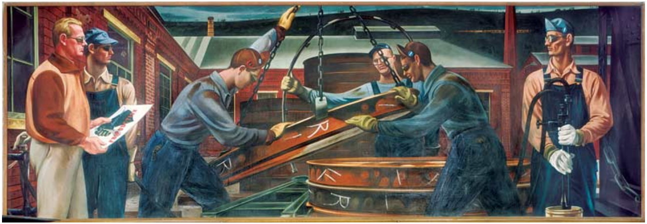


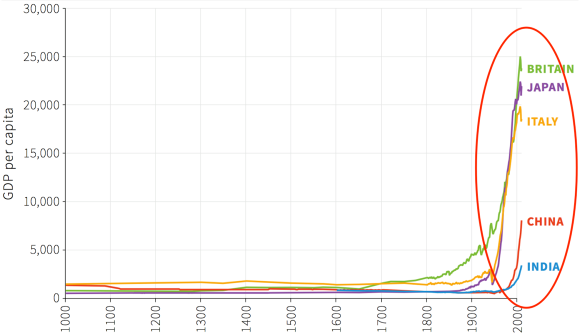
# Advanced Macroeconomics

## Section 2 - Growth (I): The mechanics of capital accumulation and growth

Daniele Girardi
King's College London

AY 2024-25, Semester I

## The hockey stick of history



**Section 2: Growth (I)**

## The Plan

1. Harrod-Domar
2. Solow
3. Ramsey-Cass-Koopmans
4. Diamond’s overlapping-generations (OLG)

- Variable $t$ represents time
  - continuous (except in the OLG model)

# Dynamic analysis

- Variable $t$ represents time
  - continuous (except in the OLG model)

- $X(t) \rightarrow$ X can change over time.

# Dynamic analysis

- Variable $t$ represents time
  - continuous (except in the OLG model)

- $X(t) \rightarrow$ X can change over time.

- $\frac{dX}{dt} \rightarrow$ rate of change of $X$.

# Dynamic analysis

- Variable $t$ represents time
  - continuous (except in the OLG model)
- $X(t) \rightarrow$ X can change over time.
- $\frac{dX}{dt} \rightarrow$ rate of change of $X$.
- $\dot{X}(t) \rightarrow$ shorthand for $\frac{dX}{dt}$

# Dynamic analysis

- Variable $t$ represents time
  - continuous (except in the OLG model)
- $X(t) \rightarrow$ X can change over time.
- $\frac{dX}{dt} \rightarrow$ rate of change of $X$.
- $\dot{X}(t) \rightarrow$ shorthand for $\frac{dX}{dt}$
- $\frac{\dot{X}(t)}{X(t)}$ = *growth rate* of $X$

- Variable $t$ represents time
  - continuous (except in the OLG model)
- $X(t) \rightarrow$ X can change over time.
- $\frac{dX}{dt} \rightarrow$ rate of change of $X$.
- $\dot{X}(t) \rightarrow$ shorthand for $\frac{dX}{dt}$
- $\frac{\dot{X}(t)}{X(t)}$ = *growth rate* of $X$
- $g_X$ is a shorthand for $\frac{\dot{X}(t)}{X(t)}$

# Intertemporal equilibrium

- Static analysis: equilibrium condition → equilibrium relations.
  - I=S → equilibrium level of Y
  - MRS=MRT → optimal quantity consumed
  - ...
- Growth theory is *dynamic*: intertemporal equilibrium.

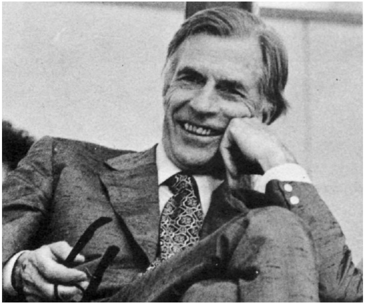
# Intertemporal equilibrium

- Static analysis: equilibrium condition → equilibrium relations.
  - I=S → equilibrium level of Y
  - MRS=MRT → optimal quantity consumed
  - ...
- Growth theory is *dynamic*: intertemporal equilibrium.

**Main concepts:**

- Intertemporal equilibrium
- Steady state
- Dynamic stability

# The Harrod-Domar model



*There's an old joke. Two elderly women are at a Catskill restaurant. One of them says, 'Boy, the food at this place is just terrible.' The other one says, 'Yeah I know. And such small portions.'*

(Woody Allen, 'Annie Hall')

## The Harrod-Domar model

- 'Grandfather' of modern growth theory.
- Premise 1: aggregate investment has a dual effect
    1. multiplier effect (demand side)
    2. capacity-creating effect (supply side)
- Premise 2: investment depends on output (accelerator)

# The Harrod-Domar model

- ‘Grandfather’ of modern growth theory.

