Chapter 1

Formule varie

$$\begin{cases} \partial_t u = \epsilon^2 \Delta u + \alpha(x) u^2 v - u + (\tau\gamma)^{-1} v \\ \partial_t v = \frac{D}{\tau}\Delta v - \frac{1}{\tau} v + \frac{1}{\tau} - \gamma\left(\alpha(x) u^2 v - u\right) - \frac{\beta\gamma}{\tau} u \end{cases} \tag{1.1}$$

Sintetic Operators without coefficients

$$\mathscr{L} = \begin{bmatrix} \Delta_s + \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \Delta_s + \mathbb{1} \end{bmatrix}$$

$$\mathscr{N}(U) = \begin{bmatrix} u^2 v \\ u^2 v \end{bmatrix}$$

1^{st} **option**: Implicit Euler + Newton's method (complete)

$$\begin{aligned} & U = (u, v) \\ U^{n+1} = U^n + \Delta t \mathscr{L} U^{n+1} + \ \Delta t \mathscr{N}\left(U^{n+1}\right) \Leftrightarrow G(U) = (\mathbf{1} - \mathscr{L}) U - \Delta t \mathscr{N}(U) - U^n = 0 \quad & (1.2) \\ & Newton\ method...\ full\ matrix \end{aligned}$$

2^{nd} **option**: Implicit Euler + semi-implicit (these morning hyp.)

$$\begin{gathered} U = (u, v) \\ (\mathbb{1} - \Delta t \mathscr{L}) U^{n+1} - \ \Delta t \mathscr{N}\left(U^{n+1}, U^n\right) - U^n = 0 \\ with \\ \mathscr{N}\left(U^{n+1}, U^n\right) = \left((U^n)^2 V^{n+1}, (U^n)^2 V^{n+1}\right)' \\ Linear\ but\ still\ not\ sparse \end{gathered}$$

$$
\begin{cases}
\left(1-\Delta t\epsilon^2\Delta_s\right)u^{n+1}+\Delta t u^{n+1}-\Delta t\alpha(x)\left(u^n\right)^2 v^{n+1}-\Delta t\left(\tau\gamma\right)^{-1}v^{n+1}=u^n \\
\left(1-\Delta t\frac{D}{\tau}\Delta_s\right)v^{n+1}+\frac{\Delta t}{\tau}v^{n+1}+\Delta t\gamma\alpha(x)\left(u^n\right)^2 v^{n+1}-\Delta t\gamma u^{n+1}+\Delta t\frac{\beta\gamma}{\tau}u^{n+1}=\frac{\Delta t}{\tau}+v^n
\end{cases}
\tag{1.3}
$$

$$
\begin{cases}
\left(\frac{1}{\Delta t}-\epsilon^2\Delta_s+1\right)u^{n+1}-\left(\tau\gamma\right)^{-1}v^{n+1}-\alpha(x)\left(u^n\right)^2 v^{n+1}=\frac{u^n}{\Delta t} \\
\left(\frac{1}{\Delta t}-\frac{D}{\tau}\Delta_s+\frac{1}{\tau}\right)v^{n+1}+\left(-\gamma+\frac{\beta\gamma}{\tau}\right)u^{n+1}+\gamma\alpha(x)\left(u^n\right)^2 v^{n+1}=\frac{1}{\tau}+\frac{v^n}{\Delta t}
\end{cases}
\tag{1.4}
$$

$$
\begin{bmatrix} A_u & B_u \\ B_v & A_v \end{bmatrix} * \begin{bmatrix} \mathbf{U}^{n+1} \\ \mathbf{V}^{n+1} \end{bmatrix} + \begin{bmatrix} C_u\left(\mathbf{U}^n\right)\mathbf{V}^{n+1} \\ C_v\left(\mathbf{U}^n\right)\mathbf{V}^{n+1} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\tag{1.5}
$$

