UNIVERSITÀ DELLA CALABRIA

Computational Social Choice

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Computational social choice

- Collective decision making
  - Voting
  - Preference aggregation
- Computational aspects
  - Algorithmic development of voting systems
  - Manipulability of voting systems

How should we vote?

Suppose we have 3 alternatives, 11 voters and the following votes:

5 voters think: $A \succ B \succ C$

4 voters think: $C \succ B \succ A$

2 voters think: $B \succ C \succ A$

Which of the alternatives should be chosen?

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Non-dictatorship: there is no individual for which the outcome is always equal to the preference supplied by said individual

Independence on irrelevant alternatives: if society prefers x to y, then this should not change when an individual changes their ranking of z.

Notation

- $N = \{1,2,\ldots,n\}$ individuals, with $n \geq 2$
- $X = \{X_1, X_2, \ldots, X_m\}$ alternatives
- $\mathcal{L}(X)$ is the set of permutations of X
- $R = \{R_1, R_2, \ldots, R_n\} \in \mathcal{L}(X)^N$ is a preference profile
  - $R_i \in \mathcal{L}(X)$ is the vector of preference of the i-th individual
- A social welfare function (SWF) is a function $F: \mathcal{L}(X)^N \longrightarrow \mathcal{L}(X)$

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- G is decisive iff G is decisive for all ordered pairs
- G is weakly decisive over (x, y) iff $x \succ_i y, y \succ_j x \Rightarrow x \succ y,\ i \in G, j \notin G$

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<u>Proof plan:</u>

- ► Pareto condition implies N is decisive for all pairs
- ► Lemma: if $G \subseteq N, |G| \geq 2$ is decisive for all pairs, then so is $G' \subset G$
- ► Thus, by induction, there is a coalition of size 1 (i.e. a dictator)

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- Members of G: $x' \succ x \succ y \succ y'$
- Others: $x' \succ x, y \succ y'$, and $y \succ x$

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Hence, if G is weakly decisive on one pair, G is decisive on all pairs

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Proof: Let $G_1 \cup G_2 = G, G_1 \cap G_2 = \emptyset$. Consider the following:

- Members of G_1: $x \succ y \succ z \succ \cdots$
- Members of G_2: $y \succ z \succ x \succ \cdots$

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As $G_1 \cup G_2 = G$ is decisive, society ranks $y \succ z$

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- Members of G_1: $x \succ y \succ z \succ \cdots$
- Members of G_2: $y \succ z \succ x \succ \cdots$

If society ranks $x \succ z$, exactly G_1 ranks $x \succ z$. By IIA, in any profile where G_1 ranks $x \succ z$, so does society. Hence, G_1 is weakly decisive on (x, z). So (see previous slide) G_1 is decisive on all pairs.

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Claim: $G \subseteq N, |G| \geq 2$ decisive for all pairs $\Rightarrow G' \subset G$ decisive as well

Proof: Let $G_1 \cup G_2 = G, G_1 \cap G_2 = \emptyset$. Consider the following:

- Members of G_1: $x \succ y \succ z \succ \cdots$
- Members of G_2: $y \succ z \succ x \succ \cdots$

If society ranks $z \succ x$, exactly G_2 ranks $z \succ x$. By IIA, in any profile where G_1 ranks $z \succ x$, so does society. Hence, G_2 is weakly decisive on (z, x). So G_2 is decisive on all pairs.

Hence, one of G_1 and G_2 will always be decisive.

Other desidered properties of voting systems

Majority: if a candidate is ranked first by a majority of people, then that candidate should be elected.

Condorcet winner criterion: if a candidate wins the majority vote in every head-to-head election against each other candidate, then that candidate should be elected

Many more…

Voting systems

- Plurality voting
- Borda counting
- Instant-runoff voting
- Schulze Method

Plurality voting

The candidate which is ranked first most often wins

5 voters think: $A \succ B \succ C$

4 voters think: $C \succ B \succ A$

2 voters think: $B \succ C \succ A$

Winner: A

Borda counting

It is a positional voting system. Assigns $m - 1$ to the candidate ranked first, $m - 2$ to the second, and so on, for each voter's preference vector.

5 voters think: $A \succ B \succ C$

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2 voters think: $B \succ C \succ A$

$A: 5 \times 2 + 4 \times 0 + 2 \times 0 = 10$

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$B: 2 \times 2 + 4 \times 1 + 5 \times 1 = 13$

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$C: 4 \times 2 + 2 \times 1 + 5 \times 0 = 10$

Instant-runoff voting

Extends the plurality method. Counts the times each candidate is ranked first. If the candidate with the highest number of votes has the absolute majority elects that candidate. Otherwise removes the last candidate, reassigns their votes and repeats the procedure.

