UNIVERSITÀ DELLA CALABRIA

Tree Decomposition

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Roadmap

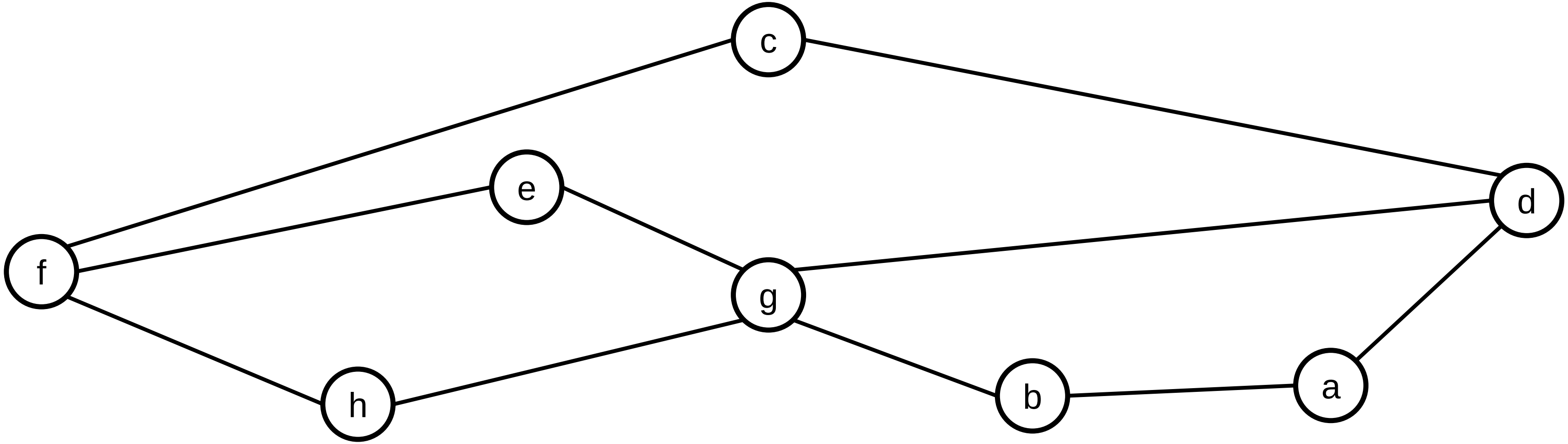
1. NP-Hard problems on trees
2. Treewidth and Tree Decomposition
3. Cops and robbers
4. Computing a Tree Decomposition

NP-Hard problems on trees

- NP-Hard problems on graphs, generally, cannot be solved exactly because of their prohibitive computational cost
  - Greedy Approximation
  - Heuristics: Simulated Annealing, Genetic Algorithm, ACO, …
  - Polynomial Time Approximation Scheme (PTAS)
- On trees, many NP-Hard problems can be solved exactly in polynomial time
  - Trees have a computationally convenient structure
    - Sub-trees are independent

Maximum Independent Set (MIS)

Let $G = (V, E)$ be a graph, an independent set is a subset of nodes such that there is no edge between them: $S = \{C \subset V, (u, v) \notin E\ \forall u, v \in C\}$



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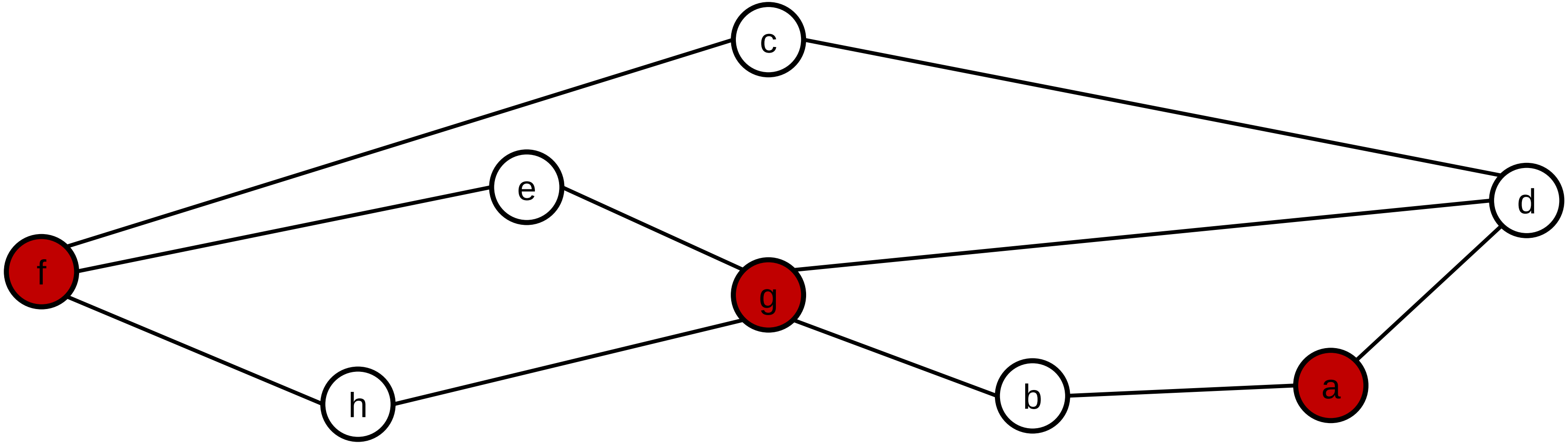

$S = \{f, g, a\}$ is an independent set

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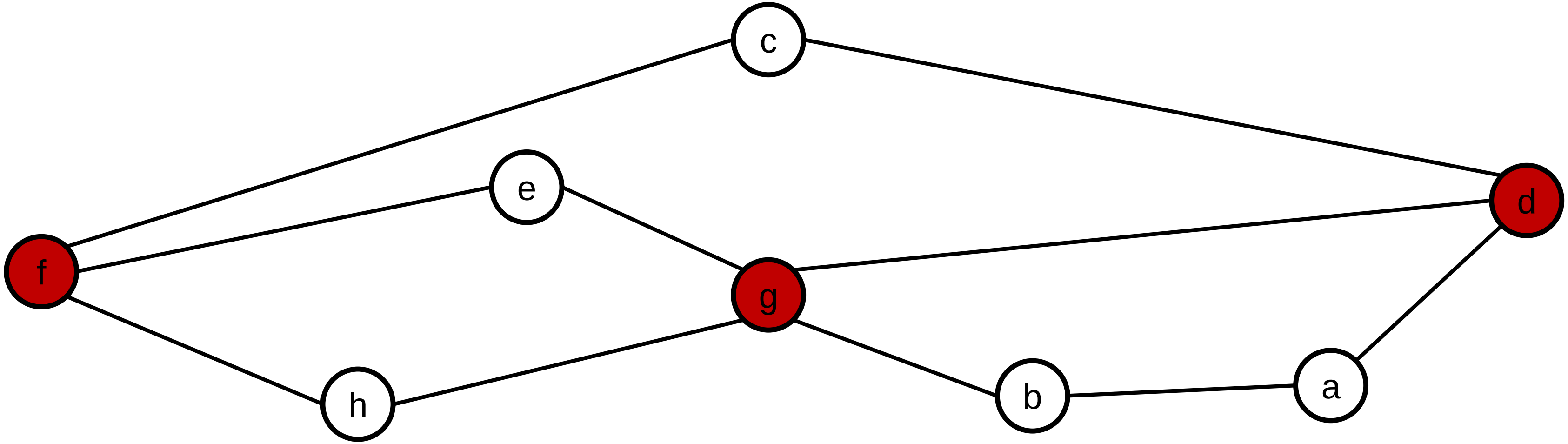

$S = \{f, g, d\}$ is not an independent set

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$$MIS = \underset{S}{\operatorname{argmax}} |S|, \qquad \forall S \in IS(G)$$

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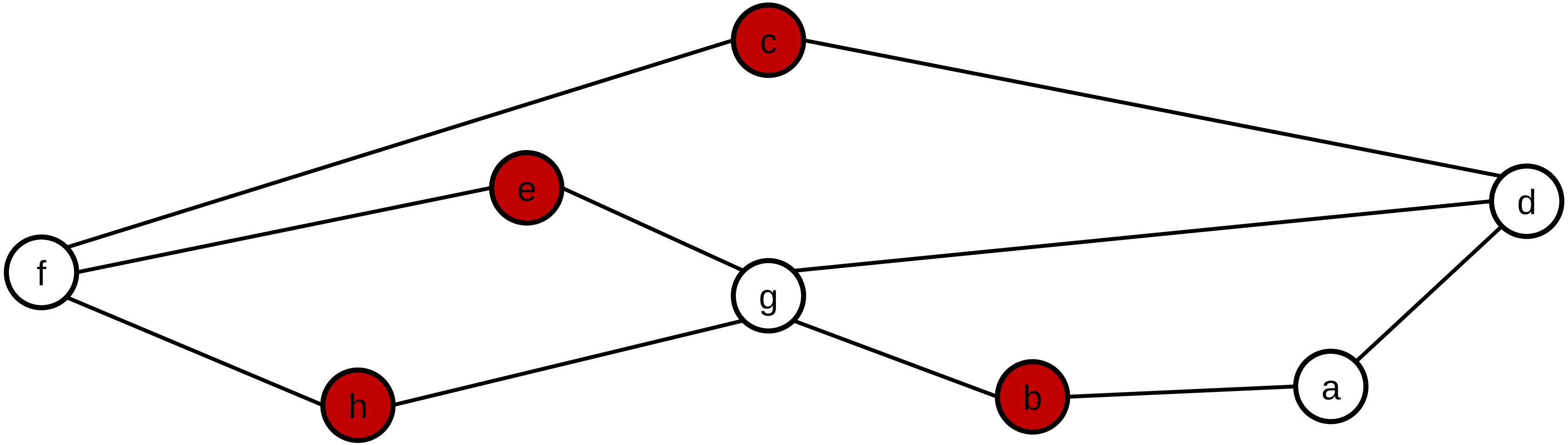
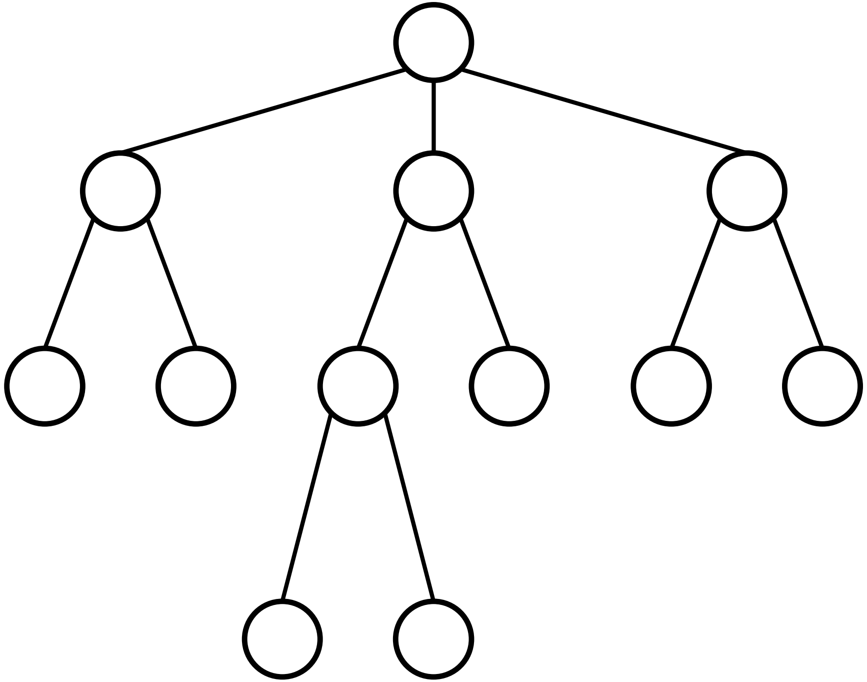


Maximum Independent Set (MIS)

- MIS is NP-Complete
- Naive implementation requires $\mathcal{O}(n^2 2^n)$ computational time
  - "Fast" implementation in $\mathcal{O}(1.1996^n)$

- On trees, we can solve MIS in linear time!

Maximum Independent Set on Trees



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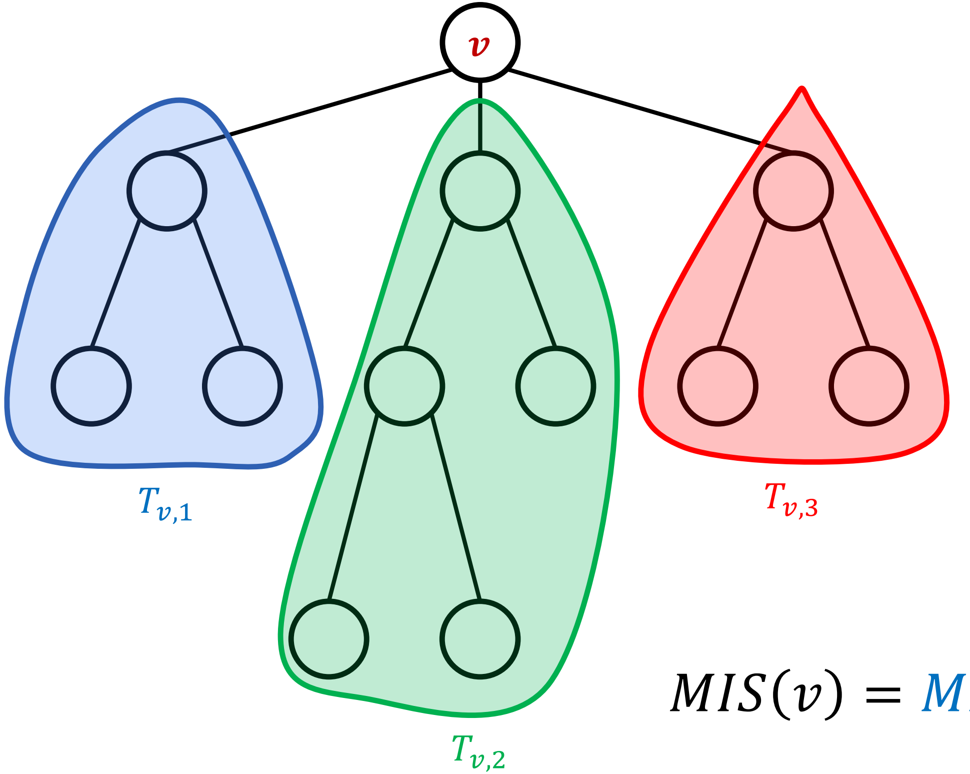
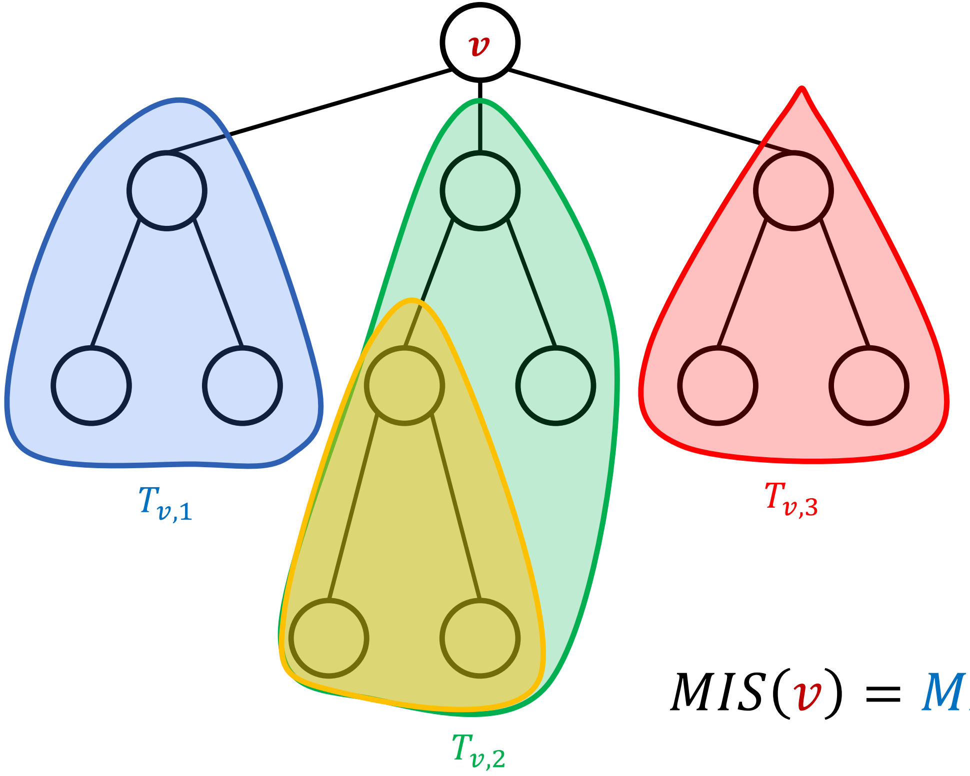


