Chapter 1

Formule varie

This is the matlab equation taken from the paper intra.2, with a proper rescaling coefficients.

$$\begin{cases} \partial_t u = & \epsilon^2 \Delta_s u + \alpha(x) u^2 v - u + (\tau\gamma)^{-1} v \\ \partial_t v = & \frac{D}{\tau} \Delta_s v - \frac{1}{\tau} v + \frac{1}{\tau} - \gamma \left(\alpha(x) u^2 v - u\right) - \frac{\beta\gamma}{\tau} u \end{cases} \tag{1.1}$$

This is the equation extrapolated from the Matlab code (it differs from the one before for the coefficients used, probably it is the real one not scaled):

$$\begin{cases} \partial_t u = & D_1 \Delta_s u + \alpha(x) u^2 v - (c + r)\, u + k_1 v \\ \partial_t v = & D_2 \Delta_s v - \alpha(x) u^2 v - k_1 v + c u + b \end{cases} \tag{1.2}$$

Both can be rewritten sintethically as

$$\begin{cases} \partial_t u = & \tilde{D_1} \Delta_s u + \tilde{a_1} u + \tilde{b_1} v + \tilde{c_1} u^2 v \\ \partial_t v = & \tilde{D_2} \Delta_s v + \tilde{a_2} v + \tilde{b_2} u + \tilde{c_2} u^2 v + f_2 \end{cases} \tag{1.3}$$

where for the first case, the "PAPER" CASE:

$$\begin{aligned} \tilde{D_1} &= \epsilon^2 & \tilde{D_2} &= \frac{D}{\tau} \\ \tilde{a_1} &= -1 & \tilde{a_2} &= -\frac{1}{\tau} \\ \tilde{b_1} &= \frac{1}{\tau\gamma} & \tilde{b_2} = \gamma - \frac{\beta\gamma}{\tau} &= \gamma\left(1 - \frac{\beta}{\tau}\right) \\ \tilde{c_1} &= \alpha(x) & \tilde{c_2} &= -\gamma\alpha(x) \\ & - & f_2 &= \frac{1}{\tau} \end{aligned} \tag{1.4}$$

whereas for the second system of equation, the "MATLAB" CASE:

$$
\begin{aligned}
\tilde{D_1} &= D_1 & \tilde{D_2} &= D_2 \\
\tilde{a_1} &= -(c+r) & \tilde{a_2} &= -k_1 \\
\tilde{b_1} &= k_1 & \tilde{b_2} &= c \\
\tilde{c_1} &= \alpha(x) & \tilde{c_2} &= -\alpha(x) \\
&- & f_2 &= b
\end{aligned}
\tag{1.5}
$$

Sintetic Operators without coefficients

$$
\mathscr{L} = \begin{bmatrix} \Delta_s + \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \Delta_s + \mathbb{1} \end{bmatrix}
$$

$$
\mathscr{N}(U) = \begin{bmatrix} u^2 v \\ u^2 v \end{bmatrix}
$$

1.1 Test of stability of the solution

Rewriting (1.3) schematically as:

$$
\mathring{U} = G(U) = LU + N(U)
$$

we can find stationary solutions solving the non linear problem $G(U) = LU + N(U) = \mathscr{F} = 0$ with Newtons' method. Given an initial guess $U^0 = [u^0, v^0]'$, $\forall k$ upto convergence solve:

$$
\begin{cases}
D_U G(U)|_{U^k}(\delta U) = \mathscr{F} - G\left(U^k\right) \Leftrightarrow D_{(u,v)} G((u,v))|_{\left(u^k, v^k\right)}((\delta u, \delta v)) = \mathscr{F} - G\left(\left(u^k, v^k\right)\right) \\
U^{k+1} = U^k + \delta U \Leftrightarrow \left(u^{k+1}, v^{k+1}\right) = \left(u^k, v^k\right)(\delta u, \delta v)
\end{cases}
$$

$$
\begin{aligned}
G(U) &= \begin{bmatrix} \tilde{D_1}\Delta_s + \tilde{a_1} & \tilde{b_1} \\ \tilde{b_2} & \tilde{D_2}\Delta_s + \tilde{a_2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \tilde{c_1} u^2 v \\ \tilde{c_2} u^2 v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\tau} \end{bmatrix} \\
D_U G(U) &= \begin{bmatrix} \tilde{D_1}\Delta_s + \tilde{a_1} & \tilde{b_1} \\ \tilde{b_2} & \tilde{D_2}\Delta_s + \tilde{a_2} \end{bmatrix} + D_U N(U)
\end{aligned}
\tag{1.6}
$$

