6 – NONLINEAR REGRESSION FUNCTIONS

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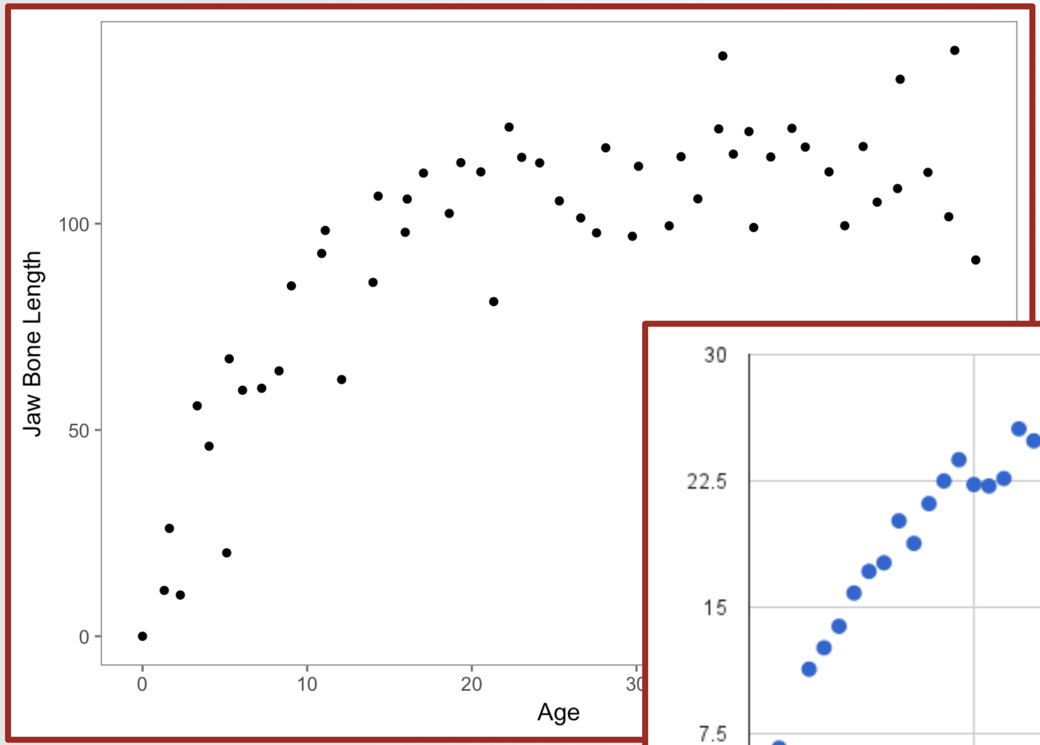
SECTION 6 – NONLINEAR REGRESSION FUNCTIONS
THE PLAN

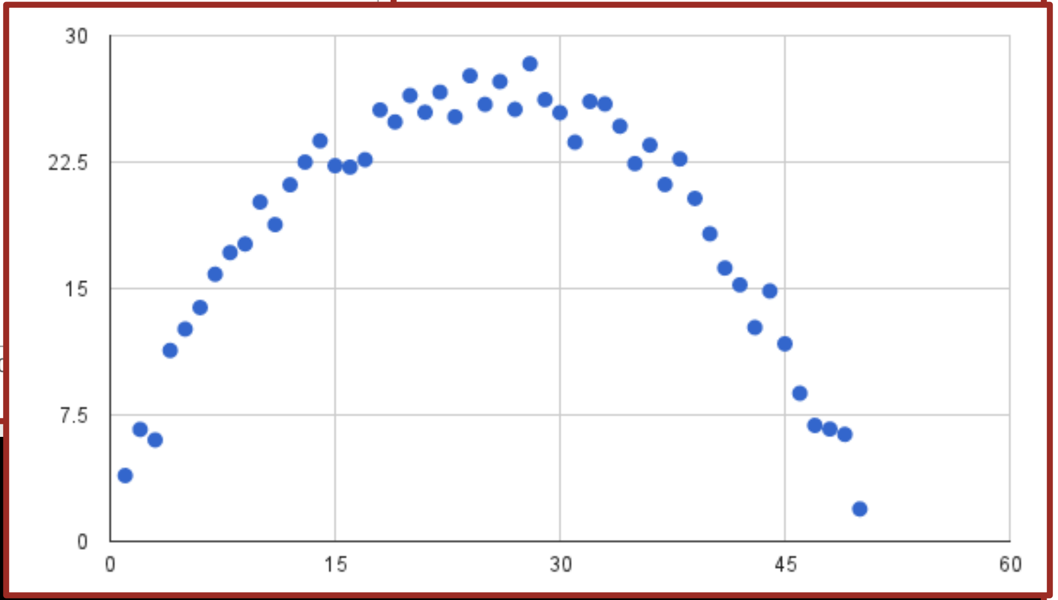
1. Nonlinear functions of a single independent variable.
2. Polynomial regression functions.
3. Logarithmic regression functions.
4. Interactions between regressors.

