



# Macroeconomic Theory I

## Section 3 - Growth (II): Ideas, history, geography and institutions

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## Section 3 - Growth (II): The Plan

1. Endogenous Growth Theory: key ideas.
2. EG with a fixed saving rate and share of R&D.
3. Learning-by-doing: The AK model
4. The Romer (1990) model: endogenous R&D investment.
5. Fundamental determinants of growth

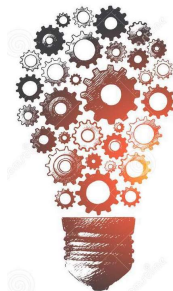
## New growth theory: Key ideas

- ▶ Production function for innovation

$$\dot{A}(t) = f(A(t), x(t))$$

$x$  = some measure of R&D efforts.

- ▶ *A is non-rival but potentially excludable*
- ▶ *Growth as a result of market-based incentives* requires that innovators enjoy market power.



## Determinants of innovation

1. Public support for basic research.
2. Private incentives for R&D investment
  - o requires some excludability
  - o rate of return on R&D.
3. Alternative opportunities for talented individuals
  - o Baumol (1990); Murphy, Shleifer & Vishny (1991)
4. Learning-by-doing
  - o Innovation as a side-effect of economic activity
  - o AK models

## EG with fixed R&D share

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Model dynamics:

$$g_A = B a_L^\gamma L(t)^\gamma A(t)^{\theta-1}$$

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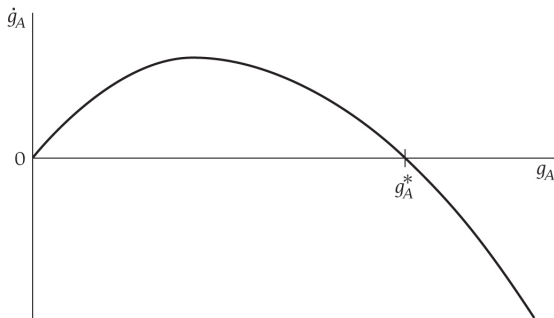
- ▶ Value of  $\theta$  determines the behavior of this model.



## Endogenous growth

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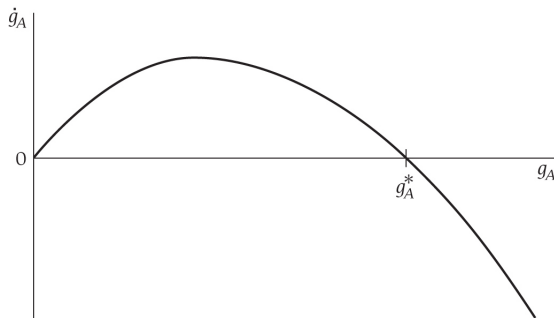
Case (1): decreasing returns to A ( $\theta < 1$ )



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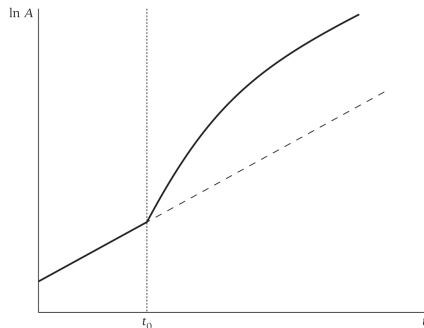
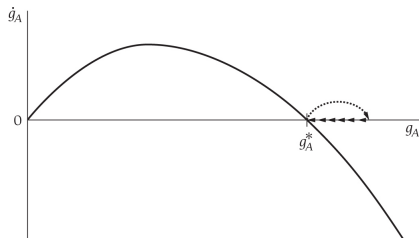
- ▶ stable equilibrium
- ▶  $g_A^* = \frac{\gamma}{1-\theta} n$ ;
- ▶ no growth effect of  $a_L$  and  $L$ ;
- ▶  $g_{Y/L}$  depends (positively) on population growth;
- ▶ *semi-endogenous growth.*

