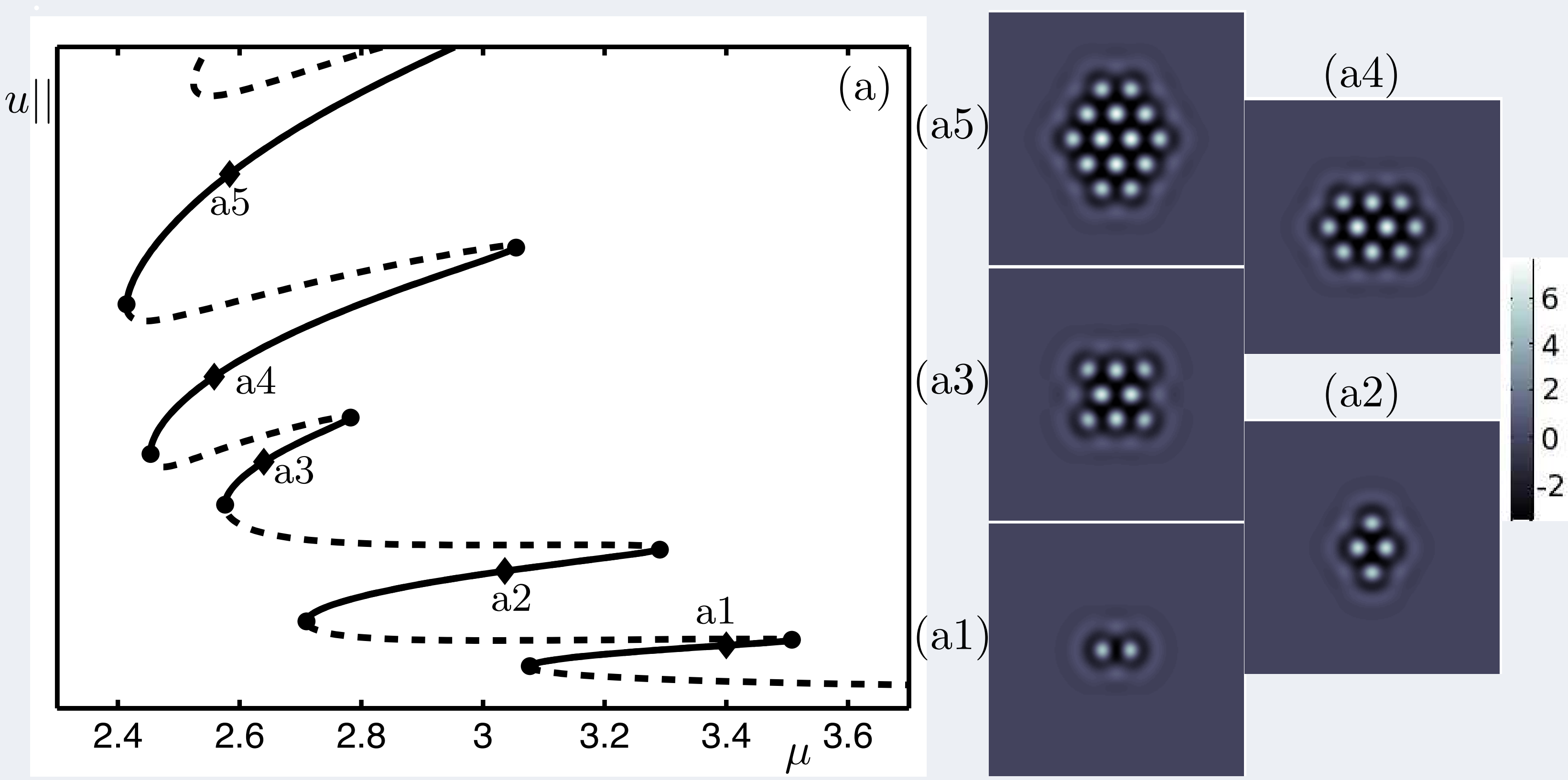
$v \mapsto D_u N(u) v$ evaluates in $O(n \log n)$ operations