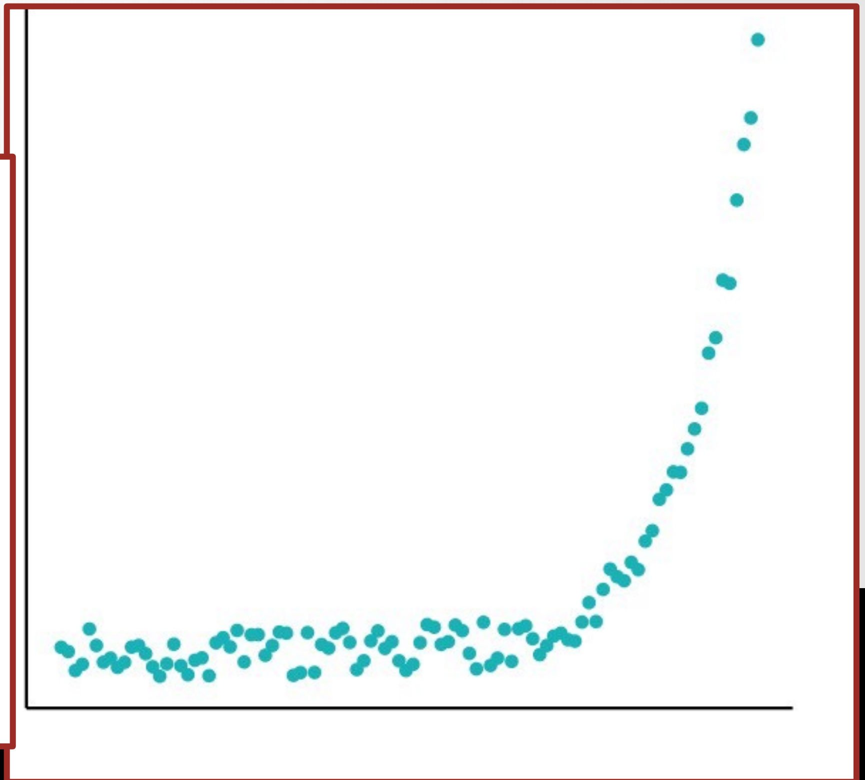
OVERVIEW

We considered *linear* regression functions so far (constant slope)…

…but what if your data looks like one of these?





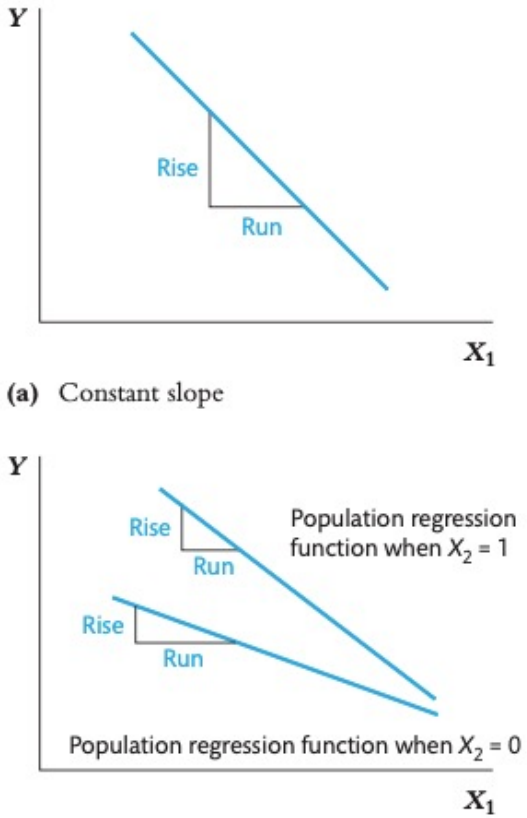


Two main ways of being nonlinear:

1. The "effect" of one regressor is nonlinear.

2. Interaction: the "effect" of a regressor depends on the value taken by the other.

FIGURE 8.1 Population Regression Functions with Different Slopes



(a) Constant slope

(b) Slope depends on the value of X_1

(c) Slope depends on the value of X_2

In Figure 8.1(a), the population regression function has a constant slope. In Figure 8.1(b), the slope of the population regression function depends on the value of X_1. In Figure 8.1(c), the slope of the population regression function depends on the value of X_2.