# Endogenous growth

## Effect of a increase in $a_L$ with $\theta < 1$

$$g_A(t) = B a_L^\gamma L(t)^\gamma A(t)^{\theta-1}$$

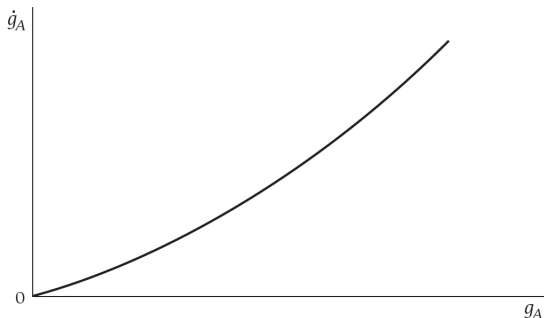
$$\dot{g}_A(t) = \gamma n g_A(t) + (\theta - 1)[g_A(t)]^2 \Rightarrow g_A^* = \frac{\gamma}{1 - \theta} n$$



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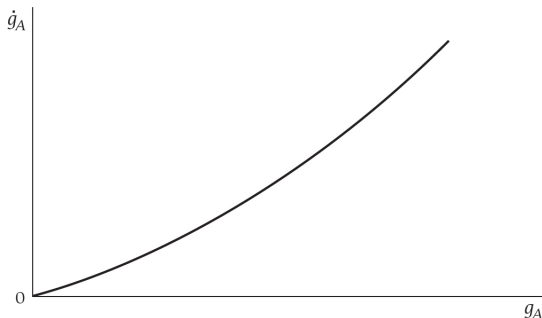
Case (2): increasing returns to A ( $\theta > 1$ )



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Case (2): increasing returns to A ( $\theta > 1$ )



- ▶ no equilibrium;
- ▶ ever-increasing ('explosive') growth;
- ▶ intuition: every marginal addition to A results in a bigger increase in A.

Case (3): constant returns to A ( $\theta = 1$ )

$$g_A(t) = Ba_L^\gamma L(t)^\gamma$$

$$\dot{g}_A(t) = \gamma n g_A(t)$$

- ▶ if  $n > 0$ ,  $g_A$  is ever-increasing ('explosive' growth);
- ▶ If  $n = 0$ ,  $g_A$  fixed.
  - ▶ no transitions, always in equilibrium
  - ▶ fully endogenous growth: growth depends on  $a_L$
  - ▶ example of a *linear growth model* ( $\dot{A}$  linear in A)

## Learning-by-doing: The AK model

Assumptions:

$$Y(t) = K(t)^\alpha [A(t)L(t)]^{1-\alpha}; \quad A(t) = BK(t);$$

$$\dot{K}(t) = sY(t); \quad L(t) = \bar{L}.$$



*Kenneth Arrow*

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*PS: where have you seen  $Y = bK$  and  $g_K = sb$  before??*

EGT and the *linearity* assumption

- ▶ *AK model*
  - $A=BK \Rightarrow g_Y$  depends on  $s$ .
- ▶ *EG model with fixed R&D share and  $\theta = 1$* 
  - $\dot{A} = [B(a_L L)^\gamma]A \Rightarrow g_Y$  depends on  $a_L$  and  $L$ .
- ▶ *Romer (1990) is also a linear growth model*
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  - $\dot{A} = (DL_A)A \Rightarrow g_Y$  depends on  $L$ .
- ▶ Linearity  $\rightarrow$  stable endogenous growth.
- ▶ *The 'trick' of EGT:*
  - if  $\dot{A}$  is linear in  $A$ , it means that the other factors that multiply  $A$  in the knowledge-production function affect the rate of growth of technology (so they will affect growth).
  - $\dot{A} = f(x)A \Rightarrow \frac{\dot{A}}{A} = f(x)$

## The Romer model

*(a simplified version)*



- o Output produced from intermediate inputs.
- o Technical progress = increasing variety of intermediate inputs.
- o Innovation arises from  $R\&D$  investment by private actors.
- o Market power: inventor has permanent patent rights.

## Assumptions about production

- Production function:

$$Y = \left[ \int_{i=0}^A L(i)^\phi di \right]^{1/\phi}, \quad 0 < \phi < 1$$

- $L(i)$  = quantity of input  $i$  = labor employed in producing  $i$ ;
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- When all existing inputs are produced in equal quantities:

$$Y = \left[ A \left( \frac{L_Y}{A} \right)^\phi \right]^{1/\phi} = A^{\frac{1-\phi}{\phi}} L_Y$$

- $L_Y$  = workers in inputs production = tot. amount of inputs

## Demand for inputs

- ▶ Patent-holder hires workers to produce the input associated with her idea
- ▶ Inputs then sold to final output producers
- ▶ Downward-sloping demand curve for input  $i$ :

$$L(i) = \left[ \frac{\lambda}{p(i)} \right]^{\frac{1}{1-\phi}}$$

$p(i)$  = price of input  $i$ .