Replace Newton with Newton-GMRES

Efficient Numerical Bifurcation Analysis

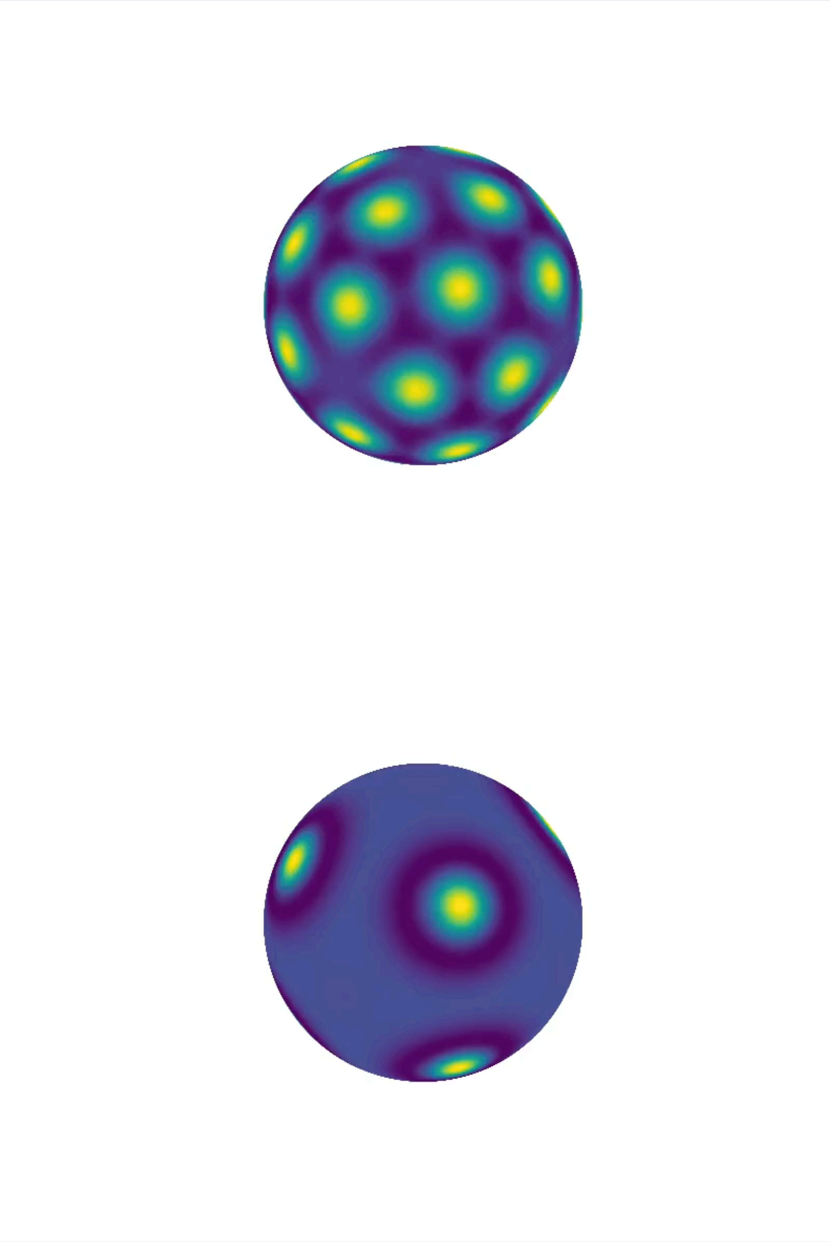
[Rankin et al]



