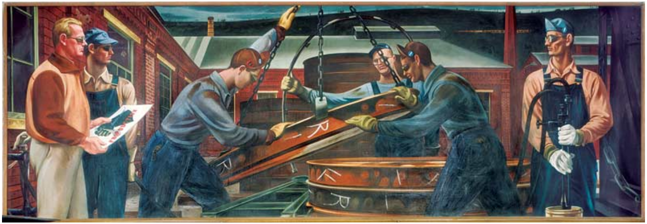


Macroeconomic Theory I

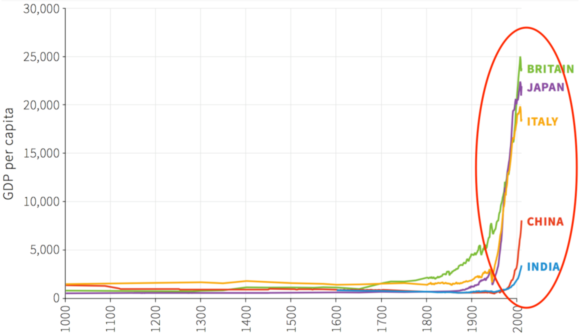
Section 2 - Growth (I)

Daniele Girardi
University of Massachusetts Amherst

Spring 2021

UMASS AMHERST

The hockey stick of history



Section 2: Growth (I)

The Plan

1. Harrod-Domar
2. Solow
3. Ramsey-Cass-Koopmans
4. Diamond’s overlapping-generations (OLG)

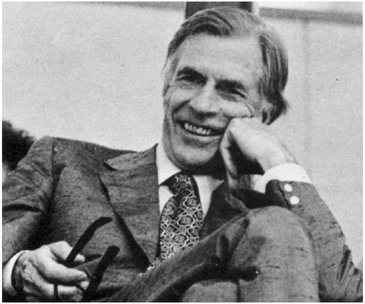
Key idea: Intertemporal equilibrium

- Static analysis: equilibrium condition –> equilibrium relations.
  - I=S
  - supply=demand
  - ...
- Growth theory is *dynamic*: intertemporal equilibria.

Main concepts:

- Intertemporal equilibrium
- Steady state
- Dynamic stability

The Harrod-Domar model



There's an old joke. Two elderly women are at a Catskill restaurant. One of them says, 'Boy, the food at this place is just terrible.' The other one says, 'Yeah I know. And such small portions.'

(Woody Allen, 'Annie Hall')

The Harrod-Domar model

- 'Grandfather' of modern growth theory.
- Premise 1: aggregate investment has a dual effect
  1. multiplier effect (demand side)
  2. capacity-creating effect (supply side)
- Premise 2: investment depends on output (accelerator)

The Harrod-Domar model

- 'Grandfather' of modern growth theory.
- Premise 1: aggregate investment has a dual effect
  1. multiplier effect (demand side)
  2. capacity-creating effect (supply side)
- Premise 2: investment depends on output (accelerator)
- Main findings:
  - unique equilibrium path: $g_W = sa$ (*warranted rate*)
  - warranted rate does not guarantee full (nor stable) employment
  - instability: economy won't converge to g_W, except by a fluke

Harrod-Domar model

Assumptions:

$$Y = C + I; \qquad S = sY; \qquad Y^* = aK; \qquad Y = uY^*$$

$$g_K = \frac{\dot{K}}{K} = \frac{I}{K}; \qquad \frac{\partial g_K}{\partial t} = \dot{g}_K = \alpha(u-1) \qquad \text{with } \alpha > 0$$

Harrod-Domar model

Assumptions:

$$Y = C + I; \qquad S = sY; \qquad Y^* = aK; \qquad Y = uY^*$$

$$g_K = \frac{\dot{K}}{K} = \frac{I}{K}; \qquad \frac{\partial g_K}{\partial t} = \dot{g}_K = \alpha(u-1) \qquad \text{with } \alpha > 0$$

Actual investment rate:

$$I = S \rightarrow g_K(t) = \frac{S(t)}{K(t)} = s\frac{Y(t)}{K(t)} = s\frac{Y^*(t)}{K(t)}\frac{Y(t)}{Y^*(t)} = sa[u(t)]$$

Harrod-Domar model

Assumptions:

$$Y = C + I; \qquad S = sY; \qquad Y^* = aK; \qquad Y = uY^*$$

$$g_K = \frac{\dot{K}}{K} = \frac{I}{K}; \qquad \frac{\partial g_K}{\partial t} = \dot{g}_K = \alpha(u-1) \qquad \text{with } \alpha > 0$$

Actual investment rate:

$$I = S \rightarrow g_K(t) = \frac{S(t)}{K(t)} = s\frac{Y(t)}{K(t)} = s\frac{Y^*(t)}{K(t)}\frac{Y(t)}{Y^*(t)} = sa[u(t)]$$

Warranted growth:

$$g_W = sa$$

The equilibrium ('warranted') rate of growth

- Warranted vs natural growth rate:

$$g_Y = sa \neq n$$

The equilibrium ('warranted') rate of growth

- Warranted vs natural growth rate:

$$g_Y = sa \neq n$$

- Dynamic instability:

$$g_K = u(sa) > g_W = sa \quad \Rightarrow \quad u > 1 \quad \Rightarrow \quad \dot{g}_K > 0$$

$$g_K = u(sa) < g_W = sa \quad \Rightarrow \quad u < 1 \quad \Rightarrow \quad \dot{g}_K < 0$$

The equilibrium (‘warranted’) rate of growth

- Warranted vs natural growth rate:

$$g_Y = sa \neq n$$

- Dynamic instability:

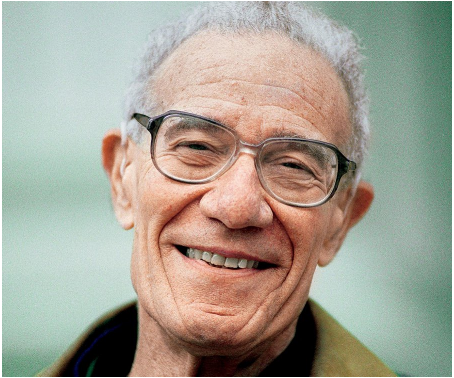
$$g_K = u(sa) > g_W = sa \quad \Rightarrow \quad u > 1 \quad \Rightarrow \quad \dot{g}_K > 0$$

$$g_K = u(sa) < g_W = sa \quad \Rightarrow \quad u < 1 \quad \Rightarrow \quad \dot{g}_K < 0$$

- More formally:

$$\dot{g}_K = \alpha \left(\frac{g_K}{g_W} - 1 \right) \quad \text{with } \alpha > 0$$

The Solow model



Solow growth model

Key premises:

- *neoclassical* production function
- Say's law: $S \rightarrow I$.