$$
\begin{aligned}
D_U N(U)|_U(\delta U) &= D_{(u,v)} N((u,v))|_{\left(u^k, v^k\right)}((\delta u, \delta v)) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \begin{bmatrix} \tilde{c_1}(u + \epsilon \delta u)^2 (v + \epsilon \delta v) - \tilde{c_1} u^2 v \\ \tilde{c_2}(u + \epsilon \delta u)^2 (v + \epsilon \delta v) - \tilde{c_2} u^2 v \end{bmatrix} = \ldots = \begin{bmatrix} \tilde{c_1} 2uv\delta u + \tilde{c_1} u^2 \delta v \\ \tilde{c_2} 2uv\delta u + \tilde{c_2} u^2 \delta v \end{bmatrix}
\end{aligned}
$$

where the coefficients are one int eq.(1.4).

$$\begin{aligned}
&\begin{bmatrix} \tilde{D_1}\Delta_s + \tilde{a_1} & \tilde{b_1} \\ \tilde{b_2} & \tilde{D_2}\Delta_s + \tilde{a_2} \end{bmatrix} \begin{bmatrix} \delta u \\ \delta v \end{bmatrix} + \begin{bmatrix} \tilde{c_1} 2u^k v^k \delta u + \tilde{c_1} {u^k}^2 \delta v \\ \tilde{c_2} 2u^k v^k \delta u + \tilde{c_2} {u^k}^2 \delta v \end{bmatrix} = \\
&\qquad - \begin{bmatrix} \tilde{D_1}\Delta_s + \tilde{a_1} & \tilde{b_1} \\ \tilde{b_2} & \tilde{D_2}\Delta_s + \tilde{a_2} \end{bmatrix} \begin{bmatrix} u^k \\ v^k \end{bmatrix} - \begin{bmatrix} \tilde{c_1} {u^k}^2 v^k \\ \tilde{c_2} {u^k}^2 v^k \end{bmatrix}, \begin{bmatrix} u^{k+1} \\ v^{k+1} \end{bmatrix} = \begin{bmatrix} u^k \\ v^k \end{bmatrix} + \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}
\end{aligned} \tag{1.7}$$

Therefore Newton's method written explicitly:

$$\begin{aligned}
&\begin{bmatrix} \tilde{D_1}\Delta_s + \tilde{a_1} & \tilde{b_1} \\ \tilde{b_2} & \tilde{D_2}\Delta_s + \tilde{a_2} \end{bmatrix} \begin{bmatrix} u^{k+1} - u^k \\ v^{k+1} - v^k \end{bmatrix} + \begin{bmatrix} \tilde{c_1} 2u^k v^k \left(u^{k+1} - u^k\right) + \tilde{c_1} {u^k}^2 \left(v^{k+1} - v^k\right) \\ \tilde{c_2} 2u^k v^k \left(u^{k+1} - u^k\right) + \tilde{c_2} {u^k}^2 \left(v^{k+1} - v^k\right) \end{bmatrix} = \\
&\qquad - \begin{bmatrix} \tilde{D_1}\Delta_s + \tilde{a_1} & \tilde{b_1} \\ \tilde{b_2} & \tilde{D_2}\Delta_s + \tilde{a_2} \end{bmatrix} \begin{bmatrix} u^k \\ v^k \end{bmatrix} - \begin{bmatrix} \tilde{c_1} {u^k}^2 v^k \\ \tilde{c_2} {u^k}^2 v^k \end{bmatrix} \\
&\Leftrightarrow \begin{bmatrix} \tilde{D_1}\Delta_s + \tilde{a_1} & \tilde{b_1} \\ \tilde{b_2} & \tilde{D_2}\Delta_s + \tilde{a_2} \end{bmatrix} \begin{bmatrix} u^{k+1} \\ v^{k+1} \end{bmatrix} + \begin{bmatrix} \tilde{c_1} 2u^k v^k u^{k+1} + \tilde{c_1} {u^k}^2 v^{k+1} \\ \tilde{c_2} 2u^k v^k u^{k+1} + \tilde{c_2} {u^k}^2 v^{k+1} \end{bmatrix} = \begin{bmatrix} 2\tilde{c_1} {u^k}^2 v^k \\ 2\tilde{c_2} {u^k}^2 v^k \end{bmatrix}
\end{aligned} \tag{1.8}$$