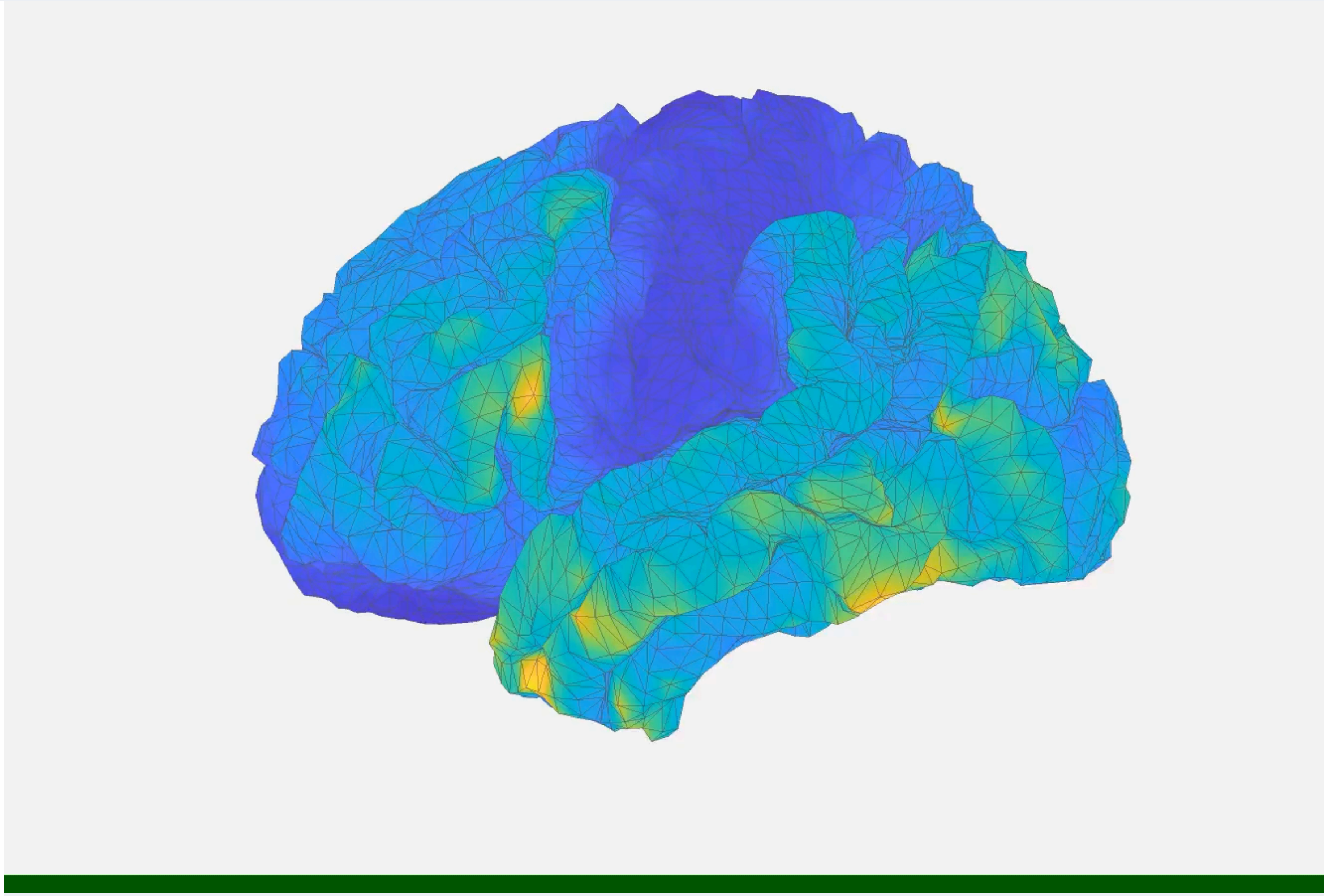


More generally

[Smith et al]



[Sammy Petros]



Need scalable, provably accurate methods for NFEs

Collocation scheme (heuristic)