Main implications:

- stable steady-state with $g_Y = n + g$;
- saving rate determines output level but not growth rate;
- K accumulation cannot explain long-run growth nor cross-country income differences;

Assumptions about production

- One-good economy
- Say's law: full employment of production factors at each t.
- K & L can be used in different proportions to produce same Y level.

$$Y(t) = F[K(t), A(t)L(t)] \qquad \text{[Aggregate production function]}$$

- $A(t)$ = state of knowledge/technology;
- labor-augmenting technological progress.

Assumptions about production

$$Y(t) = F[K(t), A(t)L(t)] \qquad \text{[Aggregate production function]}$$

$$F(cK, cAL) = cF(K, AL) \qquad \text{[CRS]}$$

UMASS AMHERST

Assumptions about production

$$Y(t) = F[K(t), A(t)L(t)] \qquad \text{[Aggregate production function]}$$

$$F(cK, cAL) = cF(K, AL) \qquad \text{[CRS]}$$

- CRS implies (setting $c = 1/AL$)

$$\frac{Y}{AL} = \frac{1}{AL}F(K, AL) = F(\frac{K}{AL}, 1)$$

- which we can write as:

$$y = f(k) \qquad \textit{[intensive form]}$$

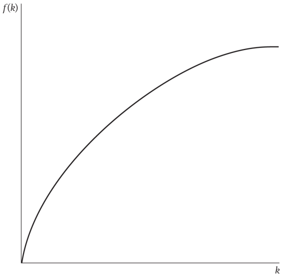
with $k = \dfrac{K}{AL}$, $y = \dfrac{Y}{AL}$ and $f(k) = F(k, 1)$

Assumptions about production

Assume that $y = f(k)$ satisfies the following properties:

$$f(0) = 0, \qquad f'(k) > 0, \qquad f''(k) < 0 \qquad \text{[positive but decreasing MPK]}$$

$$lim_{k \to 0} f'(k) = \infty, \qquad lim_{k \to \infty} f'(k) = 0 \qquad \text{[Inada conditions]}$$



UMASS AMHERST

Assumptions about the evolution of production inputs

$$\dot{L}(t) = nL(t) \rightarrow g_L = n \qquad \text{[Constant population growth]}$$

$$\dot{A}(t) = gA(t) \rightarrow g_A = g \qquad \text{[Constant ‘knowledge’ growth]}$$

$$\dot{K}(t) = sY(t) - \delta K(t), \qquad 0 < s \leq 1 \qquad \text{[Constant saving rate]}$$

The dynamics of the model

- Strategy: focus on $k = \frac{K}{AL}$
- Take the derivative of k wrt time

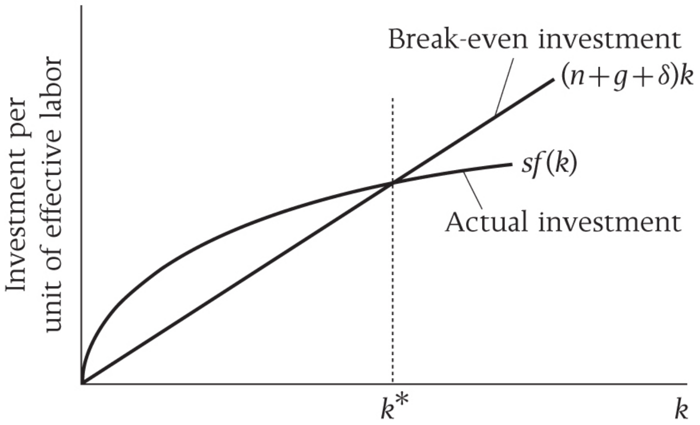
$$\dot{k}(t) = \frac{d(K/AL)}{dt} = \frac{\dot{K}}{AL} - \frac{K}{(AL)^2}(A\dot{L} + \dot{A}L) = \frac{\dot{K}}{AL} - \frac{K}{AL}\frac{\dot{L}}{L} - \frac{K}{AL}\frac{\dot{A}}{A}$$

- using $k = \frac{A}{AL}$, $y = \frac{Y}{AL}$ and the assumptions about inputs rewrite as:

$$\dot{k}(t) = sf[k(t)] - (n+g+\delta)k(t)$$

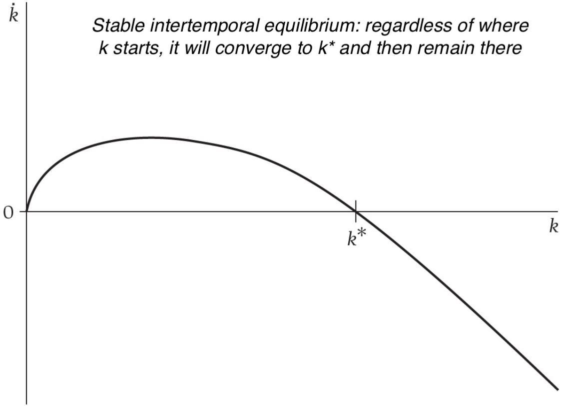
change in k = investment − breakeven investment

The dynamics of the model: actual vs. break-even investment



UMASS AMHERST

The dynamics of the model: phase diagram



The dynamics of the model: the 'balanced growth path'

Implications for the variables we actually care about?

- by assumption, $g_L = n$ and $g_A = g$;
- $K = ALk \rightarrow g_K = n + g$
- $Y = ALf(k) \rightarrow g_Y = n + g$
- $\frac{K}{L} = Ak \rightarrow g_{\frac{K}{L}} = g$
- $\frac{Y}{L} = Af(k) \rightarrow g_{\frac{Y}{L}} = g$

balanced growth path: all variables grow at constant rates.

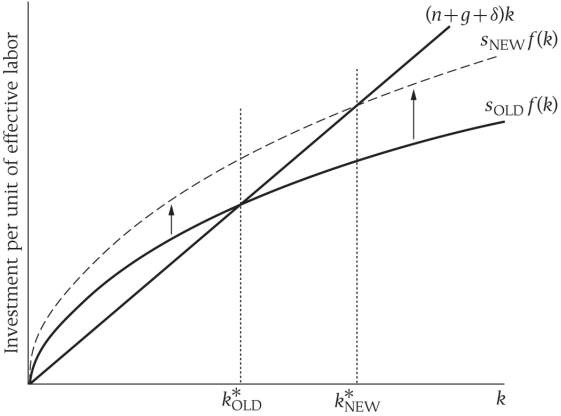
Other things we want to know:

1. Qualitative effect of an increase in s (*direction*)
2. What level of k maximizes consumption (*golden-rule* k^*)
3. Size of the effect of an increase in s (*how big*)
4. Speed of convergence: *how long* does transition take?

