Chapter 1

Formule varie

$$
\begin{cases}
\partial_t u = \epsilon^2 \Delta_s u + \alpha(x) u^2 v - u + (\tau\gamma)^{-1} v \\
\partial_t v = \frac{D}{\tau} \Delta_s v - \frac{1}{\tau} v + \frac{1}{\tau} - \gamma \left(\alpha(x) u^2 v - u \right) - \frac{\beta\gamma}{\tau} u
\end{cases}
$$
$$
\begin{cases}
\partial_t u = \tilde{D_1} \Delta_s u + \tilde{a_1} u + \tilde{b_1} v + \tilde{c_1} u^2 v \\
\partial_t v = \tilde{D_2} \Delta_s v + \tilde{a_2} v + \tilde{b_2} u + \tilde{c_2} u^2 v + \frac{1}{\tau}
\end{cases}
\tag{1.1}
$$

where

$$
\begin{aligned}
\tilde{D_1} &= \epsilon^2 & \tilde{D_2} &= \frac{D}{\tau} \\
\tilde{a_1} &= -1 & \tilde{a_2} &= -\frac{1}{\tau} \\
\tilde{b_1} &= \frac{1}{\tau\gamma} & \tilde{b_2} = \gamma - \frac{\beta\gamma}{\tau} &= \gamma \left(1 - \frac{\beta}{\tau} \right) \\
\tilde{c_1} &= \alpha(x) & \tilde{c_2} &= -\gamma\alpha(x)
\end{aligned}
\tag{1.2}
$$

Sintetic Operators without coefficients

$$
\mathscr{L} = \begin{bmatrix} \Delta_s + \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \Delta_s + \mathbb{1} \end{bmatrix}
$$

$$
\mathscr{N}(U) = \begin{bmatrix} u^2 v \\ u^2 v \end{bmatrix}
$$

1.1 Implicit Euler + Newton's method (complete)

$$
\begin{aligned}
U &= (u, v) \\
U^{n+1} = U^n + \Delta t \mathscr{L} U^{n+1} + \ \Delta t \mathscr{N}\left(U^{n+1}\right) \Leftrightarrow G(U) &= (\mathbf{1} - \Delta t \mathscr{L}) U - \Delta t \mathscr{N}(U) - U^n = 0 \\
&\mathit{Newton\ method...\ full\ matrix}
\end{aligned}
\tag{1.3}
$$

$$
\begin{cases}
\frac{u^{k+1}}{\Delta t} - \epsilon^2 \Delta_s u^{k+1} - \alpha(x) u^{k^2} v^{k+1} - \alpha(x) 2 u^k v^k u^{k+1} + \alpha(x) 2 u^{k^2} v^k + u^{k+1} - \frac{1}{\tau\gamma} v^{k+1} = \frac{u^n}{\Delta t} \\
\frac{v^{k+1}}{\Delta t} - \frac{D}{\tau} \Delta_s v^{k+1} + \gamma\alpha(x) u^{k^2} v^{k+1} + \gamma\alpha(x) 2 u^k v^k u^{k+1} - \gamma\alpha(x) 2 u^{k^2} v^k - \gamma u^{k+1} + \frac{\beta\gamma}{\tau} u^{k+1} + \frac{1}{\tau} v^{k+1} = \frac{v^n}{\Delta t} + \frac{1}{\tau}
\end{cases}
\tag{1.4}
$$

1.2 Implicit Euler + semi-implicit

$$
\begin{aligned}
U &= (u, v) \\
\left(\mathbb{1} - \Delta t \mathscr{L}\right) U^{n+1} - \ \Delta t \mathscr{N}\left(U^{n+1}, U^n\right) - U^n &= 0 \\
&\mathit{with} \\
\mathscr{N}\left(U^{n+1}, U^n\right) &= \left(\left(U^n\right)^2 V^{n+1}, \left(U^n\right)^2 V^{n+1}\right)' \\
\mathit{Linear\ but\ still\ not\ sparse}
\end{aligned}
$$