$$\partial_t u(x,t) = -u(x,t) + \int_D w(x,x')f(u(x',t))dx'$$

$\downarrow$ Evaluate at $x = x_i$

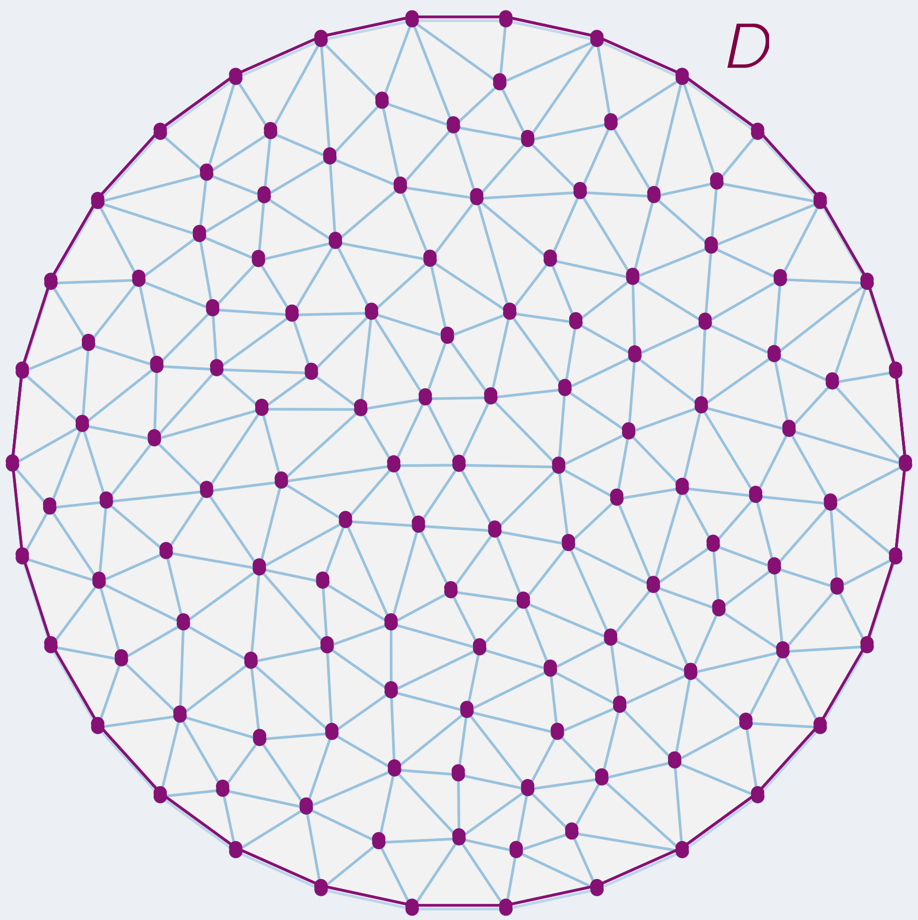
$$\partial_t u(x_i,t) = -u(x_i,t) + \int_D w(x_i,x')f(u(x',t))dx'$$

