Some statistics on λ-terms

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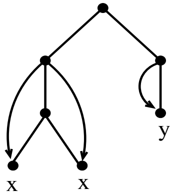
Motzkin trees and λ-terms

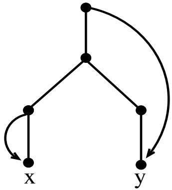
$$T ::= a \mid (T * T) \mid \lambda a.T$$

$(T * T)$: application $\qquad$ $\lambda a.T$: abstraction

$(\lambda x.(x * x) * \lambda y.y)$

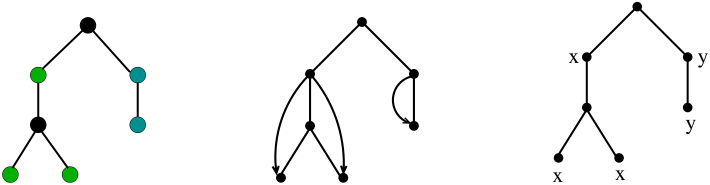


$\lambda y.(\lambda x.x * \lambda x.y)$



These λ-terms are closed (no free variable)

λ-terms as enriched Motzkin trees

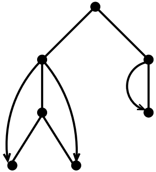


Labelling rules:

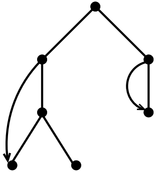
- Binary nodes are unlabelled
- Unary nodes get distinct labels (colors)
- Leaves get the label (color) of one of their unary ancestors

Free and bound variables in leaves

- Here all variables are bound



- Some variables may be free



- Recursive definition for λ-terms?
  - $\mathcal{L}$: class of λ-terms with free variables
  - $\mathcal{N}$ atomic class of binary node
  - $\mathcal{U}$ atomic class of unary node
  - $\mathcal{F}$ atomic class of free leaf
  - $\mathcal{B}$ atomic class of bound leaf

$$\mathcal{L} = \mathcal{F} + \left(\mathcal{N} \times \mathcal{L}^2\right) + (\mathcal{U} \times \mathit{subs}(\mathcal{F} \to \mathcal{F} + \mathcal{B}, \mathcal{L}))$$

- Generating function

$$L(z, f) = fz + z\, L(z, f)^2 + z\, L(z, f + 1).$$

with $z \leftrightarrow$ size of the λ-term and $f \leftrightarrow$ free leaves (size = total number of nodes)

- Generating function enumerating closed λ-terms (without free variables): $L(z,0)$
- Generating function enumerating all λ-terms: $L(z,1) = \frac{1}{z}L(z,0) - L(z,0)^2$
- $L(z,0) = \frac{1}{2z}\left(1 - \sqrt{\Lambda(z)}\right)$ with $\Lambda(z)$ equal to

$$1-2z+2z\sqrt{1-2z-4z^2+2z\sqrt{....\sqrt{1-2z-4nz^2+2z\sqrt{...}}}}$$

- $L(z,0)$ has null radius of convergence $\Rightarrow$ standard tools of analytic combinatorics fail

What can we do?

- Try to find a way to deal with null radius of convergence?
- *Ad hoc* methods?

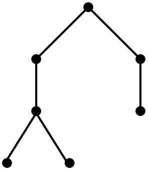
$$\left(\frac{(4-\epsilon)n}{\log n}\right)^{n(1-1/\log n)} \leq L_n \leq \left(\frac{(12+\epsilon)n}{\log n}\right)^{n(1-1/3\log n)}$$

[David et al. 10; here leaves have size 0]

- Consider sub-classes of terms?
  - Restrict the *total* number of abstractions [this talk]
  - Restrict the number of abstractions *in a path from the root towards a leaf:* bounded unary height [Analco'11, this talk]
  - Restrict the number of pointers from an abstraction to a leaf [AofA11]

How do restricted λ-terms compare with Motzkin trees?