- Optimal substructure
  - optimal solution implies optimal solutions of subproblems

$$MIS(v) = MIS(T_{v,1}) + MIS(T_{v,2}) + MIS(T_{v,3}) \pm v$$

Maximum Independent Set on Trees



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  - subproblems are computed (reused) several times

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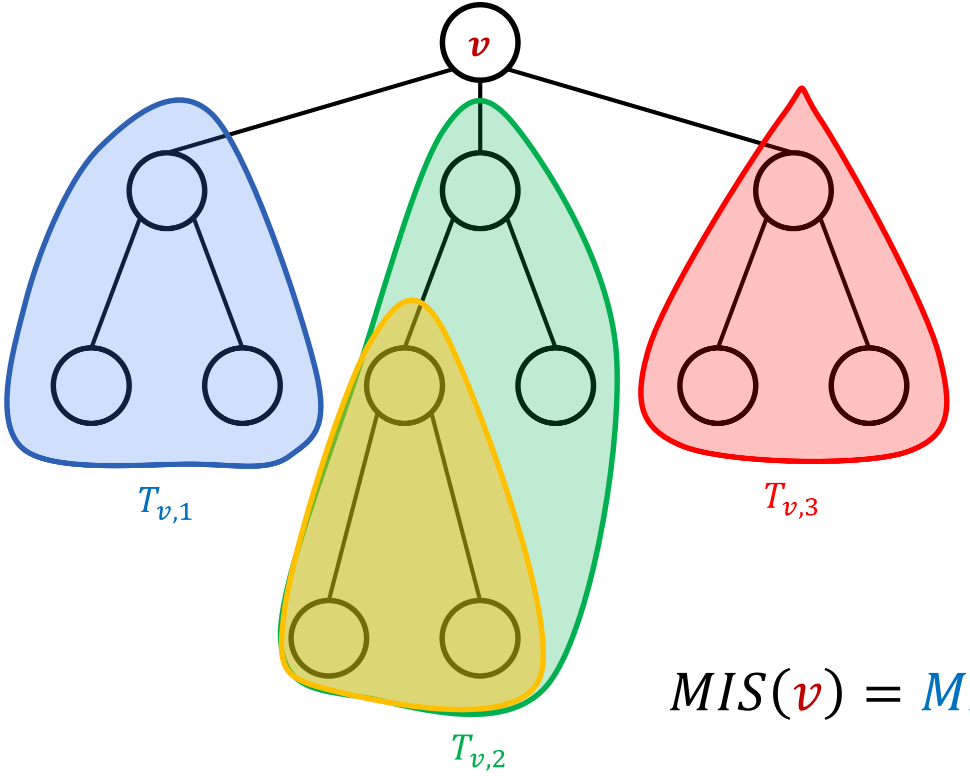

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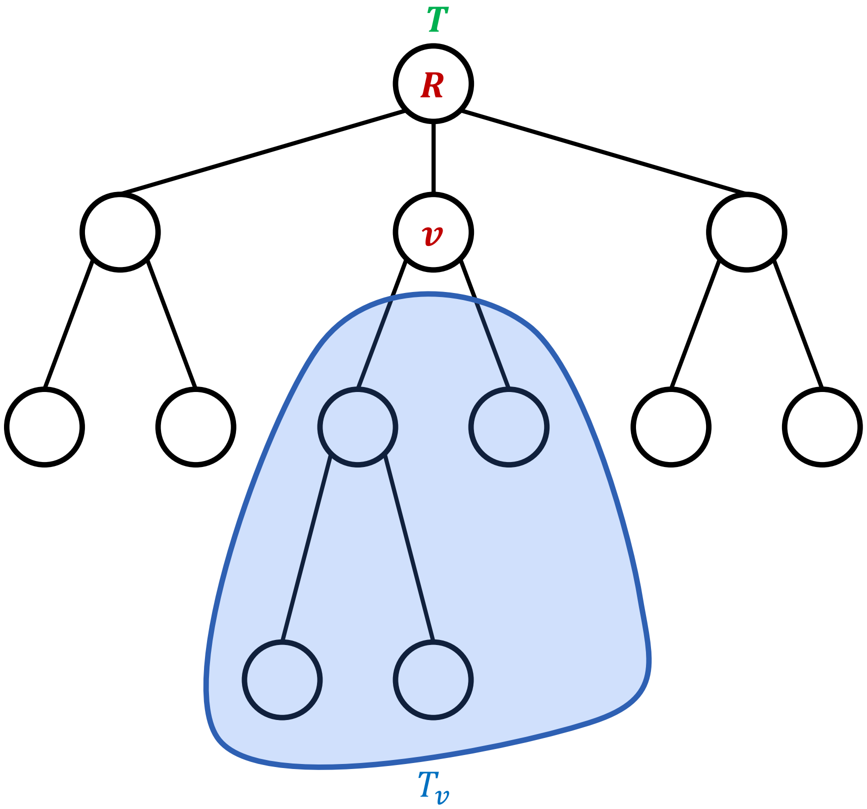
- **Dynamic Programming**

$$MIS(v) = MIS(T_{v,1}) + MIS(T_{v,2}) + MIS(T_{v,3}) \pm v$$

Maximum Independent Set on Trees



For each vertex v we compute:

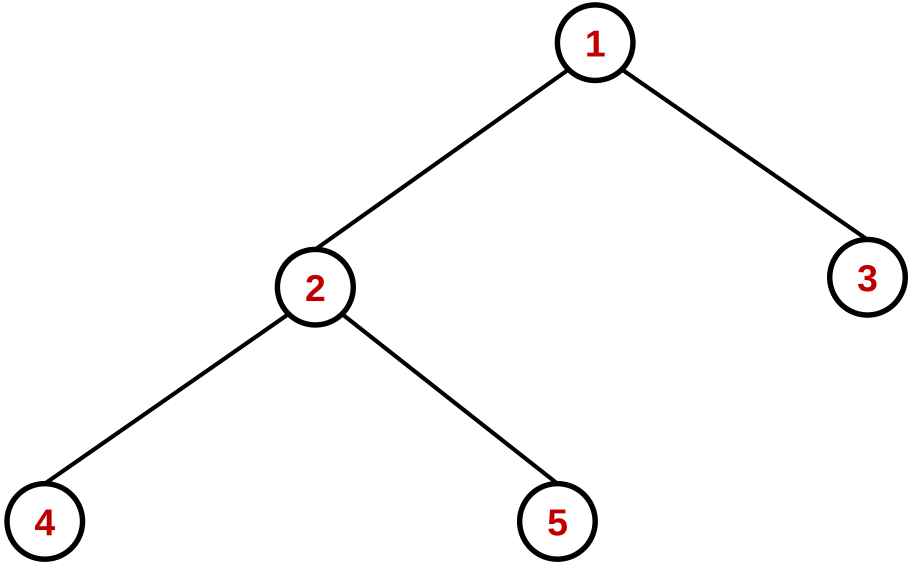
- $M^+[v] = |MIS(T_v) \cup \{v\}|$
- $M^-[v] = |MIS(T_v) \setminus \{v\}|$

For a vertex v with children $w_1, \dots, w_d$

- $M^+[v] = 1 + \sum_{i=1}^{d} M^-[w_i]$
- $M^-[v] = \sum_{i=1}^{d} \max\{M^+[w_i], M^-[w_i]\}$

$$MIS(T) = \max\{M^+[R], M^-[R]\}$$

Maximum Independent Set on Trees



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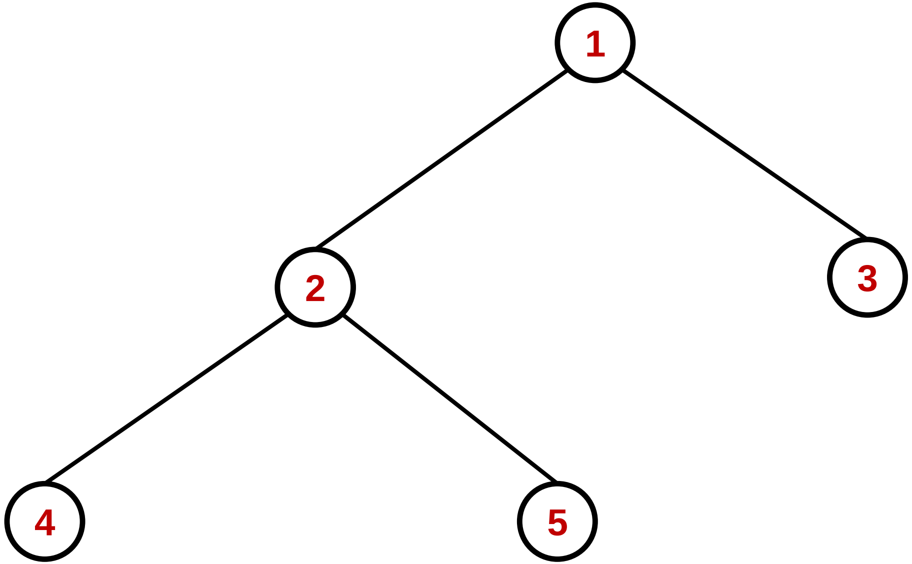
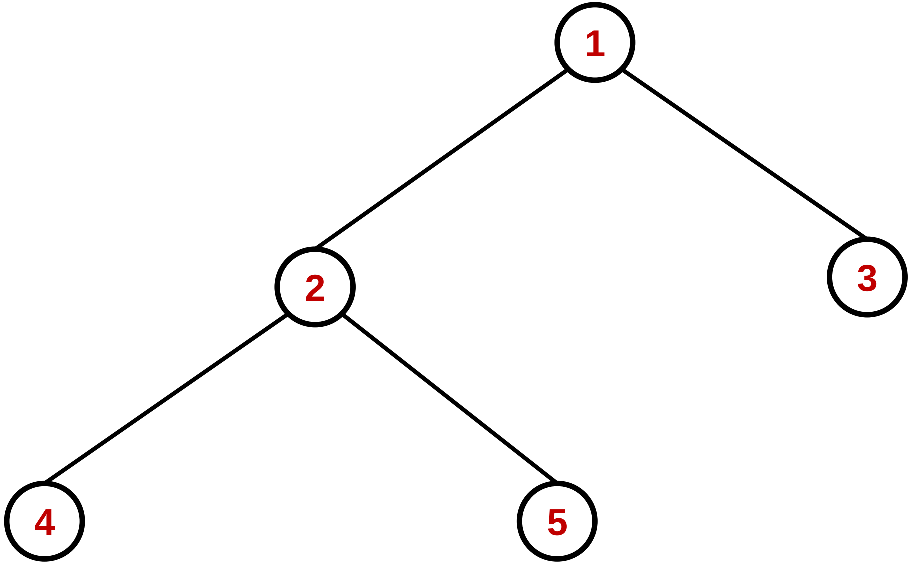


$M^{+}[4] = 1$
$M^{-}[4] = 0$

Maximum Independent Set on Trees



$M^+[4] = 1$
$M^-[4] = 0$

$M^+[5] = 1$
$M^-[5] = 0$

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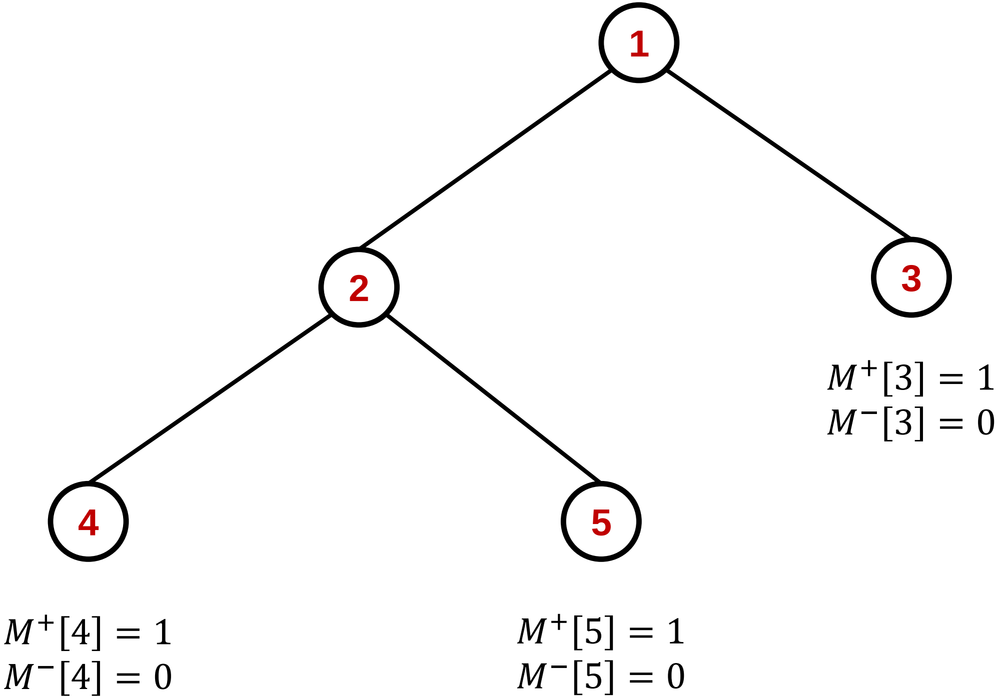

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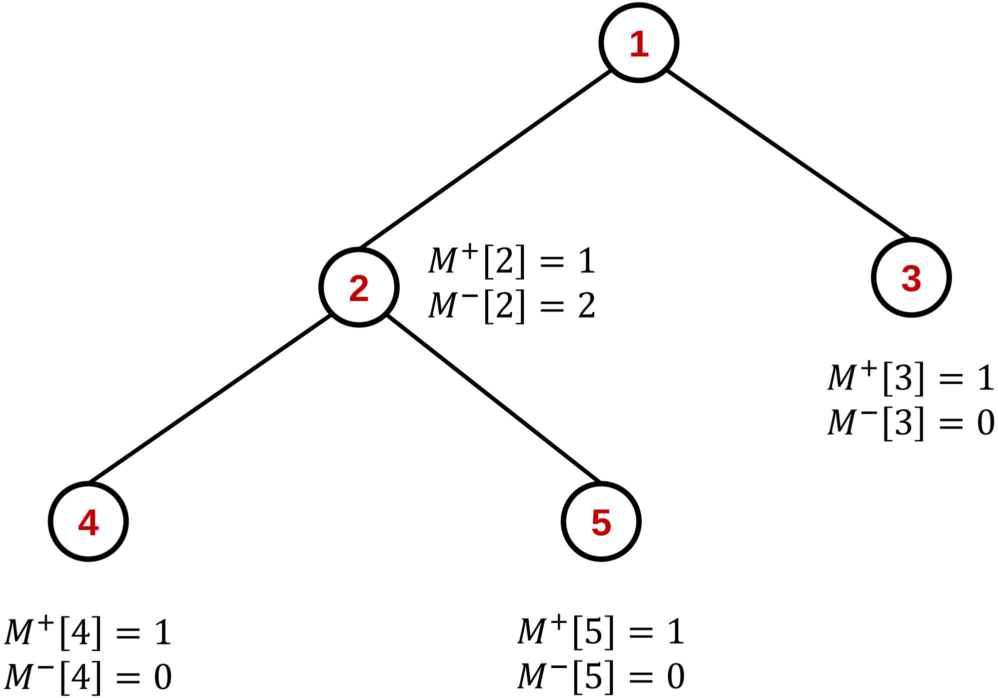