$\downarrow$ Approximate integral $\int_D h(x') \approx \sum_j h(x_j)\rho_j$

$$U_i'(t) = -U_i(t) + \sum_j w(x_i,x_j)f(U_j(t))\rho_j$$

$\downarrow$

Time step, Error Analysis

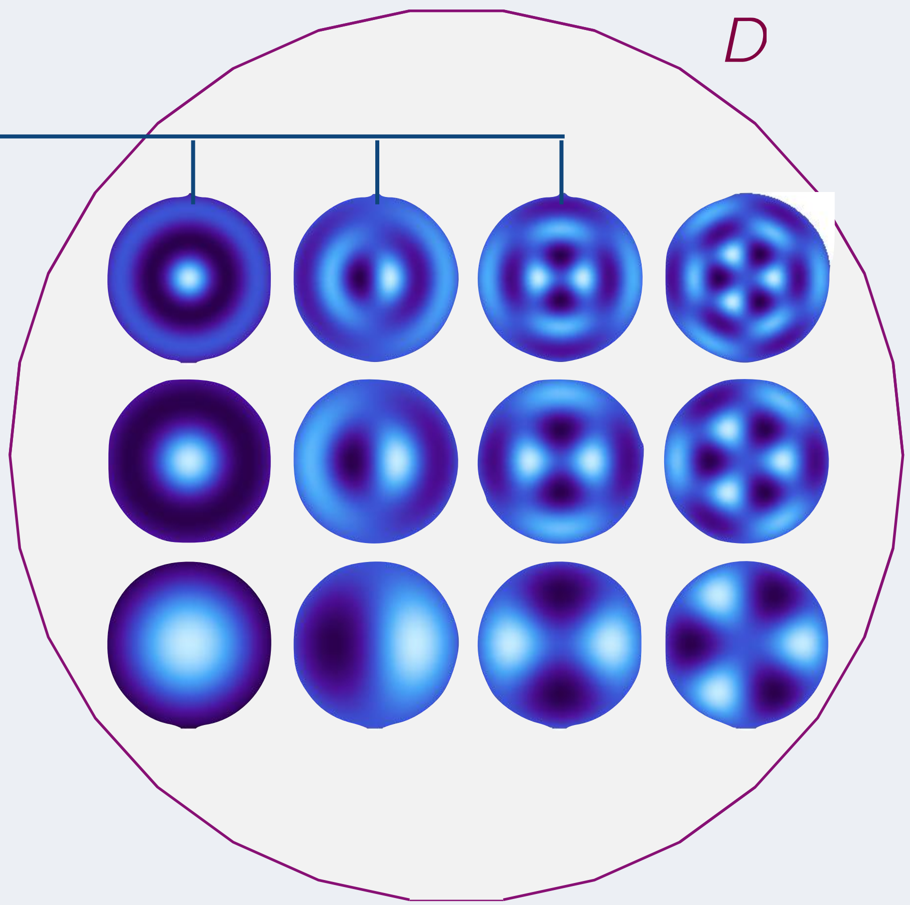


[Lima, Buckwar]

[Avitabile, Lima, Coombes]

Galerkin scheme (heuristic)

$$\partial_t u(x,t) = N(u)(x,t)$$



$$u(x,t) = a_1(t)\varphi_1(x) + a_2(t)\varphi_2(x) + a_3(t)\varphi_3(x) + \cdots$$

$\downarrow$ Truncate $u(x,t) \approx \sum_j^n a_j(t)\varphi_j(x)$

$$\sum_j \langle \varphi_i, \varphi_j \rangle a_j'(t) = \Big\langle \varphi_i, N\big(\sum_j a_j(t)\varphi_j\big)\Big\rangle$$

$\downarrow$ Approximate integrals $\langle \varphi_i, \varphi_j \rangle \approx \sum_k \varphi_i(x_k)\varphi_j(x_k)\rho_k$

