Mass Lumped triangular elements for the Wave Equation

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The Scalar Wave Equation

Let Ω be a domain in $\mathbb{R}^2$, and let $(0, T)$ be the time domain. The **scalar wave equation** can be written as:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\mathbf{x}, t) - \nabla \cdot (k(\mathbf{x}) \nabla u(\mathbf{x}, t)) = f(\mathbf{x}, t) & (\mathbf{x}, t) \in \Omega \times (0, T) \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & \mathbf{x} \in \Omega \\ \dfrac{\partial u}{\partial t}(\mathbf{x}, 0) = u_1(\mathbf{x}) & \mathbf{x} \in \Omega \\ u(\mathbf{x}, t) = g(\mathbf{x}, t) & (\mathbf{x}, t) \in \partial\Omega \times (0, T) \end{cases}$$

where $u : \Omega \times (0, T) \to \mathbb{R}$.

Fully-discrete formulation

find $U^{n+1} \in \mathbb{R}^{N_h}$ s.t.

$$MU^1 = (M - \frac{\Delta t^2}{2}A)U_0 + \Delta t M \dot{U}_0 + \frac{\Delta t^2}{2}F^0$$
$$MU^{n+1} = (2M - \Delta t^2 A)U^n - MU^{n-1} + \Delta t^2 F^n \;\; \forall n = 1, ..., N_t$$

with $U^n \approx U(t^n)$ and $F^n \approx F(t^n)$.
Apart from the first step, the solution of this system is formally given by

$$U^{n+1} = 2U^n - U^{n-1} + \Delta t^2 M^{-1}(F^n - AU^n)\,.$$

Different ML techniques

- **row sum method**:

$$M_{ii}^{lumped} = \sum_j M_{ij} \; ;$$

- **diagonal scaling**:

$$M_{ii}^{lumped} = cM_{ii} \; ,$$

  with c scaling constant;

- **nodal points method**: use a quadrature rule in the nodal points; to avoid numerical problems such as negative quadrature weights we need to **enrich** the FE space.

The fundamental lemma

Lemma 1

If the nodes of the finite element space V_h and the quadrature points **coincide***, i.e., if*

$$\{\mathbf{a}_i\} = \{\hat{\mathbf{a}}_l\} \tag{1}$$

then one has **mass lumping***.*

Conditions on quadrature weights

In order to have a Lumped Mass matrix that makes sense we need to have **strictly positive quadrature weights**.

Let $\{\hat{\mathbf{a}}_l, \omega_l\}$ be our quadrature rule, we need:

$$\omega_l > 0, \qquad \forall l = 1, \dots, n$$

The case degree $= 2$

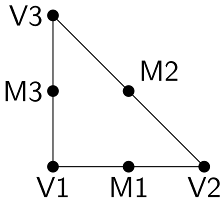

Figure 1: Ref. Triangle of P_2

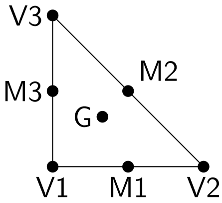

Figure 2: Ref. Triangle of $\tilde{P}_2$

$$w_V = 0 \text{ and } w_M = \frac{1}{3}$$

$$w_V = \frac{1}{20},\ w_M = \frac{2}{15} \text{ and } w_G = \frac{9}{20}$$

The mesh

- triangular mesh adopted to handle **complex domains**;
- mesh designed with `pdetool`;
- a whole section of the code designed to **incorporate** the `pdetool` mesh in our code.

Error estimate

Theorem 2

Under suitable conditions on the solution u (at least $u \in H^r$) and on the Ω, f, g, u_0 and u_1, we have that:

$$||u - u_h||_{H^m} \leq C(h^{r+1-m} + dt), \quad m = 0, 1$$

where u_h the discrete solution.