where

$$
\begin{aligned}
\left[A_u\right]_{i,j} &= \int_\Omega \left(\frac{1}{\Delta t}+1\right)\phi_j\phi_i + \int_\Omega \epsilon^2\nabla_s\phi_j\cdot\nabla_s\phi_j \\
\left[A_v\right]_{i,j} &= \int_\Omega \left(\frac{1}{\Delta t}+\frac{1}{\tau}\right)\phi_j\phi_i + \int_\Omega \frac{D}{\tau}\nabla_s\phi_j\cdot\nabla_s\phi_j \\
\left[B_u\right]_{i,j} &= \int_\Omega \left(\frac{-1}{\tau\gamma}\right)\phi_j\phi_i \\
\left[B_v\right]_{i,j} &= \int_\Omega \left(\frac{\beta\gamma}{\tau}-\gamma\right)\phi_j\phi_i \\
\left[C_u\left(u^n\right)\right]_{i,j} &= \int_\Omega -\alpha(x)\left(u^n\right)^2\phi_j\phi_i \\
\left[C_v\left(u^n\right)\right]_{i,j} &= \int_\Omega \gamma\alpha(x)\left(u^n\right)^2\phi_j\phi_i \\
\left[F_1\right]_i &= \int_\Omega \frac{u^n}{\Delta t}\phi_i \\
\left[F_2\right]_i &= \int_\Omega \frac{v^n}{\Delta t}\phi_i + \frac{1}{\tau}\phi_i
\end{aligned}
\tag{1.6}
$$

Initialization of the variables: in paper of year 2015 at page 5 is stated that:

"As initial conditions for our time-dependent computations, we take a small random perturbation to:

$$
U_0 \equiv \frac{1}{\gamma\beta} \qquad V_0 \equiv \frac{\tau\beta\gamma}{\tau+\beta^2\gamma},
\tag{1.7}
$$

"

Since you told me to initialize the system as in Fig 2(a) in 1.1, I decided upto now to set::

V0-eps-converted-to.pdf U0-eps-converted-to.pdf

Figure 1.1: Hyp. initialization from Fig.2(a), 2015 paper

$$
\begin{aligned}
\mathbf{U}^0 &= (U_0 - (x - 0.05))\, \mathbb{1}_{x<0.05} \text{ active ROPs solution at istant } t_k = 0 \\
\mathbf{V}^0 &= (V_0 - (x - 0.05))\, \mathbb{1}_{x<0.05} \text{ inactive ROPs solution at istant } t_k = 0
\end{aligned}
\tag{1.8}
$$

(sort of step function, but continuous, that near the left border assume the values U_0, V_0). This option results in `FreeFem++` as in Fig 1.2

Fig2.png

Figure 1.2: Fig.2 take from 2015 paper

3^{rd} **option NOT TO CONSIDER**: Innmplicit Euler + "semi-implicit" (decoupled eqs)
Non linear problem

$$
\begin{aligned}
(\mathbb{1} - \Delta t \mathscr{L})\, U^{n+1} - \ \Delta t \mathscr{N}\left(U^{n+1}\right) - U^n &= 0 \\
\begin{bmatrix} 1 - \Delta t\ \Delta_s \\ 1 - \Delta t\ \Delta_s \end{bmatrix} * \begin{bmatrix} u^{n+1} \\ v^{n+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} * \begin{bmatrix} v^{n+1} \\ u^{n+1} \end{bmatrix} - \ \Delta t \mathscr{N}\left(U^{n+1}\right) - U^n &= 0
\end{aligned}
\tag{1.9}
$$

Take explicit terms in the Non linear part and in the second contribute:

$$
\begin{aligned}
\begin{bmatrix} 1 - \Delta t\ \Delta_s \\ 1 - \Delta t\ \Delta_s \end{bmatrix} * \begin{bmatrix} u^{n+1} \\ v^{n+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} * \begin{bmatrix} v^{n} \\ u^{n} \end{bmatrix} - \ \Delta t \hat{\mathscr{N}}\left(U^{n+1}, U^n\right) - U^n &= 0 \\
&\mathit{where} \\
\hat{\mathscr{N}}\left(U^{n+1}, U^n\right) &= \begin{bmatrix} u^n v^n u^{n+1} \\ \left(u^n\right)^2 v^{n+1} \end{bmatrix}
\end{aligned}
\tag{1.10}
$$

Explicitly with all the coefficients

$$
\begin{cases}
u^{n+1} - \Delta t\ \epsilon^2\ \Delta u^{n+1} + \Delta t\ u^{n+1} - \Delta t\ \alpha(x) u^n v^n u^{n+1} = +\Delta t\ \ (\tau\gamma)^{-1} v^n + u^n \\
v^{n+1} - \Delta t\ \frac{D}{\tau} \Delta v^{n+1} + \frac{\Delta t}{\tau} v^{n+1} + \Delta t\ \gamma\alpha(x) \left(u^n\right)^2 v^{n+1} = \frac{\Delta t}{\tau} + \Delta t \gamma u^n - \Delta t \beta\gamma u^n + v^n
\end{cases}
\tag{1.11}
$$