$$
\begin{cases}
\left(1 - \Delta t \epsilon^2 \Delta_s\right) u^{n+1} + \Delta t u^{n+1} - \Delta t \alpha(x) \left(u^n\right)^2 v^{n+1} - \Delta t \left(\tau\gamma\right)^{-1} v^{n+1} = u^n \\
\left(1 - \Delta t \frac{D}{\tau} \Delta_s\right) v^{n+1} + \frac{\Delta t}{\tau} v^{n+1} + \Delta t \gamma \alpha(x) \left(u^n\right)^2 v^{n+1} - \Delta t \gamma u^{n+1} + \Delta t \frac{\beta\gamma}{\tau} u^{n+1} = \frac{\Delta t}{\tau} + v^n
\end{cases}
\tag{1.5}
$$

$$
\begin{cases}
\left(\frac{1}{\Delta t} - \epsilon^2 \Delta_s + 1\right) u^{n+1} - \left(\tau\gamma\right)^{-1} v^{n+1} - \alpha(x) \left(u^n\right)^2 v^{n+1} = \frac{u^n}{\Delta t} \\
\left(\frac{1}{\Delta t} - \frac{D}{\tau} \Delta_s + \frac{1}{\tau}\right) v^{n+1} + \left(-\gamma + \frac{\beta\gamma}{\tau}\right) u^{n+1} + \gamma \alpha(x) \left(u^n\right)^2 v^{n+1} = \frac{1}{\tau} + \frac{v^n}{\Delta t}
\end{cases}
\tag{1.6}
$$

$$
\begin{bmatrix} A_u & B_u \\ B_v & A_v \end{bmatrix} * \begin{bmatrix} \mathbf{U}^{n+1} \\ \mathbf{V}^{n+1} \end{bmatrix} + \begin{bmatrix} C_u \left(\mathbf{U}^n\right) \mathbf{V}^{n+1} \\ C_v \left(\mathbf{U}^n\right) \mathbf{V}^{n+1} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\tag{1.7}
$$

where

$$
\begin{aligned}
[A_u]_{i,j} &= \int_\Omega \left(\frac{1}{\Delta t} + 1\right) \phi_j \phi_i + \int_\Omega \epsilon^2 \nabla_s \phi_j \cdot \nabla_s \phi_j \\
[A_v]_{i,j} &= \int_\Omega \left(\frac{1}{\Delta t} + \frac{1}{\tau}\right) \phi_j \phi_i + \int_\Omega \frac{D}{\tau} \nabla_s \phi_j \cdot \nabla_s \phi_j \\
[B_u]_{i,j} &= \int_\Omega \left(\frac{-1}{\tau\gamma}\right) \phi_j \phi_i \\
[B_v]_{i,j} &= \int_\Omega \left(\frac{\beta\gamma}{\tau} - \gamma\right) \phi_j \phi_i \\
[C_u(u^n)]_{i,j} &= \int_\Omega -\alpha(x) (u^n)^2 \phi_j \phi_i \\
[C_v(u^n)]_{i,j} &= \int_\Omega \gamma\alpha(x) (u^n)^2 \phi_j \phi_i \\
[F_1]_i &= \int_\Omega \frac{u^n}{\Delta t} \phi_i \\
[F_2]_i &= \int_\Omega \frac{v^n}{\Delta t} \phi_i + \frac{1}{\tau} \phi_i
\end{aligned}
\tag{1.8}
$$

1.2.1 Initialization of the variables

In paper of year 2015 at page 5 is stated that:

"As initial conditions for our time-dependent computations, we take a small random perturbation to:

$$U_0 \equiv \frac{1}{\gamma\beta} \qquad V_0 \equiv \frac{\tau\beta\gamma}{\tau + \beta^2\gamma}, \tag{1.9}$$

"

I initialized the solutions therefore in FreeFem++ code as:

```
//Initialization: random perturbation or prev res
real u0 = 1./(gamma*beta);
real v0 = tau*beta*gamma/(tau + beta^2*gamma);
// ...
{
  U = u0;
  V = v0;
  srandom(seed);
  // ranreal 1 generates a random number in [0,1]
    for(int ii = 0; ii < Xh.ndof; ii++)
    {
      U[][ii] += randreal1() / 3e5;
      V[][ii] += randreal1() / 3e5;
    }
}
```