h-convergence analysis

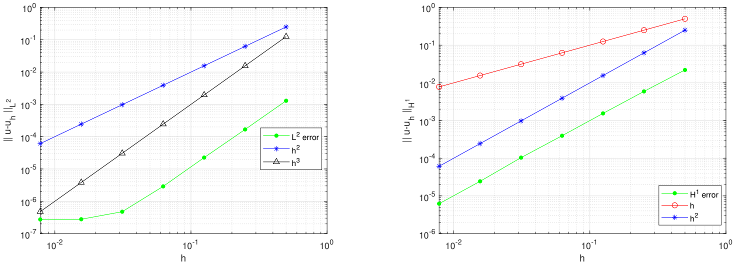

Figure 3: h-convergence analysis for $d = 2$

dt-convergence analysis

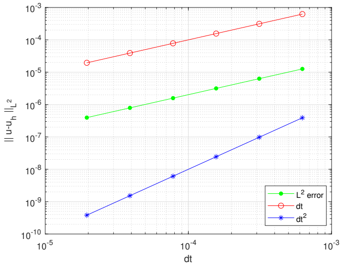

Figure 4: dt-convergence analysis for $d = 2$

Comparison

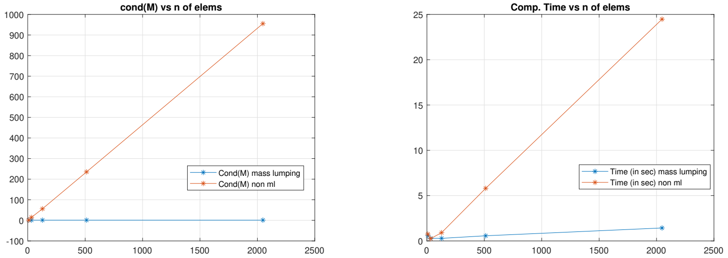

Figure 5: cond(M) and computational time vs n_{els}

A non homogeneous plate

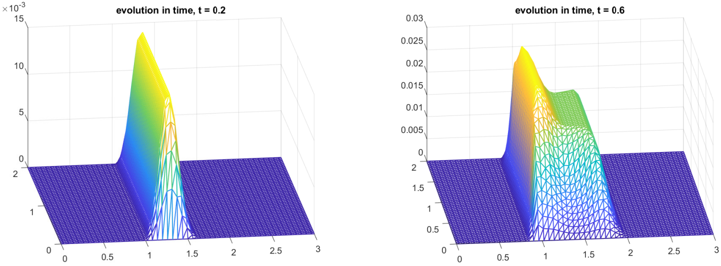

Figure 6: Plot of the wave at $t = 0.2$ and $t = 0.6$

A non homogeneous plate

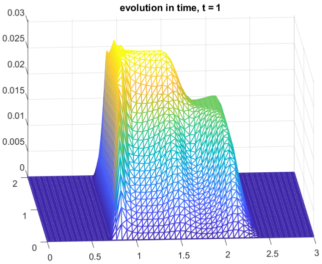

Figure 7: Plot of the wave at $t = 1$ and $t = 1.6$

The tear-drop

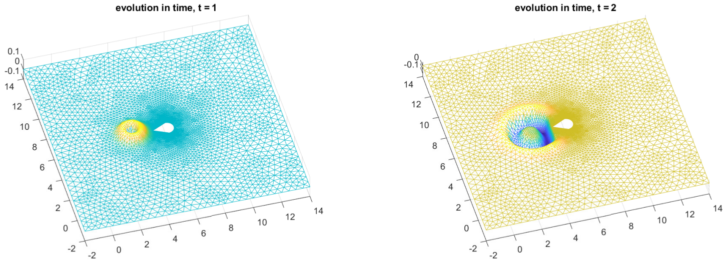

Figure 8: Plot of the wave at $t = 1$ and $t = 2$

The tear-drop

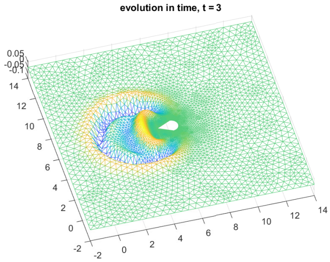

Figure 9: Plot of the wave at $t = 3$ and $t = 4$

The narrow channel

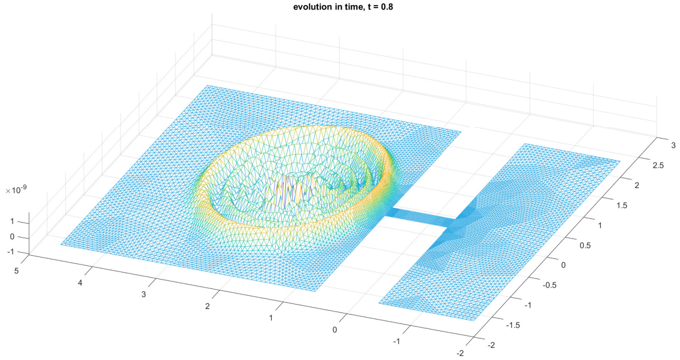

Figure 10: Plot of the wave at $t = 0.8$

The narrow channel

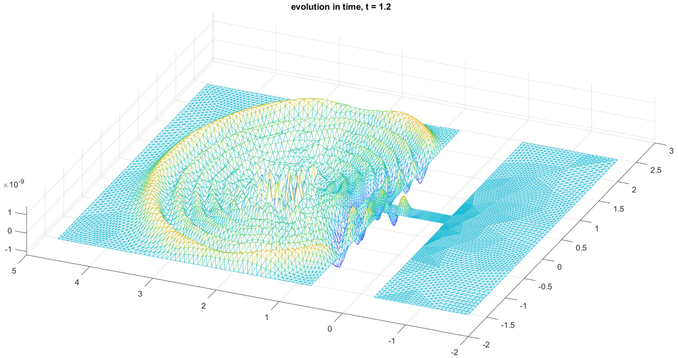

Figure 11: Plot of the wave at $t = 1.2$

The narrow channel

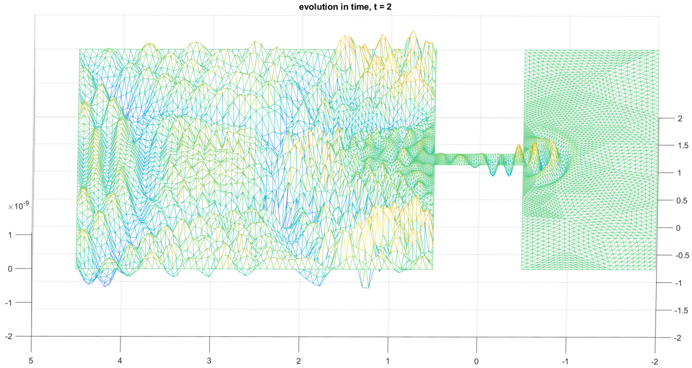