Other things we want to know:

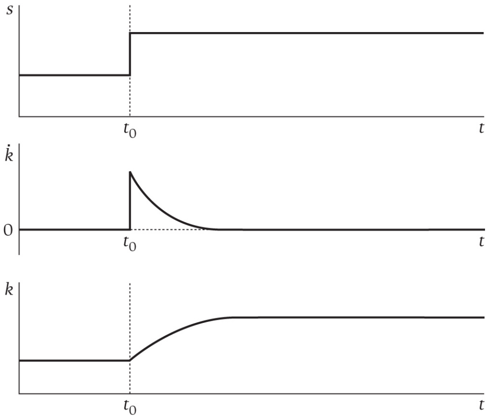
1. Qualitative effect of an increase in s (*direction*)
2. What level of k maximizes consumption (*golden-rule* k^*)
3. Size of the effect of an increase in s (*how big*)
4. Speed of convergence: *how long* does transition take?

An increase in the saving rate



UMASS AMHERST

An increase in the saving rate

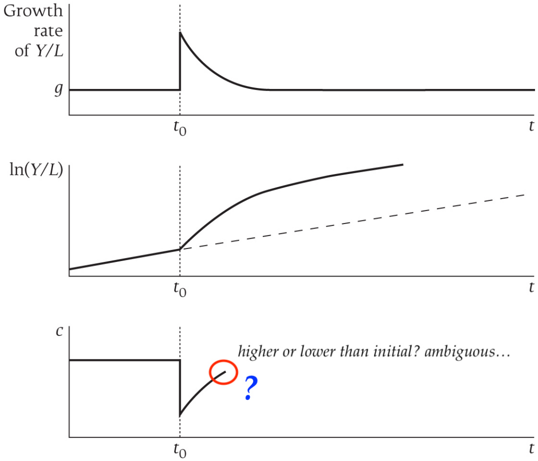


UMASS AMHERST

An increase in the saving rate

$Y/L = Af(k)$

$c = f(k)(1-s)$



UMASS AMHERST

Other things we want to know:

1. Qualitative effect of an increase in s (*direction*)
2. What level of k maximizes consumption (*golden-rule* k^*)
3. Size of the effect of an increase in s (*how big*)
4. Speed of convergence: *how long* does transition take?

UMASS AMHERST

Consumption and the golden-rule

- $c^* = f(k^*) - (n+g+\delta)k^*$
- $k^* = k^*(s, n, g, \delta)$

For what value of s is c^* maximized?

UMASS AMHERST

Consumption and the golden-rule

- $c^* = f(k^*) - (n+g+\delta)k^*$
- $k^* = k^*(s,n,g,\delta)$

For what value of s is c^* maximized?

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - (n+g+\delta)]\frac{\partial k^*}{\partial s} = 0$$

$$\Downarrow$$

$$f'(k^*) = (n+g+\delta)$$

Consumption and the golden-rule

- $c^* = f(k^*) - (n+g+\delta)k^*$
- $k^* = k^*(s,n,g,\delta)$

For what value of s is c^* maximized?

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - (n+g+\delta)]\frac{\partial k^*}{\partial s} = 0$$
$$\Downarrow$$
$$f'(k^*) = (n+g+\delta)$$

- golden-rule k^*: maximizes c^*;
- characterized by $MPK = (n+g+\delta)$;
- but no reason for s to be exactly at the level which implies the golden-rule k^*.

Other things we want to know:

1. Qualitative effect of an increase in s (*direction*)
2. What level of k maximizes consumption (*golden-rule* k^*)
3. Size of the effect of an increase in s (*how big*)
4. Speed of convergence: *how long* does transition take?

UMASS AMHERST

How big is the effect of the saving rate on output?

$$\frac{\partial y^*}{\partial s} = \frac{\partial f(k^*)}{\partial s} = f'(k^*)\frac{\partial k^*(s,n,g,\delta)}{\partial s}$$

UMASS AMHERST

How big is the effect of the saving rate on output?

$$\frac{\partial y^*}{\partial s} = \frac{\partial f(k^*)}{\partial s} = f'(k^*)\frac{\partial k^*(s,n,g,\delta)}{\partial s}$$

- from implicit differentiation $\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n+g+\delta)-sf'(k^*)}$

UMASS AMHERST

How big is the effect of the saving rate on output?

$$\frac{\partial y^*}{\partial s} = \frac{\partial f(k^*)}{\partial s} = f'(k^*)\frac{\partial k^*(s,n,g,\delta)}{\partial s}$$

- from implicit differentiation $\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n+g+\delta)-sf'(k^*)}$

$$\Rightarrow \frac{\partial y^*}{\partial s} = \frac{f'(k^*)f(k^*)}{(n+g+\delta)-sf'(k^*)}$$

UMASS AMHERST

How big is the effect of the saving rate on output?

$$\frac{\partial y^*}{\partial s} = \frac{\partial f(k^*)}{\partial s} = f'(k^*)\frac{\partial k^*(s,n,g,\delta)}{\partial s}$$

- from implicit differentiation $\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n+g+\delta)-sf'(k^*)}$

$$\Rightarrow \frac{\partial y^*}{\partial s} = \frac{f'(k^*)f(k^*)}{(n+g+\delta)-sf'(k^*)}$$

- convert this to an elasticity, use $sf(k^*) = (n+g+\delta)k^*$ and rearrange to get:

$$\frac{s}{y^*}\frac{\delta y^*}{\delta s} = \frac{\alpha_k(k^*)}{1-\alpha_k(k^*)}$$

where $\alpha_K(k^*) = \frac{k^*}{f(k^*)}f'(k^*)$ is the elasticity of y w.r.t. k.

Effect of the saving rate on output

$$\frac{s}{y^*}\frac{\delta y^*}{\delta s} = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)}$$

$$\text{Assume} \quad \alpha \approx 1/3 \quad \Rightarrow \quad \frac{s}{y^*}\frac{\delta y^*}{\delta s} \approx \frac{1}{2} \tag{1}$$

- A 1 percent increase in s increases y^* by 0.5 percent;
- ex: raising s from 0.20 to 0.22 (+10%) increases y^* by 5%;
- significant but not very big.

