UNIVERSITÀ DELLA CALABRIA

VCG Mechanism

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A quick recap on mechanism design

- Mechanism Design is inverse game theory
  - Game theory: we are given a game and we want to predict what happens
  - Mechanism design: we want to create games that incentivise a given behaviour.

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  - Game theory: we are given a game and we want to predict what happens
  - Mechanism design: we want to create games that incentivise a given behaviour.
- Desired behaviours
  - Game is strategy proof
  - Individual rationality
  - Efficiency
  - Fairness

A quick recap on mechanism design

- Mechanism design
  - Without money → Matching problems, facility location
  - With money → Auctions

Auctions

- We are selling a given item x
- There are N bidders
- Bidders have report valuations $v_i(x)$
- Bidders pay a price $p_i(x)$
- Bidders have *quasi-linear* utilities → $u_i(x) = v_i(x) - p_i(x)$

First price auctions

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The winner has a utility equal to zero. Therefore, the first price auction gives an incentive to lie and report a lower value in order to pay less.

Second price auctions

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$v_w \rightarrow$ reported valuation of the winner
$v_s \rightarrow$ reported valuation of the second

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Second price auctions

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v_w → reported valuation of the winner

v_s → reported valuation of the second

$$u_w(x) = v_w(x) - v_s(x) \geq 0$$

Theorem: Bidding truthfully, in second price auctions, is a weakly dominant strategy.

Applications of Second Price Auctions

- Ebay auctions!
- Online advertisement (through generalised second price auctions)
- Network routing

Extension to multiple items

- m items
- n bidders

- We want a mechanism that incentivates people to bid truthfully

VCG Auctions

In general auctions we have two problems:

- Determining the winner
- Charging the winner an appropriate price

By charging the *right* appropriate price, we can ensure that bidding truthfully is a weakly dominant strategy for the bidders.

VCG auctions are one of such mechanisms.

Combinatorial Auctions

- m Items
- n bidders
- Bidders value combinations of items

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Example:

$n = \{A, B, C\}, m = \{a, b\}$

$$v(A) = \{\{a\} = 2, \{b\} = 2, \{a, b\} = 5\}$$
$$v(B) = \{\{a\} = 5, \{b\} = 0, \{a, b\} = 0\}$$
$$v(C) = \{\{a\} = 3, \{b\} = 2, \{a, b\} = 6\}$$

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How should we allocate the items?

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How should we allocate the items?

Maximise social welfare

B ← a and C ← b, for a total welfare of 7

Combinatorial Auctions

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B $\leftarrow a$ and C $\leftarrow b$

How should we charge B and C?

Combinatorial Auctions

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B ← a and C ← b

How should we charge B and C?

Each bidder pays the externality they impose on others

$$p_A = 7 - 7 = 0$$
$$p_B = 6 - 3 = 3$$
$$p_C = 6 - 5 = 1$$

VCG Mechanism: formalization

- m Items
- n bidders
- Bidders value combinations of items (Ω)

Optimal allocation: $\omega^* = \arg\max_{\omega \in \Omega} \sum_{i \in N} v_i(\omega)$

Each bidder pays: $p_i(\omega^*) = \max\limits_{\omega \in \Omega} \sum_{j \neq i \in N} v_i(\omega) - \sum_{j \neq i \in N} v_i(\omega^*)$

The optimal allocation problem

Let's consider this formula:

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This is the NP-Hard problem called maximum weighted set packing.