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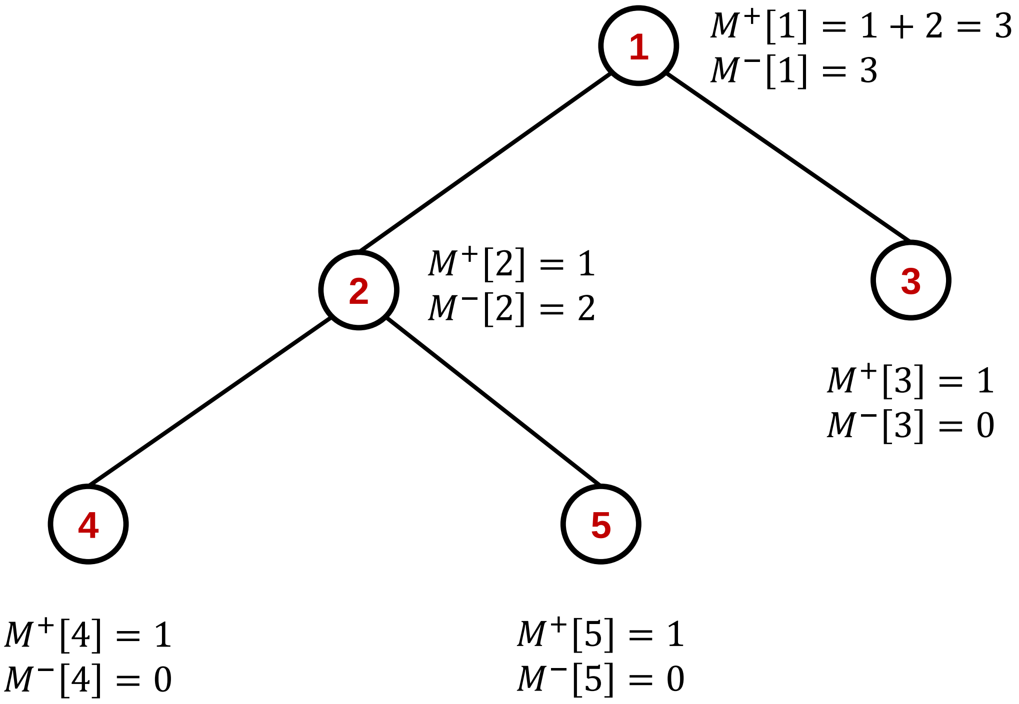
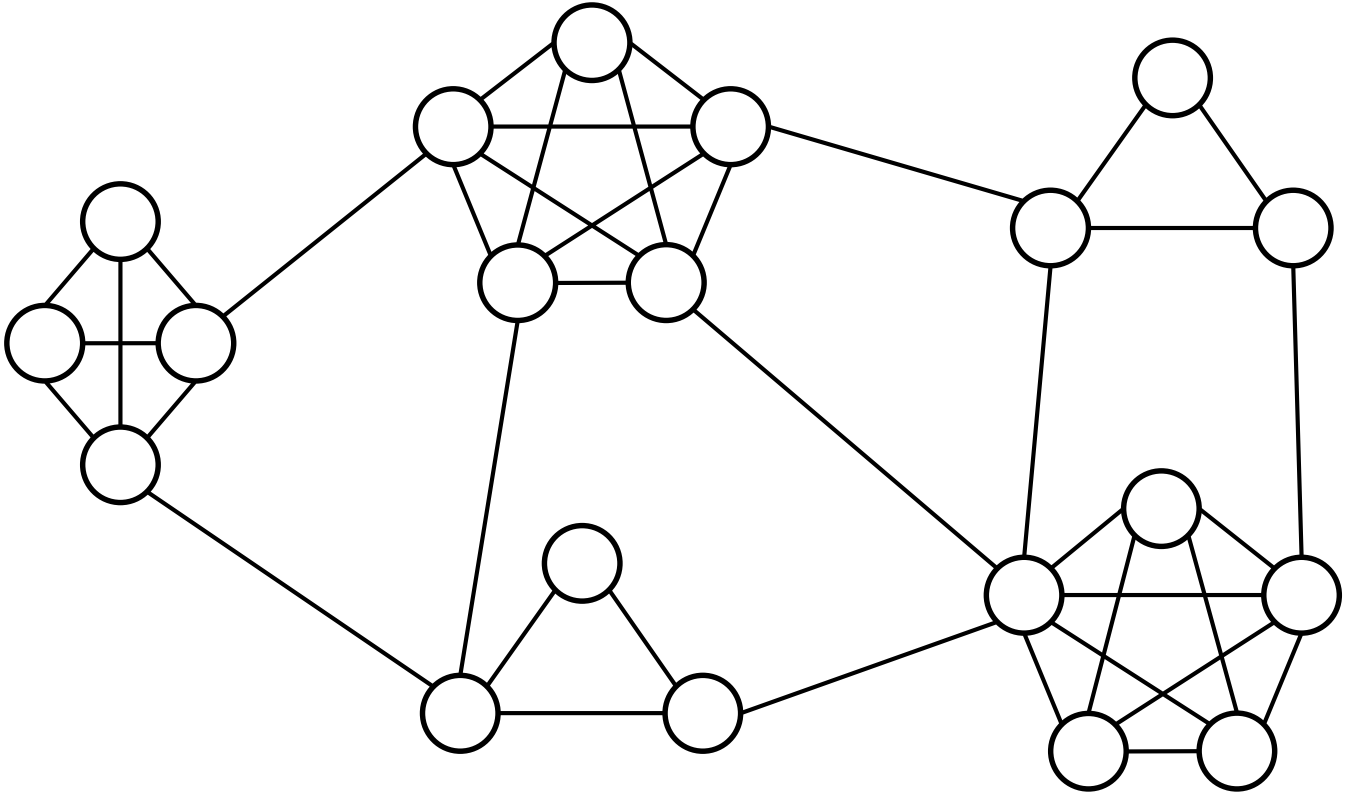


What is the MIS on this graph?



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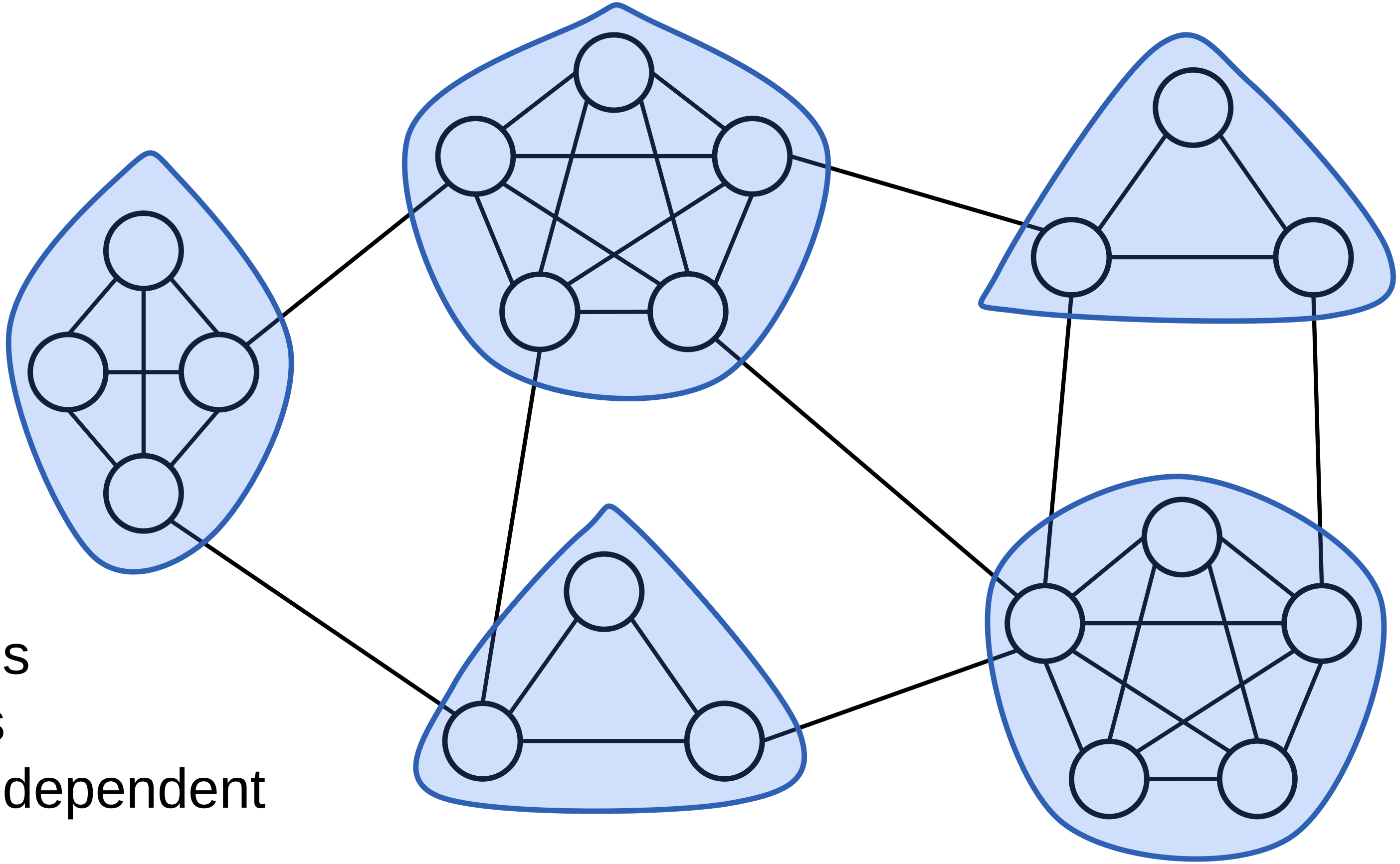

5 Modules
5 Cliques
Almost independent

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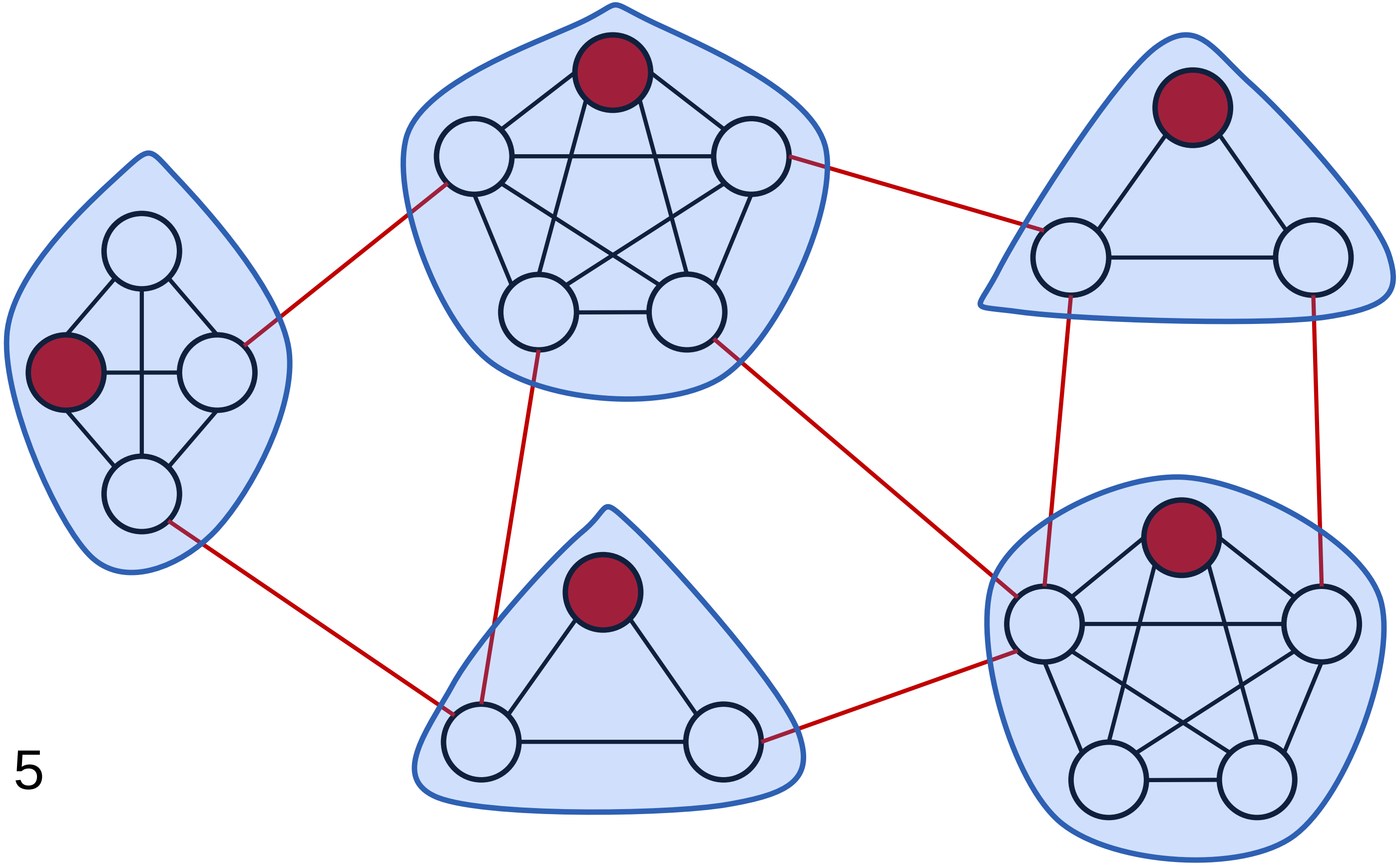
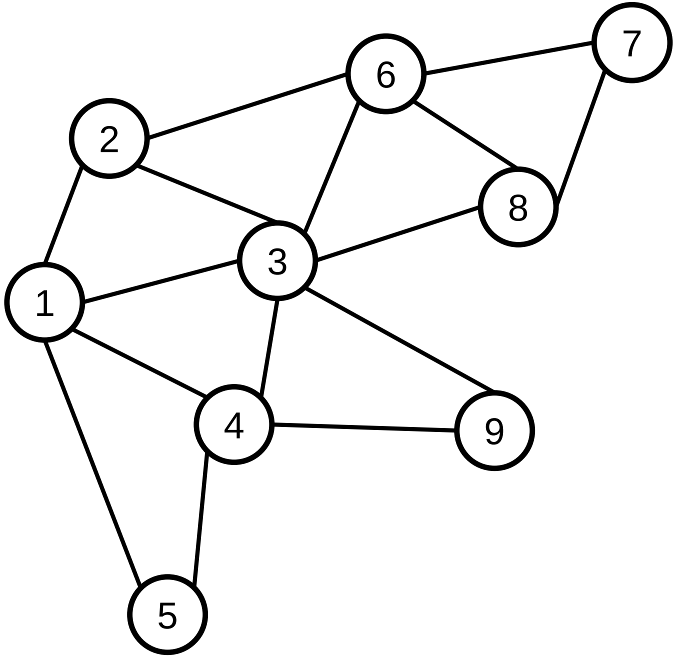


MIS = 5

Tree decomposition: intuition

- Representing a graph as a tree T
  - Nodes of T are small *modules*, called bags
  - Bags form subproblems
- We can apply dynamic programming on a tree decomposition

Tree decomposition: intuition



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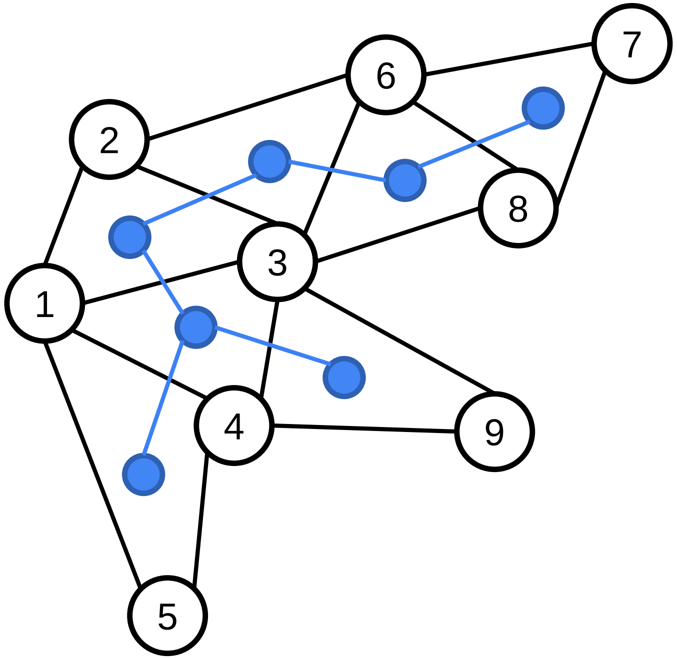