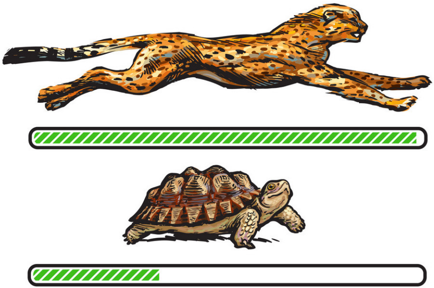
UMASS AMHERST

Other things we want to know:

1. Qualitative effect of an increase in s (*direction*)
2. What level of k maximizes consumption (*golden-rule* k^*)
3. Size of the effect of an increase in s (*how big*)
4. Speed of convergence: *how long* does transition take?

Speed of convergence

- *How fast* will k reach k^* when starting out of equilibrium?



UMASS AMHERST

Refresher: Taylor approximations

- *Taylor's theorem:* any (continuously differentiable) function $\phi(x)$ can be approximated, around a point x_0, by a n-th degree polynomial.
- n-th degree Taylor approximation around x_0:

$$\phi(x) = [\frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)}{1!}(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{\phi^{(n)}(x_0)}{n!}(x - x_0)^n] + R_n$$

  (R_n = remainder)
- linear approximation around x_0:

$$\phi(x) \approx \phi(x_0) + \phi'(x_0)(x - x_0)$$

UMASS AMHERST

Speed of convergence

- $\dot{k} = \dot{k}(k)$
- linear approximation around k^*:

$$\dot{k} \approx \left[\frac{\partial \dot{k}(k)}{\partial k}\bigg|_{k=k^*} \right](k - k^*) \quad \Rightarrow \quad \dot{k} \approx -\lambda(k - k^*) \quad \Rightarrow \quad \dot{k} + \lambda k = \lambda k^*$$

UMASS AMHERST

Speed of convergence

- $\dot{k} = \dot{k}(k)$
- linear approximation around k^*:

$$\dot{k} \approx \left[\frac{\partial \dot{k}(k)}{\partial k} \bigg|_{k=k^*} \right] (k - k^*) \quad \Rightarrow \quad \dot{k} \approx -\lambda (k - k^*) \quad \Rightarrow \quad \dot{k} + \lambda k = \lambda k^*$$

- $k(t) \approx k^* + e^{-\lambda t}(k(0) - k^*)$

Speed of convergence

- $\dot{k} = \dot{k}(k)$
- linear approximation around k^*:

$$\dot{k} \approx \left[\frac{\partial \dot{k}(k)}{\partial k} \Big|_{k=k^*} \right] (k - k^*) \quad \Rightarrow \quad \dot{k} \approx -\lambda(k - k^*) \quad \Rightarrow \quad \dot{k} + \lambda k = \lambda k^*$$

- $k(t) \approx k^* + e^{-\lambda t}(k(0) - k^*)$
- $\lambda = -\frac{\delta \dot{k}}{\delta k}|_{k=k^*} = (n + g + \delta)(1 - \alpha_K)$
- Bottom line: for plausible parameters, convergence is not fast.
- eg: $(n + g + \delta) = 6\%$ and $\alpha_K = 1/3 \rightarrow \lambda = 0.04$

Solow model: Takeaways

1. Convergence to a balanced growth path, where $g_Y = n + g$.
   - $g_Y > n + g$ only during convergence.

Solow model: Takeaways

1. Convergence to a balanced growth path, where $g_Y = n + g$.
   - $g_Y > n + g$ only during convergence.

2. s affects Y but not g_Y (in equilibrium).

UMASS AMHERST

Solow model: Takeaways

1. Convergence to a balanced growth path, where $g_Y = n + g$.
   - $g_Y > n + g$ only during convergence.

2. s affects Y but not g_Y (in equilibrium).

3. Two sources of cross-country variation in Y/L: s and A.
   - BUT implausibly huge differences in s would be needed to produce sizable differences in Y/L.

Solow model: Takeaways

1. Convergence to a balanced growth path, where $g_Y = n + g$.
   - $g_Y > n + g$ only during convergence.

2. s affects Y but not g_Y (in equilibrium).

3. Two sources of cross-country variation in Y/L: s and A.
   - BUT implausibly huge differences in s would be needed to produce sizable differences in Y/L.

4. Technology (A) is the only possible explanation of vast cross-country differences in Y/L.

Solow model: the conventional criticisms

- Main driver of growth (A) falls from the sky and has a vague definition;

Solow model: the conventional criticisms

- Main driver of growth (A) falls from the sky and has a vague definition;
- No microfoundations (not robust to Lucas critique);

Solow model: the conventional criticisms

- Main driver of growth (A) falls from the sky and has a vague definition;
- No microfoundations (not robust to Lucas critique);
- Can't say anything about welfare;

Solow model: the conventional criticisms

- Main driver of growth (A) falls from the sky and has a vague definition;
- No microfoundations (not robust to Lucas critique);
- Can't say anything about welfare;
- Subsequent developments of neoclassical growth theory address these issues.

Solow model: more radical criticisms

- Overcomes Harrodian instability by assuming it away
  - continuous full employment by assumption.
  - $g = g_W$ is assumed, not demonstrated.

Solow model: more radical criticisms

- Overcomes Harrodian instability by assuming it away
  - continuous full employment by assumption.
  - $g = g_W$ is assumed, not demonstrated.
- K as a single homogenous and 'malleable' factor, substitutable for L according to relative factor prices.
  - This notion of K is incorrect in a world of multiple capital goods.
  - Sraffa (1960) & Cambridge capital controversy – also see Felipe & Fisher (MECA, 2003)
  - One-good economy: *'Venerable Solow may make peculiar assumptions, but he never makes a mistake'* (A. Sen, 1974, Economica)
  - Recent mainstream 'rediscovery' of the problem: Baqaee & Fahri (JEEA, forthcoming).
- Subsequent developments of neoclassical growth theory do *not* address these issues.

Growth accounting

Given the production function, we can decompose $g_{Y/L}(t)$ into:

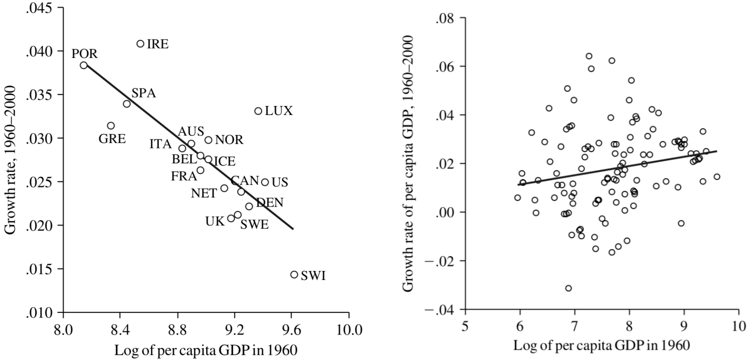
$$g_{Y/L} = \alpha_K g_{K/L} + R$$

- α_K estimated using P/Y;
- residual R interpreted as the contribution of unobservable technological progress.