$$\sum_j M_{ij} a_j'(t) = F(a_1(t), \ldots, a_n(t))$$

$\downarrow$

Time step, Error Analysis

Abstract approximation strategy

- Residual at a Neural field solution

$$R(u) = u' - N(\cdot, u) = 0, \qquad \text{on } J = [0, T]$$

- Approximation strategy
  - Select

$$\mathbb{X}_n = \mathrm{Span}\{\varphi_1, \dots, \varphi_n\}$$

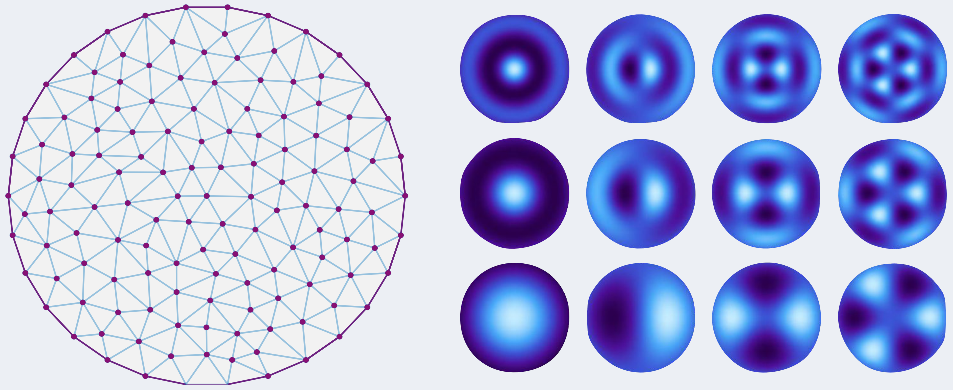
  - Set

$$u_n(t) = \sum_{j=1}^{n} U_j(t)\varphi_j \in \mathbb{X}_n$$

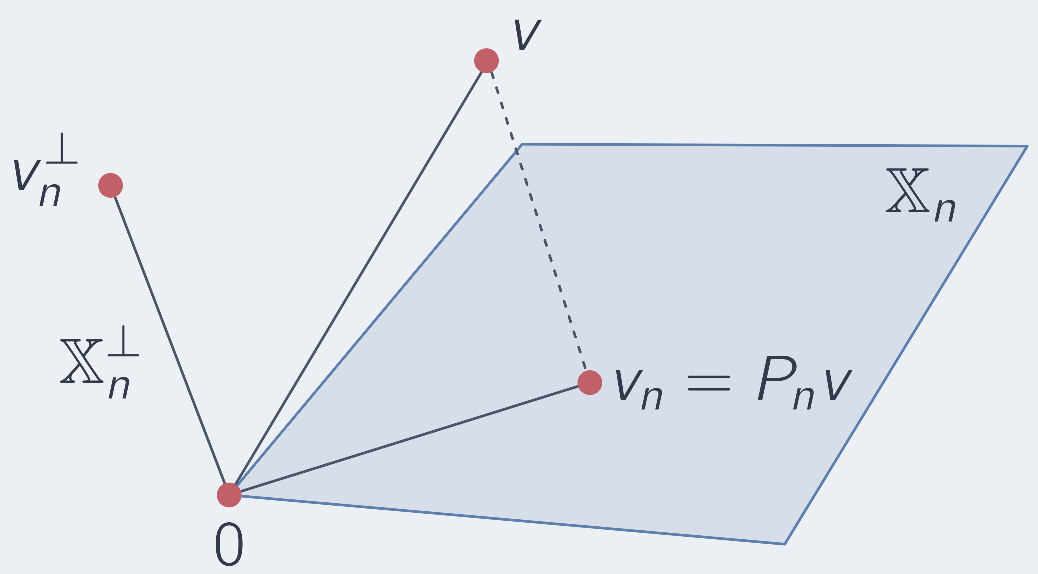
  - Make the residual “small”

$$R(u_n(t)) \approx 0, \qquad t \in J$$

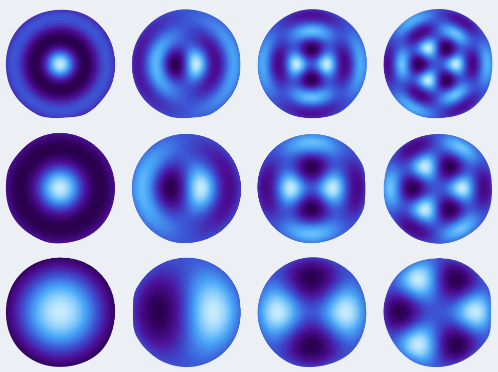
“Approximate” u, $N(u)$



Residual conditions - abstract schemes



Galerkin



$\mathbb{X} = L^2(D)$

Orthogonal projector

$$P_n \colon \mathbb{X} \to \mathbb{X}_n, \ v \mapsto \sum_j^n \langle v, \varphi_j \rangle \varphi_j$$

- Characterisation $\quad P_n v = 0 \iff \langle v, \varphi_j \rangle = 0 \quad \text{for all } j$

- Scheme

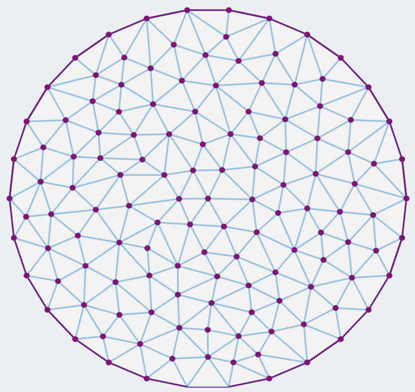
$$P_n R(u_n) = 0 \text{ for all } x,\ t \iff u_n'(t) = P_n N(t, u_n(t))$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\langle R(u_n), \varphi_j \rangle = 0 \text{ for all } j,\ t \iff \sum_j \langle \varphi_i, \varphi_j \rangle a_j'(t) = \Big\langle \varphi_i, N\Big(\sum_j a_j(t)\varphi_j\Big)\Big\rangle$$