6.1 NONLINEAR FUNCTIONS OF A SINGLE INDEPENDENT VARIABLE

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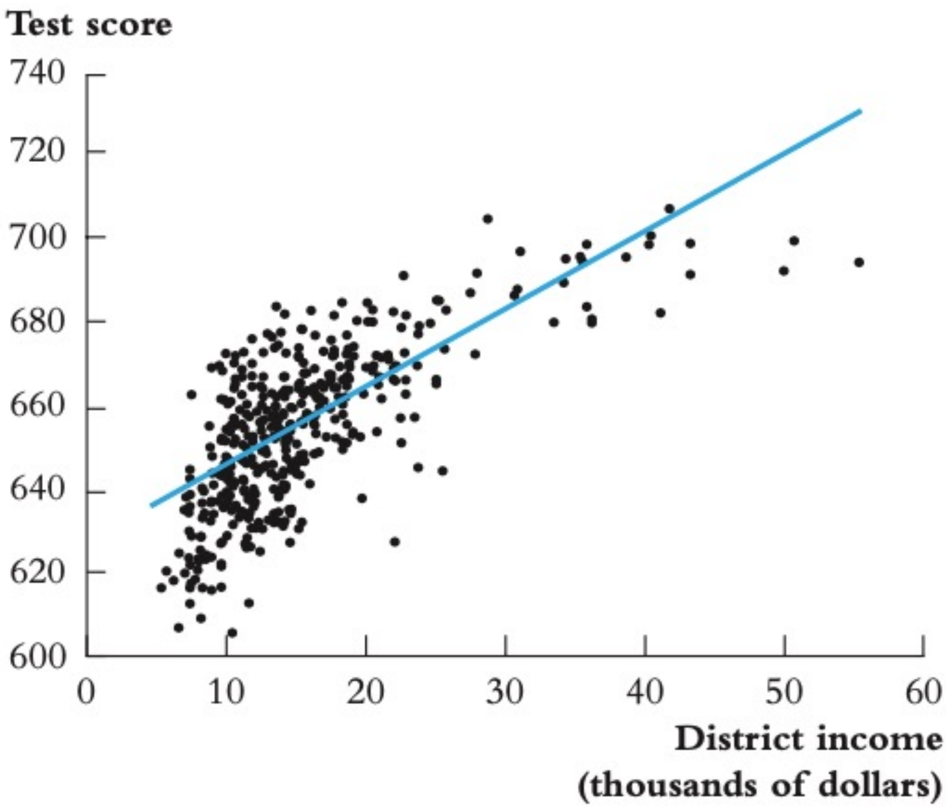
TEST SCORES & DISTRICT INCOME

FIGURE 8.2 Scatterplot of Test Scores vs. District Income with a Linear OLS Regression Function

There is a positive correlation between test scores and district income (correlation = 0.71), but the linear OLS regression line does not adequately describe the relationship between these variables.



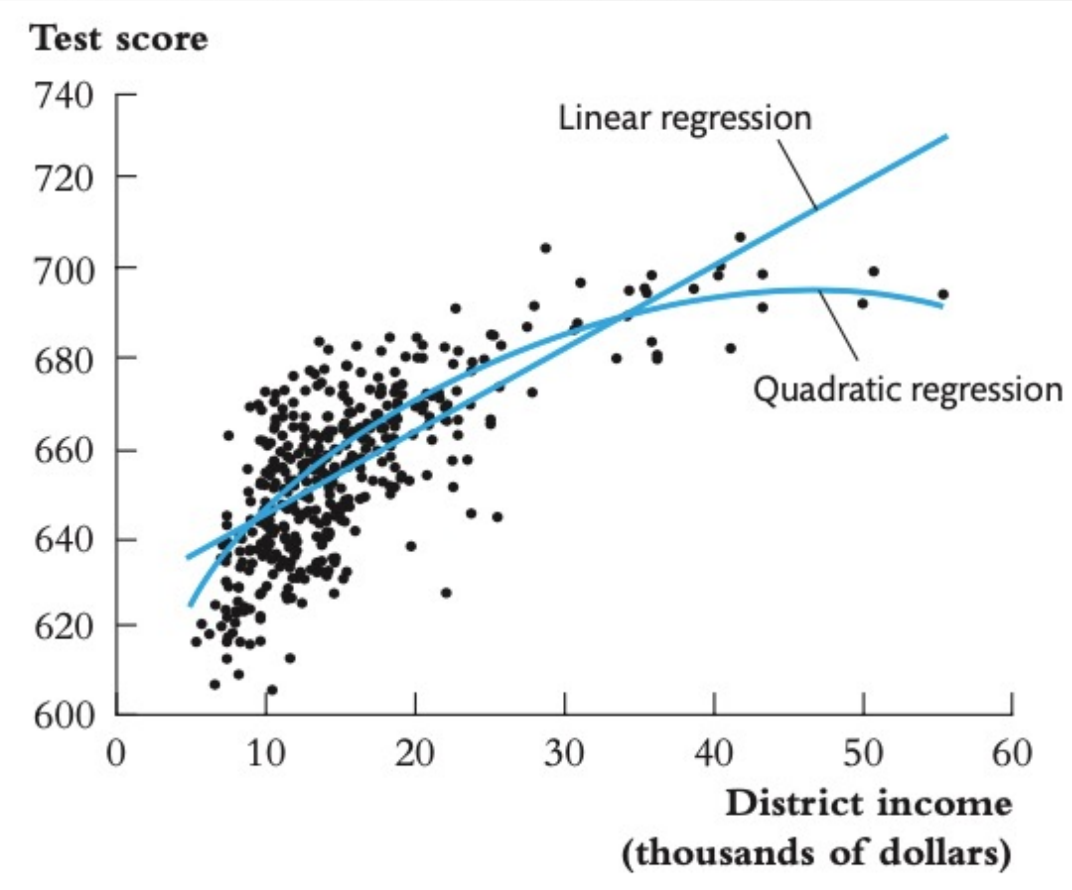
TEST SCORES & DISTRICT INCOME

- Nonlinear function: the slope is not constant.
- Quadratic regression function:

$$TestScore = \beta_0 + \beta_1 Income_i + \beta_2 Income_i^2 + u_i$$

- Technically, just a multiple regression with $Income_i^2$ as an additional regressor.
- OLS estimate:

$$\widehat{TestScores_i} = \underset{(2.9)}{607.4} + \underset{(0.27)}{3.85} Income - \underset{(0.005)}{0.04} Income^2$$