After writing it in weak formulation, multiplying for proper test function and writing the solutions as linear combinations of the elements of the space's basis:

$$\begin{bmatrix} A_u & B_u \\ B_v & A_v \end{bmatrix} \begin{bmatrix} \mathbf{U}^{k+1} \\ \mathbf{V}^{k+1} \end{bmatrix} + \begin{bmatrix} C_{uu}\mathbf{U}^{k+1} + C_{uv}\mathbf{V}^{k+1} \\ C_{vu}\mathbf{U}^{k+1} + C_{vv}\mathbf{V}^{k+1} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{1.9}$$

where

$$
\begin{aligned}
[A_u]_{i,j} &= \int_\Omega \left(-\tilde{D_1}\nabla_s\phi_j\cdot\nabla_s\phi_i + \tilde{a_1}\phi_j\phi_i\right)\\
[A_v]_{i,j} &= \int_\Omega \left(-\tilde{D_2}\nabla_s\phi_j\cdot\nabla_s\phi_i + \tilde{a_2}\phi_j\phi_i\right)\\
[B_u]_{i,j} &= \int_\Omega \left(\tilde{b_1}\phi_j\phi_i\right)\\
[B_v]_{i,j} &= \int_\Omega \left(\tilde{b_2}\phi_j\phi_i\right)\\
[C_{uu}]_{i,j} &= \int_\Omega \left(2\tilde{c_1}u^0v^0\phi_j\phi_i\right)\\
[C_{uv}]_{i,j} &= \int_\Omega \left(\tilde{c_1}u^0u^0\phi_j\phi_i\right)\\
[C_{vu}]_{i,j} &= \int_\Omega \left(2\tilde{c_2}u^0v^0\phi_j\phi_i\right)\\
[C_{vv}]_{i,j} &= \int_\Omega \left(\tilde{c_2}u^0u^0\phi_j\phi_i\right)\\
[F_1]_i &= \int_\Omega \left(2\tilde{c_1}u^0u^0v^0\phi_i\right)\\
[F_2]_i &= \int_\Omega \left(2\tilde{c_2}u^0u^0v^0\phi_i\right)
\end{aligned}
\tag{1.10}
$$

Check Jacobian found for Newton in test of the stability

$$
\begin{aligned}
J(U^*)\,dU &= \frac{G(U^*+\epsilon dU) - G(U^*-\epsilon dU)}{2\epsilon}\\
dU &= \begin{bmatrix} du \\ dv \end{bmatrix}, U^* = \begin{bmatrix} u^0 \\ v^0 \end{bmatrix}\\
J(U^*)\,du = D_U G(U)\,|_{U^*} dU &= \begin{bmatrix} \tilde{D_1}\Delta_s + \tilde{a_1} & \tilde{b_1} \\ \tilde{b_2} & \tilde{D_2}\Delta_s + \tilde{a_2} \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix} + \begin{bmatrix} \tilde{c_1}2u^0v^0du + \tilde{c_1}{u^0}^2dv \\ \tilde{c_2}2u^0v^0du + \tilde{c_2}{u^0}^2dv \end{bmatrix}\\
G(U^*+\epsilon dU) &= \begin{bmatrix} \tilde{D_1}\Delta_s + \tilde{a_1} & \tilde{b_1} \\ \tilde{b_2} & \tilde{D_2}\Delta_s + \tilde{a_2} \end{bmatrix} \begin{bmatrix} U^*+\epsilon du \\ V^*+\epsilon dv \end{bmatrix} + \begin{bmatrix} \tilde{c_1}(U^*+\epsilon du)^2(V^*+\epsilon dv) \\ \tilde{c_2}(U^*+\epsilon du)^2(V^*+\epsilon dv) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\tau} \end{bmatrix}
\end{aligned}
$$

1.2 Implicit Euler + Newton's method (Fully implicit)

The systems of equations (1.3) can be rewritten synthetically as:

$$
\mathring{U} = G(U) \;\; \textit{where}\; U = (u, v) \tag{1.11}
$$

Use implicit Euler method to discretize the time derivative

$$\frac{U^{n+1}-U^{n}}{\Delta t}=\mathscr{L}U^{n+1}+\ \mathscr{N}\left(U^{n+1}\right)=G\left(U^{n+1}\right) \tag{1.12}$$