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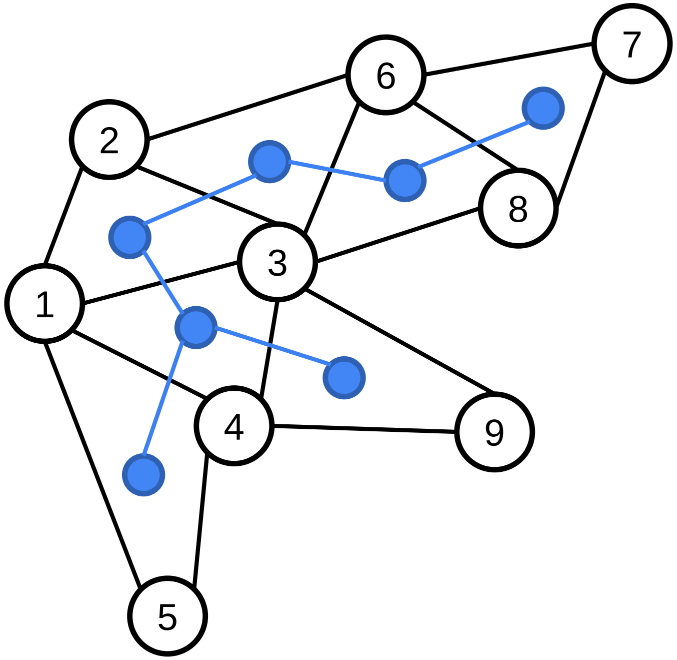

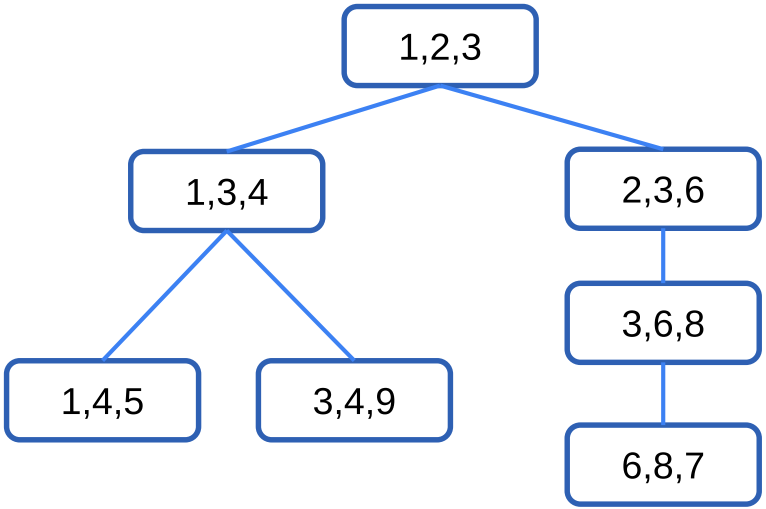

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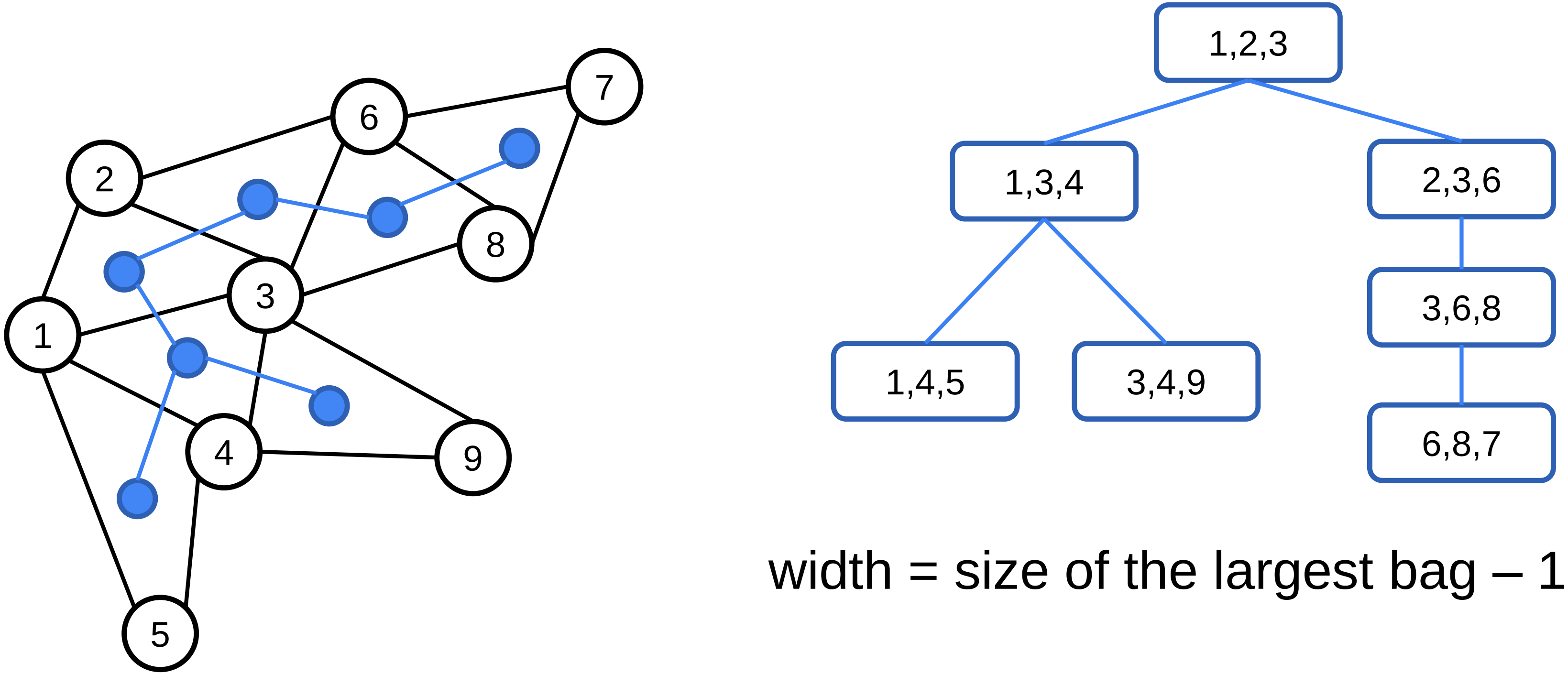


width = size of the largest bag – 1

Tree decomposition: definition

A tree decomposition of $G = (V, E)$ is a tree T of bags X such that:

- if $(u, v) \in E$ then u and v are together in some bag
- $\forall v \in V$ the bags containing v are connected in T

A graph can admit many tree decompositions

Treewidth

The treewidth is the smallest possible width among all the tree decompositions admitted by a graph

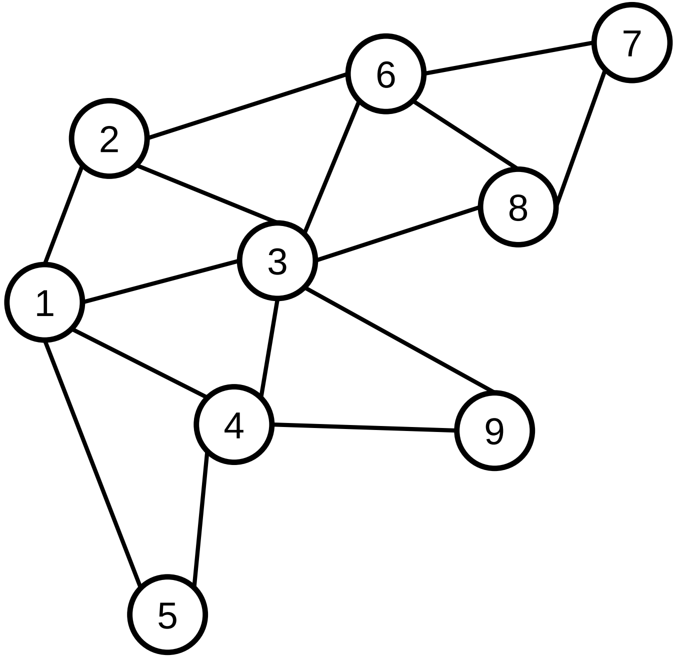
- tw(G) = 1 iff. G is a forest
- tw(G) = 2 iff. G is a series-parallel graph
- Deleting edges from G does not increase the treewidth
- Contracting edges does not increse treewidth
- Any clique in G must be in a bag

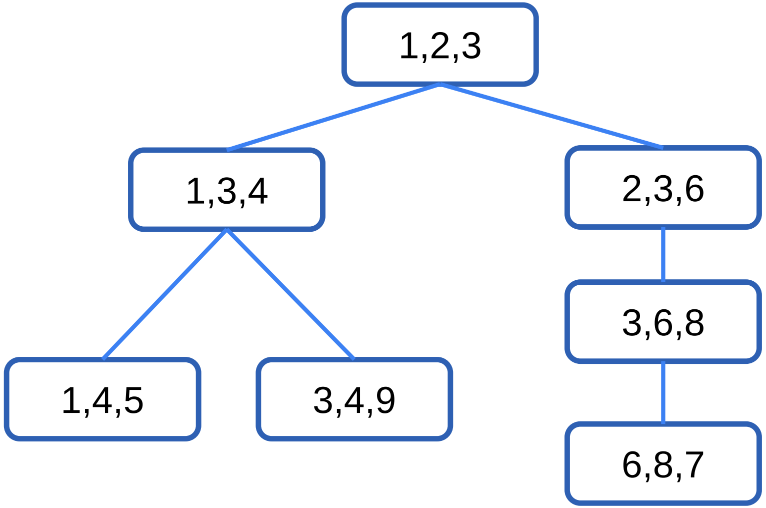
Cops and robbers

- One robber: very fast, can move on the graph
- k cops: assumed to be in helicopters (can jump through nodes)
- In order to win, cops need to corner the robber (blocking all the escape routes) and land on the same node in which the robber is

- Theorem: $tw(g) \leq k \Leftrightarrow k + 1$ cops can win the game
- Strategy given by the tree decomposition

Cops and robbers





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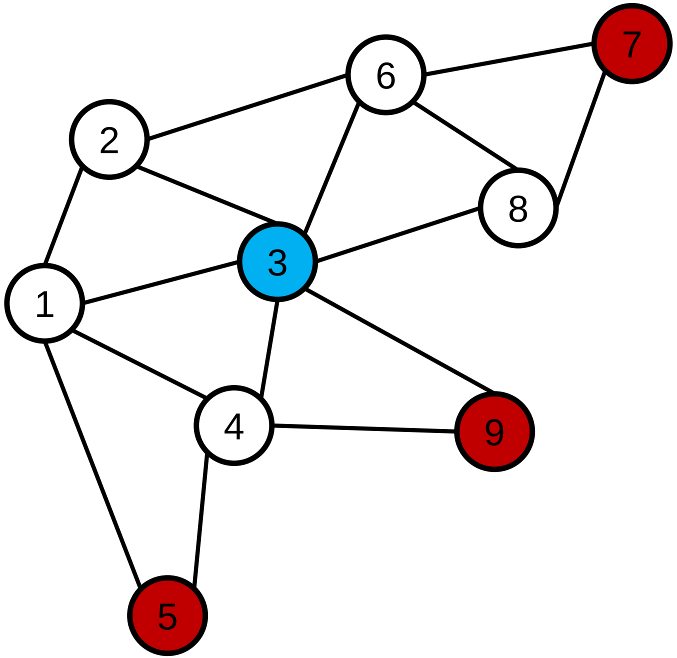

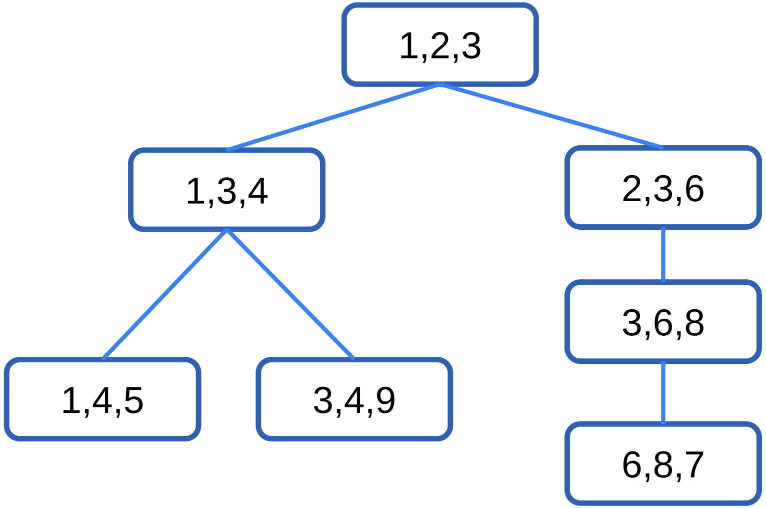

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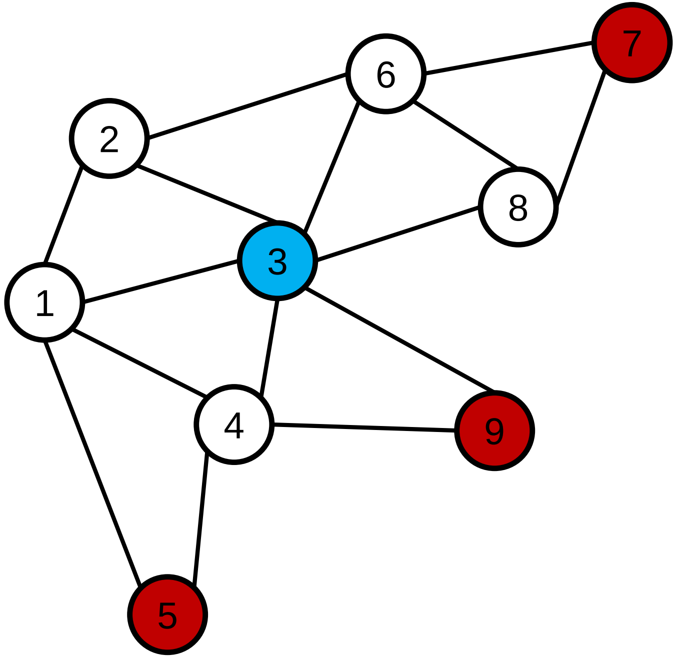

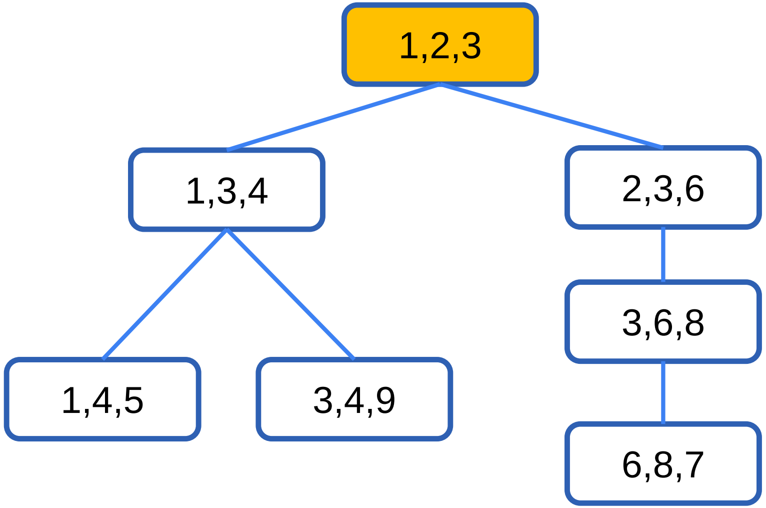

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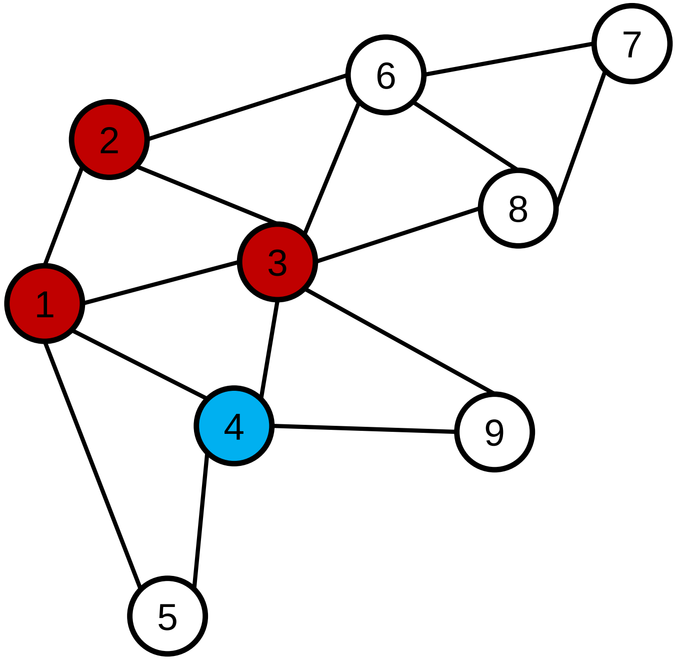