NONLINEAR FUNCTIONS

$$E(Y_i) = f(X_{1i}, X_{2i}, \dots, X_{ki})$$

$$\Downarrow$$

$$Y_i = f(X_{1i}, X_{2i}, \dots, X_{ki}) + u_i$$

- Effect of a change in X_1 by ΔX_1 units:

$$E(\Delta Y) = f(X_1 + \Delta X_1, X_2, \dots, X_k) - f(X_1, X_2, \dots, X_k)$$

- If X_1 is continuous and ΔX_1 small, can use the partial derivative:

$$E(\Delta Y) \approx \Delta X_1 \frac{df(X_1, X_2, \dots, X_k)}{dX_1}$$

TEST SCORES & DISTRICT INCOME

$$TestScore = \beta_0 + \beta_1 Income_i + \beta_2 Income_i^2 + u_i$$

$$\hat{\beta}_0 = 607.4; \quad \hat{\beta}_1 = 3.85; \quad \hat{\beta}_2 = -0.0423$$

- Effect of district income going from 10 to 11 (thousand dollars):

$$E(\Delta Y) = (\beta_0 + \beta_1(11) + \beta_2(11)^2) - (\beta_0 + \beta_1(10) + \beta_2(10)^2) =$$

$$= (\beta_1(11 - 10) + \beta_2(11^2 - 10^2)) =$$

$$= (\beta_1 + 21\beta_2) = 3.85 - 0.89 = 2.96$$

TEST SCORES & DISTRICT INCOME

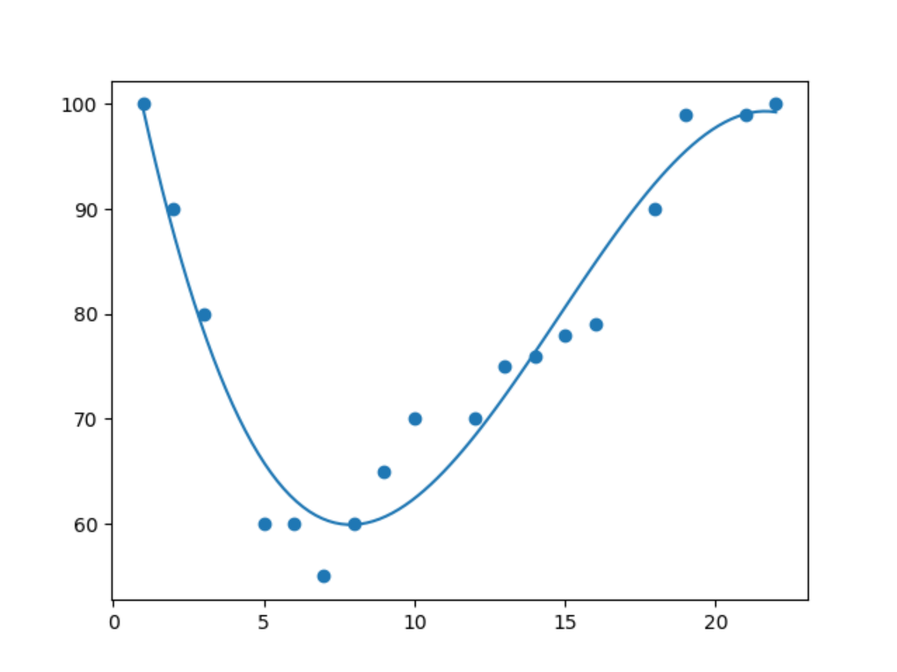
- $TestScore = \beta_0 + \beta_1 Income_i + \beta_2 Income_i^2 + u_i$
- Effect of district income going from 10 to 11 (thousand dollars):