Given $U^0 = U^k$ at each time step is necessary to apply Newton's method to the non linear problem:

$$\begin{aligned}&Find\ U=[u,v]\ \ s.t.\\ &F\left(U\right)=\frac{\mathbf{1}}{\Delta t}U-G\left(U\right)-\frac{\mathbf{1}}{\Delta t}U^{n}=\left(\frac{\mathbf{1}}{\Delta t}-\mathscr{L}\right)U-\mathscr{N}\left(U\right)-\frac{\mathbf{1}}{\Delta t}U^{n}=0\end{aligned} \tag{1.13}$$

and the solution U obtained at convergence is U^{n+1}. (Newton 's method would have a full matrix to solve at each sub iteration of the Newtons' loop solve at each time step, to0 much big computational effort so actually this is not the method chosen)
Newton's method applied to this non linear problem means solve the system:

$$\begin{aligned}&\begin{cases}D_{U}F\left(U\right)|_{U^{k}}\left(\delta U\right)=-F\left(U^{k}\right)\\ U^{k+1}=U^{k}+\delta U\Leftrightarrow\left(u^{k+1},v^{k+1}\right)=\left(u^{k},v^{k}\right)\left(\delta u,\delta v\right)\end{cases}\\ &\begin{cases}\left[\frac{1}{\Delta t}\mathbf{1}-J|_{U^{k}}\right]\left(\delta U\right)=-\frac{\mathbf{1}}{\Delta t}U^{k}+G\left(U^{k}\right)+\frac{\mathbf{1}}{\Delta t}U^{n}\\ U^{k+1}=U^{k}+\delta U\end{cases}\end{aligned} \tag{1.14}$$

where J is the Jacobian matrix computed in section 1.1.
Written explicitly, using synthetic coefficients found in (1.4) and in (1.5) is:

$$\begin{cases}\dfrac{u^{k+1}}{\Delta t}-\tilde{D}_{1}\Delta_{s}u^{k+1}-\tilde{c}_{1}{u^{k}}^{2}v^{k+1}-\tilde{c}_{1}2u^{k}v^{k}u^{k+1}+\tilde{c}_{1}2{u^{k}}^{2}v^{k}-\tilde{a}_{1}u^{k+1}-\tilde{b}_{1}v^{k+1}=\dfrac{u^{n}}{\Delta t}\\ \dfrac{v^{k+1}}{\Delta t}-\tilde{D}_{2}\Delta_{s}v^{k+1}-\tilde{c}_{2}{u^{k}}^{2}v^{k+1}-\tilde{c}_{2}2u^{k}v^{k}u^{k+1}+\tilde{c}_{2}2{u^{k}}^{2}v^{k}-\tilde{a}_{2}v^{k+1}-\tilde{b}_{2}u^{k+1}=\dfrac{v^{n}}{\Delta t}+f_{2}\end{cases} \tag{1.15}$$

1.3 Implicit Euler + semi-implicit

Implicit in the linear part and explicit in the non-linear part.

$$\begin{aligned}&U=(u,v)\\ &\left(\mathbb{1}-\Delta t\mathscr{L}\right)U^{n+1}-\ \Delta t\mathscr{N}\left(U^{n+1},U^{n}\right)-U^{n}=0\\ &with\\ &\mathscr{N}\left(U^{n+1},U^{n}\right)=\left(\left(U^{n}\right)^{2}V^{n+1},\left(U^{n}\right)^{2}V^{n+1}\right)'\\ &Linear\ but\ still\ not\ sparse\end{aligned}$$

$$
\begin{cases}
\left(1-\Delta t\epsilon^{2}\Delta_{s}\right)u^{n+1}+\Delta tu^{n+1}-\Delta t\alpha(x)\left(u^{n}\right)^{2}v^{n+1}-\Delta t\left(\tau\gamma\right)^{-1}v^{n+1}=u^{n}\\
\left(1-\Delta t\frac{D}{\tau}\Delta_{s}\right)v^{n+1}+\frac{\Delta t}{\tau}v^{n+1}+\Delta t\gamma\alpha(x)\left(u^{n}\right)^{2}v^{n+1}-\Delta t\gamma u^{n+1}+\Delta t\frac{\beta\gamma}{\tau}u^{n+1}=\frac{\Delta t}{\tau}+v^{n}
\end{cases}
\tag{1.16}
$$