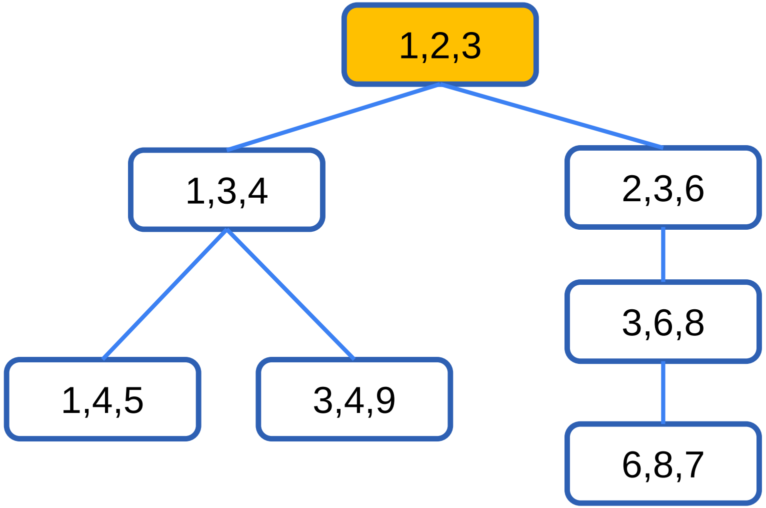

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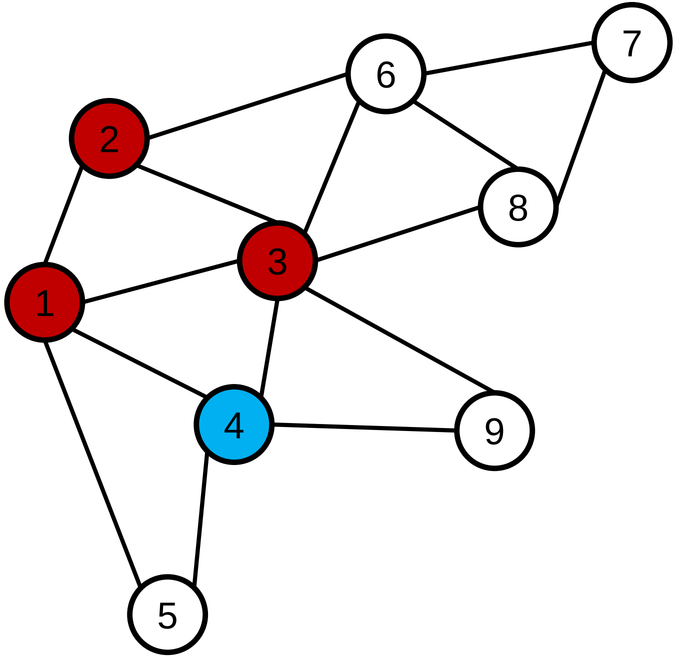

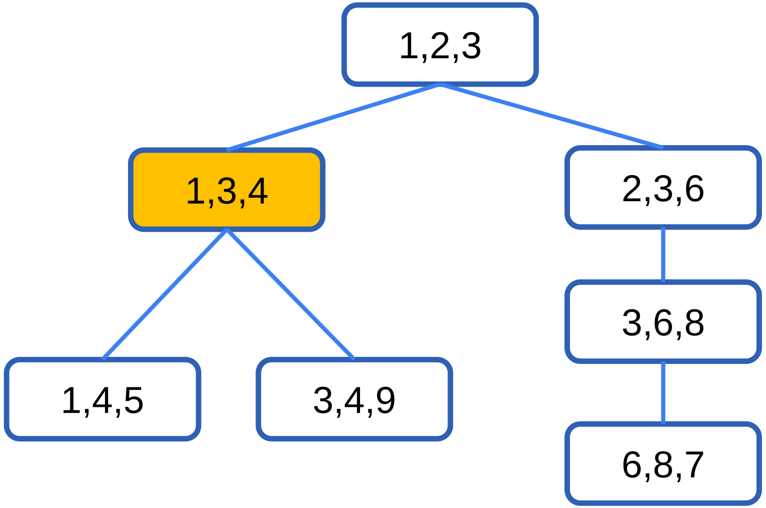

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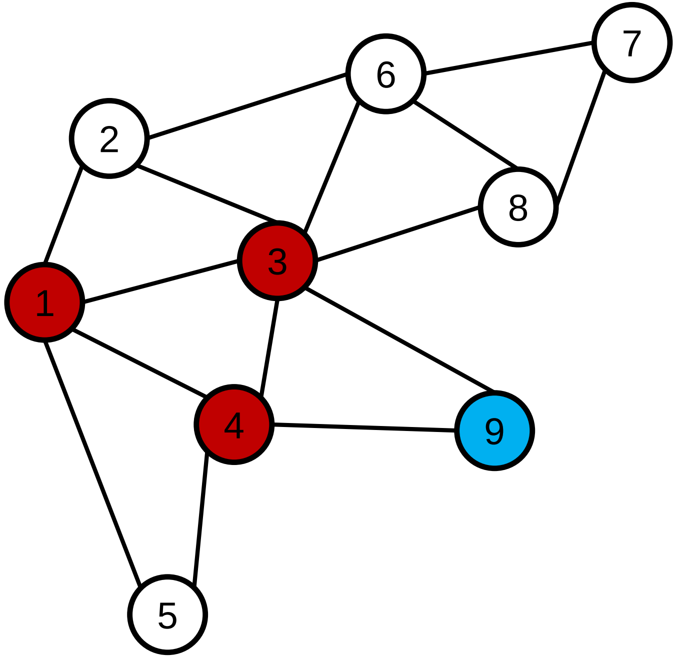

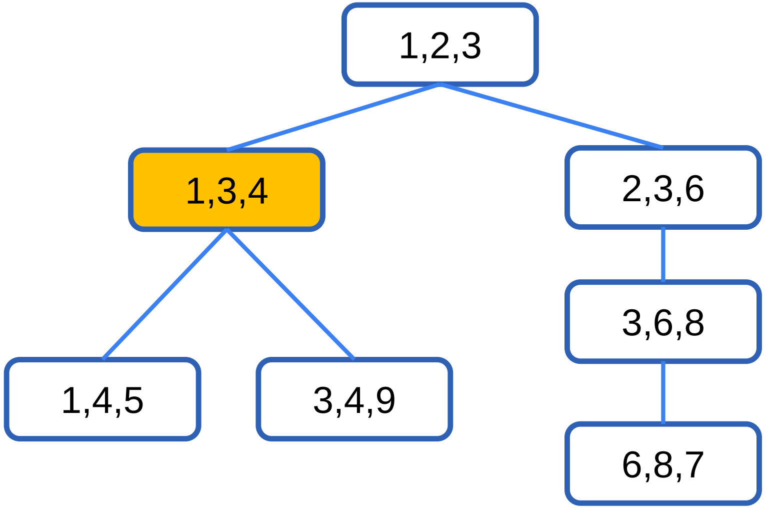

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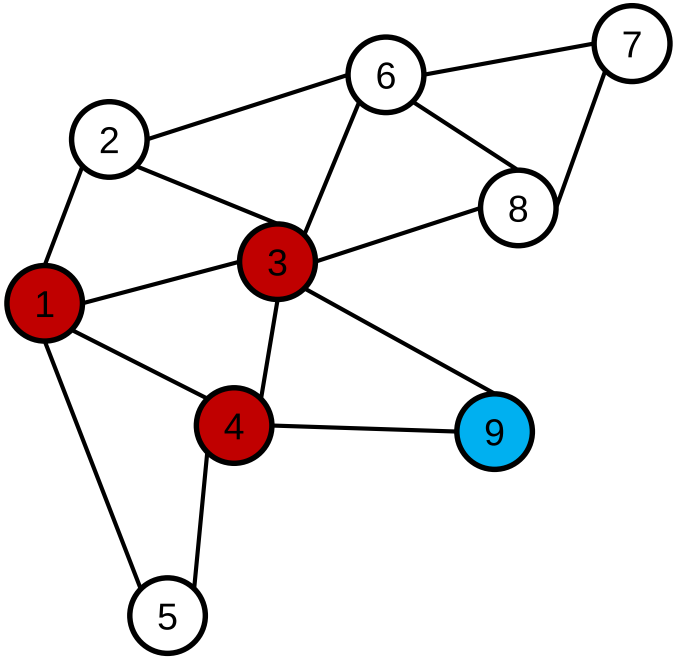

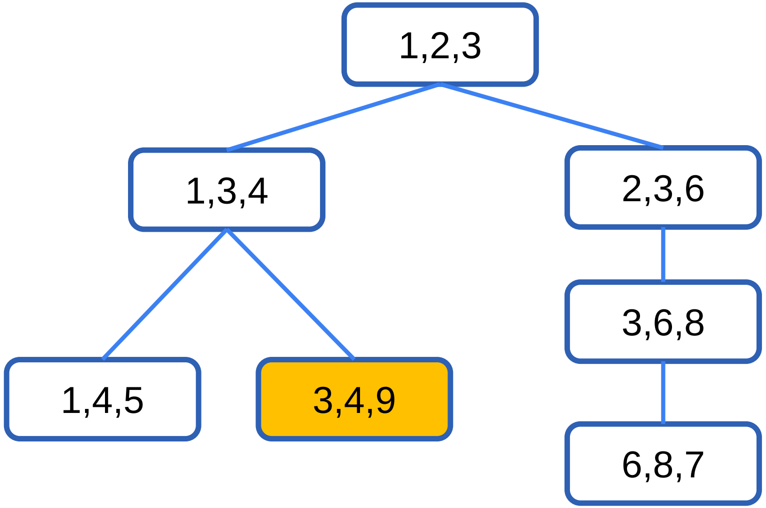

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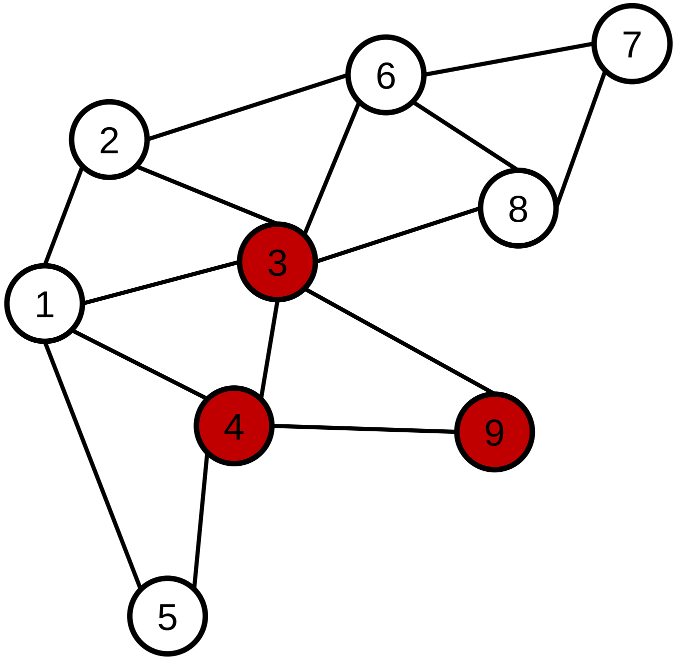

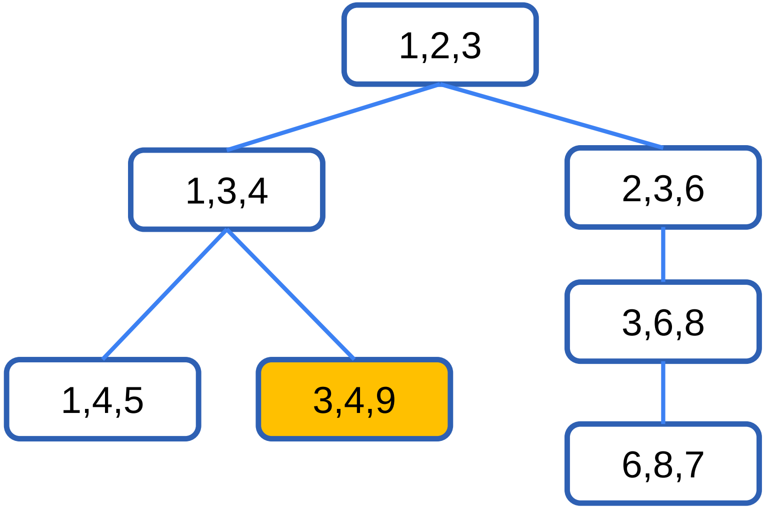

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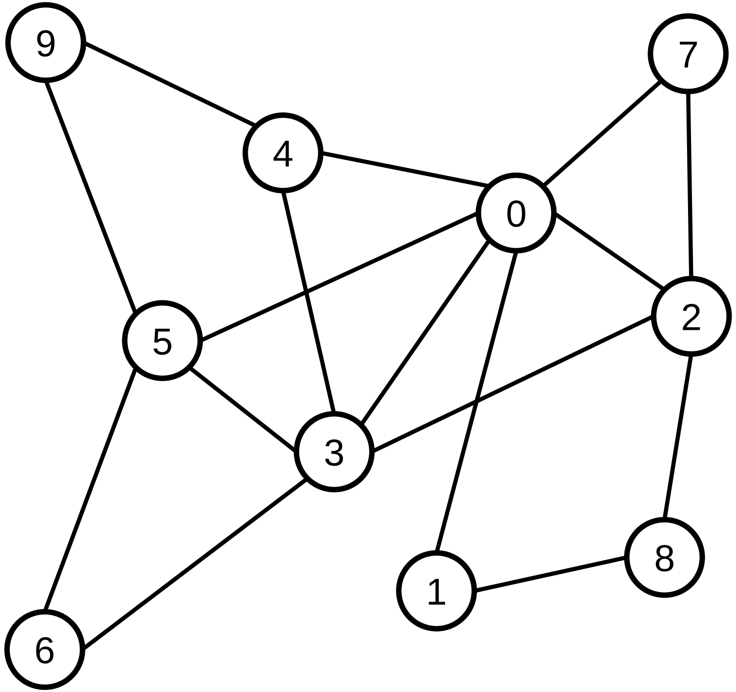

Remove a node, triangulate its neighbours, and make a bag

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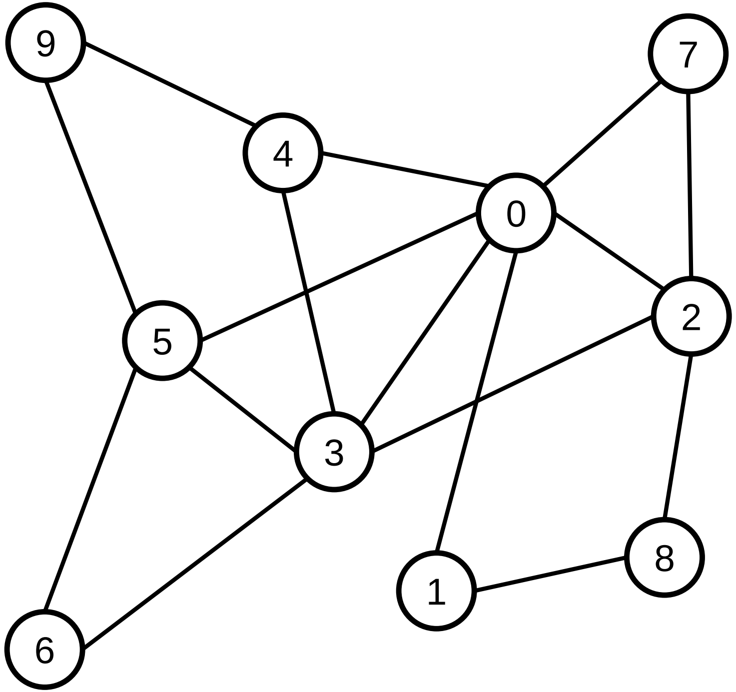

