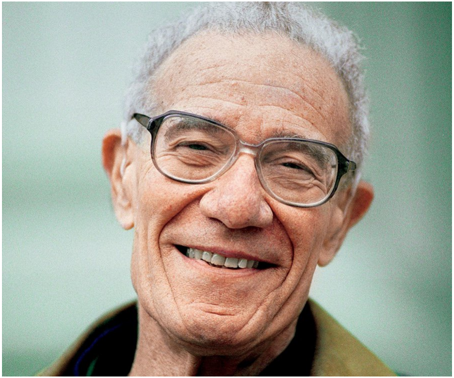
# The Solow model



# Solow growth model

## Key premises:

- *neoclassical* production function
- Say's law: full employment at all times.

## Main implications:

- stable steady-state with $g_Y = n + g$
- saving rate determines output level but not growth rate
- K accumulation cannot explain long-run growth or cross-country income differences.

# Production function

- One-good economy
- 4 variables: $Y$, $K$, $L$, $A$.
- Say's law: full employment of $L$ & $K$ at each $t$.
- Neoclassical aggregate production function

$$Y(t) = F[K(t), A(t)L(t)]$$

  - AL: labor-augmenting technological progress.

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- *Intensive form* of the production function:

$$\frac{Y}{AL} = F(\frac{K}{AL}, \frac{AL}{AL}) = F(\frac{K}{AL}, 1)$$
$$\Downarrow$$

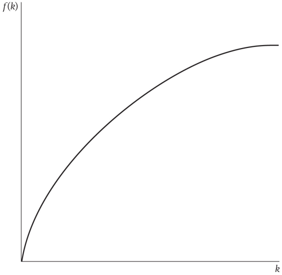
$$y = f(k)$$

with $k = \dfrac{K}{AL}$ , $y = \dfrac{Y}{AL}$ and $f(k) = F(k, 1)$

## Other assumptions about the production function

$$f(0) = 0, \qquad f'(k) > 0, \qquad f''(k) < 0$$

$$lim_{k\to 0} f'(k) = \infty, \qquad lim_{k\to\infty} f'(k) = 0$$



## Evolution of production inputs

$$g_L = n \quad \text{(therefore } \dot{L}(t) = nL(t)\text{)}$$

$$g_A = g \quad \text{(therefore } \dot{A}(t) = gA(t)\text{)}$$

$$\dot{K}(t) = sY(t) - \delta K(t), \qquad 0 < s \leq 1$$

# The dynamics of the model

- Strategy: focus on $k = \frac{K}{AL}$
- Take the derivative of $k$ wrt time

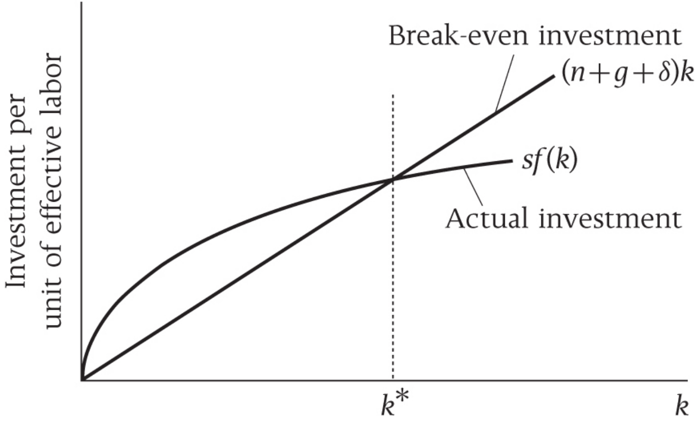
$$\dot{k}(t) = \frac{d(K/AL)}{dt} = \frac{\dot{K}}{AL} - \frac{K}{AL}\frac{\dot{L}}{L} - \frac{K}{AL}\frac{\dot{A}}{A}$$

- using $k = \frac{K}{AL}$, $y = \frac{Y}{AL}$ & the assumptions about inputs:

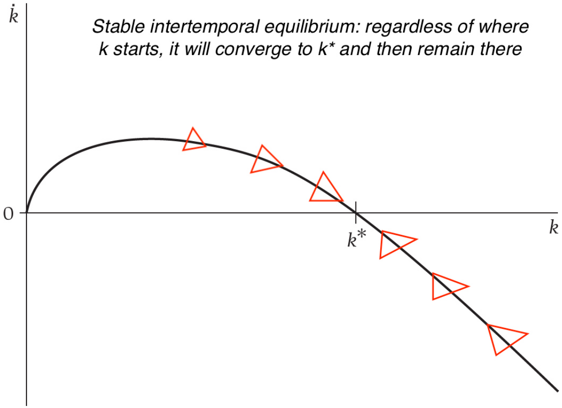
$$\dot{k}(t) = sf[k(t)] - (n + g + \delta)k(t)$$

change in $k$ = investment − breakeven investment

# Actual vs. break-even investment



# Phase diagram



# The steady state

In the intertemporal equilibrium...