$$
\begin{cases}
\left(\frac{1}{\Delta t}-\epsilon^{2}\Delta_{s}+1\right)u^{n+1}-\left(\tau\gamma\right)^{-1}v^{n+1}-\alpha(x)\left(u^{n}\right)^{2}v^{n+1}=\frac{u^{n}}{\Delta t}\\
\left(\frac{1}{\Delta t}-\frac{D}{\tau}\Delta_{s}+\frac{1}{\tau}\right)v^{n+1}+\left(-\gamma+\frac{\beta\gamma}{\tau}\right)u^{n+1}+\gamma\alpha(x)\left(u^{n}\right)^{2}v^{n+1}=\frac{1}{\tau}+\frac{v^{n}}{\Delta t}
\end{cases}
\tag{1.17}
$$

$$
\begin{bmatrix} A_u & B_u \\ B_v & A_v \end{bmatrix} * \begin{bmatrix} \mathbf{U}^{n+1} \\ \mathbf{V}^{n+1} \end{bmatrix} + \begin{bmatrix} C_u\left(\mathbf{U}^{n}\right)\mathbf{V}^{n+1} \\ C_v\left(\mathbf{U}^{n}\right)\mathbf{V}^{n+1} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\tag{1.18}
$$

$$
\begin{bmatrix} A_u & B_u + C_u\left(\mathbf{U}^{n}\right) \\ B_v & A_v + C_v\left(\mathbf{U}^{n}\right) \end{bmatrix} * \begin{bmatrix} \mathbf{U}^{n+1} \\ \mathbf{V}^{n+1} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\tag{1.19}
$$

where, specifically for "PAPER" case:

$$
\begin{aligned}
\left[A_u\right]_{i,j} &= \int_\Omega \left(\frac{1}{\Delta t}+1\right)\phi_j\phi_i + \int_\Omega \epsilon^2 \nabla_s\phi_j\cdot\nabla_s\phi_j \\
\left[A_v\right]_{i,j} &= \int_\Omega \left(\frac{1}{\Delta t}+\frac{1}{\tau}\right)\phi_j\phi_i + \int_\Omega \frac{D}{\tau} \nabla_s\phi_j\cdot\nabla_s\phi_j \\
\left[B_u\right]_{i,j} &= \int_\Omega \left(\frac{-1}{\tau\gamma}\right)\phi_j\phi_i \\
\left[B_v\right]_{i,j} &= \int_\Omega \left(\frac{\beta\gamma}{\tau}-\gamma\right)\phi_j\phi_i \\
\left[C_u\left(u^n\right)\right]_{i,j} &= \int_\Omega -\alpha(x)\left(u^n\right)^2\phi_j\phi_i \\
\left[C_v\left(u^n\right)\right]_{i,j} &= \int_\Omega \gamma\alpha(x)\left(u^n\right)^2\phi_j\phi_i \\
\left[F_1\right]_i &= \int_\Omega \frac{u^n}{\Delta t}\phi_i \\
\left[F_2\right]_i &= \int_\Omega \frac{v^n}{\Delta t}\phi_i + \frac{1}{\tau}\phi_i
\end{aligned}
\tag{1.20}
$$

More generically referring synthetic coefficient in (1.3):

$$
\begin{aligned}
[A_u]_{i,j} &= \int_\Omega \left(\frac{1}{\Delta t} - \tilde{a}_1\right) \phi_j \phi_i + \int_\Omega \tilde{D}_1 \nabla_s \phi_j \cdot \nabla_s \phi_j \\
[A_v]_{i,j} &= \int_\Omega \left(\frac{1}{\Delta t} - \tilde{a}_2\right) \phi_j \phi_i + \int_\Omega \tilde{D}_2 \nabla_s \phi_j \cdot \nabla_s \phi_j \\
[B_u]_{i,j} &= \int_\Omega -\tilde{b}_1 \phi_j \phi_i \\
[B_v]_{i,j} &= \int_\Omega -\tilde{b}_2 \phi_j \phi_i \\
[C_u(u^n)]_{i,j} &= \int_\Omega -\tilde{c}_1 (u^n)^2 \phi_j \phi_i \\
[C_v(u^n)]_{i,j} &= \int_\Omega -\tilde{c}_2 (u^n)^2 \phi_j \phi_i \\
[F_1]_i &= \int_\Omega \frac{u^n}{\Delta t} \phi_i \\
[F_2]_i &= \int_\Omega \frac{v^n}{\Delta t} \phi_i + f_2 \phi_i
\end{aligned}
\tag{1.21}
$$