## Other key assumptions

- ▶ Full-employment and fixed labor force:

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$$g_C = \dot{C}(t)/C(t) = r(t) - \rho$$

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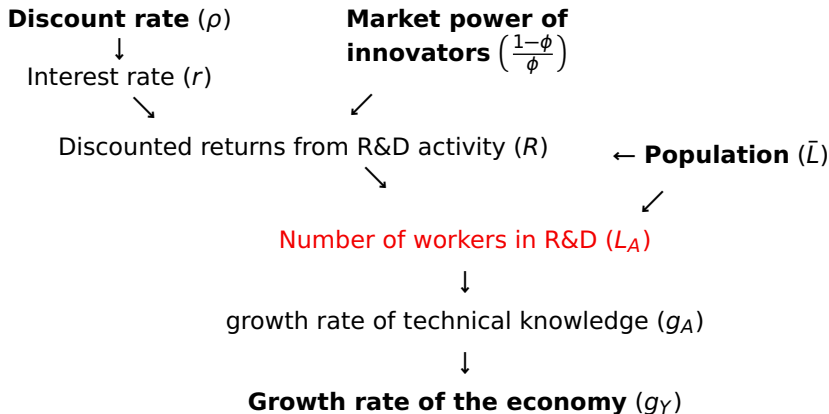
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- ▶ Free-entry condition in the R&D sector:

$$\int_{\tau=t}^{\infty} e^{-r(\tau-t)} \pi(i, \tau) d\tau = \frac{w(t)}{BA(t)}$$

PV of profits from an idea = production cost

## The logic of the Romer model



## Solving the model

- ▶  $g_Y = \frac{1-\phi}{\phi} g_A + g_{L_Y} = \frac{1-\phi}{\phi} B L_A + g_{(\bar{L}-L_A)}.$
- ▶ Steady state  $\rightarrow$  constant  $L_A$ .
- ▶ Use R&D free-entry condition to infer  $L_A^*$  and thus  $g_Y^*$ .

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- ▶ Use R&D free-entry condition to infer  $L_A^*$  and thus  $g_Y^*$ .
- ▶ Steps:
  1. Calculate  $\pi(t)$  and  $g_\pi = g_\pi(g_W)$
  2. Figure out  $r$  and  $g_W$
  3. Calculate PV of profits from a new idea  $R(t)$  using  $R(t) = \frac{\pi(t)}{r-g_\pi}$
  4. Set PV of profits from idea = production cost, to obtain  $L_A^*$  &  $g_Y^*$ .

## Step 1: find $\pi(t)$ and $g_\pi$

- ▶ Monopolist patent-holder sets

$$p(i, t) = \frac{\eta}{\eta - 1} w(t)$$

- ▶ From demand curve we have:

$$\eta = - \frac{\partial L(i)}{\partial p(i)} \frac{p(i)}{L(i)} = \frac{1}{\phi - 1} \rightarrow p(i, t) = \frac{w(t)}{\phi}$$

- ▶ Profits at each point in time:

$$\pi(t) = \frac{\bar{L} - L_A}{A(t)} \left[ \frac{w(t)}{\phi} - w(t) \right] = \frac{1 - \phi}{\phi} \frac{\bar{L} - L_A}{A(t)} w(t)$$

- ▶ Growth rate of profits:

$$g_\pi = g_W - g_A$$

## Step 2: find $r$ and $g_W$

- All output is consumed and we are assuming constant  $L_A$ , so

$$g_C = g_Y = \frac{1-\phi}{\phi} BL_A$$

- Having  $g_C$ , we can derive interest rate  $r(t)$  from Euler equation:

$$r(t) = \rho + \frac{\dot{C}(t)}{C(t)} = \rho + \frac{1-\phi}{\phi} BL_A$$

- Constant monopoly mark-up implies constant wage share, so

$$g_W = g_Y = \frac{1-\phi}{\phi} BL_A \quad \rightarrow \quad g_\pi = g_W - g_A = \frac{1-\phi}{\phi} BL_A - BL_A$$