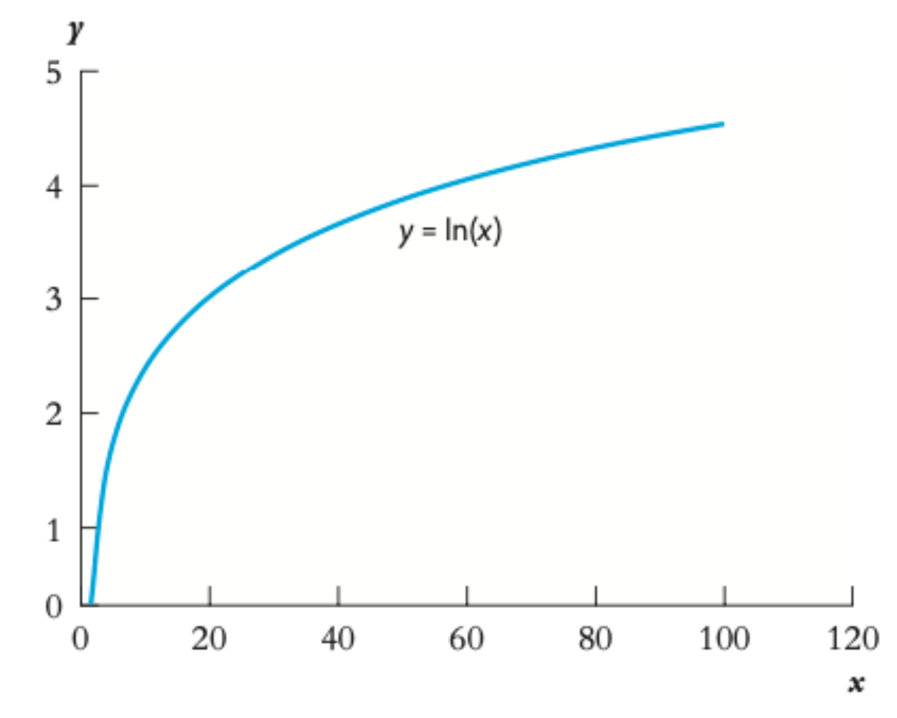
$$\Delta\hat{Y} = \left(\widehat{\beta_1} + 21\widehat{\beta_2}\right) = 3.85 - 0.89 = 2.96$$

- $SE\left(\Delta\hat{Y}\right) = SE(\beta_1 + 21\beta_2)$
- STATA will compute it for you (if you know how to ask!)
- → *SE for predicted effect can be computed through a test of a single restriction involving multiple coefficients.*

NONLINEAR FUNCTIONS OF A SINGLE INDEPENDENT VARIABLE

1. Polynomial regression model.
2. Logarithmic regression model(s).





6.2 POLYNOMIAL REGRESSION FUNCTIONS

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POLYNOMIALS

- Polynomial regression model of degree r:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + \cdots + \beta_r X_i^r$$

  - $r = 2$ → quadratic regression model
  - $r = 3$ → cubic regression model

- Estimation & inference: just like a multivariate regression.

POLYNOMIALS

- Polynomial regression model of degree r:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + \cdots + \beta_r X_i^r$$

- Testing the null hypothesis of linearity:
  - $H_0: \beta_2, \dots, \beta_r = 0$
  - $H_1: at\ least\ one\ of\ these\ coefficients\ is\ not\ 0$

- How to choose the degree r?
  - *Theory; Plot the data; F- & t-tests.*

POLYNOMIALS: INCOME & TEST SCORES

- Quadratic specification:

$$TestScore = \beta_0 + \beta_1 Income_i + \beta_2 Income_i^2 + u_i$$

- Cubic specification:

$$TestScore = \beta_0 + \beta_1 Income_i + \beta_2 Income_i^2 + \beta_3 Income_i^3 + u_i$$

QUADRATIC SPECIFICATION IN STATA

```
generate avginc2 = avginc*avginc          Create a new regressor
reg testscr avginc avginc2, r

Regression with robust standard errors                 Number of obs =     420
                                                       F(  2,   417) =  428.52
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.5562
                                                       Root MSE      =  12.724

------------------------------------------------------------------------------
             |               Robust
     testscr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      avginc |   3.850995   .2680941    14.36   0.000      3.32401    4.377979
     avginc2 |  -.0423085   .0047803    -8.85   0.000     -.051705   -.0329119
       _cons |   607.3017   2.901754   209.29   0.000     601.5978    613.0056
------------------------------------------------------------------------------
```

CUBIC SPECIFICATION IN STATA

```
gen avginc3 = avginc*avginc2                Create the cubic regressor
reg testscr avginc avginc2 avginc3, r

Regression with robust standard errors               Number of obs =      420
                                                     F(  3,   416) =   270.18
                                                     Prob > F      =   0.0000
                                                     R-squared     =   0.5584
                                                     Root MSE      =   12.707

------------------------------------------------------------------------------
             |               Robust
     testscr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      avginc |   5.018677   .7073505     7.10   0.000     3.628251    6.409104
     avginc2 |  -.0958052   .0289537    -3.31   0.001    -.1527191   -.0388913
     avginc3 |   .0006855   .0003471     1.98   0.049     3.27e-06    .0013677
       _cons |    600.079   5.102062   117.61   0.000     590.0499     610.108
------------------------------------------------------------------------------
```

TESTING THE NULL OF LINEARITY AGAINST A CUBIC ALTERNATIVE

```
test avginc2 avginc3  Use the test command after running the regression

 ( 1)   avginc2 = 0.0
 ( 2)   avginc3 = 0.0

      F(   2,    416) =    37.69
      Prob > F =      0.0000
```

The hypothesis that the population regression is linear is rejected at the 1% significance level against the alternative that it is a polynomial of up to degree 3.

