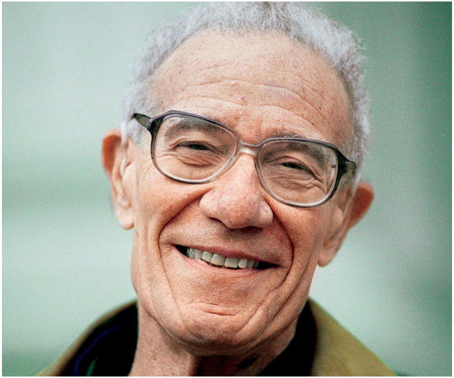
The Solow model



Solow growth model

Key premises:

- *neoclassical* production function
- Say's law: full employment at all times.

Main implications:

- stable steady-state with $g_Y = n + g$
- saving rate determines output level but not growth rate
- K accumulation cannot explain long-run growth or cross-country income differences.

Production function

- One-good economy
- 4 variables: Y, K, L, A.
- Say's law: full employment of L & K at each t.
- Neoclassical aggregate production function

$$Y(t) = F[K(t), A(t)L(t)]$$

  - labor-augmenting technological progress.
  - AL = 'effective labor'.

Constant returns to scale (CRS)

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- *Intensive form* of the production function:

$$\frac{Y}{AL} = F(\frac{K}{AL}, \frac{AL}{AL}) = F(\frac{K}{AL}, 1)$$
$$\Downarrow$$

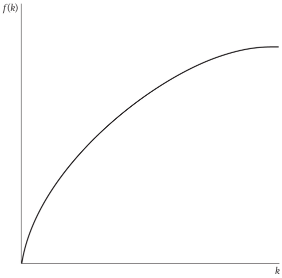
$$y = f(k)$$

with $k = \frac{K}{AL}$, $y = \frac{Y}{AL}$ and $f(k) = F(k, 1)$

Other assumptions about the production function

$$f(0) = 0, \qquad f'(k) > 0, \qquad f''(k) < 0$$

$$lim_{k\to 0} f'(k) = \infty, \qquad lim_{k\to\infty} f'(k) = 0$$



Evolution of production inputs

$$g_L = n \quad \text{(therefore } \dot{L}(t) = nL(t)\text{)}$$

$$g_A = g \quad \text{(therefore } \dot{A}(t) = gA(t)\text{)}$$

$$\dot{K}(t) = sY(t) - \delta K(t), \qquad 0 < s \leq 1$$

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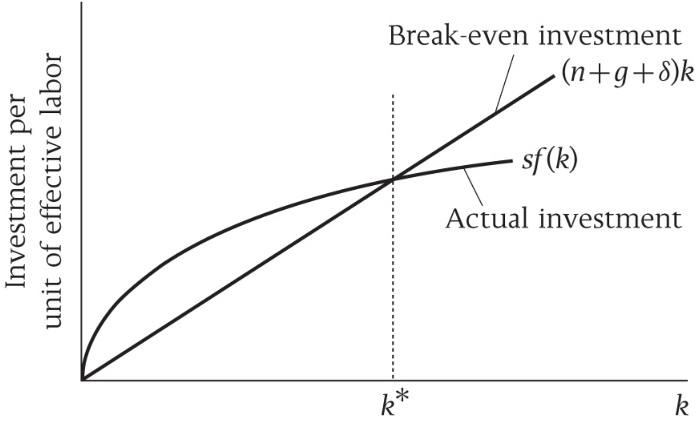
- using $k = \frac{K}{AL}$, $y = \frac{Y}{AL}$, & the assumptions about inputs:

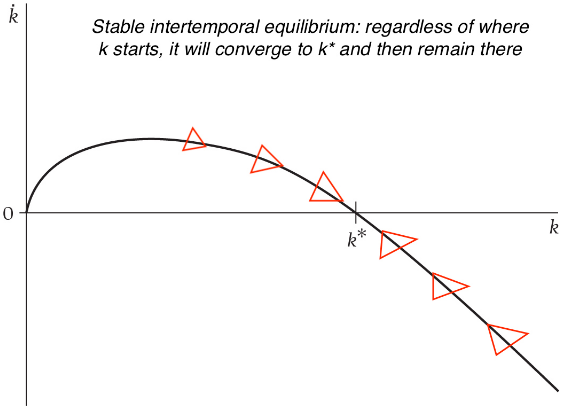
$$\dot{k}(t) = sf[k(t)] - (n + g + \delta)k(t)$$

change in k = investment − breakeven investment

Actual vs. break-even investment



Phase diagram



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- $\frac{K}{L} = Ak \rightarrow g_{\frac{K}{L}} = g$
- $\frac{Y}{L} = Af(k) \rightarrow g_{\frac{Y}{L}} = g$

balanced growth path: all variables grow at constant rates.