UMASS AMHERST

Convergence regressions

- Do poor countries catch up?
- Solow model suggests convergence, if A non-excludable.
- Empirical test:
$$\Delta ln(Y/L)_{i,1} = \alpha + \beta ln(Y/N)_{i,0} + \epsilon_i$$
- with $t = 0$ and $t = 1$ usually quite apart in time (40/50 years).
- $\beta = -1$ = perfect convergence
- evidence: some convergence among core-OECD countries ($\beta \approx -1$), but little or no convergence overall ($\beta \approx 0$).



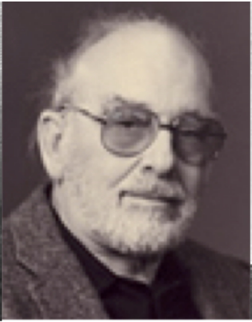
(a) 18 original OECD members

(b) world (114 countries)

Figure: International evidence on convergence: 1960 income and subsequent growth

Ramsey-Cass-Koopmans model

Frank Ramsey



David Cass

Tjalling Koopmans

Assumptions about production

- Production technology and inputs evolution exactly the same as in Solow, but $\delta = 0$ for simplicity.
- $F(K, AL)$; CRS; $f(0) = 0$; $f'(k) > 0$; $f''(k) < 0$;
  $lim_{k \to 0} f'(k) = \infty$; $lim_{k \to \infty} f'(k) = 0$.
- $\frac{\dot{A}}{A} = g_A = g$; $\frac{\dot{L}}{L} = g = n$
- $\dot{K}(t) = Y(t) - \zeta(t)$ where ζ is total consumption.
- *Representative firm* assumption.

Assumptions about households

Large but fixed number of identical households:

- each grows at rate n;
- household members are infinitely lived, forward-looking *and* have perfect foresight into the infinite future;
- they supply 1 unit of L at each point in time and earn wages;
- own K, that they rent to firms, earning K income;
- divide their income between C and I in such a way as to maximize utility over their (infinite) lifetime.
- *representative household* assumption.

The Euler equation

- Assumed households' behavior implies the **Euler equation** – the fundamental driver of this model:

$$g_C = \frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta}$$

- Higher interest rate induces to postpone consumption, so it contributes *positively* to its growth in time.
- Higher discount rate induces to anticipate consumption, so it contributes *negatively* to its growth in time.
- Let's now study formally how this equation follows from the assumptions of the model...

UMASS AMHERST

Household's utility function

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt$$

- gives PV of total utility enjoyed by household members over their lifetime.
- C = consumption per person.
- L/H = number of household members.
- ρ = discount rate.
- instantaneous utility $u()$:

$$u[C(t)] = \frac{C(t)^{1-\theta}}{1-\theta} \qquad \theta > 0; \qquad \rho - n - (1-\theta)g > 0$$

Firms & factors' prices

Perfectly competitive firms in single-good economy, therefore

- interest rate: $r(t) = f'[k(t)]$
- wage per unit of eff. labor: $w(t) = \frac{W(t)}{A} = [f(k) - kf'(k)]$

No-Ponzi condition

- Household consumption is constrained by the PV of their wealth:

$$\lim_{s\to\infty} e^{-R(s)} K(s) \geq 0$$

- Or, in 'intensive form' (scaled by AL):

$$\lim_{s\to\infty} e^{-R(s)} e^{(n+g)s} k(s) \geq 0$$

- *No-Ponzi condition:* household's asset holdings cannot be negative in the limit.

- Will be satisfied with equality.

Households' dynamic optimization problem

- Household maximizes PV of lifetime utility (intensive form):

$$\text{Max } U = B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt$$

with $B = A(0)^{1-\theta} \frac{L(0)}{H}$ and $\beta = \rho - n - (1-\theta)g$

- subject to the state equation:

$$\dot{k}(t) = (r - n - g)k(t) + w(t) - c(t)$$

- and the transversality (no-Ponzi w/ equality) condition:

$$\lim_{s \to \infty} e^{-R(s)} e^{(n+g)s} k(s) = 0$$

The Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

- This is the **Euler equation** we have seen before, but scaled by AL (intensive form). Same meaning and interpretation.
- $r > \rho \rightarrow$ households postpone consumption $\rightarrow c(t)$ increases in time.
- $r < \rho \rightarrow$ households anticipate consumption $\rightarrow c(t)$ decreases in time.

The dynamics of the economy: c & k

- Dynamics of c (Euler equation):

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

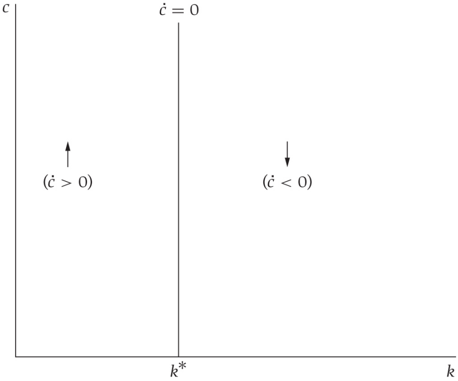
- Dynamics of k (like in Solow but w/o depreciation):

$$\dot{k} = f(k(t)) - c(t) - (n+g)k(t)$$

- intertemporal equilibrium: $\dot{c} = 0$ and $\dot{k} = 0$;
- two variables phase diagram

The dynamics of the economy: consumption

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

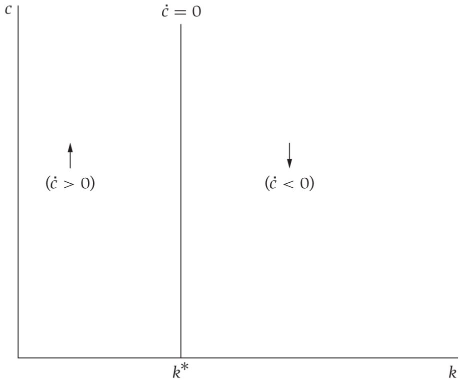


- what's going on in this graph?