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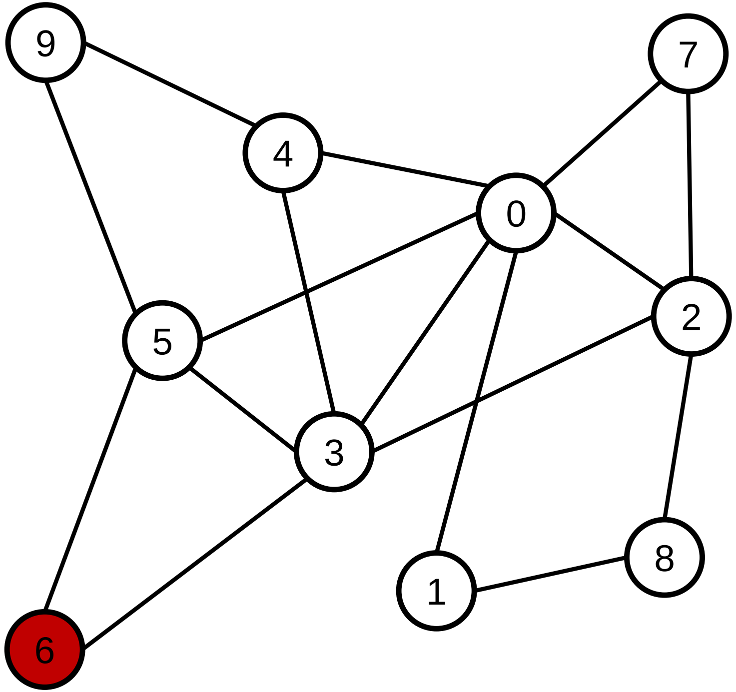

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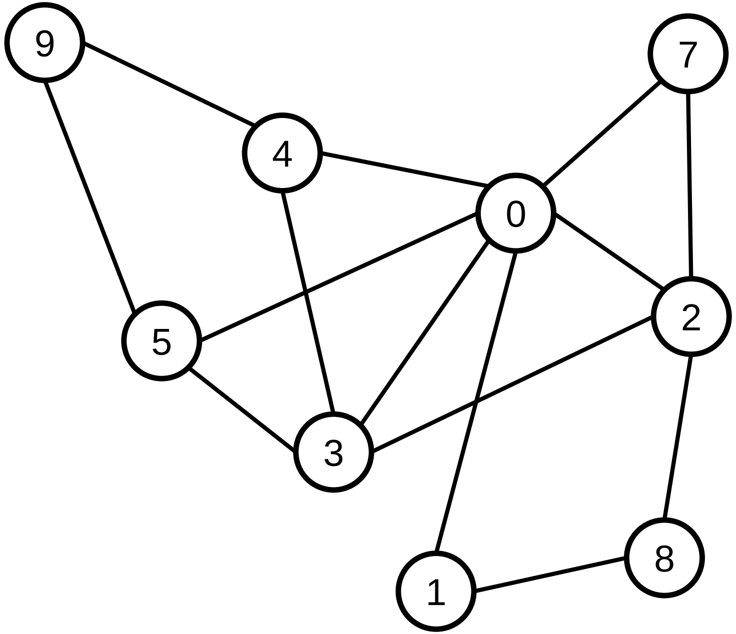

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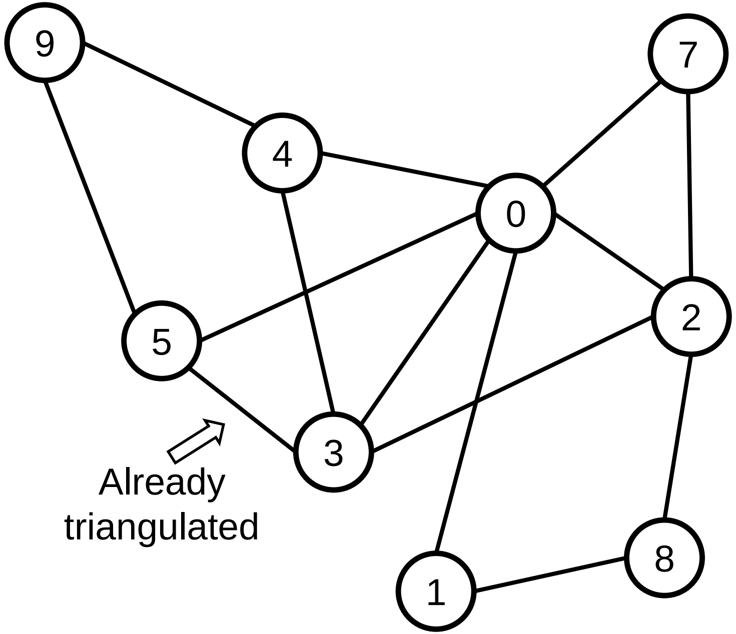

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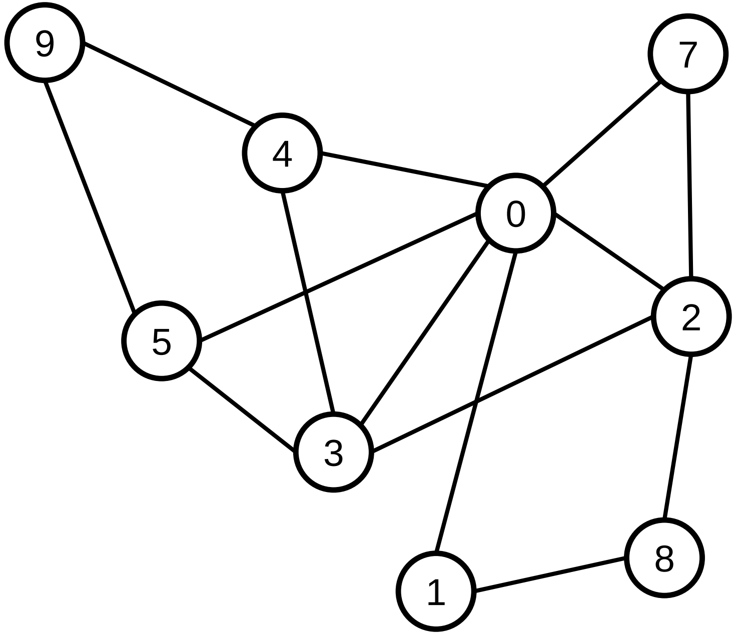

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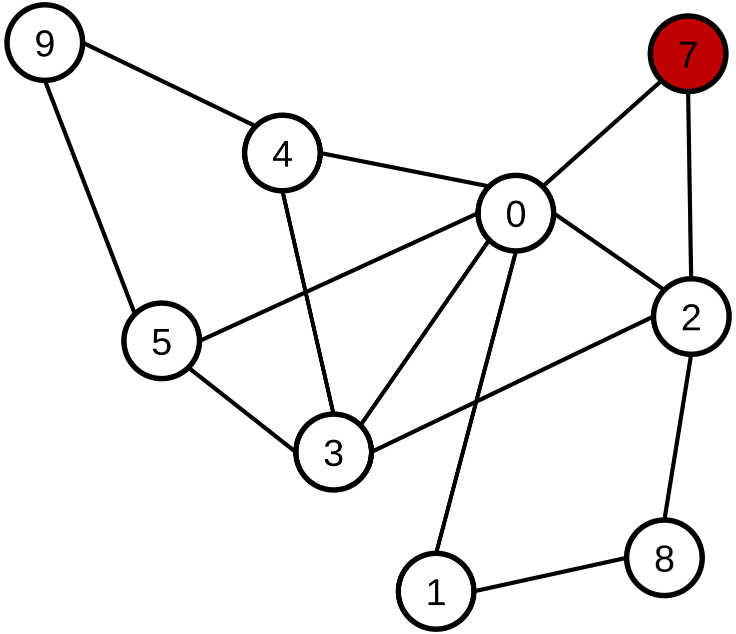

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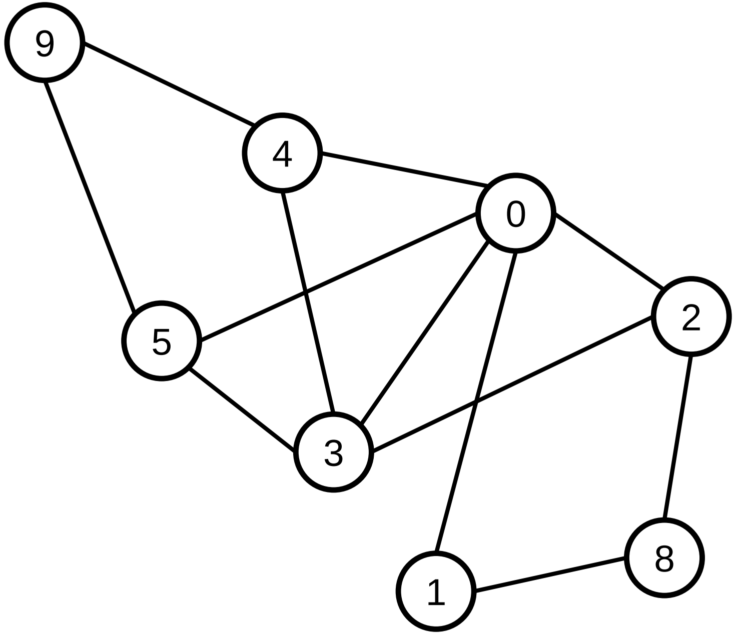

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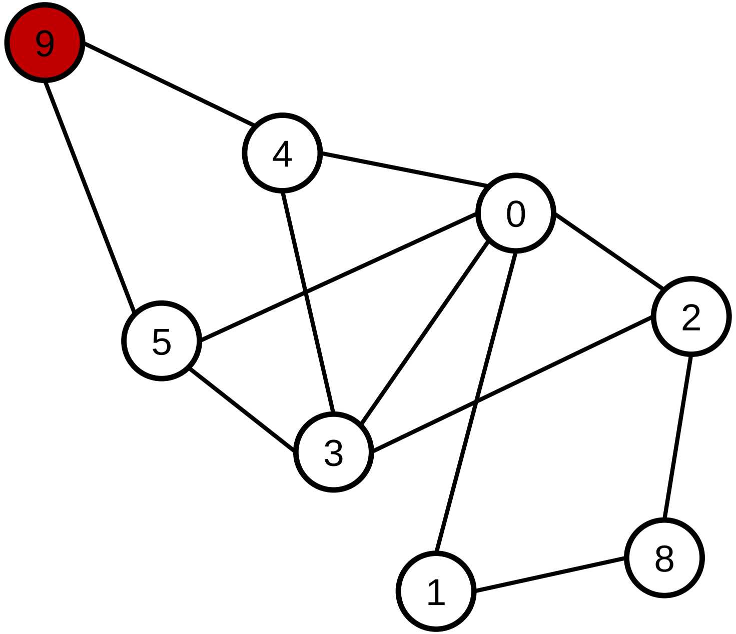

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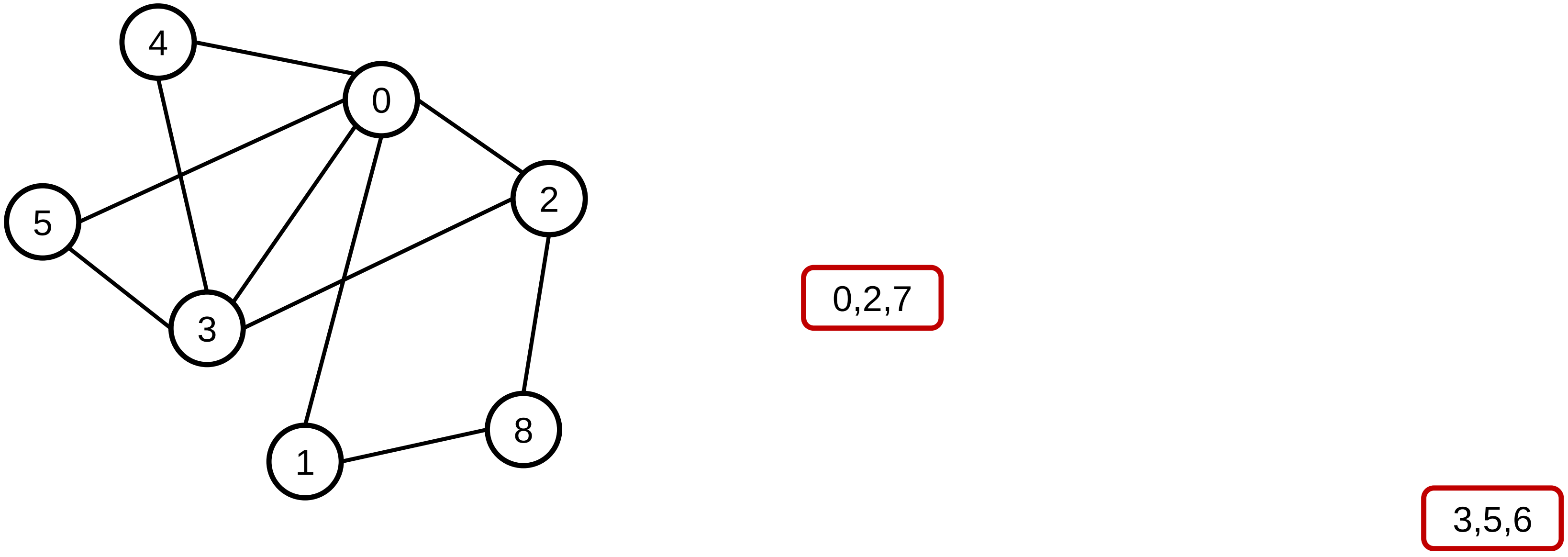

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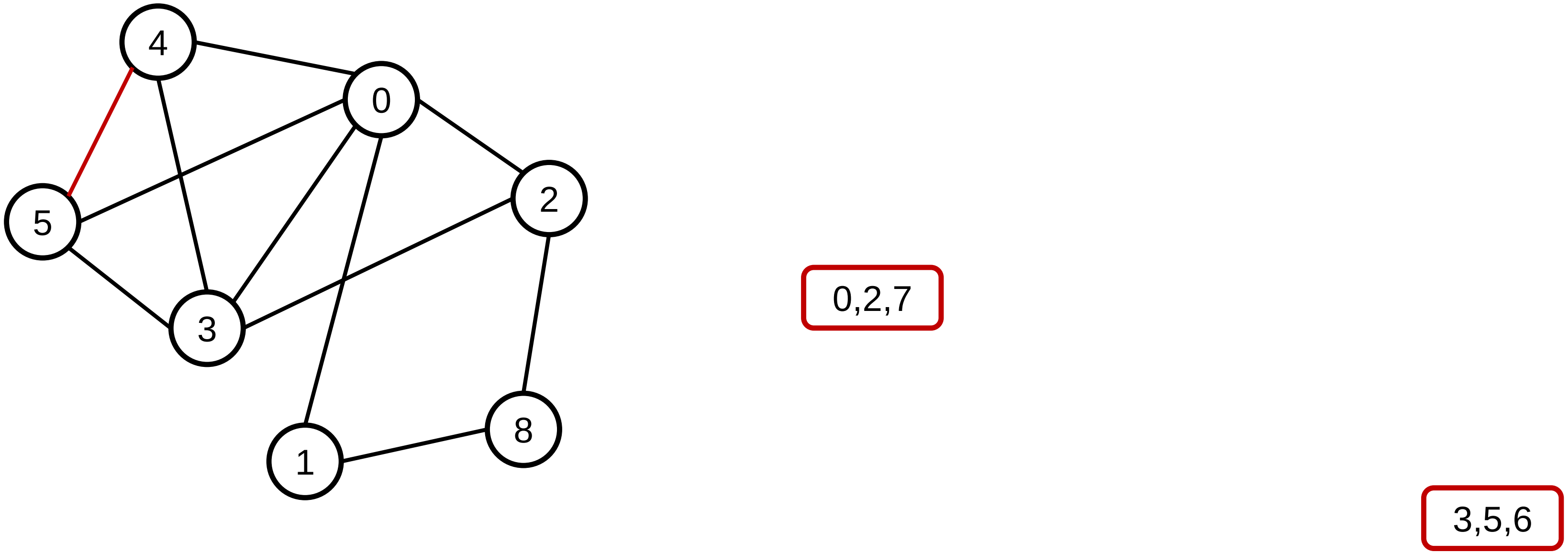