1.3.1 Initialization of the variables

In paper of year 2015 at page 5 is stated that:

"As initial conditions for our time-dependent computations, we take a small random perturbation to:

$$U_0 \equiv \frac{1}{\gamma\beta} \qquad V_0 \equiv \frac{\tau\beta\gamma}{\tau + \beta^2\gamma}, \tag{1.22}$$

"

I initialized the solutions therefore in FreeFem++ code as:

```
//Initialization: random perturbation or prev res
real u0 = 1./(gamma*beta);
real v0 = tau*beta*gamma/(tau + beta^2*gamma);
// ...
{
  U = u0;
  V = v0;
  srandom(seed);
  // ranreal 1 generates a random number in [0,1]
    for(int ii = 0; ii < Xh.ndof; ii++)
    {
      U[][ii] += randreal1() / 3e5;
      V[][ii] += randreal1() / 3e5;
    }
}
```

1.3.2 Adimensionalization

QUESTA PARTE IN RELTÀ NON SERVE PERCHÈ GIÀ EQUAZIONE (1.1) DEL PAPER, CIOÈ (1.1) È ADIMENSIONALIZZATA? DICEVA CHE CHIARAMENTE LO È LA PRIMA, LA SECONDA HA DI CONSEGUENZA QUEL TAU E che differenzia i tempi di diffusione (molto lenti perchè proporzionali a 1/coeff diffusione)

- Physical reference scales (scale di riferimento) L_x,L_y,U_0,V_0, T.
- Dimensions M(mass), L (length), t(time).

The system seems to be not consistent regarding the dimensions, because it is stated in the paper that all coefficients are already scaled, dimensionless, and also space derivative are already scaled with respect to the aspect ratio s (so dimentionless):

$$x^* = \frac{x}{L_x} \quad y^* = \frac{y}{L_y} \textit{ space variables in the square} [0,1]^2$$
$$\Delta = \partial_{xx} + \partial_{yy} = \frac{1}{L_x^2}\partial_{x^*x^*} + \frac{1}{L_y^2}\partial_{y^*y^*} \textit{ as if variables are rescaled wtr 2 different characteristic lengths}$$
$$\Delta_s = L_x^2 \left(\frac{1}{L_x^2}\partial_{x^*x^*} + \frac{1}{L_y^2}\partial_{y^*y^*} \right) = L_x^2 \Delta$$

For dimension consistency

$$[\alpha(x)] = \left(\frac{L^2}{M} \right)^2$$

maybe, but still time derivative add time that doesn't appear in the dimensionless coefficients such as $\epsilon^2, 1, \frac{1}{\tau\gamma}$
(trascuro questa non consistenza)
Define the dimensionless variables $u^* = \frac{u}{U_0}, v^* = \frac{v}{V_0}$. Being $\partial_{t^*} = \frac{1}{T}\partial_t$, we rewrite (1.1) in adimensional form:

$$\begin{cases} \partial_t u = \epsilon^2 \Delta_s u + \alpha(x) u^2 v - u + (\tau\gamma)^{-1} v \\ \partial_t v = \frac{D}{\tau}\Delta_s v - \frac{1}{\tau} v + \frac{1}{\tau} - \gamma \left(\alpha(x) u^2 v - u \right) - \frac{\beta\gamma}{\tau} u \end{cases}$$

$$\begin{cases} \frac{U_0}{T}\partial_{t^*} u^* = \epsilon^2 U^0 \Delta_s u^* + \alpha(x) U_0^2 V_0 u^{*2} v^* - U_0 u^* + V_0 (\tau\gamma)^{-1} v^* \\ \frac{V_0}{T}\partial_{t^*} v^* = \frac{DV_0}{\tau}\Delta_s v^* - \frac{V_0}{\tau} v^* + \frac{1}{\tau} - \gamma \left(\alpha(x) U_0^2 V_0 u^{*2} v^* - U_0 u^* \right) - \frac{\beta\gamma U_0}{\tau} u^* \end{cases}$$

$$\begin{cases} \partial_{t^*} u^* = \epsilon^2 T \Delta_s u^* + \alpha(x) U_0 V_0 T u^{*2} v^* - T u^* + \frac{V_0}{U_0}\frac{T}{\tau\gamma} v^* \\ \partial_{t^*} v^* = \frac{DT}{\tau}\Delta_s v^* - \frac{T}{\tau} v^* + \frac{T}{\tau V_0} - \gamma T \left(\alpha(x) U_0^2 u^{*2} v^* - \frac{U_0}{V_0} u^* \right) - \frac{\beta\gamma T}{\tau}\frac{U_0}{V_0} u^* \end{cases}$$