6.3 LOGARITHMIC REGRESSION FUNCTIONS

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LOGARITHMIC REGRESSION FUNCTIONS

- Y or X (or both) transformed in their *natural logarithm*.
- **Three types:**

1. Linear-log model

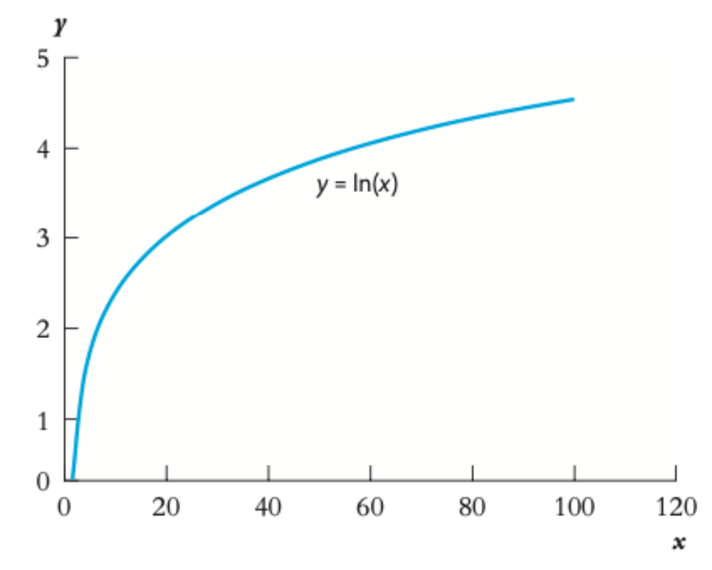
$$Y_i = \beta_0 + \beta_1 \ln(X_i) + u_i$$

2. Log-linear model

$$\ln(Y_i) = \beta_0 + \beta_1 X_i + u_i$$

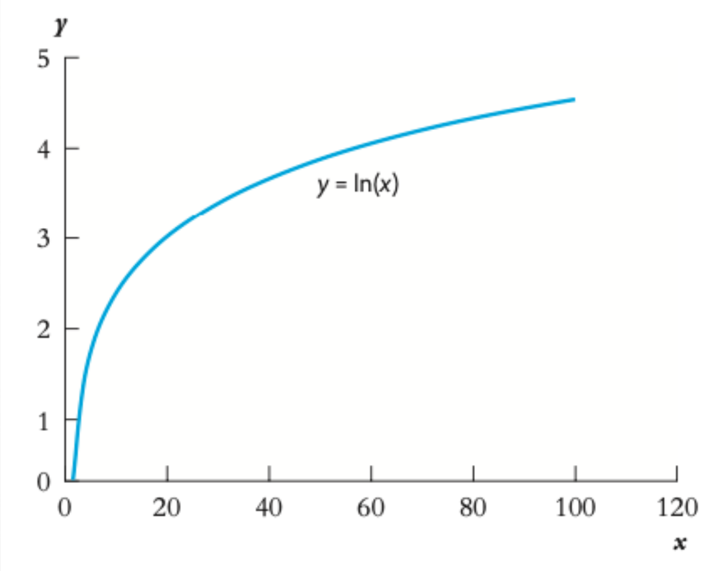
3. Log-log model

$$\ln(Y_i) = \beta_0 + \beta_1 \ln(X_i) + u_i$$



LOGS & PERCENTAGES

- Log changes are approximately equal to percentage changes.
- $\ln(x_2) - \ln(x_1) \approx \frac{x_2 - x_1}{x_1}$
- The approximation gets worse as $x_2 - x_1$ gets larger.



1) LINEAR-LOG MODEL

$$Y_i = \beta_0 + \beta_1 \ln(X_i) + u_i$$

- Constant effect of a % change in X.
- 1% increase in X → $\beta_1/100$ change in E(Y).
- $\Delta\hat{Y} = \beta_1[\ln(X + \Delta X) - \ln(X)] \approx \beta_1 \frac{\Delta X}{X}$
- Test score example:

$$\widehat{TestScore} = 557.8 + 36.42 \ln(Income)$$

  - *A 1% increase in income increases test scores by 0.36 points.*

2) LOG-LINEAR MODEL

$$\ln(Y_i) = \beta_0 + \beta_1 X_i + u_i$$

- The % change in E(Y) caused by a unit change in X is constant.
- unit change in X → $(100\times\beta_1)\%$ change in E(Y).
- Age-earnings example:

$$\ln(\widehat{Earnings}) = 2.876 + 0.0095\, Age$$

  - *Earnings increase by 0.95% for each additional year of age*

3) LOG-LOG MODEL

$$\ln(Y_i) = \beta_0 + \beta_1 \ln(X_i) + u_i$$

- Constant *elasticity* of Y with respect to X.
- 1% change in X → β_1% change in E(Y).
- Test score example:

$$\ln(\widehat{TestScore}) = 6.4 + 0.06 \ln(Income)$$

  - *A 1% increase in income increases test scores by 0.06%.*

HOW TO CHOOSE A LOG SPECIFICATION?