Residual conditions - abstract schemes

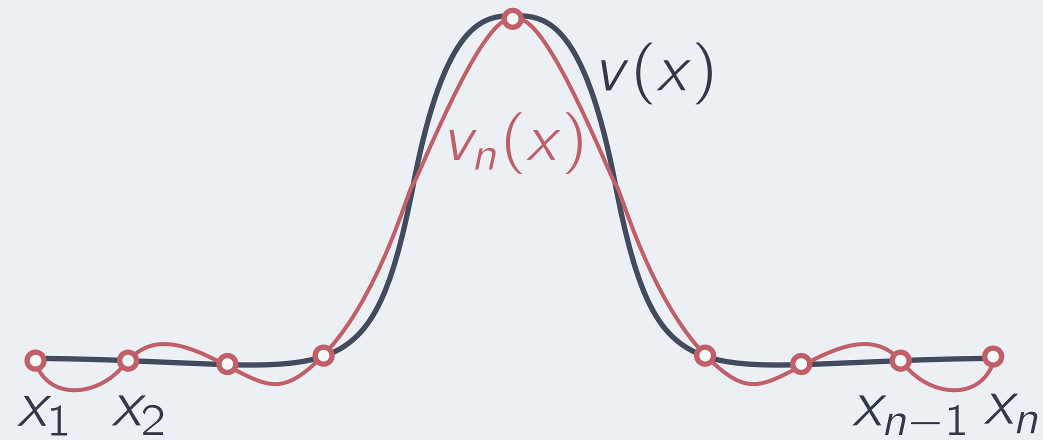
Collocation



$\mathbb{X} = C(D)$

Interpolating projector

$$P_n : \mathbb{X} \to \mathbb{X}_n,\ v \mapsto \sum_j^n v(x_j)\varphi_j$$



- Characterisation $\quad P_n v = 0 \iff v(x_j) = 0, \quad \text{for all } j$

- Scheme

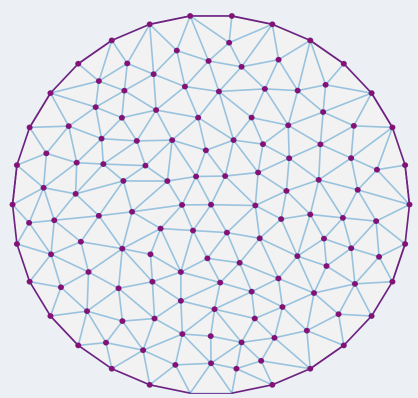
$$\begin{array}{ccc} P_n R(u_n) = 0 \text{ for all } x,\ t & \iff & u_n'(t) = P_n N(t, u_n(t)) \\ \Updownarrow & & \Updownarrow \\ R(u_n) = 0 \text{ for all } x_j,\ t & \iff & a_j'(t) = N\left(\sum_j a_j(t)\varphi_j\right) \end{array}$$

Schemes in operator form

- NF Equation $\quad u'(t) = N(t, u(t)), \qquad u(0) = u_0 \quad$ ODE in $\mathbb{X}$

$$\downarrow$$

Collocation



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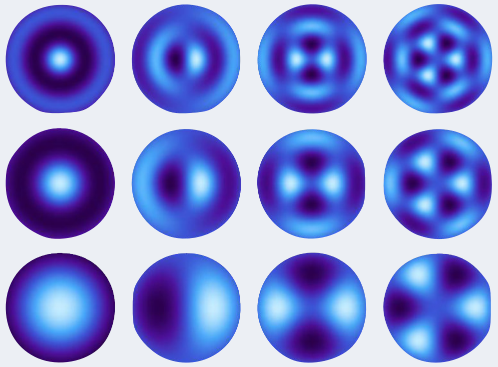
Galerkin



$$\mathbb{X} = L^2(D)$$

Orthogonal projector

$$P_n \colon \mathbb{X} \to \mathbb{X}_n,\ v \mapsto \sum_j^n \langle v, \varphi_j\rangle\varphi_j$$

$$\downarrow$$

- NF Equation $\quad u_n'(t) = P_n N(t, u_n(t)), \qquad u_n(0) = P_n u_0 \quad$ ODE in $\mathbb{X}_n$

Papers

- Rankin, J., Avitabile, D., Baladron, J., Faye, G. and Lloyd, D.J., 2014. Continuation of localized coherent structures in nonlocal neural field equations. SIAM Journal on Scientific Computing, 36(1), pp.B70-B93.
- Avitabile, D., Desroches, M. and Knobloch, E., 2017. Spatiotemporal canards in neural field equations. Physical Review E, 95(4), p.042205.
- Avitabile D, Projection methods for Neural Field Equations, 2023..SIAM Journal on Numerical AnalysisVol. 61(2).

Papers and codes available at www.danieleavitabile.com