UMASS AMHERST

The dynamics of the economy: consumption

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

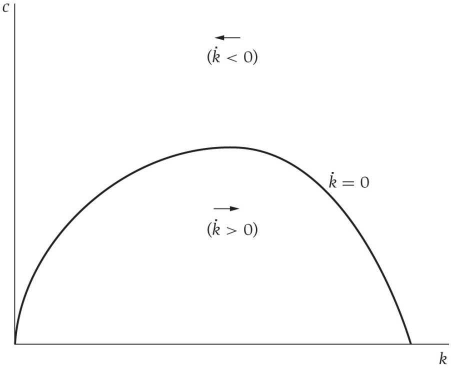


- what's going on in this graph?
- $\dot{c} = 0 \quad \rightarrow \quad f'(k) = \rho + \theta g$
- implicitly defines k^*.
- k^* is unique & independent of c (vertical line).
- $k > k^* \rightarrow f'(k) < \rho + \theta g \rightarrow \dot{c} < 0$
- $k < k^* \rightarrow f'(k) > \rho + \theta g \rightarrow \dot{c} > 0$

UMASS AMHERST

The dynamics of the economy: capital stock

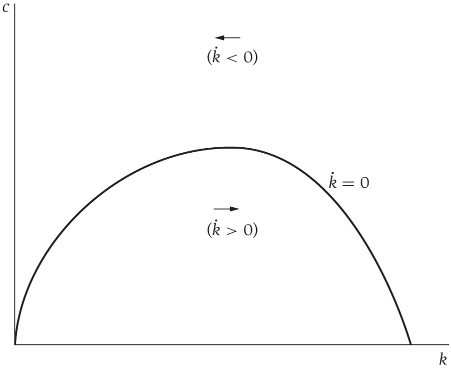
$$\dot{k} = f(k(t)) - c(t) - (n+g)k(t)$$



- what's going on in this graph?

The dynamics of the economy: capital stock

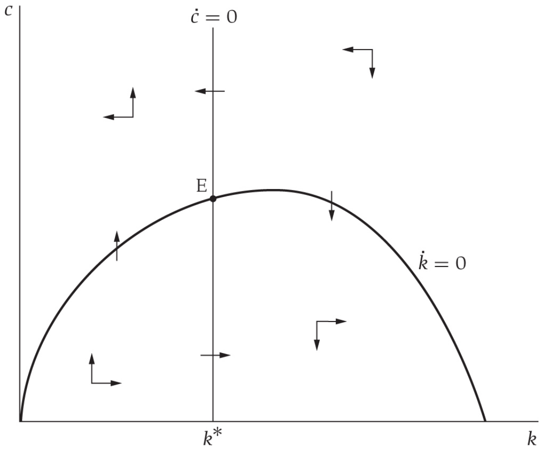
$$\dot{k} = f(k(t)) - c(t) - (n+g)k(t)$$



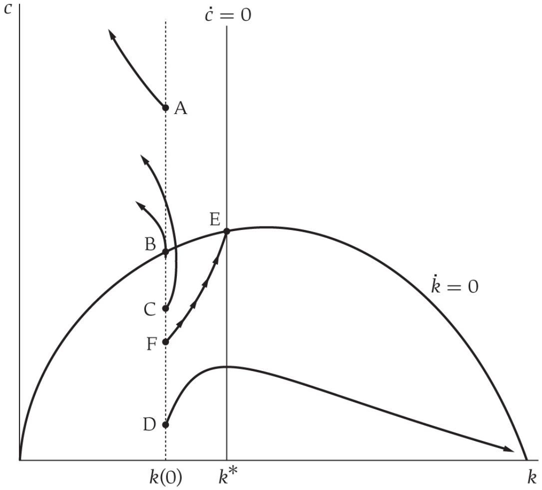
- what's going on in this graph?
- $\dot{k} = 0 \quad \rightarrow \quad c^* = f(k) - (n+g)k$
- c^* U-shaped: increasing in c as long as $f'(k) > (n+g)$.
- $c > c^* \rightarrow$, investment lower than break-even $\rightarrow \dot{k} < 0$.
- $c < c^* \rightarrow$, investment higher than break-even $\rightarrow \dot{k} > 0$.

UMASS AMHERST

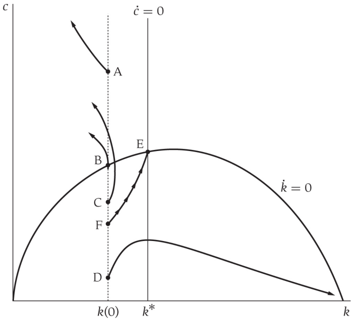
The dynamics of the economy: phase diagram



The dynamics of the economy: phase diagram



The dynamics of the economy: phase diagram



- E = intertemporal equilibrium ($\dot{k} = \dot{c} = 0$);
- given $k(0)$, only $c(0) = F$ is on the 'stable branch' that leads to E;
- $c(0) < F$ leads to zero c and infinite k: not utility-maximizing!
- $c(0) > F$ leads to negative k but positive c: not feasible!
- $c(0) = F$ is the only $c(0)$ that implies

$$\lim_{s \to \infty} e^{-R(s)} e^{(n+g)s} k(s) = 0$$

so it is the only feasible and utility-maximizing one.

The saddle path

- For any possible $k(0)$, there is a unique $c(0)$ that satisfies

$$\lim_{s\to\infty} e^{-R(s)} e^{(n+g)s} k(s) = 0$$

- this $c(0)$ is the one on the 'saddle path'[1] towards steady state.
- all other $c(0)$'s are on unstable trajectories, but are ruled out by the no-Ponzi condition or by intertemporal optimization;
- *saddle-path stable* equilibrium.
- (quite obviously) it is Pareto-efficient.

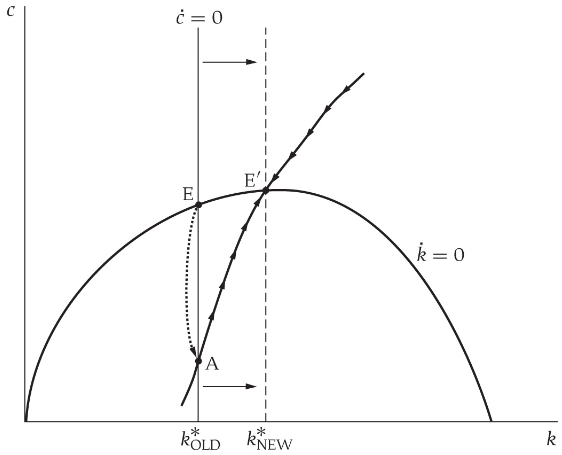
[1] The reason for this name is the analogy with a marble left on top of a saddle. There is one point on the saddle where, if left there, the marble does not move. This point corresponds to the steady state. There is a trajectory on the saddle with the property that if the marble is left at any point on that trajectory, it rolls toward the steady state. But if the marble is left at any other point, the marble falls to the ground. (Barro & Sala i-Martin, *Economic growth*, 1990).