- Warning: Can't compare R^2 if the dependent variable is different
    - *ln(Y) vs. Y*
- Does it make sense to think in terms of % changes?
    - *Usually it does for income or wages.*
- Does it make results easier to interpret?
    - *Percentage changes don't depend on the unit of measure.*

6.4 INTERACTIONS BETWEEN REGRESSORS

INTERACTION TERMS

- What if the effect of X_j on Y is different in different circumstances?
- Example with binary regressors:

$$\ln(Earnings_i) = \beta_0 + \beta_1 College_i + \beta_2 Female_i + u_i$$

  - Effect of College assumed to be the same for men and women.
  - Add an *interaction term*:

$$\ln(Earnings_i) \\ = \beta_0 + \beta_1 College_i + \beta_2 Female_i + \beta_3 College_i \times Female_i + u_i$$

INTERACTION TERMS

$$\ln(Earnings_i) = \beta_0 + \beta_1 College_i + \beta_2 Female_i + \beta_3 College_i \times Female_i + u_i$$

- $E(Earnings|College = 0, Female = 0) = \beta_0$
- $E(Earnings|College = 1, Female = 0) = \beta_0 + \beta_1$
- $E(Earnings|College = 0, Female = 1) = \beta_0 + \beta_2$
- $E(Earnings|College = 1, Female = 1) = \beta_0 + \beta_1 + \beta_2 + \beta_3$

Effect of college for a man: β_1

Effect of college for a woman: $\beta_1 + \beta_3$

INTERACTION BETWEEN A CONTINUOUS AND A BINARY VARIABLE

- Regression with 1 continuous & 1 binary variable:

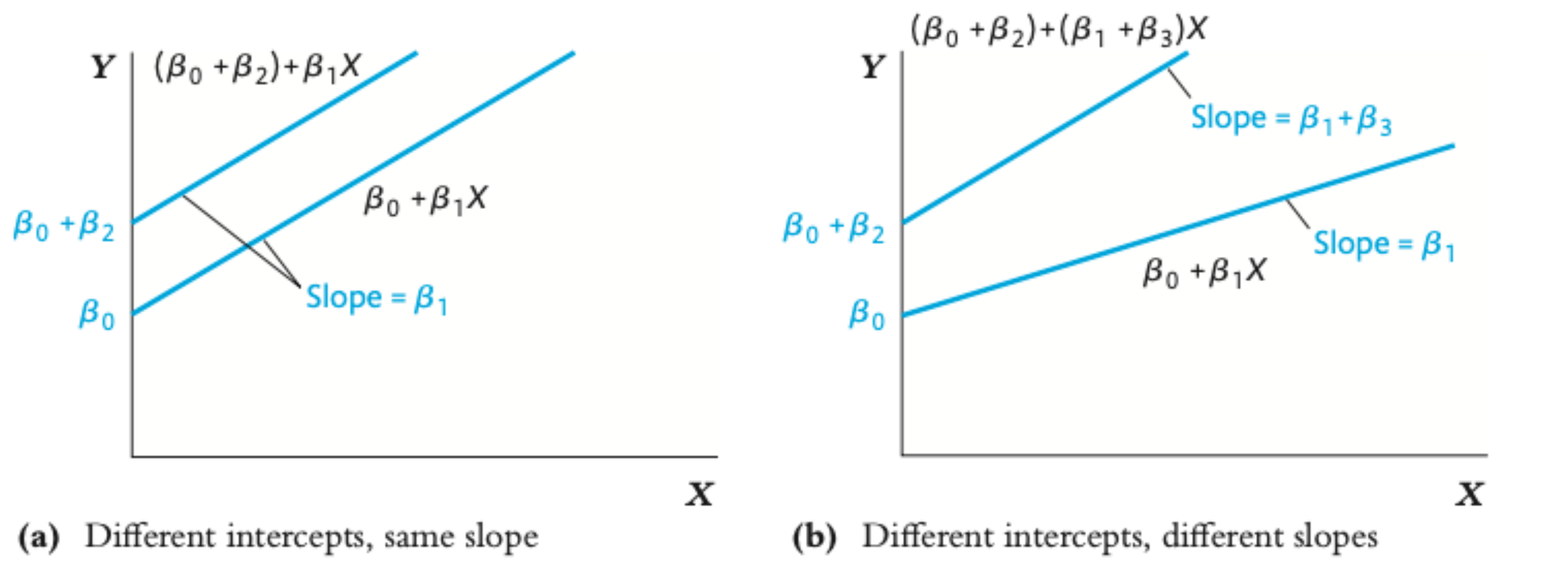
$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + u_i$$

- With interaction term:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + \beta_3 (X_i \times D_i) + u_i$$

  - Slope if D=0: β_1
  - Slope if D=1: $\beta_1 + \beta_3$

FIGURE 8.8 Regression Functions Using Binary and Continuous Variables



INTERACTION: TWO CONTINOUS VARIABLES

- Regression with two continuous variables:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i$$

- With interaction:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 (X_{1i} \times X_{2i}) + u_i$$

  - $\frac{\Delta Y}{\Delta X_1} = \beta_1 + \beta_3 X_2$
  - $\frac{\Delta Y}{\Delta X_2} = \beta_2 + \beta_3 X_1$

EXAMPLE: CALIFORNIA SCHOOLS DATASET

```
. reg testscr str el_pct str_el_pct, robust

Linear regression                               Number of obs     =        420
                                                F(3, 416)         =     155.05
                                                Prob > F          =     0.0000
                                                R-squared         =     0.4264
                                                Root MSE          =     14.482

------------------------------------------------------------------------------
             |               Robust
     testscr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         str |  -1.117018   .5875135    -1.90   0.058    -2.271884    .0378468
      el_pct |  -.6729116   .3741231    -1.80   0.073    -1.408319    .0624958
  str_el_pct |   .0011618   .0185357     0.06   0.950    -.0352736    .0375971
       _cons |   686.3385   11.75935    58.37   0.000     663.2234    709.4537
------------------------------------------------------------------------------
```

The interaction term (str*el_pct) is not significantly different from zero - effect of class sizes does not depend on share of English language learners.