Other things we want to know:

1. Qualitative effect of an increase in s (*direction*)
2. What level of k maximizes consumption (*golden-rule* k^*)
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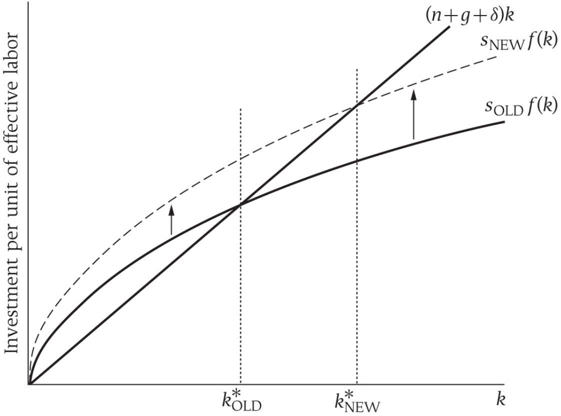

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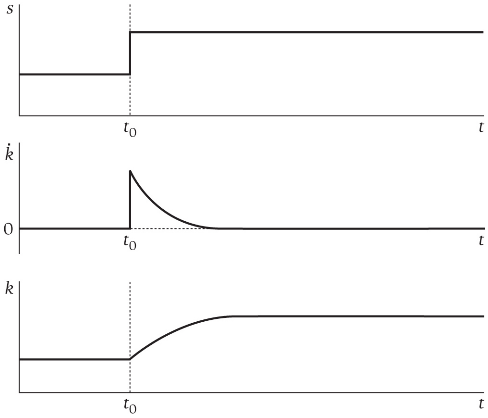
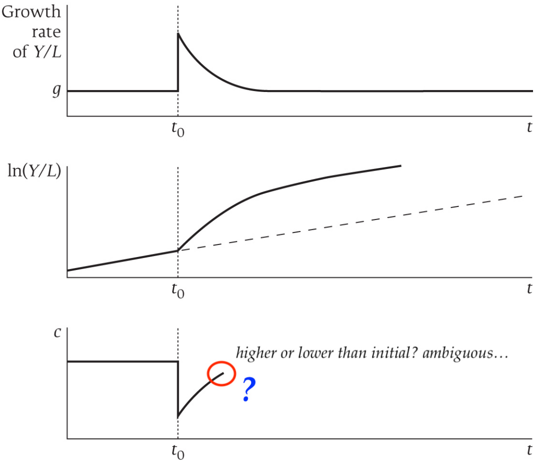


An increase in the saving rate

$$Y/L = Af(k)$$

$$c = f(k)(1-s)$$



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Consumption and the golden-rule

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- golden-rule k^*
- characterized by $MPK = (n+g+\delta)$.
- but no reason for s to be exactly at the level which implies the golden-rule k^*!

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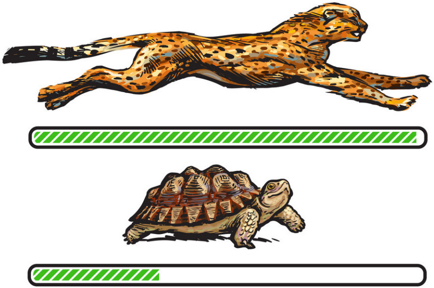
Speed of convergence

- *How fast* will k reach k^* when starting out of equilibrium?



Refresher: Taylor approximations

- *Taylor's theorem:* any (continuously differentiable) function $\phi(x)$ can be approximated, around a point x_0, by a n-th degree polynomial.
- n-th degree Taylor approximation around x_0:

$$\phi(x) = [\frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)}{1!}(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{\phi^{(n)}(x_0)}{n!}(x - x_0)^n] + R_n$$

(R_n = remainder)

- linear approximation around x_0:

$$\phi(x) \approx \phi(x_0) + \phi'(x_0)(x - x_0)$$

Speed of convergence

- $\dot{k} = \dot{k}(k)$
- linear approximation around k^*:

$$\dot{k} \approx \left[\frac{\partial \dot{k}(k)}{\partial k} \Big|_{k=k^*} \right] (k - k^*) \quad \Rightarrow \quad \dot{k} \approx -\lambda (k - k^*) \quad \Rightarrow \quad \dot{k} + \lambda k = \lambda k^*$$