Motzkin trees



$$\mathcal{M} = \mathcal{Z} + (\mathcal{U} \times \mathcal{M}) + (\mathcal{Z} \times \mathcal{M}^2)$$

$$M(z) = \frac{1}{2z}\left(1 - z - \sqrt{1 - 2z - 3z^2}\right)$$

Dominant singularity at $z = 1/3$ of square-root type

$$[z^n]M(z) \sim \frac{3^{n+\frac{1}{2}}}{2n\sqrt{\pi n}}$$

q unary nodes

$$\mathcal{M}_q = \mathcal{U} \times \mathcal{M}_{q-1} + \sum_{\ell=0}^{q} \mathcal{A} \times \mathcal{M}_\ell \times \mathcal{M}_{q-\ell}.$$

Recurrence equation on the generating functions

$$M_q(z) = \frac{zM_{q-1}(z) + z \sum_{1 \le \ell \le q-1} M_\ell(z)\, M_{q-\ell}(z)}{1 - 2zM_0(z)}.$$

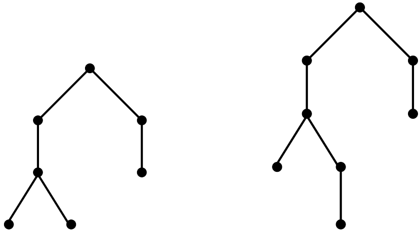
$\Rightarrow$ there exist polynomials P_q s.t.

$$M_q(z) = \frac{z^{q+1}\, P_q(z^2)}{(1-4z^2)^{q-\frac{1}{2}}},$$

Straightforward computations give

$$[z^n]M_q(z) \sim [z^n]\mathcal{M}_{\le q} \sim \frac{\sqrt{2}\, P_q(1/4)}{\Gamma(q-\frac{1}{2})}\, 4^n\, n^{q-\frac{3}{2}}$$

Leaves at same unary height



- Tree on the left: all leaves have unary height 1
- Tree on the right: leaves have unary heights 1, 2 and 1

Leaves at same unary height

$$\mathcal{MH}_k = \mathcal{U} \times \mathcal{MH}_{k-1} + \mathcal{A} \times \mathcal{MH}_k^2$$

On generating functions

$$\mathcal{MH}_k = \frac{1}{2}\left(1 - \sqrt{1 - 2z + 2z\sqrt{1 - 2z + 2z\sqrt{\ldots + 2z\sqrt{1 - 4z^2}}}}\right)$$

Two singularities

- $z = -\frac{1}{2}$ of type $(1+2z)^{\frac{1}{2}}$ (negligible)
- $z = \frac{1}{2}$ of type $(1+2z)^{\frac{1}{2^{k+1}}}$ (dominant, comes from the innermost radicand)

$$\Rightarrow [z^n]\mathcal{MH}_k(z) \sim \frac{2^{\frac{1}{2^{k+1}}}\, 2^n\, n^{-1-\frac{1}{2^{k+1}}}}{2^{k+1}\Gamma(1 - \frac{1}{2^{k+1}})}$$

Bounded unary height

Here leaves can have different unary height!

$$\mathcal{MH}_{\leq k} = \mathcal{Z} + \mathcal{U} \times \mathcal{MH}_{\leq k-1} + \mathcal{A} \times \mathcal{MH}^2_{\leq k}$$

Generating function

$$\mathcal{MH}_{\leq k} = \frac{1}{2}\left(1 - \sqrt{1 - 2z - 4z^2 + 2z\sqrt{1 - 2z - 4z^2 + 2z\sqrt{\ldots + 2z\sqrt{1 - 4z^2}}}}\right)$$

Dominant singularity ρ_k comes from outermost radicand, decreases towards $\frac{1}{3}$

$$\Rightarrow [z^n]\mathcal{MH}_{\leq k} \sim \frac{\sqrt{1 + 4\rho_k^2}}{4\,\rho_k^{n+1} n\sqrt{\pi n}}$$

λ-terms with bounded number of unary nodes

q unary nodes

$$\mathcal{S}_q = \left(\mathcal{U} \times \textit{subs}(\mathcal{F} \rightarrow \mathcal{F} + \mathcal{B}, \mathcal{S}_{q-1})\right) + \sum_{\ell=0}^{q} (\mathcal{A}, \mathcal{S}_\ell, \mathcal{S}_{q-\ell})$$

Generating function

$$S_q(z,f) = zS_{q-1}(z,f+1) + z\sum_{\ell=0}^{q} S_\ell(z,f)\ S_{q-\ell}(z,f).$$