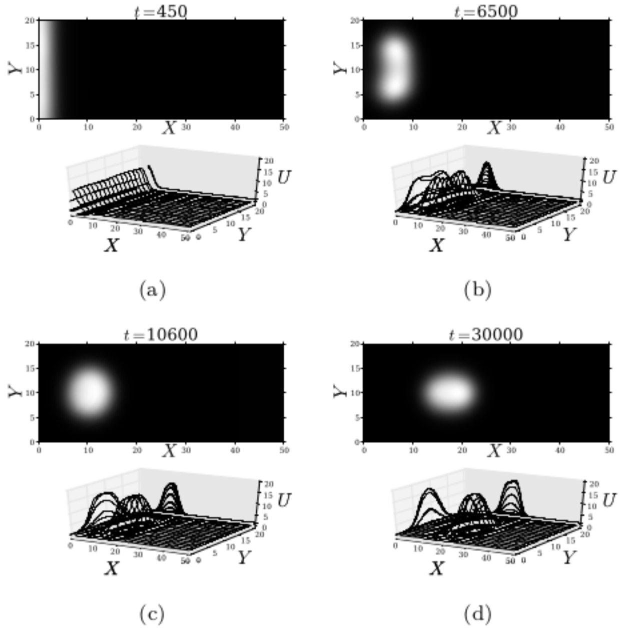

FIG. 2. *Snapshots of a traveling front breaking up into a slowly traveling spot that gets pinned after a long time.* (a) *Front formed at the boundary.* (b) *Break-up into a peanut-shaped form.* (c) *Traveling spot.* (d) *Final pinned spot. Original parameter set one as given in Table* 1 *with* $k_2 = 0.1$. *Notice that the spot drifts very slowly in time.*

Figure 1.1: Fig.2 take from 2015 paper

1.2.2 Adimensionalization

QUESTA PARTE IN RELTÀ NON SERVE PERCHÈ GIÀ EQUAZIONE (1.1) DEL PAPER, CIOÈ (1.1) È ADIMENSIONALIZZATA? DICEVA CHE CHIARAMENTE LO È LA PRIMA, LA SECONDA HA DI CONSEGUENZA QUEL TAU E che differenzia i tempi di diffusione (molto lenti perchè proporzionali a 1/coeff diffusione)

- Physical reference scales (scale di riferimento) L_x,L_y,U_0,V_0, T.
- Dimensions M(mass), L (length), t(time).

The system seems to be not consistent regarding the dimensions, because it is stated in the paper that all coefficients are already scaled, dimensionless, and also space derivative are already scaled with respect to the aspect ratio s (so dimentionless):

$$x^* = \frac{x}{L_x} \quad y^* = \frac{y}{L_y} \; \textit{space variables in the square} \, [0,1]^2$$

$$\Delta = \partial_{xx} + \partial_{yy} = \frac{1}{L_x^2}\partial_{x^*x^*} + \frac{1}{L_y^2}\partial_{y^*y^*} \, \textit{as if variables are rescaled wtr 2 different characteristic lengths}$$

$$\Delta_s = L_x^2 \left(\frac{1}{L_x^2}\partial_{x^*x^*} + \frac{1}{L_y^2}\partial_{y^*y^*} \right) = L_x^2 \Delta$$

For dimension consistency

$$[\alpha(x)] = \left(\frac{L^2}{M}\right)^2$$

maybe, but still time derivative add time that doesn't appear in the dimensionless coefficients such as $\epsilon^2, 1, \frac{1}{\tau\gamma}$

(trascuro questa non consistenza)

Define the dimensionless variables $u^* = \frac{u}{U_0}, v^* = \frac{v}{V_0}$. Being $\partial_{t*} = \frac{1}{T}\partial_t$, we rewrite (1.1) in adimensional form:

$$\begin{cases} \partial_t u = \epsilon^2 \Delta_s u + \alpha(x) u^2 v - u + (\tau\gamma)^{-1} v \\ \partial_t v = \frac{D}{\tau}\Delta_s v - \frac{1}{\tau} v + \frac{1}{\tau} - \gamma\left(\alpha(x)u^2 v - u\right) - \frac{\beta\gamma}{\tau} u \end{cases}$$

$$\begin{cases} \frac{U_0}{T}\partial_{t*} u^* = \epsilon^2 U^0 \Delta_s u^* + \alpha(x) U_0^2 V_0 u^{*2} v^* - U_0 u^* + V_0 (\tau\gamma)^{-1} v^* \\ \frac{V_0}{T}\partial_{t*} v^* = \frac{DV_0}{\tau}\Delta_s v^* - \frac{V_0}{\tau} v^* + \frac{1}{\tau} - \gamma\left(\alpha(x) U_0^2 V_0 u^{*2} v^* - U_0 u^*\right) - \frac{\beta\gamma U_0}{\tau} u^* \end{cases}$$

