Chapter 1

Formule varie

$$\begin{cases} \partial_t u = \epsilon^2 \Delta u + \alpha(x) u^2 v - u + (\tau\gamma)^{-1} v \\ \partial_t v = \frac{D}{\tau}\Delta v - \frac{1}{\tau} v + \frac{1}{\tau} - \gamma\left(\alpha(x) u^2 v - u\right) - \frac{\beta\gamma}{\tau} u \end{cases} \tag{1.1}$$

Sintetic Operators without coefficients

$$\mathscr{L} = \begin{bmatrix} \Delta_s + \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \Delta_s + \mathbb{1} \end{bmatrix}$$

$$\mathscr{N}(U) = \begin{bmatrix} u^2 v \\ u^2 v \end{bmatrix}$$

1^{st} **option**: Implicit Euler + Newton's method (complete)

$$U = (u, v)$$

$$U^{n+1} = U^n + \Delta t \mathscr{L} U^{n+1} + \ \Delta t \mathscr{N}\left(U^{n+1}\right) \Leftrightarrow G(U) = (\mathbf{1} - \mathscr{L})\, U - \Delta t \mathscr{N}(U) - U^n = 0 \tag{1.2}$$

$$Newton\ method...\ full\ matrix$$

2^{nd} **option**: Implicit Euler + semi-implicit (these morning hyp.)

$$U = (u, v)$$

$$(\mathbb{1} - \Delta t \mathscr{L})\, U^{n+1} - \ \Delta t \mathscr{N}\left(U^{n+1}, U^n\right) - U^n = 0$$

$$with$$

$$\mathscr{N}\left(U^{n+1}, U^n\right) = \left((U^n)^2 V^{n+1}, (U^n)^2 V^{n+1}\right)'$$

$$Linear\ but\ still\ not\ sparse$$

$$
\begin{cases}
\left(1-\Delta t\epsilon^{2}\Delta_{s}\right)u^{n+1}+\Delta t u^{n+1}-\Delta t\alpha(x)\left(u^{n}\right)^{2}v^{n+1}-\Delta t\left(\tau\gamma\right)^{-1}v^{n+1}=u^{n}\\
\left(1-\Delta t\frac{D}{\tau}\Delta_{s}\right)v^{n+1}+\frac{\Delta t}{\tau}v^{n+1}+\Delta t\gamma\alpha(x)\left(u^{n}\right)^{2}v^{n+1}-\Delta t\gamma u^{n+1}+\Delta t\frac{\beta\gamma}{\tau}u^{n+1}=\frac{\Delta t}{\tau}+v^{n}
\end{cases}
\tag{1.3}
$$

$$
\begin{cases}
\left(\frac{1}{\Delta t}-\epsilon^{2}\Delta_{s}+1\right)u^{n+1}-\left(\tau\gamma\right)^{-1}v^{n+1}-\alpha(x)\left(u^{n}\right)^{2}v^{n+1}=\frac{u^{n}}{\Delta t}\\
\left(\frac{1}{\Delta t}-\frac{D}{\tau}\Delta_{s}+\frac{1}{\tau}\right)v^{n+1}+\left(-\gamma+\frac{\beta\gamma}{\tau}\right)u^{n+1}+\gamma\alpha(x)\left(u^{n}\right)^{2}v^{n+1}=\frac{1}{\tau}+\frac{v^{n}}{\Delta t}
\end{cases}
\tag{1.4}
$$

$$
\begin{bmatrix} A_u & B_u \\ B_v & A_v \end{bmatrix} * \begin{bmatrix} U^{n+1} \\ V^{n+1} \end{bmatrix} + \begin{bmatrix} C_u\left(U^{n}\right)V^{n+1} \\ C_v\left(U^{n}\right)V^{n+1} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\tag{1.5}
$$

where

$$
\begin{aligned}
\left[A_u\right]_{i,j} &= \int_\Omega \left(\frac{1}{\Delta t}+1\right)\phi_j\phi_i + \int_\Omega \epsilon^2\nabla_s\phi_j\cdot\nabla_s\phi_j\\
\left[A_v\right]_{i,j} &= \int_\Omega \left(\frac{1}{\Delta t}+\frac{1}{\tau}\right)\phi_j\phi_i + \int_\Omega \frac{D}{\tau}\nabla_s\phi_j\cdot\nabla_s\phi_j\\
\left[B_u\right]_{i,j} &= \int_\Omega \left(\frac{-1}{\tau\gamma}\right)\phi_j\phi_i\\
\left[B_v\right]_{i,j} &= \int_\Omega \left(\frac{\beta\gamma}{\tau}-\gamma\right)\phi_j\phi_i\\
\left[C_u\left(u^n\right)\right]_{i,j} &= \int_\Omega -\alpha(x)\left(u^n\right)^2\phi_j\phi_i\\
\left[C_v\left(u^n\right)\right]_{i,j} &= \int_\Omega \gamma\alpha(x)\left(u^n\right)^2\phi_j\phi_i\\
\left[F_1\right]_i &= \int_\Omega \frac{u^n}{\Delta t}\phi_i\\
\left[F_2\right]_i &= \int_\Omega \frac{v^n}{\Delta t}\phi_i + \frac{1}{\tau}\phi_i
\end{aligned}
\tag{1.6}
$$

3^{rd} **option NOT TO CONSIDER**: Innmplicit Euler + "semi-implicit" (decoupled eqs)

Non linear problem

$$
\begin{gathered}
\left(\mathbb{1}-\Delta t\mathscr{L}\right)U^{n+1}-\ \Delta t\mathscr{N}\left(U^{n+1}\right)-U^{n}=0\\
\begin{bmatrix} 1-\Delta t\ \Delta_s \\ 1-\Delta t\ \Delta_s \end{bmatrix} * \begin{bmatrix} u^{n+1} \\ v^{n+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} * \begin{bmatrix} v^{n+1} \\ u^{n+1} \end{bmatrix} -\ \Delta t\mathscr{N}\left(U^{n+1}\right)-U^{n}=0
\end{gathered}
\tag{1.7}
$$

Take explicit terms in the Non linear part and in the second contribute:

$$
\begin{bmatrix} 1-\Delta t\ \Delta_s \\ 1-\Delta t\ \Delta_s \end{bmatrix} * \begin{bmatrix} u^{n+1} \\ v^{n+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} * \begin{bmatrix} v^{n} \\ u^{n} \end{bmatrix} - \ \Delta t \hat{\mathscr{N}}\left(U^{n+1}, U^{n}\right) - U^{n} = 0
$$

$$
\text{where} \tag{1.8}
$$

$$
\hat{\mathscr{N}}\left(U^{n+1}, U^{n}\right) = \begin{bmatrix} u^{n} v^{n} u^{n+1} \\ \left(u^{n}\right)^{2} v^{n+1} \end{bmatrix}
$$

Explicitly with all the coefficients

$$
\begin{cases} u^{n+1} - \Delta t\ \epsilon^{2}\ \Delta u^{n+1} + \Delta t\ u^{n+1} - \Delta t\ \alpha(x) u^{n} v^{n} u^{n+1} = +\Delta t\ \left(\tau\gamma\right)^{-1} v^{n} + u^{n} \\ v^{n+1} - \Delta t\ \frac{D}{\tau} \Delta v^{n+1} + \frac{\Delta t}{\tau} v^{n+1} + \Delta t\ \gamma\alpha(x) \left(u^{n}\right)^{2} v^{n+1} = \frac{\Delta t}{\tau} + \Delta t \gamma u^{n} - \Delta t \beta\gamma u^{n} + v^{n} \end{cases} \tag{1.9}
$$