G.F. for closed terms $S_q(z,0)$?

$$\begin{aligned} S_1(z,0) &= \frac{1}{2} - \frac{\sqrt{1-4z^2}}{2}; \\ S_2(z,0) &= \frac{z}{2}(1-2z^2) + \frac{2z^3}{\sqrt{1-4z^2}} - \frac{z\sqrt{1-8z^2}}{2\sqrt{1-4z^2}}; \end{aligned}$$

q unary nodes

$$S_q(z,f) = -\frac{z^{q-1}\sigma_q(f)}{2\prod_{\ell=0}^{q-1}\sigma_\ell(f)} + R_q(z,\sigma_0(f),\ldots,\sigma_{q-1}(f))$$

where

- $\sigma_q(f) = \sqrt{1-4(f+q)z^2}$
- R_q rational
- denominator of R_q equal to $\prod_{0\le\ell<q}\sigma_\ell(f)^{\alpha_{\ell,q}}$
- $\alpha_{\ell,q} > 0$, either integer or $\frac{1}{2}+$ an integer

q unary nodes

$$\Rightarrow S_q(z,0) = -\frac{z^{q-1}\sqrt{1-4qz^2}}{2\prod_{\ell=0}^{q-1}\sqrt{1-4\ell z^2}} + R_q(z,1,\sqrt{1-4z^2},\ldots,\sqrt{1-4(q-1)z^2})$$

Dominant singularities at $\pm\frac{1}{2\sqrt{q}}$ of square-root type

$$\Rightarrow [z^n]S_q(z,0) \sim \frac{q^{\frac{q}{2}}}{\sqrt{q!}\,\sqrt{2\,\pi\,n^3}}\,(4q)^{\frac{n+1-q}{2}}$$

(null if $n = q\ mod\ 2$)

λ-terms of bounded unary height

The classes $\mathcal{P}^{(i,k)}$

k: maximal number of abstractions on a path from the root to a leaf

- $\mathcal{P}^{(0,k)}$: λ-terms with bound variables and unary height $\leq k$
- $\mathcal{P}^{(1,k)}$: λ-terms with bound variables, 1 kind of free variables, and unary height $\leq k-1$
- ...
- $\mathcal{P}^{(i,k)}$: λ-terms with bound variables, i kinds of free variables, and unary height $\leq k-i$
- ...
- $\mathcal{P}^{(k,k)}$: λ-terms with bound variables, k kinds of free variables, and no unary node

The classes $\mathcal{P}^{(i,k)}$

- $i = k$

$$\mathcal{P}^{(k,k)} = k\mathcal{Z} + \mathcal{Z}{\mathcal{P}^{(k,k)}}^2$$

Generating function:

$$P^{(k,k)}(z) = kz + zP^{(k,k)}(z)^2$$

- $i < k$

$$\mathcal{P}^{(i,k)} = i\mathcal{Z} + \mathcal{Z}{\mathcal{P}^{(i,k)}}^2 + \mathcal{Z}\mathcal{P}^{(i+1,k)}$$

Generating function:

$$P^{(i,k)}(z) = iz + zP^{(i,k)}(z)^2 + zP^{(i+1,k)}(z)$$

Solve in $P^{(i,k)}$ and take $H_{\leq k}(z) = P^{(0,k)}(z)$:

$$H_{\leq k} = \frac{1 - \sqrt{1 - 2z + 2z\sqrt{1 - 2z - 4z^2 + 2z\sqrt{\ldots\sqrt{1 - 4kz^2}}}}}{2z}$$

We can start the asymptotic study of its coefficients!

Solve in $P^{(i,k)}$ and take $H_{\leq k}(z) = P^{(0,k)}(z)$:

$$H_{\leq k} = \frac{1 - \sqrt{1 - 2z + 2z\sqrt{1 - 2z - 4z^2 + 2z\sqrt{\ldots\sqrt{1 - 4kz^2}}}}}{2z}$$

We can start the asymptotic study of its coefficients!

- $H_{\leq k}$ is algebraic and written with $k+1$ iterated radicands
- Its singularities are the values that cancel its radicands
- Which radicant has smallest root?