$$\begin{cases} \partial_{t*} u^* = \epsilon^2 T \Delta_s u^* + \alpha(x) U_0 V_0 T u^{*2} v^* - T u^* + \frac{V_0}{U_0}\frac{T}{\tau\gamma} v^* \\ \partial_{t*} v^* = \frac{DT}{\tau}\Delta_s v^* - \frac{T}{\tau} v^* + \frac{T}{\tau V_0} - \gamma T\left(\alpha(x) U_0^2 u^{*2} v^* - \frac{U_0}{V_0} u^*\right) - \frac{\beta\gamma T}{\tau}\frac{U_0}{V_0} u^* \end{cases}$$

Omitting the star notation * (for example, u means actually u^*), we obtain the adimensional equation:

$$\begin{cases} \partial_t u = \tilde{D_1}\Delta_s u + \tilde{a_1} u + \tilde{b_1} v + \tilde{c_1} u^2 v \\ \partial_t v = \tilde{D_2}\Delta_s v + \tilde{a_2} v + \tilde{b_2} u + \tilde{c_2} u^2 v + \frac{T}{\tau V_0} \end{cases}$$

where

$$\tilde{D_1} = \epsilon^2 T \qquad \tilde{D_2} = \frac{DT}{\tau}$$

$$\tilde{a_1} = -T \qquad \tilde{a_2} = -\frac{T}{\tau}$$

$$\tilde{b_1} = \frac{V_0}{U_0}\frac{T}{\tau\gamma} \qquad \tilde{b_2} = \gamma T\frac{U_0}{V_0} - \frac{\beta\gamma T}{\tau}\frac{U_0}{V_0} = \gamma T\frac{U_0}{V_0}\left(1 - \frac{\beta}{\tau}\right)$$

$$\tilde{c_1} = \alpha(x) U_0 V_0 T \qquad \tilde{c_2} = -\gamma T \alpha(x) U_0^2$$

1.2.3 Test of stability of the solution

Rewriting (1.1) schematically as:

$$M\mathring{U} = G(U) = LU + N(U)$$

we can find stationary solutions solving the non linear problem $G(U) = LU + N(U) = \mathscr{F} = 0$ with Newtons' method. Given an initial guess $U^0 = [u^0, v^0]'$, $\forall k$ upto convergence solve:

$$\begin{cases} D_U G(U)|_{U^k}(\delta U) = \mathscr{F} - G\left(U^k\right) \Leftrightarrow D_{(u,v)} G((u,v))|_{(u^k,v^k)}((\delta u, \delta v)) = \mathscr{F} - G\left(\left(u^k, v^k\right)\right) \\ U^{k+1} = U^k + \delta U \Leftrightarrow \left(u^{k+1}, v^{k+1}\right) = \left(u^k, v^k\right)(\delta u, \delta v) \end{cases}$$

$$G(U) = \begin{bmatrix} \tilde{D}_1 \Delta_s + \tilde{a}_1 & \tilde{b_1} \\ \tilde{b_2} & \tilde{D}_2 \Delta_s + \tilde{a}_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \tilde{c}_1 u^2 v \\ \tilde{c}_2 u^2 v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\tau} \end{bmatrix}$$

$$D_U G(U) = \begin{bmatrix} \tilde{D}_1 \Delta_s + \tilde{a}_1 & \tilde{b_1} \\ \tilde{b_2} & \tilde{D}_2 \Delta_s + \tilde{a}_2 \end{bmatrix} + D_U N(U)$$

$$\begin{aligned} D_U N(U)|_U(\delta U) &= D_{(u,v)} N((u,v))|_{(u^k,v^k)}((\delta u, \delta v)) \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \begin{bmatrix} \tilde{c_1}(u + \epsilon \delta u)^2 (v + \epsilon \delta v) - \tilde{c_1} u^2 v \\ \tilde{c_2}(u + \epsilon \delta u)^2 (v + \epsilon \delta v) - \tilde{c_2} u^2 v \end{bmatrix} = \ldots = \begin{bmatrix} \tilde{c_1} 2uv\delta u + \tilde{c_1} u^2 \delta v \\ \tilde{c_2} 2uv\delta u + \tilde{c_2} u^2 \delta v \end{bmatrix} \end{aligned}$$