UMASS AMHERST

The balanced growth path

- $\dot{k} = \dot{y} = \dot{c} = 0$
- $g_Y = g_K = g_C = n + g$
- $g_{Y/L} = g_{K/L} = g_{C/L} = g$
- exactly as in Solow model!
- here, however, $k < k_{GR}$
  - households don't maximize c (as in the *golden rule*), but $PV(c)$.
  - $\rho > 0$: bias towards the present

A fall in the discount rate



- $f'(k) = \rho + \theta g$
- $c^* = f(k) - (n+g)k$

Diamond (1965): The Overlapping Generations (OLG) model

OLG model: assumptions about households

- Time is discrete ($t = 0, 1, 2, \dots$);
- each individual lives for two periods;
- $L_t = (1+n)L_{t-1}$ individuals born at time t;
- young (1st period):
  - no K
  - supplies 1 unit of L;
  - divides resulting wage between C and S;
- old (2nd period):
  - rents her K (=1st period savings)
  - then consumes $(1+r)K$

OLG model: assumptions about production

- At each t, old people's K and young people's L are combined to produce Y;
- $Y = F(K_t, A_t L_t)$
- CRS and Inada conditions (as in Solow & Ramsey);
- $\delta = 0$ for simplicity;
- $A_t = (1+g)A_{t-1}$;
- Competitive markets
  - $r_t = f'(k_t)$
  - $w_t = f(k_t) - k_t f'(k_t)$

OLG model: The Plan

- Focus on $k = \frac{K}{AL}$.
- Intertemporal equilibrium: $k_{t+1} = k_t$

OLG model: The Plan

- Focus on $k = \frac{K}{AL}$.
- Intertemporal equilibrium: $k_{t+1} = k_t$

Our strategy:

1. U maximization → C dynamics (Euler equation);
2. C dynamics → dynamics of K & k $\Rightarrow$ k_{t+1} as a function of k_t;
3. set $k_{t+1} = k_t$ to study intertemporal equilibrium & stability.

OLG model: The Plan

- Focus on $k = \frac{K}{AL}$.
- Intertemporal equilibrium: $k_{t+1} = k_t$

Our strategy:

1. U maximization $\rightarrow$ C dynamics (Euler equation);
2. C dynamics $\rightarrow$ dynamics of K & k $\quad\Rightarrow\quad$ k_{t+1} as a function of k_t;
3. set $k_{t+1} = k_t$ to study intertemporal equilibrium & stability.

- 1. using relatively general production and utility functions;
- stronger functional form assumptions needed to do 2. & 3.

UMASS AMHERST

Consumption dynamics

- Maximization of lifetime-utility...

$$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta} \quad \text{with} \quad \theta > 0, \quad \rho > -1$$

- ...subject to the budget constraint

$$C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1} = A_t w_t$$

UMASS AMHERST

Consumption dynamics

- Maximization of lifetime-utility...

$$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta} \quad \text{with} \quad \theta > 0, \quad \rho > -1$$

- ...subject to the budget constraint

$$C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1} = A_t w_t$$

$\Downarrow$

- f.o.c. imply the Euler Equation:

$$\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}$$

Discrete-time Euler equation: intuitive derivation

- At the optimal point, a marginal reallocation of C from 1st to 2nd period does not affect utility:

$$C_{1t}^{-\theta}\Delta C = \frac{1}{1+\rho} C_{2t+1}^{-\theta}(1+r_{t+1})\Delta C \tag{2}$$

(MU of change in C_1=MU of change in C_2)

- rearrange as

$$\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}$$

Discrete-time Euler equation: 'systematic' derivation

- Set the Lagrangian for the utility-maximization problem

$$\mathcal{L} = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta} + \lambda\left[A_t w_t - \left(C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1}\right)\right]$$

- f.o.c. for C_{1t} and C_{2t}:

$$C_{1t}^{-\theta} = \lambda; \qquad \frac{1}{1+\rho} C_{2t+1}^{-\theta} = \frac{1}{1+r_{t+1}}\lambda$$

- Substitute for λ and rearrange:

$$\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}$$

Consumption dynamics

- Substitute Euler Equation into budget constraint to get

$$C_{1t} = \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} A_t w_t$$

- or more simply:

$$C_{1t} = [1 - s(r)] A_t w_t$$

$$\text{with} \quad s(r) = 1 - \frac{C_{1t}}{A_t w_t} = \frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}$$

- Implication: 1st period saving increasing in r if $\theta < 1$; decreasing if $\theta > 1$; s independent of r with logarithmic utility ($\theta = 1$)

Consumption dynamics

- Substitute Euler Equation into budget constraint to get

$$C_{1t} = \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} A_t w_t$$

- or more simply:

$$C_{1t} = [1 - s(r)] A_t w_t$$

$$\text{with} \quad s(r) = 1 - \frac{C_{1t}}{A_t w_t} = \frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}$$

- Implication: 1st period saving increasing in r if $\theta < 1$; decreasing if $\theta > 1$; s independent of r with logarithmic utility ($\theta = 1$)
- r has both an *income* and a *substitution* effect.

The dynamics of the OLG economy

- Capital accumulation in a given period is:

$$K_{t+1} = s(r_{t+1})A_t w_t L_t$$

The dynamics of the OLG economy

- Capital accumulation in a given period is:

$$K_{t+1} = s(r_{t+1})A_t w_t L_t$$

- Focus on the intensive form (divided by $A_{t+1}L_{t+1}$):

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r_{t+1}) w_t$$

UMASS AMHERST

The dynamics of the OLG economy

- Capital accumulation in a given period is:

$$K_{t+1} = s(r_{t+1})A_t w_t L_t$$

- Focus on the intensive form (divided by $A_{t+1}L_{t+1}$):

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r_{t+1}) w_t$$

- Given one-good competitive economy, we can substitute for factors' prices:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(f'(k_{t+1}))[f(k_t) - k_t f'(k_t)]$$

UMASS AMHERST

The dynamics of the OLG economy

- We now have k_{t+1} as a (implicit) function of k_t:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(f'(k_{t+1}))[f(k_t) - k_t f'(k_t)] \tag{3}$$