- $k = 1$

$$H_{\leq 1}(z) = \frac{1 - \sqrt{1 - 2z + 2z\sqrt{1 - 4z^2}}}{2z}$$

Dominant singularity: $\frac{1}{2}$ (cancels both radicands)

$$[z^n]H_{\leq 1}(z) \sim \frac{1}{4} \frac{2^{\frac{1}{4}} 2^n n^{-\frac{5}{4}}}{\Gamma(\frac{3}{4})}$$

- $k = 2$

$$H_{\leq 2}(z) = \frac{1 - \sqrt{1 - 2z + 2z\sqrt{1 - 2z - 4z^2 + 2z\sqrt{1 - 8z^2}}}}{2z}$$

Dominant singularity: $\rho = 0.3437999303$ (cancels the second innermost radicand)

$$[z^n]H_{\leq 2}(z) \sim \frac{C}{\Gamma(\frac{1}{2})} n^{-\frac{3}{2}} \rho^{-n}$$

Where is the dominant singularity when k grows?

Function	Radicand	Singularity
$H_{\leq 1}$	{1,2}	0.5
$H_{\leq 2}$	2	0.3438
$H_{\leq 3}$	2	0.2760
...	...	...
$H_{\leq 8}$	{2,3}	0.1667
$H_{\leq 9}$	3	0.1571
...	...	...
$SH_{\leq 134}$	3	0.0418
$H_{\leq 135}$	{3,4}	0.0417
$H_{\leq 136}$	4	0.0415
...	...	...

Sometimes, the same value cancels *two* consecutive radicands.

Values of k which give two dominant radicands?

- Define $(u_k)_{k\geq 0}$: $u_0 = 0$ and $u_k = u_{k-1}^2 + k$ for $k > 0$
- First values: $u_1 = 1$, $u_2 = 3$, $u_3 = 12$, $u_4 = 148$, $u_5 = 21909$, ...
- The sequence $(u_k)_{k\geq 0}$ is doubly exponential
- $\lim_{k\to\infty} u_k^{1/2^k} \simeq \chi = 1.36660956...$
- Define $N_k = u_k^2 - u_k + k = u_k^2 - u_{k-1}^2$.
  $N_1 = 1$, $N_2 = 8$, $N_3 = 135$, $N_4 = 21760$,
  $N_5 = 479982377$, ...

Theorem

Two asymptotic behaviours according to the value of k

- *For unary height N_i, the radicands with ranks i and $(i+1)$ both cancel for the same value; both are dominant; the dominant singularity is algebraic of type $1/4$, and $[z^n]H_{\leq N_i} \sim C_i n^{-5/4} \rho_i^n$, with $\rho_i = 1/2u_i$.*
- *If $k \in]N_i, N_{i+1}[$, the dominant radicand of $H_{\leq k}(z)$ is the i-th radicand; the dominant singularity is algebraic of type $1/2$, and $[z^n]H_{\leq k} \sim C_k n^{-3/2} \rho_k^n$.*

(Radicands are ranked from the innermost to the outermost)

[Bodini-G-Gittenberger, Analco'11]

Observations

- The constants C_k become small very quickly. Variation of C_k as a function of k?
  - $[z^n]H_{\leq 1}(z) \sim 0.2426128012 \cdot \left(\frac{1}{n}\right)^{5/4} \cdot 2^n$
  - $[z^n]H_{\leq 8}(z) \sim 9.318885373 \cdot 10^{-5} \left(\frac{1}{n}\right)^{5/4} 6^n$
  - $[z^n]H_{\leq 135}(z) \sim 7.116999389 \cdot 10^{-158} \left(\frac{1}{n}\right)^{5/4} 24^n$
  - Doubly-exponential decay?
- We cannot hope to observe the asymptotic behaviour of $[z^n]H_{\leq k}$ from computations for “reasonable” n
- Yet we can randomly generate lambda-terms of bounded height and observe the behaviour of parameters...