where the coefficients are one int eq.(1.2). Therefore Newton's method written explicitly:

$$\begin{aligned} &\begin{bmatrix} \tilde{D}_1 \Delta_s + \tilde{a}_1 & \tilde{b_1} \\ \tilde{b_2} & \tilde{D}_2 \Delta_s + \tilde{a}_2 \end{bmatrix} \begin{bmatrix} u^{k+1} - u^k \\ v^{k+1} - v^k \end{bmatrix} + \begin{bmatrix} \tilde{c_1} 2u^k v^k \left(u^{k+1} - u^k\right) + \tilde{c_1} {u^k}^2 \left(v^{k+1} - v^k\right) \\ \tilde{c_2} 2u^k v^k \left(u^{k+1} - u^k\right) + \tilde{c_2} {u^k}^2 \left(v^{k+1} - v^k\right) \end{bmatrix} = \\ &\quad - \begin{bmatrix} \tilde{D}_1 \Delta_s + \tilde{a}_1 & \tilde{b_1} \\ \tilde{b_2} & \tilde{D}_2 \Delta_s + \tilde{a}_2 \end{bmatrix} \begin{bmatrix} u^k \\ v^k \end{bmatrix} - \begin{bmatrix} \tilde{c_1} {u^k}^2 v^k \\ \tilde{c_2} {u^k}^2 v^k \end{bmatrix} \\ &\Leftrightarrow \begin{bmatrix} \tilde{D}_1 \Delta_s + \tilde{a}_1 & \tilde{b_1} \\ \tilde{b_2} & \tilde{D}_2 \Delta_s + \tilde{a}_2 \end{bmatrix} \begin{bmatrix} u^{k+1} \\ v^{k+1} \end{bmatrix} + \begin{bmatrix} \tilde{c_1} 2u^k v^k u^{k+1} + \tilde{c_1} {u^k}^2 v^{k+1} \\ \tilde{c_2} 2u^k v^k u^{k+1} + \tilde{c_2} {u^k}^2 v^{k+1} \end{bmatrix} = \begin{bmatrix} 2\tilde{c_1} {u^k}^2 v^k \\ 2\tilde{c_2} {u^k}^2 v^k \end{bmatrix} \end{aligned}$$

After writing it in weak formulation, multiplying for proper test function and writing the solutions as linear combinations of the elements of the space's basis:

$$\begin{bmatrix} A_u & B_u \\ B_v & A_v \end{bmatrix} \begin{bmatrix} \mathbf{U}^{k+1} \\ \mathbf{V}^{k+1} \end{bmatrix} + \begin{bmatrix} C_{uu} \mathbf{U}^{k+1} + C_{uv} \mathbf{V}^{k+1} \\ C_{vu} \mathbf{U}^{k+1} + C_{vv} \mathbf{V}^{k+1} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{1.10}$$

where

$$
\begin{aligned}
\left[A_u\right]_{i,j} &= \int_\Omega \left(-\tilde{D_1}\nabla_s\phi_j\cdot\nabla_s\phi_i + \tilde{a_1}\phi_j\phi_i\right)\\
\left[A_v\right]_{i,j} &= \int_\Omega \left(-\tilde{D_2}\nabla_s\phi_j\cdot\nabla_s\phi_i + \tilde{a_2}\phi_j\phi_i\right)\\
\left[B_u\right]_{i,j} &= \int_\Omega \left(\tilde{b_1}\phi_j\phi_i\right)\\
\left[B_v\right]_{i,j} &= \int_\Omega \left(\tilde{b_2}\phi_j\phi_i\right)\\
\left[C_{uu}\right]_{i,j} &= \int_\Omega \left(2\tilde{c_1}u^0v^0\phi_j\phi_i\right)\\
\left[C_{uv}\right]_{i,j} &= \int_\Omega \left(\tilde{c_1}u^0u^0\phi_j\phi_i\right)\\
\left[C_{vu}\right]_{i,j} &= \int_\Omega \left(2\tilde{c_2}u^0v^0\phi_j\phi_i\right)\\
\left[C_{vv}\right]_{i,j} &= \int_\Omega \left(\tilde{c_2}u^0u^0\phi_j\phi_i\right)\\
\left[F_1\right]_{i} &= \int_\Omega \left(2\tilde{c_1}u^0u^0v^0\phi_i\right)\\
\left[F_2\right]_{i} &= \int_\Omega \left(2\tilde{c_2}u^0u^0v^0\phi_i\right)
\end{aligned}
\tag{1.11}
$$