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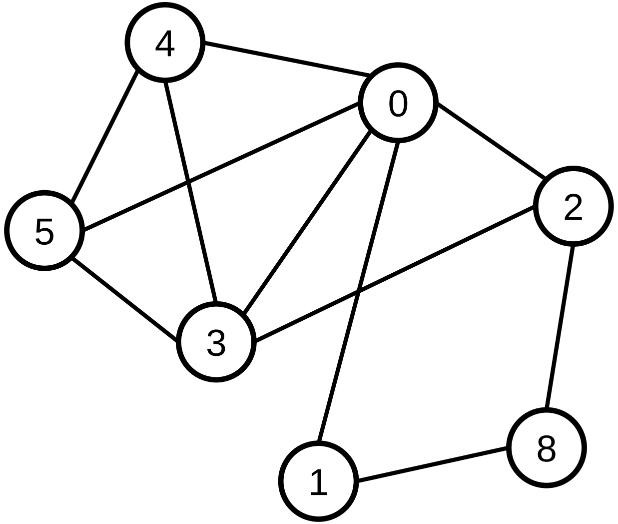

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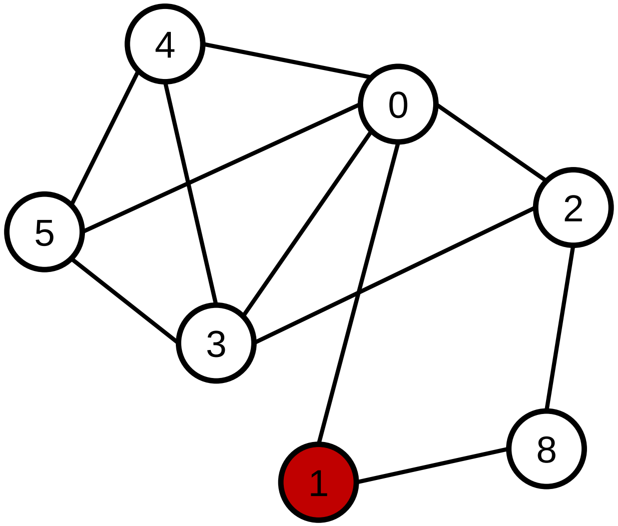

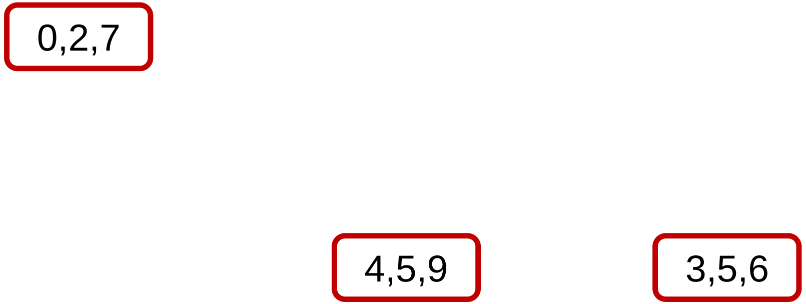

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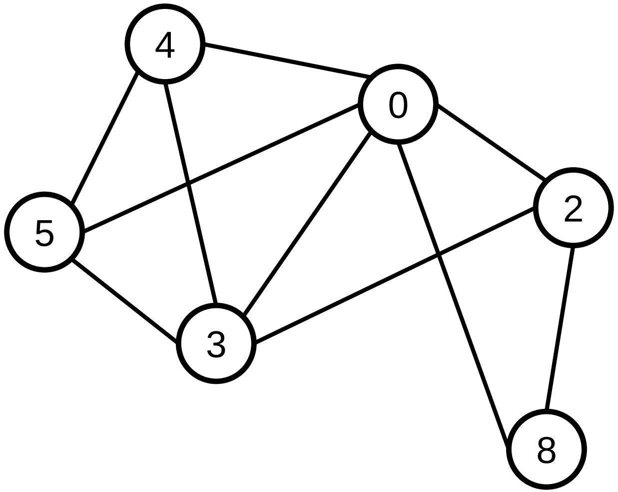

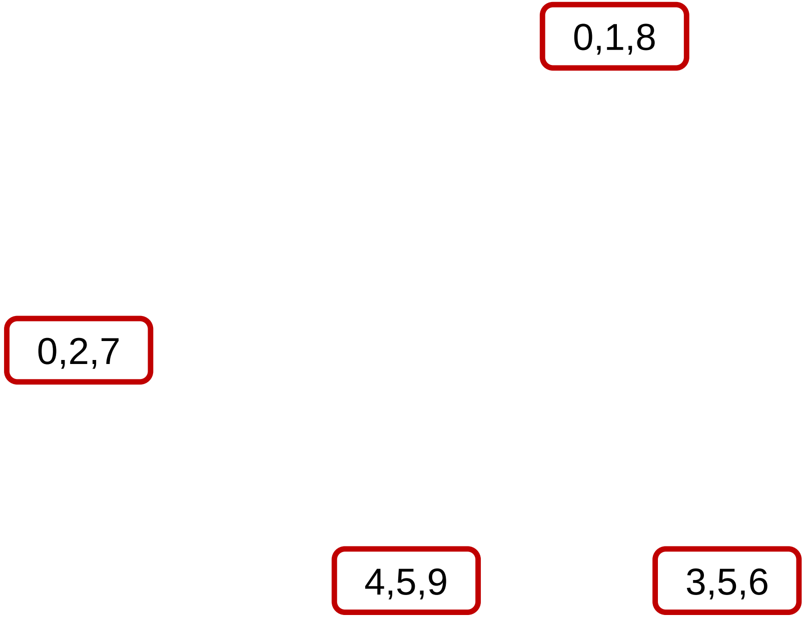

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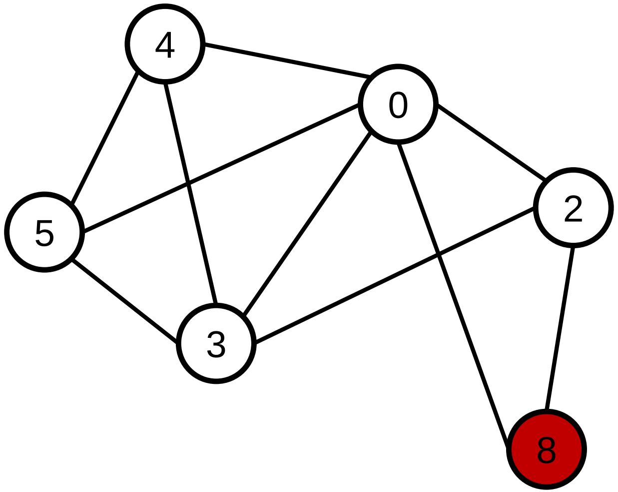

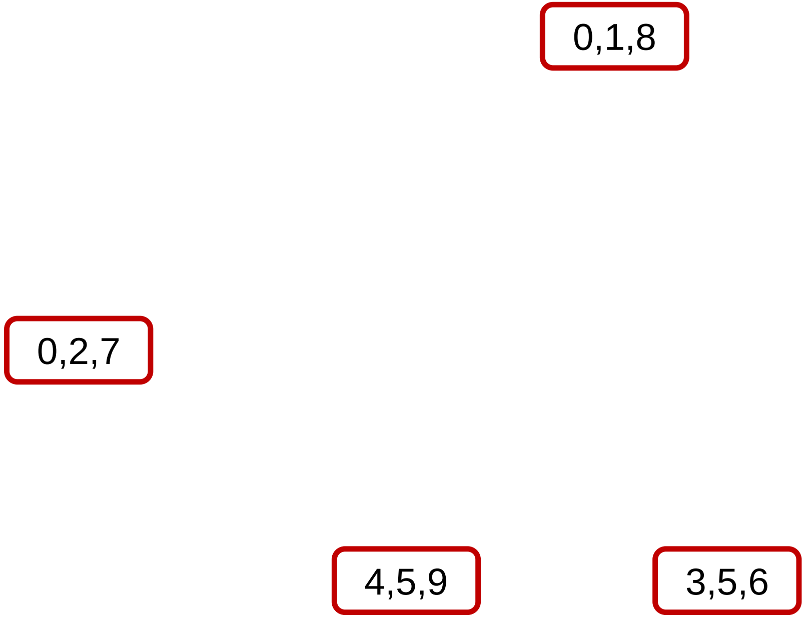

Remove a node, triangulate its neighbours, and make a bag

Computing a tree decomposition

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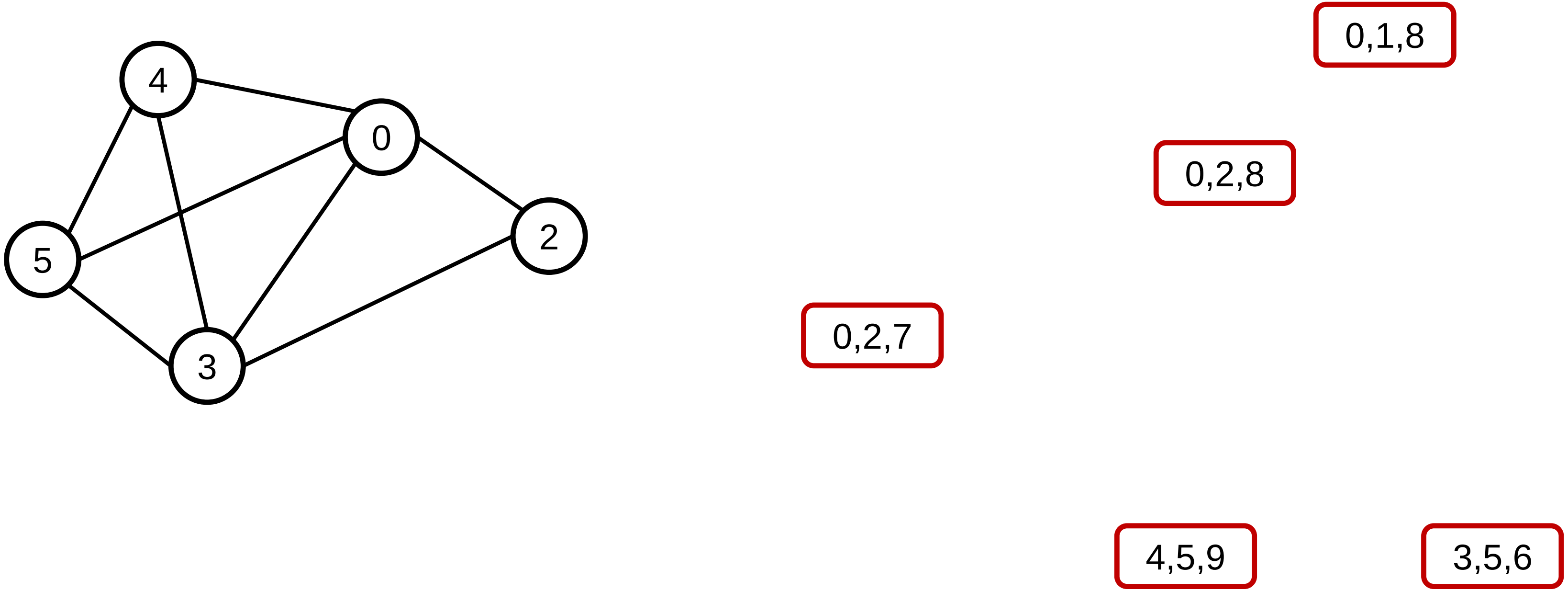

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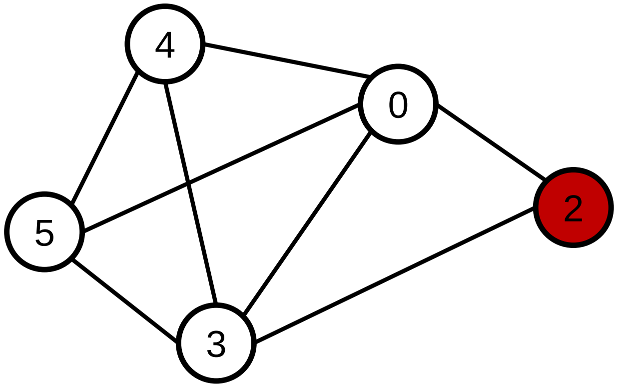

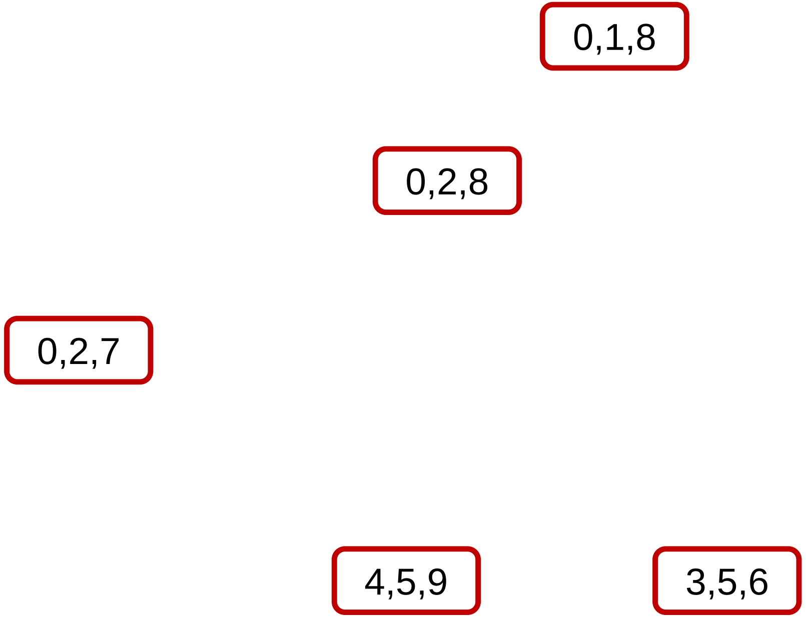

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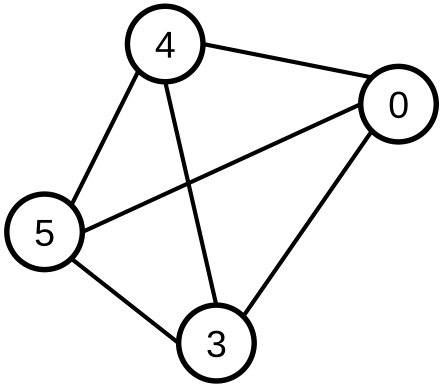

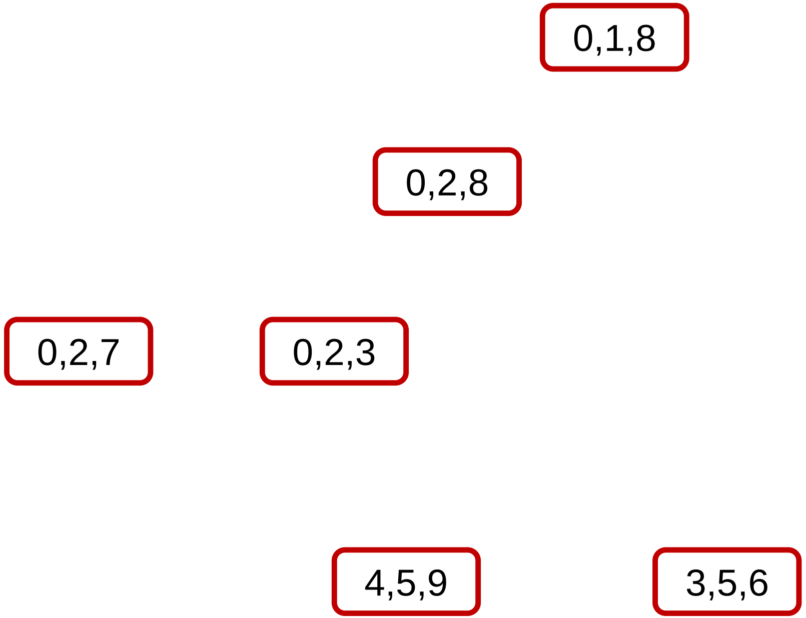

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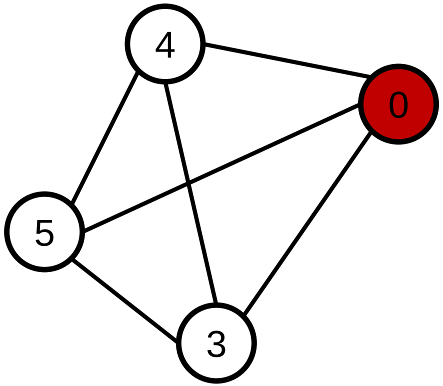