Omitting the star notation * (for example, u means actually u^*), we obtain the adimensional equation:

$$\begin{cases} \partial_t u = \tilde{D_1}\Delta_s u + \tilde{a_1} u + \tilde{b_1} v + \tilde{c_1} u^2 v \\ \partial_t v = \tilde{D_2}\Delta_s v + \tilde{a_2} v + \tilde{b_2} u + \tilde{c_2} u^2 v + \frac{T}{\tau V_0} \end{cases}$$

where

$$\tilde{D_1} = \epsilon^2 T \qquad \tilde{D_2} = \frac{DT}{\tau}$$

$$\tilde{a_1} = -T \qquad \tilde{a_2} = -\frac{T}{\tau}$$

$$\tilde{b_1} = \frac{V_0}{U_0}\frac{T}{\tau\gamma} \qquad \tilde{b_2} = \gamma T \frac{U_0}{V_0} - \frac{\beta\gamma T}{\tau}\frac{U_0}{V_0} = \gamma T \frac{U_0}{V_0}\left(1 - \frac{\beta}{\tau}\right)$$

$$\tilde{c_1} = \alpha(x) U_0 V_0 T \qquad \tilde{c_2} = -\gamma T \alpha(x) U_0^2$$

1.4 Implicit Euler + semi-implicit 2.0

The original problem in (1.3) can be rewritten schematically as:

$$\begin{bmatrix} \mathring{U} \\ \mathring{V} \end{bmatrix} = \begin{bmatrix} \Delta_s + \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \Delta_s + \mathbb{1} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} + \begin{bmatrix} U^2 V \\ U^2 V \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{1.23}$$

Applying implicit Euler scheme, taking implicitly only the diffusive part, explicitly the reaction part, :

$$\begin{bmatrix} \frac{U^{n+1} - U^n}{\Delta t} \\ \frac{V^{n+1} - V^n}{\Delta t} \end{bmatrix} = \begin{bmatrix} \Delta_s & 0 \\ 0 & \Delta_s \end{bmatrix} \begin{bmatrix} U^{n+1} \\ V^{n+1} \end{bmatrix} + \begin{bmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} U^n \\ V^n \end{bmatrix} + \begin{bmatrix} U^{n2} V^n \\ U^{n2} V^n \end{bmatrix} + \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \tag{1.24}$$

$$\begin{cases} \left(\frac{1}{\Delta t} - \tilde{D_1}\Delta_s\right) u^{n+1} = \frac{u^n}{\Delta t} + \tilde{a_1} u^n + \tilde{b_1} v^n + \tilde{c_1} \left(u^n\right)^2 v^n \\ \left(\frac{1}{\Delta t} - \tilde{D_2}\Delta_s\right) v^{n+1} = \frac{v^n}{\Delta t} + \tilde{a_2} v^n + \tilde{b_2} u^n + \tilde{c_2} \left(u^n\right)^2 v^n + f_2 \end{cases} \tag{1.25}$$

specifically in the "PAPER" case:

$$\begin{cases} \left(\frac{1}{\Delta t} - \epsilon^2\Delta_s\right) u^{n+1} = \frac{u^n}{\Delta t} - u^n + \left(\tau\gamma\right)^{-1} v^n + \alpha(x) \left(u^n\right)^2 v^n \\ \left(\frac{1}{\Delta t} - \frac{D}{\tau}\Delta_s\right) v^{n+1} = \frac{v^n}{\Delta t} - \frac{1}{\tau} v^n + \left(\gamma - \frac{\beta\gamma}{\tau}\right) u^n - \gamma\alpha(x) \left(u^n\right)^2 v^n + \frac{1}{\tau} \end{cases} \tag{1.26}$$