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- $k(t) \approx k^* + e^{-\lambda t}(k(0) - k^*)$
- $\lambda = -\frac{\delta \dot{k}}{\delta k}|_{k=k^*} = (n + g + \delta)(1 - \alpha_K)$
- Bottom line: for plausible parameters, convergence is not fast.
- eg: $(n + g + \delta) = 6\%$ and $\alpha_K = 1/3 \rightarrow \lambda = 0.04$

Solow model: Takeaways

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3. Two sources of cross-country variation in Y/L: s and A.
   - BUT implausibly huge differences in s would be needed to produce sizable differences in Y/L.
   - AND A is non-rival, so big differences cannot last.
   - *Long-run convergence across countries.*

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Subsequent developments of neoclassical growth theory address these 3 issues.

Solow model: more radical criticisms

- Overcomes Harrodian instability by assuming it away
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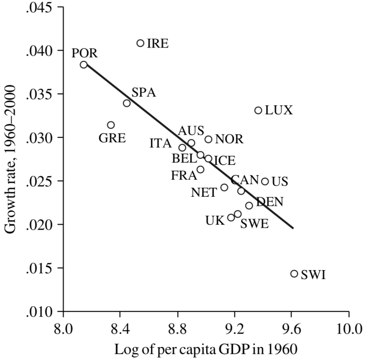
- Overcomes Harrodian instability by assuming it away
  - continuous full employment by assumption.
  - $g = g_W$ is assumed, not demonstrated.
- Relies on a fictional aggregate production function $Y(K, AL)$.
  - In reality, many production processes and many types of inputs
  - They do not add up to an aggregate production function with the properties assumed by Solow model (Cambridge capital controversy)
  - One-good economy: *'Venerable Solow may make peculiar assumptions, but he never makes a mistake'* (A. Sen, 1974)
  - Recent discussion of the problem: Baqaee & Fahri (2019).
- Subsequent developments of neoclassical growth theory do *not* address these issues.

Convergence regressions

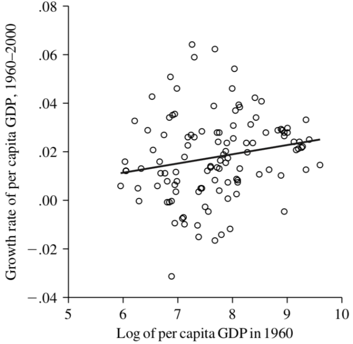
- Do poor countries catch up, as Solow model suggests?
- Empirical test:

$$\Delta ln(Y/L)_{i,1} = \alpha + \beta ln(Y/N)_{i,0} + \epsilon_i$$

- with $t = 0$ and $t = 1$ quite apart in time.
- $\beta = -1$ would signal perfect convergence
- Evidence from 1960-2000: convergence among Western countries ($\beta \approx -1$), but little or no convergence overall ($\beta \approx 0$).
- Since 2000, there is some sign of overall convergence.



(a) 18 original OECD members



(b) world (114 countries)

Figure: International evidence on convergence: 1960 income and subsequent growth

But: some sign of increasing convergence in the last 2 decades

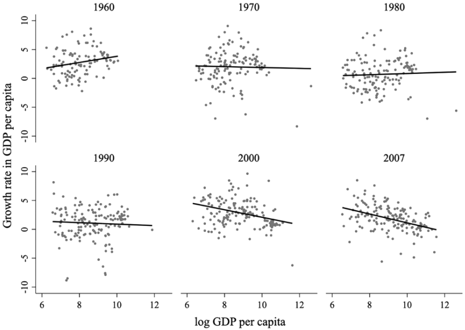


Fig. 1. Income convergence by decade. This figure plots, by decade, the raw scatter plots for the decade's β-convergence regression, as well as the regression line itself. $100\,(\log(\mathrm{GDPpc})_{i,t+10} - \log(\mathrm{GDPpc})_{i,t}/10) = \alpha_t + \beta_t \log(\mathrm{GDPpc})_{i,t} + \epsilon_{i,t}$. The income measure is income per capita, adjusted for PPP, from the Penn World Tables v10.0. The sample is all countries for which data are available, excluding those with a population less than 200,000 or for whom natural resources account for >75% of their GDP. Data availability means that the number of countries is growing over time. For 2007, the period considered is 2007–17. A color version of this figure is available online.

Source: Kremer, Willis and You (2022)