Figure 12: Plot of the wave at $t = 2$

The narrow channel

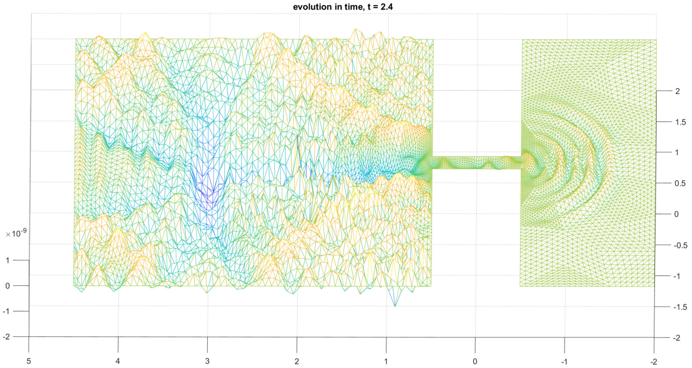

Figure 13: Plot of the wave at $t = 2.4$

Pros and cons

- ML algorithm much **less expensive** than standard FE method (trivial inversion of the mass matrix), even if the number of degrees of freedom is bigger;
- **same accuracy** in the error analysis, despite less accuracy of the quadrature rule (very bad conditioning of the mass matrix in the standard case);
- comparison with other mass-lumping techniques, FE implementations with preconditioning features and SE methods

Further developments

- higher order approximations in time and space;
- generalize the problem (different b.c., vectorial wave, $\Omega \subseteq \mathbb{R}^3$);
- improve the handling of the mesh for complex domains;
- consider the case of anisotropic diffusion: second order tensor, $\mathbf{A} = \mathbf{A}(\mathbf{x}, t)$, instead of the scalar k;
- include an advection term or a reaction term.

References

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