

## Roadmap

- 1. Basic concepts (alternatives, preferences, ...)
- 2. Representing ordinal preferences (ordinal utility)
- 3. Beyond ordinal preferences (cardinal utility, VNM utility, ...)
- 4. Attitude towards risk
- 5. Applications and introduction to decision making

## Basic Concepts: Alternatives

An agent chooses between a set *X* of alternatives

Alternatives are

- ► Mutually exclusive
- ▶ Exhaustive

### Basic Concepts: Alternatives

#### **Example:**

```
Options = {Deep Learning, Algorithmic Game Theory}
```

```
X = \{
     DL = Deep Learning,
     AGT = Algorithmic Game Theory,
     DLAGT = Deep Learning and Algorithmic Game Theory,
     N = None
```

## ■ Basic Concepts: Preferences

Preferences are a relation  $\geq$  on X, which is a subset of  $X \times X$ .

 $\geqslant$  is complete iff  $\forall x,y \in X$ ,  $x \geqslant y$  or  $y \geqslant x$ 

 $\geq$  is transitive iff  $\forall x,y,z \in X$ ,  $[x \geq y \text{ and } y \geq z] \Rightarrow x \geq z$ 

## Basic Concepts: Preference Relation

A preference is a preference relation iff it is complete and transitive.

► Strict preference:

$$x > y \Leftrightarrow [x \ge y \text{ and } y \ge x]$$

➤ Indifference:

$$x \sim y \Leftrightarrow [x \geqslant y \text{ and } y \geqslant x]$$

# Representing preferences as utilities

A preference relation can be represented by a utility function  $u: X \rightarrow \mathbb{R}$  in the following sense:

$$x \geqslant y \Leftrightarrow u(x) \ge u(y) \quad \forall x, y \in X$$

If a player finds x at least as good as y then u(x) must be at least as high as u(y).

In this sense, a player acts as if they are trying to maximise the value of  $u(\cdot)$ .

## Theorem: Ordinal Representation

**Theorem 1:** Let X be finite. A preference can be represented by a utility function if and only if it is complete and transitive. Moreover, if  $u: X \rightarrow \mathbb{R}$  represents  $\geq$ , and if  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a strictly increasing function, then  $f \circ u$  also represents  $\geq$ .

By the last statement of the theorem, such utilities are called ordinal, i.e. only the order information is relevant.



#### Transitivity is a necessary condition

Let  $X = \{a, b, c\}$ , suppose  $a > b > c > a \Rightarrow u(a) > u(b) > u(c) > u(a)$ . This is absurd.

#### Completeness is a necessary condition

If we have incomplete preferences, at most we can construct an order for a subset of X.

#### Proof of Theorem 1

#### Transitivity and completeness are necessary and sufficient

Let  $X = \{X_1, ..., X_n\}$ , we can partition the elements of X into k indifference classes  $C_1, ..., C_k$  such that  $C_1 > C_2 > ... > C_k$ .

Thus, we can define u so that:

$$u(x) = k$$
  $\forall x \in C_1$ ,  
 $u(x) = k-1$   $\forall x \in C_2$ ,  
...,  
 $u(x) = 1$   $\forall x \in C_k$ 

## Ordinal utilities: examples

$$X = \{a, b, c\}$$

Preferences:  $a \ge b \ge c$ 

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X deve essere finito perché (me lo spiega alex)

$$u(a) = -1$$

$$u(a) = 30$$

$$u(a) = 90$$

$$u(b) = -2$$

$$u(b) = 20$$

$$u(b) = 2$$

$$u(c) = -3$$

$$u(c) = 10$$

$$u(b) = 1$$

The numerical value of ordinal utility is not interpretable: ordinal utility does not tell us the magnitude with which a player prefers a to b.

## Beyond ordinal utilities: lotteries

A simple lottery is a tuple  $L = (p_1, x_1; p_2, x_2; ...; p_n, x_n)$ 

- ► Monetary prizes  $x_1, x_2, ..., x_n \in X \subseteq \mathbb{R}$
- ightharpoonup Probability distribution  $(p_1, p_2, ..., p_n)$

Let £ denote the set of simple lotteries

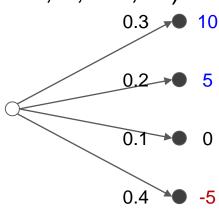
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Example L = (0.3, 10; 0.2, 5; 0.1, 0; 0.4, -5)



## Expected value

Consider the lottery L = (0.3, 10; 0.2, 5; 0.1, 0; 0.4, -5) we can compute the expected value to know its worth, as follows:

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 non giocare al lotto gioca alle slot!

In our case we have:

$$\mathbb{E}(L)=0.3 imes10+0.2 imes5-0.4 imes5=2$$

The expected value of L is 2. This means that, each time we play such a lottery, we expect to win a value of 2.

## **Expected utility**

Given a preference relation  $\geq$  on  $\mathcal{L}$ , a utility function  $U: \mathcal{L} \rightarrow \mathbb{R}$  is an expected utility function if it can written as:

$$U(L) = \sum_{i=1}^n p_i u(x_i)$$

for some function  $u : \mathbb{R} \rightarrow \mathbb{R}$ .