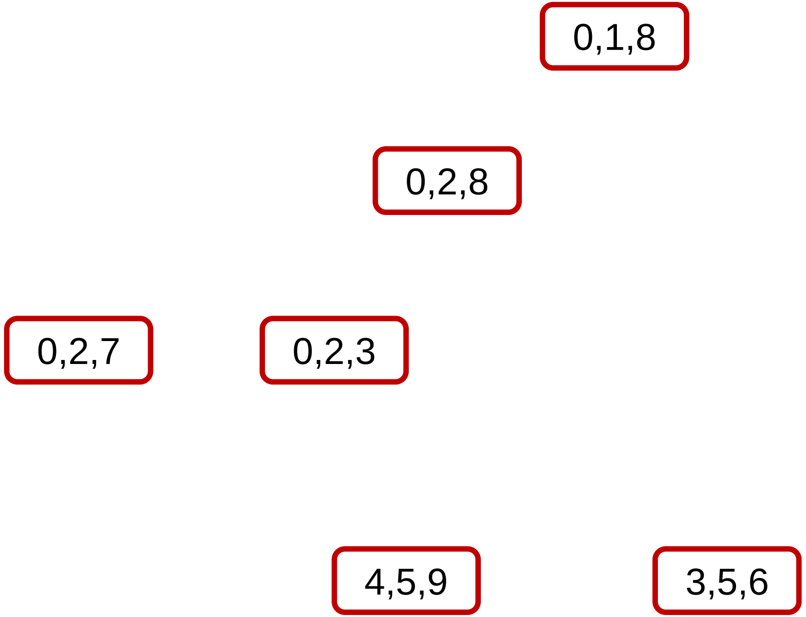

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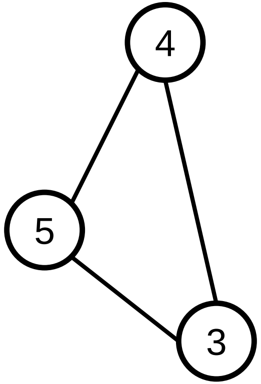

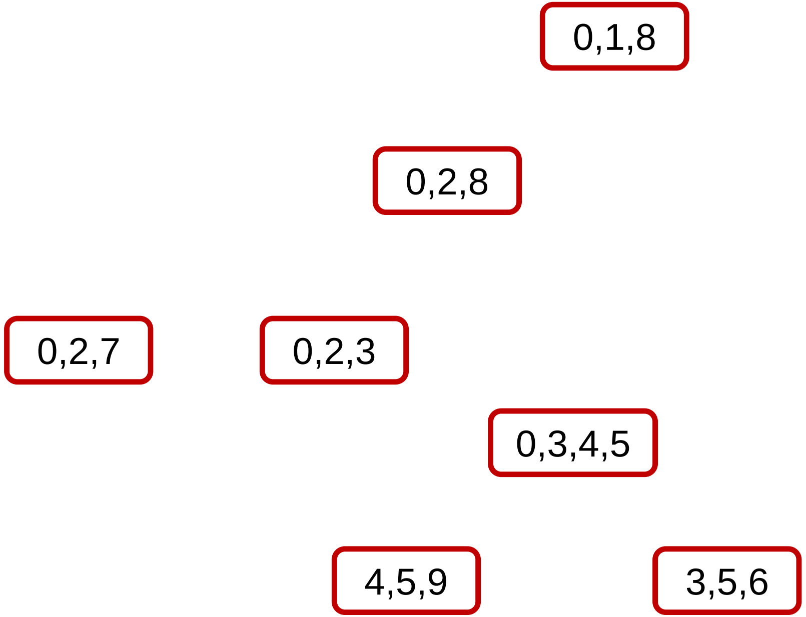

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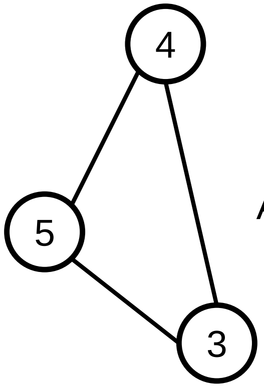
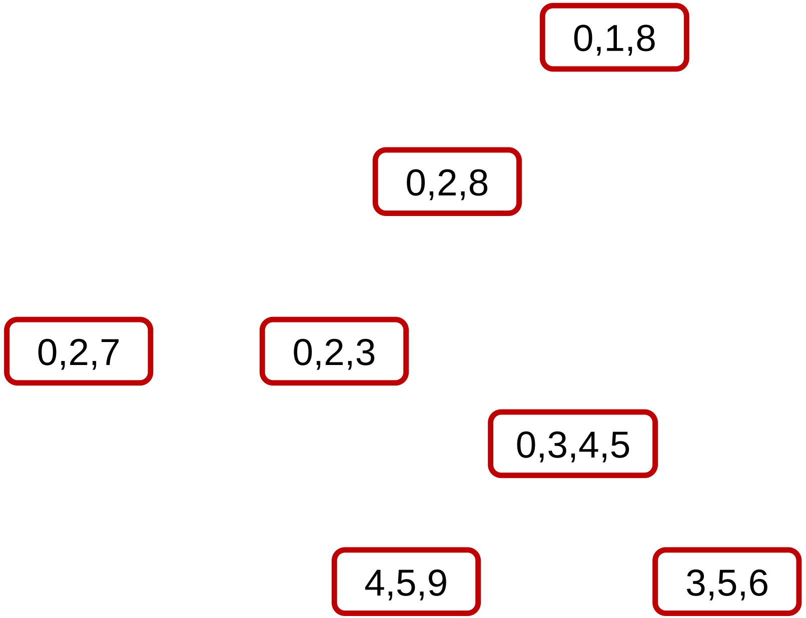


Already in bag
{0,3,4,5}



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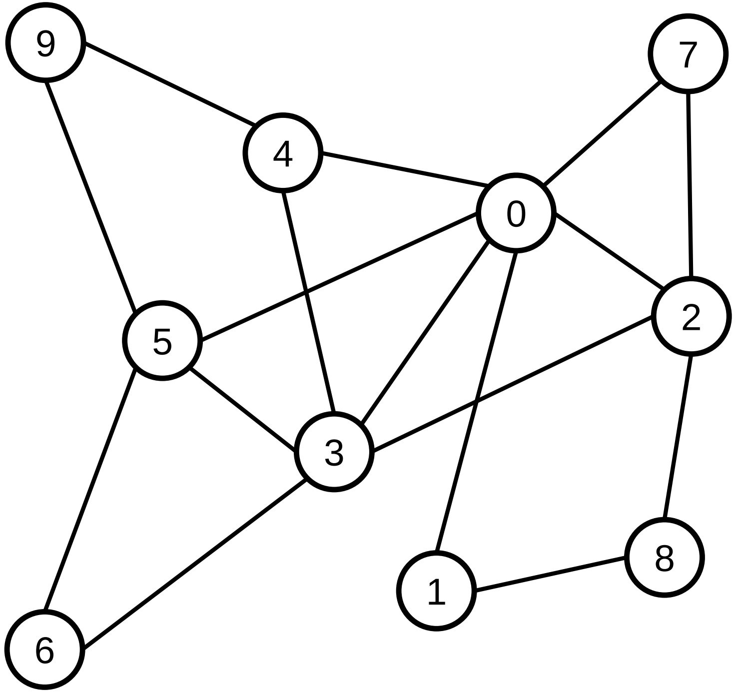

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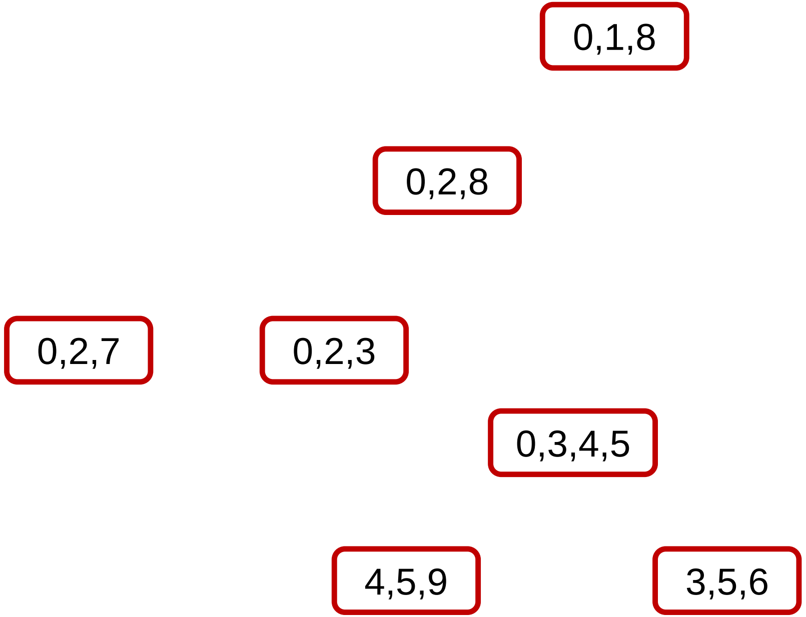
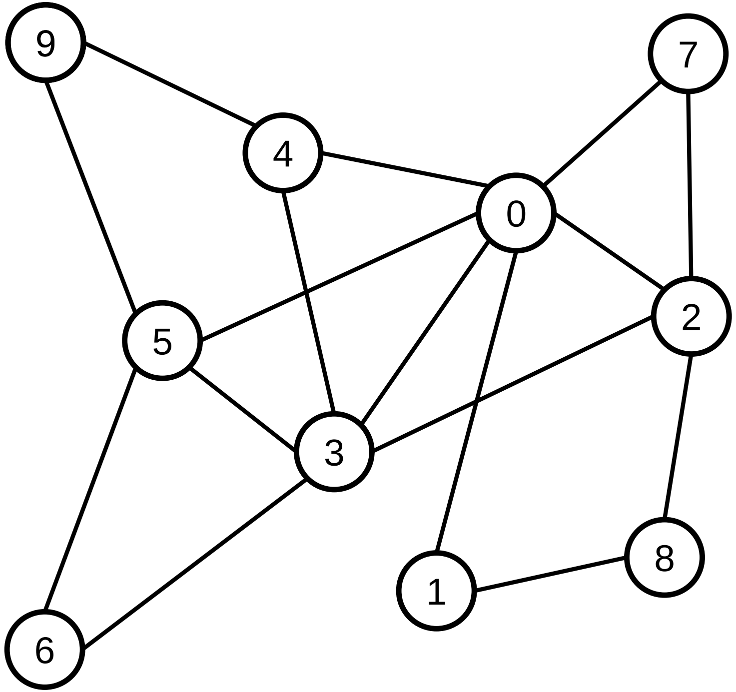


Connect each bag with the one for which the intersection is maximal

Computing a tree decomposition



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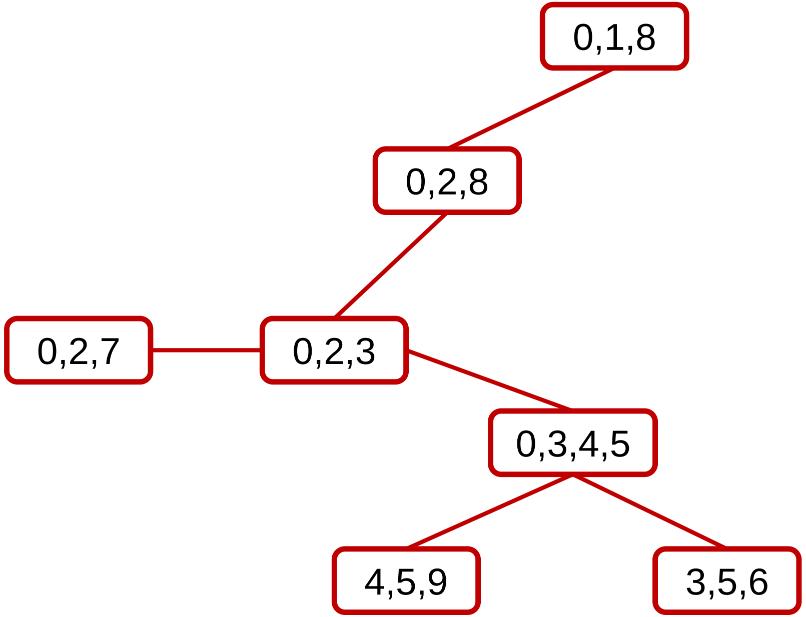

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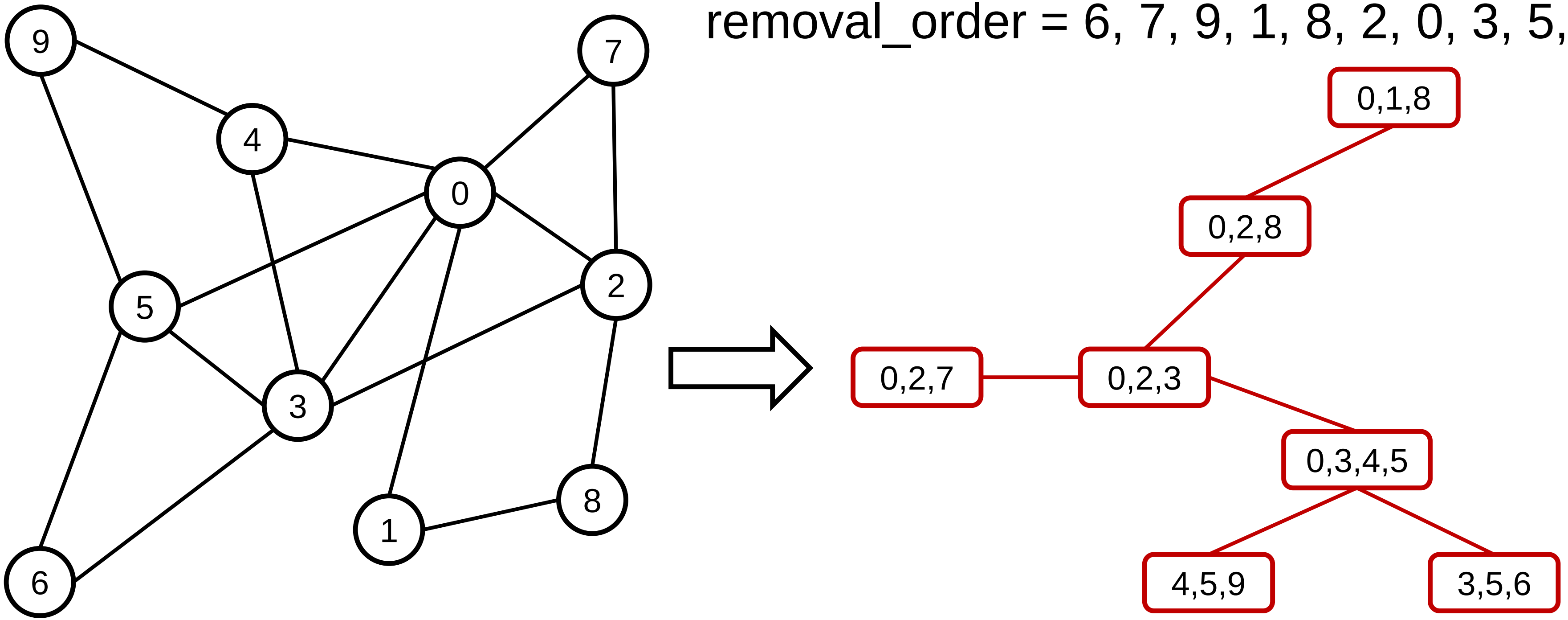

We have a tree decomposition!

Which node removal order?

- We assumed a node removal order
  - Results vary with the removal order
- Computing the treewidth (and the relative decomposition) is NP-Hard
  - $N!$ possible orderings
  - $O(2^n)$ with dynamic programming
- We use a heuristic which gives a good solution

Minimum degree heuristic

- When we remove a node to compute the tree decomposition, we remove the node with the minimum degree
- Since removing a node changes the degree of its neighbours, we compute the minimum degree every iteration