Instant-runoff voting: example

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4 voters think: $C \succ B \succ A$

2 voters think: $B \succ C \succ A$

A	5
B	2
C	4

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A	5	5
B	2	
C	4	6

Winner: C

Schulze Method

A method base on pairwise matrix and graphs. Used by many organizations including Wikimedia, Debian, Ubuntu and Gentoo.

- Compute the pairwise matrix
- Find the all pairs widest path on the pairwise matrix
- Count how many times each candidate beats all the others on the all pairs widest path matrix

Schulze Method: example

5 voters think: $A \succ B \succ C$

4 voters think: $C \succ B \succ A$

2 voters think: $B \succ C \succ A$

	A	B	C
A	0	5	5
B	5	0	7
C	6	4	0

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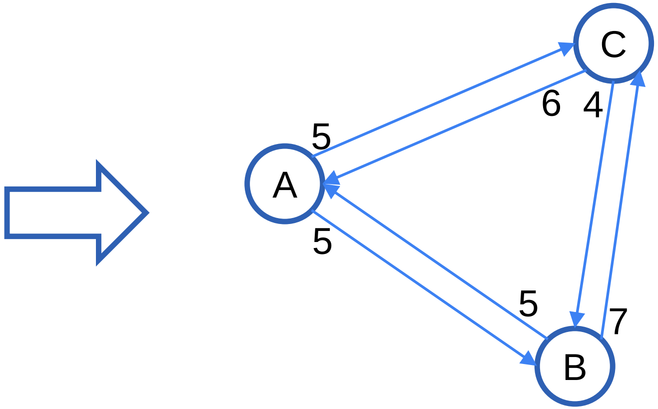

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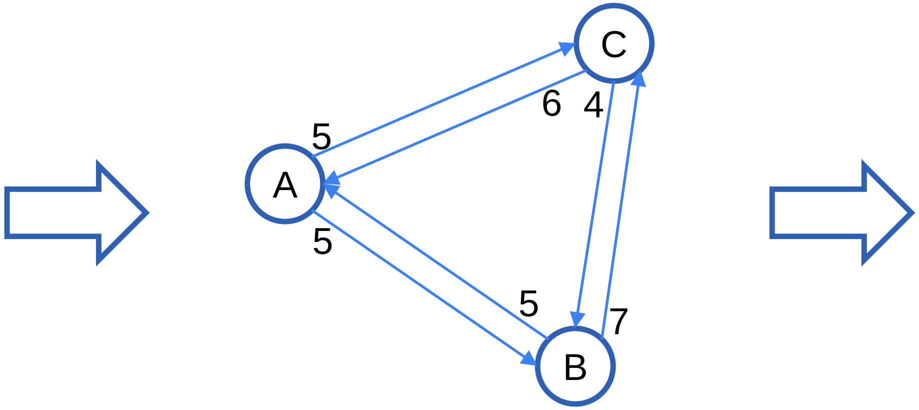

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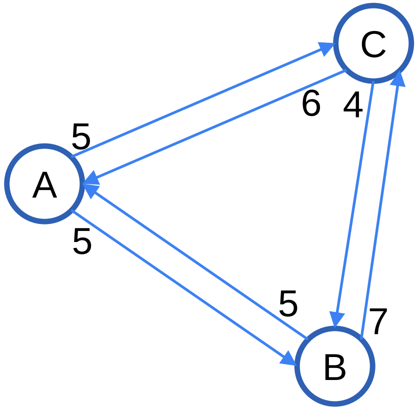

	A	B	C
A	0	5	5
B	6	0	7
C	6	5	0

Winner: B

All pairs widest path pseudocode

Input: d[i,j], the number of voters who prefer candidate i to candidate j.

Output: p[i,j], the strength of the strongest path from candidate i to candidate j.

```
for i from 1 to C
    for j from 1 to C
        if i ≠ j then
            if d[i,j] > d[j,i] then
                p[i,j] := d[i,j]
            else
                p[i,j] := 0

for i from 1 to C
    for j from 1 to C
        if i ≠ j then
            for k from 1 to C
                if i ≠ k and j ≠ k then
                    p[j,k] := max (p[j,k], min (p[j,i], p[i,k]))
```