- by assumption, $g_L = n$ and $g_A = g$;
- $K = ALk \rightarrow g_K = n + g$
- $Y = ALf(k) \rightarrow g_Y = n + g$
- $\frac{K}{L} = Ak \rightarrow g_{\frac{K}{L}} = g$
- $\frac{Y}{L} = Af(k) \rightarrow g_{\frac{Y}{L}} = g$

balanced growth path: all variables grow at constant rates.

## Other things we want to know:

1. Qualitative effect of an increase in $s$ (*direction*)
2. What level of $k$ maximizes consumption (*golden-rule* $k^*$)
3. Speed of convergence: *how long* does transition take?

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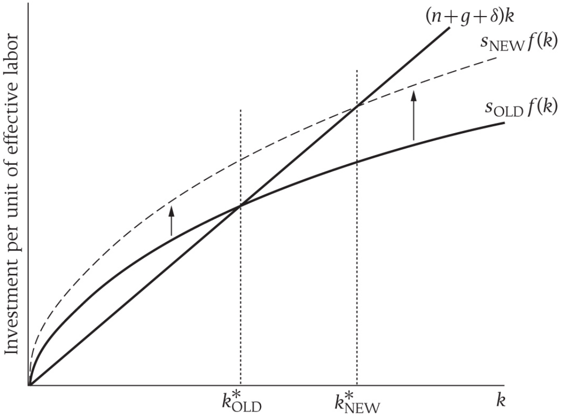

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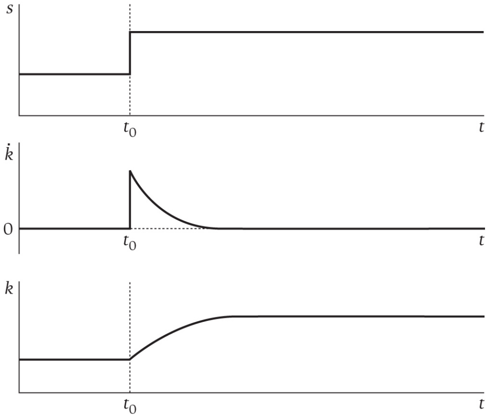
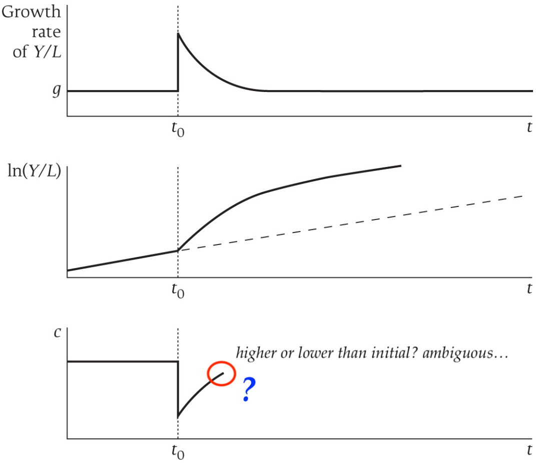


## An increase in the saving rate

$$Y/L = Af(k)$$

$$c = f(k)(1-s)$$



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## Consumption and the golden-rule

- $c^* = f(k^*) - (n+g+\delta)k^*$
- $k^* = k^*(s, n, g, \delta)$

*What value of $s$ maximizes $c^*$?*

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$$\frac{\partial c^*}{\partial s} = [f'(k^*) - (n+g+\delta)]\frac{\partial k^*}{\partial s} = 0$$

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- golden-rule $k^*$
- characterized by $MPK = (n+g+\delta)$.
- but no reason for $s$ to be exactly at the level which implies the golden-rule $k^*$!