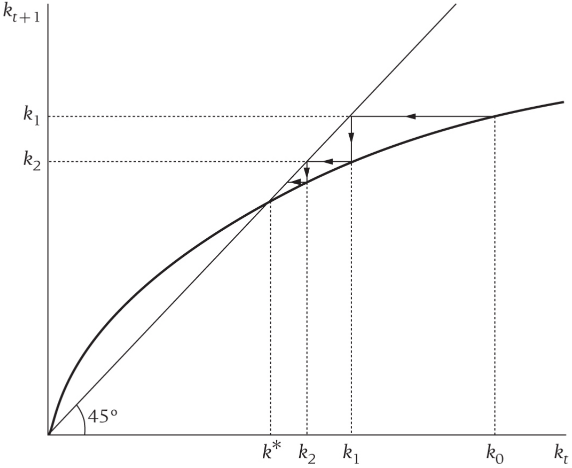
- Assume Cobb-Douglas production & logarithmic U:

$$f(k) = k^{\alpha}; \quad f'(k) = \alpha k^{\alpha-1}; \quad s = 1/(2+\rho)$$

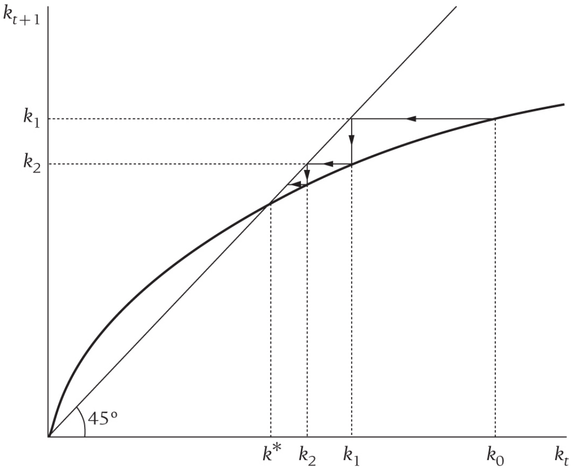
- So the equation of motion for k is:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho} (1-\alpha) k_t^{\alpha}$$

The dynamics of the OLG economy



The dynamics of the OLG economy



- Equilibrium: $k_{t+1} = k_t = k^*$;
- decreasing MPK & Inada conditions ensure existence & uniqueness (except $k = 0$);
- dynamically stable;
- steady state à la Solow-Ramsey: constant s and k; Y/L grows at rate g.

The dynamics of the OLG economy

- In intertemporal equilibrium, $k_t = k_{t+1} = k^*$

$$k^* = \frac{1}{(1+n)(1+g)(2+\rho)}(1-\alpha){k^*}^{\alpha}$$

- Solving for k^*

$$k^* = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{1}{1-\alpha}}$$

UMASS AMHERST

How fast is convergence in the OLG economy?

- *Linear approximation* around steady state

$$k_{t+1} - k^* \approx \lambda(k_t - k^*) \qquad \text{with } \lambda = \left.\frac{dk_{t+1}}{dk_t}\right|_{k_t = k^*}$$

- Solving the (1st order linear) difference equation:

$$k_t - k^* \approx \lambda^t (k_0 - k^*)$$

UMASS AMHERST

How fast is convergence in the OLG economy?

- *Linear approximation* around steady state

$$k_{t+1} - k^* \approx \lambda (k_t - k^*) \qquad \text{with } \lambda = \left. \frac{dk_{t+1}}{dk_t} \right|_{k_t = k^*}$$

- Solving the (1st order linear) difference equation:

$$k_t - k^* \approx \lambda^t (k_0 - k^*)$$

- w/ log U and Cobb-Douglas production: $0 < \lambda = \alpha < 1$;
- With $\alpha = 1/3$, two-thirds of the 'gap' removed in one period (=half a lifetime).

UMASS AMHERST

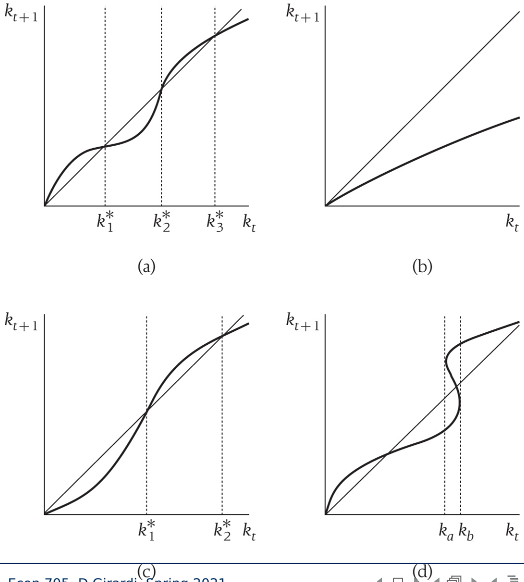
The general case

- With more general utility and production functions?
- Equation of motion for k:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r) \frac{f(k_t) - k_t f'(k_t)}{f(k_t)} f(k_t)$$

4 components: [AL_t/AL_{t+1}] [saving rate] [wage share] [Y/AL]

- k_{t+1} depends on k_t through three channels;
- (almost) anything goes.



- (a) and (c): multiple equilibria (s or W/Y increasing);
- (b): stable zero-output equilibrium (either s or W/Y approach 0 when $k = 0$);
- (d): indeterminacy (s 'very increasing' in k_t).

Welfare & dynamic inefficiency

- OLG equilibrium can be Pareto-inefficient ($k^* > k^{GR}$);

Welfare & dynamic inefficiency

- OLG equilibrium can be Pareto-inefficient ($k^* > k^{GR}$);
- Assume $g = 0$, Cobb-Douglass production and log utility:
  - k^{GR} implies $f'(k) = n$.
  - $f'(k^*) = \frac{\alpha}{1-\alpha}(1+n)(2+\rho)$
  - α small $\rightarrow f'(k^*) < n \rightarrow k^* > k^{GR}$
  - This possible *dynamic inefficiency* arises from relaxing the assumption of a finite number of agents.

OLG model: Takeaways

- Same conclusions as Solow/Ramsey on sources of long-run growth;
- but possibility of multiple equilibria and dynamic inefficiency (overaccumulation)

OLG model: Takeaways

- Same conclusions as Solow/Ramsey on sources of long-run growth;
- but possibility of multiple equilibria and dynamic inefficiency (overaccumulation)
- Is neoclassical growth theory fragile even within its ‘one-good economy with perfect markets’ assumptions?
  - However, Barro (1974) shows that a bequest motive (intergenerational altruism) may make OLG practically equivalent to Ramsey (no inefficiencies).
  - Moreover, OLG dynamic inefficiency (over-accumulation) does not seem empirically relevant: too much accumulation??