Check Jacobian found for Newton in test of the stability

$$
\begin{aligned}
J\left(U^*\right)dU &= \frac{G\left(U^*+\epsilon dU\right)-G\left(U^*-\epsilon dU\right)}{2\epsilon}\\
dU &= \begin{bmatrix} du\\ dv\end{bmatrix}, U^* = \begin{bmatrix} u^0\\ v^0\end{bmatrix}\\
J\left(U^*\right)du &= D_U G\left(U\right)|_{U*}dU = \begin{bmatrix} \tilde{D_1}\Delta_s+\tilde{a_1} & \tilde{b_1}\\ \tilde{b_2} & \tilde{D_2}\Delta_s+\tilde{a_2}\end{bmatrix}\begin{bmatrix} du\\ dv\end{bmatrix} + \begin{bmatrix} \tilde{c_1}2u^0v^0du+\tilde{c_1}{u^0}^2dv\\ \tilde{c_2}2u^0v^0du+\tilde{c_2}{u^0}^2dv\end{bmatrix}\\
G\left(U^*+\epsilon dU\right) &= \begin{bmatrix} \tilde{D_1}\Delta_s+\tilde{a_1} & \tilde{b_1}\\ \tilde{b_2} & \tilde{D_2}\Delta_s+\tilde{a_2}\end{bmatrix}\begin{bmatrix} U^*+\epsilon du\\ V^*+\epsilon dv\end{bmatrix} + \begin{bmatrix} \tilde{c_1}\left(U^*+\epsilon du\right)^2\left(V^*+\epsilon dv\right)\\ \tilde{c_2}\left(U^*+\epsilon du\right)^2\left(V^*+\epsilon dv\right)\end{bmatrix} + \begin{bmatrix} 0\\ \frac{1}{\tau}\end{bmatrix}
\end{aligned}
$$

1.3 Implicit Euler + "semi-implicit" (decoupled eqs)

NOT TO CONSIDER Non linear problem

$$
\begin{aligned}
\left(\mathbb{1}-\Delta t\mathscr{L}\right)U^{n+1} - \ \Delta t\mathscr{N}\left(U^{n+1}\right) - U^n = 0\\
\begin{bmatrix} 1-\Delta t\ \Delta_s\\ 1-\Delta t\ \Delta_s\end{bmatrix} * \begin{bmatrix} u^{n+1}\\ v^{n+1}\end{bmatrix} + \begin{bmatrix} 1\\ 1\end{bmatrix} * \begin{bmatrix} v^{n+1}\\ u^{n+1}\end{bmatrix} - \ \Delta t\mathscr{N}\left(U^{n+1}\right) - U^n = 0
\end{aligned}
\tag{1.12}
$$

Take explicit terms in the Non linear part and in the second contribute:

$$\begin{bmatrix} 1 - \Delta t \ \Delta_s \\ 1 - \Delta t \ \Delta_s \end{bmatrix} * \begin{bmatrix} u^{n+1} \\ v^{n+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} * \begin{bmatrix} v^n \\ u^n \end{bmatrix} - \ \Delta t \hat{\mathscr{N}} \left(U^{n+1}, U^n \right) - U^n = 0$$

$$where \qquad \hat{\mathscr{N}} \left(U^{n+1}, U^n \right) = \begin{bmatrix} u^n v^n u^{n+1} \\ \left(u^n\right)^2 v^{n+1} \end{bmatrix} \tag{1.13}$$

Explicitly with all the coefficients

$$\begin{cases} u^{n+1} - \Delta t \ \epsilon^2 \ \Delta u^{n+1} + \Delta t \ u^{n+1} - \Delta t \ \alpha(x) u^n v^n u^{n+1} = +\Delta t \ \ (\tau\gamma)^{-1} v^n + u^n \\ v^{n+1} - \Delta t \ \frac{D}{\tau} \Delta v^{n+1} + \frac{\Delta t}{\tau} v^{n+1} + \Delta t \ \gamma\alpha(x) \left(u^n\right)^2 v^{n+1} = \frac{\Delta t}{\tau} + \Delta t \gamma u^n - \Delta t \beta \gamma u^n + v^n \end{cases} \tag{1.14}$$