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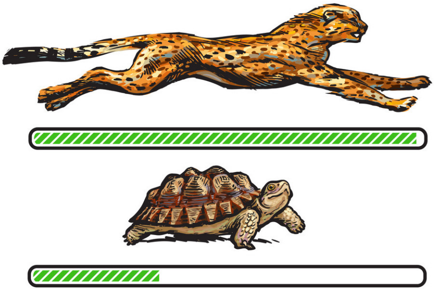
## Speed of convergence

- *How fast* will $k$ reach $k^*$ when starting out of equilibrium?



## Refresher: Taylor approximations

- *Taylor's theorem:* any (continuously differentiable) function $\phi(x)$ can be approximated, around a point $x_0$, by a n-th degree polynomial.
- n-th degree Taylor approximation around $x_0$:

$$\phi(x) = [\frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)}{1!}(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{\phi^{(n)}(x_0)}{n!}(x - x_0)^n] + R_n$$

($R_n$ = remainder)

- linear approximation around $x_0$:

$$\phi(x) \approx \phi(x_0) + \phi'(x_0)(x - x_0)$$

## Speed of convergence

- $\dot{k} = \dot{k}(k)$

- linear approximation around $k^*$:

$$\dot{k} \approx \left[ \frac{\partial \dot{k}(k)}{\partial k} \bigg|_{k=k^*} \right] (k - k^*) \quad \Rightarrow \qquad \dot{k} \approx -\lambda(k - k^*) \quad \Rightarrow \quad \dot{k} + \lambda k = \lambda k^*$$

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- $k(t) \approx k^* + e^{-\lambda t}(k(0) - k^*)$
- $\lambda = -\frac{\delta \dot{k}}{\delta k}|_{k=k^*} = (n + g + \delta)(1 - \alpha_K)$
- Bottom line: for plausible parameters, convergence is not fast.
- eg: $(n + g + \delta) = 6\%$ and $\alpha_K = 1/3 \rightarrow \lambda = 0.04$

## Solow model: Takeaways

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3. Two sources of cross-country variation in $Y/L$: $s$ and $A$.
   - BUT implausibly huge differences in $s$ would be needed to produce sizable differences in $Y/L$.

4. Technology ($A$) is the only possible explanation of vast cross-country differences in $Y/L$.

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- Subsequent developments of neoclassical growth theory address these issues.

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- Overcomes Harrodian instability by assuming it away
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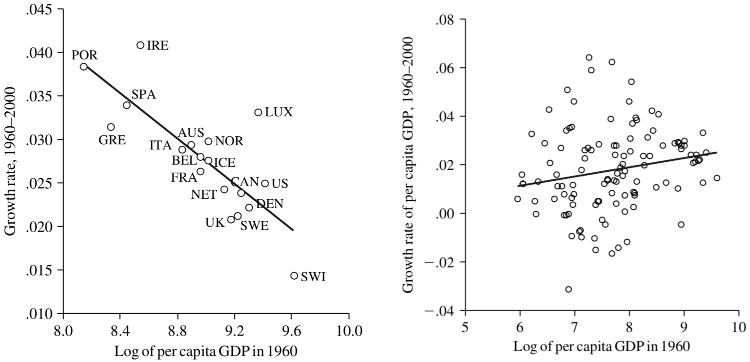
- Overcomes Harrodian instability by assuming it away
  - continuous full employment by assumption.
  - $g = g_W$ is assumed, not demonstrated.
- Relies on a fictional aggregate production function $Y(K, AL)$.
  - In reality, many production processes and many types of inputs
  - They do not add up to an aggregate production function with the properties assumed by Solow model (Cambridge capital controversy)
  - One-good economy: *'Venerable Solow may make peculiar assumptions, but he never makes a mistake'* (A. Sen, 1974)
  - Recent discussion of the problem: Baqaee & Fahri (2019).
- Subsequent developments of neoclassical growth theory do *not* address these issues.

## Convergence regressions

- Do poor countries catch up?
- Solow model suggests convergence, if $A$ non-excludable.
- Empirical test:
$$\Delta ln(Y/L)_{i,1} = \alpha + \beta ln(Y/N)_{i,0} + \epsilon_i$$
- with $t = 0$ and $t = 1$ usually quite apart in time (40/50 years).
- $\beta = -1$ = perfect convergence
- Evidence from 1960-2000: some convergence among core-OECD countries ($\beta \approx -1$), but little or no convergence overall ($\beta \approx 0$).




(a) 18 original OECD members (b) world (114 countries)

Figure: International evidence on convergence: 1960 income and subsequent growth