EXAMPLE: AGE AND EARNINGS (CPS DATA)

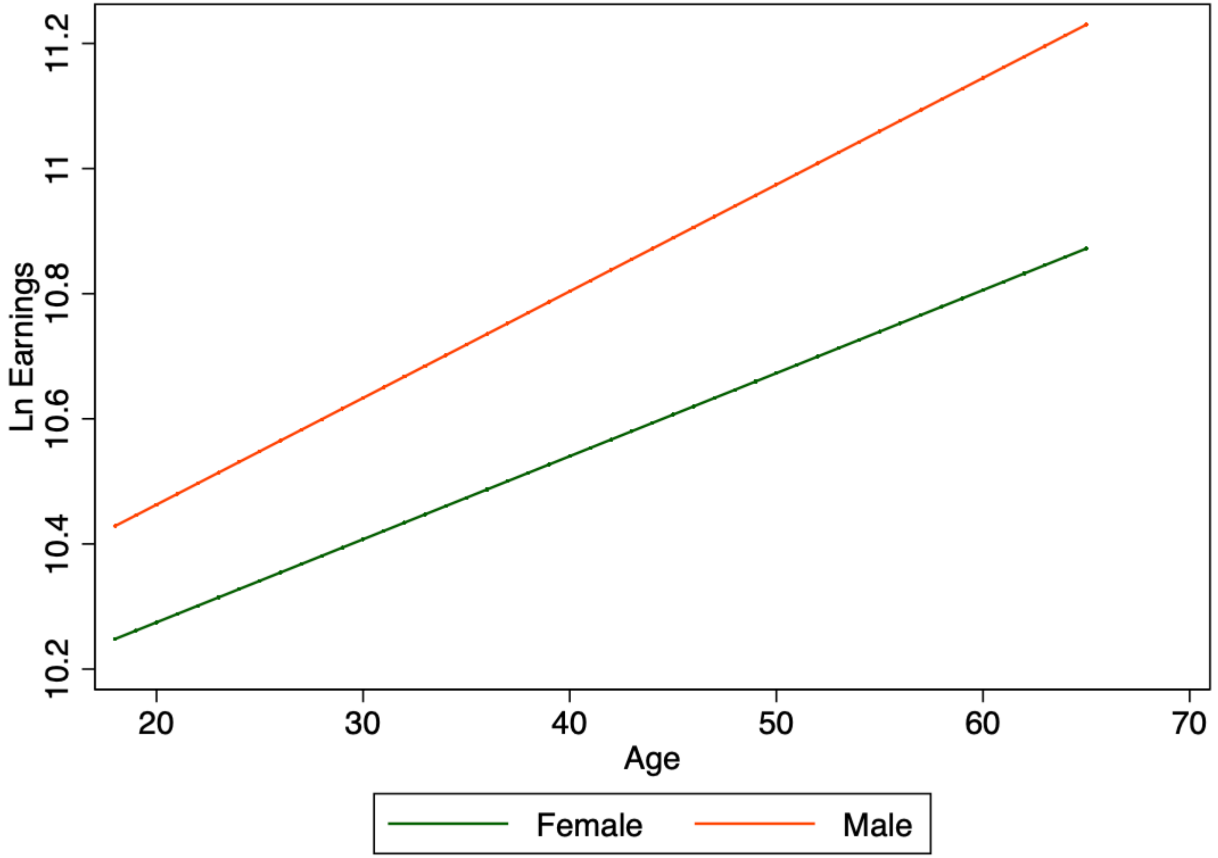
```
. reg ln_incwage female age female_age, r

Linear regression                               Number of obs     =     55,527
                                                F(3, 55523)       =    1315.58
                                                Prob > F          =     0.0000
                                                R-squared         =     0.0726
                                                Root MSE          =     .81175
```

ln_incwage	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
female	-.1126975	.0265112	-4.25	0.000	-.1646596	-.0607355
age	.017048	.0004219	40.40	0.000	.016221	.017875
female_age	-.0037716	.0006105	-6.18	0.000	-.0049682	-.0025751
_cons	10.12187	.0181315	558.25	0.000	10.08633	10.1574

The interaction term (female*age) is negative and significantly different from zero - effect of age on earnings does depend on gender.

EXAMPLE: AGE AND EARNINGS (CPS DATA)



The line for men is steeper than for women - women get less of an increase in earnings each year they get older compared to men.