The constant C_k for $k \in \{N_i\}$

$$[z^n]H_{\leq N_k} \sim \frac{D_k}{\Gamma(3/4)\,A_k} n^{-5/4}(2u_k)^n$$

Asymptotics for $k \to +\infty$?

$$\begin{aligned}
D_k &\sim \gamma u_{k-1} \qquad \textit{with} \qquad \gamma = 1.2952778 \\
A_k &= \prod_{i=k}^{N_k-1} \sqrt{i + \sqrt{i-1+\sqrt{i-2+\sqrt{\ldots+\sqrt{1}}}}} \\
&\sim \frac{\varphi(N_k)}{\varphi(k)} \qquad \textit{with} \qquad \varphi(k) = \frac{e^{\sqrt{k}}}{\sqrt{k}}.\left(\frac{2k}{e}\right)^k
\end{aligned}$$

What next?

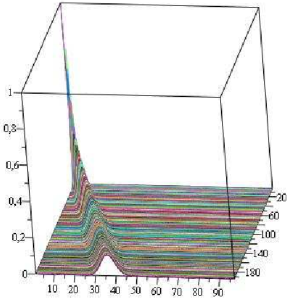
Different types of asymptotic behaviour when enumerating restricted Motzkin trees and λ-terms

1. Motzkin trees
   - Number of unary nodes = q: one radical, $C_q\, 4^n n^{q-\frac{3}{2}}$
   - Shared unary height of leaves = k: iterated radicals; *innermost* radical dominates; $C_k\, 2^n\, n^{-1-\frac{1}{2^{k+1}}}$
   - Bounded unary height = k: iterated radicals, *outermost* radical dominates; $C_k\, \rho_k^n\, n^{-\frac{3}{2}}$
2. λ-terms
   - Number of unary nodes = q: product of radicals; $C_q\,(4q)^{\frac{n+1-q}{2}}\; n^{-\frac{3}{2}}$
   - Bounded unary height = k: iterated radicals; dominant radical *fluctuates*
     - Standard case: $C_k n^{-\frac{3}{2}} \rho_k^n$
     - Special values: *two* dominant radicals; $C_k n^{-\frac{5}{4}} \rho_k^n$

Enumeration and properties of λ-terms

- How does a random restricted λ-term look like?
    - Number of nodes of each type?
    - Unary height? Total height?
    - Width? Profile?
- Enumerate (unrestricted) λ-terms according to their size
- Allow for free leaves; give a specific weight to the leaves
- Caracterize the parameters of a random λ-term, their logical properties (strong normalizing, ...)

Number of λ-terms: $n \in [1, ..., 198]$; unary height $k \in [1, ..., 98]$

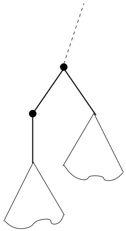


λ-terms and normal form

A term is in normal form

$$\Updownarrow$$

there is no pattern $\mathcal{A} \times (\mathcal{U}, \mathcal{T}) \times \mathcal{T}$



Forbidden pattern

Number of terms in normal form?

Normal form, bounded number of unary nodes

- Asymptotic number of closed, normal-form λ-terms with exactly q unary nodes and size n, $n \neq q \mod 2$

$$\frac{1}{2^q \sqrt{2\pi n^3}} \prod_{\ell=1}^{q} \frac{\sqrt{q}+\sqrt{\ell}}{\sqrt{\ell}} (4q)^{\frac{n+1-q}{2}}$$

- Asymptotic probability of closed, normal-form term with exactly q unary nodes and size n ($n \to +\infty$)

$$\pi_q = 2^{-q} \prod_{\ell=1}^{q} \left(1 + \sqrt{\frac{\ell}{q}}\right)$$

Normal form, bounded number of unary nodes

Asymptotic probability of closed normal-form term with exactly q unary nodes and size n for large q

$$\pi_q = \sqrt{2}\left(\frac{\sqrt{e}}{2}\right)^q (1 + o(1)) = \sqrt{2}\ \ 0.82436^q\ (1 + o(1))\,.$$

q	5	10	50	100	1000
Exact	0.496	0.193	$8.79\ 10^{-5}$	$5.67\ 10^{-9}$	$1.84\ 10^{-84}$
Large q	0.538	0.205	$9.04\ 10^{-5}$	$5.78\ 10^{-9}$	$1.85\ 10^{-84}$

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What about bounded unary height?

To be continued...

Thanks for your attention