$$\begin{bmatrix} A_u & 0 \\ 0 & A_v \end{bmatrix} \begin{bmatrix} \mathbf{U}^{n+1} \\ \mathbf{V}^{n+1} \end{bmatrix} = \begin{bmatrix} B_{uu} & B_{uv} \\ B_{vu} & B_{vv} \end{bmatrix} \begin{bmatrix} \mathbf{U}^n \\ \mathbf{V}^n \end{bmatrix} + \begin{bmatrix} C_u\left(\mathbf{U}^n\right)\mathbf{V}^n \\ C_v\left(\mathbf{U}^n\right)\mathbf{V}^n \end{bmatrix} + \begin{bmatrix} \frac{1}{\Delta t}\mathbf{U}^n \\ F_2 + \frac{1}{\Delta t}\mathbf{V}^n \end{bmatrix} \tag{1.27}$$

where

$$
\begin{aligned}
[A_u]_{ij} &= \int_\Omega \left(\frac{1}{\Delta t}\phi_j\phi_i + \tilde{D}_1\nabla_s\phi_j\cdot\nabla_s\phi_i\right)\\
[A_v]_{ij} &= \int_\Omega \left(\frac{1}{\Delta t}\phi_j\phi_i + \tilde{D}_2\nabla_s\phi_j\cdot\nabla_s\phi_i\right)\\
[B_{uu}]_{ij} &= \int_\Omega \tilde{a}_1\phi_j\phi_i\\
[B_{uv}]_{ij} &= \int_\Omega \tilde{b}_1\phi_j\phi_i\\
[B_{vu}]_{ij} &= \int_\Omega \tilde{b}_2\phi_j\phi_i\\
[B_{vv}]_{ij} &= \int_\Omega \tilde{a}_2\phi_i\phi_j\\
\left[C_u\left(\mathbf{U}^n\right)\mathbf{V}^n\right]_i &= \int_\Omega \tilde{c}_1\left(u^n\right)^2 v^n\phi_i\\
\left[C_v\left(\mathbf{U}^n\right)\mathbf{V}^n\right]_i &= \int_\Omega \tilde{c}_2\left(u^n\right)^2 v^n\phi_i\\
[F_2]_i &= \int_\Omega f_2\phi_i
\end{aligned}
$$

1.5 Implicit Euler + "semi-implicit" (decoupled eqs)

NOT TO CONSIDER Non linear problem

$$
\begin{gathered}
\left(\mathbb{1} - \Delta t\mathscr{L}\right)U^{n+1} - \ \Delta t\mathscr{N}\left(U^{n+1}\right) - U^n = 0\\
\begin{bmatrix} 1-\Delta t\ \Delta_s \\ 1-\Delta t\ \Delta_s \end{bmatrix} * \begin{bmatrix} u^{n+1} \\ v^{n+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} * \begin{bmatrix} v^{n+1} \\ u^{n+1} \end{bmatrix} - \ \Delta t\mathscr{N}\left(U^{n+1}\right) - U^n = 0
\end{gathered}
\tag{1.28}
$$

Take explicit terms in the Non linear part and in the second contribute:

$$
\begin{aligned}
\begin{bmatrix} 1-\Delta t\ \Delta_s \\ 1-\Delta t\ \Delta_s \end{bmatrix} * \begin{bmatrix} u^{n+1} \\ v^{n+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} * \begin{bmatrix} v^{n} \\ u^{n} \end{bmatrix} - \ \Delta t\hat{\mathscr{N}}\left(U^{n+1}, U^n\right) - U^n = 0&\\
where&\\
\hat{\mathscr{N}}\left(U^{n+1}, U^n\right) = \begin{bmatrix} u^n v^n u^{n+1} \\ \left(u^n\right)^2 v^{n+1} \end{bmatrix}&
\end{aligned}
\tag{1.29}
$$

Explicitly with all the coefficients

$$\begin{cases} u^{n+1} - \Delta t\ \epsilon^2\ \Delta u^{n+1} + \Delta t\ u^{n+1} - \Delta t\ \alpha(x) u^n v^n u^{n+1} = +\Delta t\ \left(\tau\gamma\right)^{-1} v^n + u^n \\ v^{n+1} - \Delta t\ \frac{D}{\tau} \Delta v^{n+1} + \frac{\Delta t}{\tau} v^{n+1} + \Delta t\ \gamma\alpha(x) \left(u^n\right)^2 v^{n+1} = \frac{\Delta t}{\tau} + \Delta t \gamma u^n - \Delta t \beta\gamma u^n + v^n \end{cases} \tag{1.30}$$