The function u is called a Bernoulli utility function.

## ■ Von Neumann and Morgenstern Utility

Von Neumann and Morgenstern (VNM) provided the conditions by which a preference relation  $\geq$  on  $\mathcal{L}$  can be represented by a utility function.

**Axiom 1:** (Preference order)

≥ is complete and transitive

Axiom 2: (Continuity)

if L > M > N, there exists  $p \in [0,1]$  such that  $pL + (1-p)N \sim M$ 

Axiom 3: (Independence)

for any lottery N and  $p \in [0,1], L \ge M \Leftrightarrow pL + (1-p)N \ge pM + (1-p)N$ 

## **VNM** Theorem

**Theorem 2:** (VNM) A binary relation > over  $\mathcal{L}$  has an expected utility representation if and only if it satisfies axioms 1–3. Moreover, if U and V are expected utility representations of >, then there exist constants a,  $b \in \mathbb{R}$ , a > 0, such that  $U(\cdot) = aV(\cdot) + b$ .

The last statement tells us that the VNM utility representation is unique up to a affine transformations.

### Proof of VNM Theorem

We divide the proof in two parts:

$$lacksquare$$
 Part 1: we show that  $U(L) = \sum_{i=1}^n p_i u(x_i)$ 

▶ Part 2: we show that  $L > M \Leftrightarrow U(L) > U(M)$ ,  $L,M \in \mathcal{L}$ 

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It follows that, the utility of a lottery  $M = \sum_i p_i o_i$  is the expectation of u

$$u(M) = u\left(\sum_i p_i o_i
ight) = \sum_i p_i u(o_i) = \sum_i p_i q_i$$

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As such, a player is indifferent between the following two lotteries:

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However,

$$M' = \left(\sum_i p_i q_i
ight) \cdot o_n + \left(\sum_i p_i (1-q_i)
ight) \cdot o_1 = U(M) \cdot o_n + (1-U(M)) \cdot o_1$$

Suppose L > M, we can define L' and M' as follows

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Since  $L' > M' \Rightarrow U(L) > U(M)$ . Hence,  $L > M \Leftrightarrow U(L) > U(M)$ .

Let's consider a fair lottery  $L = p \cdot x + (1-p) \cdot y = 0$ 

lotteria fair significa che il valore atteso è 0

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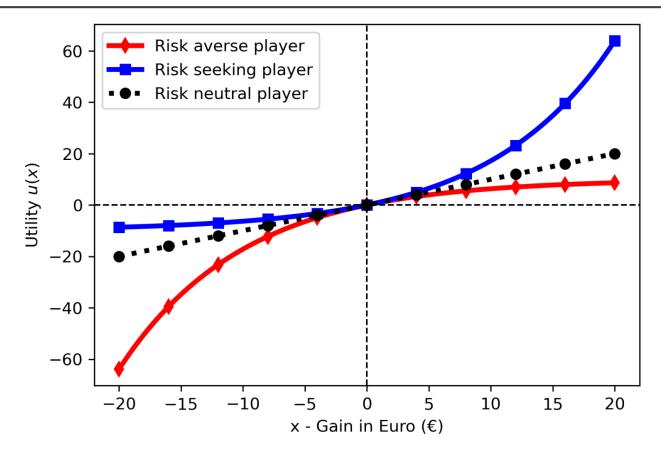
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- ► A player is risk averse iff their utility function is concave
  - ► A strictly risk averse player does not play any fair lottery

- ► A player is risk seeking iff their utility function is convex
  - ► A strictly risk seeking player plays all the fair lotteries



Suppose we have two risk averse players with  $u(x) = \sqrt{x}$  and two risky assets  $A_1$ ,  $A_2 = (0.5, 100; 0.5, 0)$ . Suppose  $A_1$  and  $A_2$  are independent.

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$$u(A_m) = 0.25 imes \sqrt{100} + 0.5 imes \sqrt{50} pprox 6$$

## Applications: Insurance

We have a risk averse player with  $u(x) = \sqrt{x}$  and an asset A = (0.5, 100; 0.5, 0). We have a risk neutral insurance company with lots of money.

What premium P would the player pay to insure their asset?

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$$u(100-P) \geq 0.5 \times u(100) + 0.5 \times u(0) \Rightarrow \sqrt{100-P} \geq 5 \Rightarrow 100-P \geq 25 \Rightarrow -P \geq -100+25 \Rightarrow P \leq 75$$

## Applications: Insurance

What premium would the insurance company require to ensure the player's asset?

$$P \geq 0.5 \times 100 + 0.5 \times 0 \Rightarrow P \geq 50$$

Hence, both parties would gain if the company insures the player's asset for a premium  $P \in [50, 75]$ .

## Recap on VNM utility theory

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  - ► Numerical values are interpretable on an interval scale
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  - Highly used in financial and economic applications

- VNM utility theory does not describe human behaviour
  - ► An alternative is not preferred because it is associated with a higher utility. Rather, it is associated with a higher utility because it is the preferred alternative.