## Step 3 - Figure out the PV of profits from a new idea

- PV of profits from a new idea:

$$R(t) = \frac{\pi(t)}{r - g_{\pi}}$$

- From previous steps:

$$\pi(t) = \frac{1 - \phi}{\phi} \frac{\bar{L} - L_A}{A(t)} w(t); \quad r = \rho + \frac{1 - \phi}{\phi} BL_A; \quad g_{\pi} = \frac{1 - \phi}{\phi} BL_A - BL$$

- Plugging-in:

$$R(t) = \frac{\pi(t)}{r - g_{\pi}} = \frac{\frac{1 - \phi}{\phi} \frac{\bar{L} - L_A}{A(t)} w(t)}{\rho + BL_A} = \frac{1 - \phi}{\phi} \frac{\bar{L} - L_A}{\rho + BL_A} \frac{w(t)}{A(t)}$$

Step 4 - Set  $R(t)$  = production cost and infer  $L_A^*$

$$\frac{1-\phi}{\phi} \frac{\bar{L}-L_A}{\rho+BL_A} \frac{w(t)}{A(t)} = \frac{w(t)}{BA(t)} \quad \rightarrow \quad L_A^* = (1-\phi)\bar{L} - \frac{\phi\rho}{B}$$

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↓

$$L_A^* = \max\{(1-\phi)\bar{L} - \frac{\phi\rho}{B}, 0\}$$

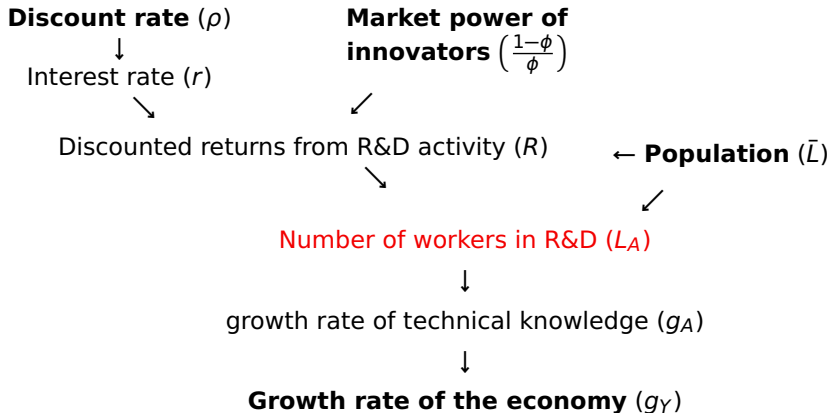
↓

$$g_Y^* = \max\left\{\frac{(1-\phi)^2}{\phi} B\bar{L} - (1-\phi)\rho, 0\right\}$$

(note: economy always on equilibrium path–no transition dynamics)

# Endogenous growth

(A second look at) [The logic of the model](#)



## Welfare: optimal vs. actual growth

1. Write PV of lifetime utility as a function of  $L_A$

$$U = \int_{t=0}^{\infty} e^{-\rho t} \ln C(t) dt \quad \Rightarrow \quad U = \int_{t=0}^{\infty} e^{-\rho t} \ln [C(0)e^{g_c}] dt$$

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$$U = \frac{1}{\rho} \left( \ln \frac{\bar{L} - L_A}{\bar{L}} + \frac{1-\phi}{\phi} \ln A(0) + \frac{1-\phi}{\phi} \frac{B L_A}{\rho} \right)$$





## Extensions

- ▶ Fixed  $K$  in the production function for  $Y$  but not  $\dot{A} \rightarrow s$  has a level effect (Romer, 1990);
- ▶ but if  $K$  enters the production of  $\dot{A}$ ,  $s$  likely to have growth effects;
- ▶ decreasing returns to  $A$  in the production of  $\dot{A} \rightarrow$  semi-endogenous growth (Jones, 1995)
  - ▶ long-run growth depends only on  $n$ , while forces affecting  $L_A$  have only level effects;
- ▶ quality-ladder models: innovation as improvement of existing inputs (Grossman & Helpman, 1991; Aghion & Howitt, 1992).

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- ▶ ...but Romer-type models may explain growth at a worldwide scale in the very long-run (Kremer, 1993).
- ▶ P.Krugman: *"too much of [EGT] involves making assumptions about how unmeasurable things affect other unmeasurable things."*