- Premise 1: aggregate investment has a dual effect
  1. multiplier effect (demand side)
  2. capacity-creating effect (supply side)

- Premise 2: investment depends on output (accelerator)

▶ Main findings:
- unique equilibrium path: $g_W = sa$ (*warranted rate*)
- warranted rate does not guarantee full (nor stable) employment
- instability: economy won’t converge to $g_W$, except by a fluke

## Harrod-Domar: Assumptions

- One-good economy, closed, no government: $Y(t) = C(t) + I(t)$

## Harrod-Domar: Assumptions

- One-good economy, closed, no government: $Y(t) = C(t) + I(t)$
- Costant saving rate: $S(t) = sY(t)$

- One-good economy, closed, no government: $Y(t) = C(t) + I(t)$
- Costant saving rate: $S(t) = sY(t)$
- No depreciation: $\dot{K}(t) = I(t)$

- One-good economy, closed, no government: $Y(t) = C(t) + I(t)$
- Costant saving rate: $S(t) = sY(t)$
- No depreciation: $\dot{K}(t) = I(t)$
- Potential output $Y^*(t)$: output level if firms utilize their productive capacity at the planned/desired level

- One-good economy, closed, no government: $Y(t) = C(t) + I(t)$
- Costant saving rate: $S(t) = sY(t)$
- No depreciation: $\dot{K}(t) = I(t)$
- Potential output $Y^*(t)$: output level if firms utilize their productive capacity at the planned/desired level
- Linear technology: $Y^*(t) = aK(t)$

## Harrod-Domar: Assumptions

- One-good economy, closed, no government: $Y(t) = C(t) + I(t)$
- Costant saving rate: $S(t) = sY(t)$
- No depreciation: $\dot{K}(t) = I(t)$
- Potential output $Y^*(t)$: output level if firms utilize their productive capacity at the planned/desired level
- Linear technology: $Y^*(t) = aK(t)$
- Rate of capacity utilization: $u(t) = \frac{Y(t)}{Y^*(t)}$

## Harrod-Domar: Assumptions

- One-good economy, closed, no government: $Y(t) = C(t) + I(t)$
- Costant saving rate: $S(t) = sY(t)$
- No depreciation: $\dot{K}(t) = I(t)$
- Potential output $Y^*(t)$: output level if firms utilize their productive capacity at the planned/desired level
- Linear technology: $Y^*(t) = aK(t)$
- Rate of capacity utilization: $u(t) = \frac{Y(t)}{Y^*(t)}$
- Investment rate: $\dot{g}_K(t) = \alpha(u(t) - 1)$ with $\alpha > 0$

## Assumptions:

$$Y(t) = C(t) + I(t); \qquad S(t) = sY(t); \qquad Y^*(t) = aK(t); \qquad u(t) = \frac{Y(t)}{Y^*(t)}$$

$$g_K(t) = \frac{\dot{K}(t)}{K(t)} = \frac{I(t)}{K(t)}; \qquad \dot{g}_K(t) = \alpha(u(t) - 1) \qquad \text{with } \alpha > 0$$

# Harrod-Domar: Intertemporal equilibrium

## Assumptions:

$$Y(t) = C(t) + I(t); \qquad S(t) = sY(t); \qquad Y^*(t) = aK(t); \qquad u(t) = \frac{Y(t)}{Y^*(t)}$$

$$g_K(t) = \frac{\dot{K}(t)}{K(t)} = \frac{I(t)}{K(t)}; \qquad \dot{g}_K(t) = \alpha(u(t) - 1) \qquad \text{with } \alpha > 0$$

## Investment rate at each point in time:

$$I(t) = S(t) \quad \rightarrow g_K(t) = sa[u(t)]$$

Assumptions:

$$Y(t) = C(t) + I(t); \qquad S(t) = sY(t); \qquad Y^*(t) = aK(t); \qquad u(t) = \frac{Y(t)}{Y^*(t)}$$

$$g_K(t) = \frac{\dot{K}(t)}{K(t)} = \frac{I(t)}{K(t)}; \qquad \dot{g}_K(t) = \alpha(u(t) - 1) \qquad \text{with } \alpha > 0$$

Investment rate at each point in time:

$$I(t) = S(t) \quad \rightarrow g_K(t) = sa[u(t)]$$

Intertemporal equilibrium ('warranted' growth rate):

$$\dot{g}_K(t) = 0 \quad \rightarrow u = 1 \quad \rightarrow g_W = sa$$

## The equilibrium ('warranted') rate of growth

- Warranted vs natural growth rate:

$$g_Y = sa \neq n$$

## The equilibrium ('warranted') rate of growth

- Warranted vs natural growth rate:

$$g_Y = sa \neq n$$

- Dynamic instability:

$$g_K = u(sa) > g_W = sa \quad \Rightarrow \quad u > 1 \quad \Rightarrow \quad \dot{g}_K > 0$$

$$g_K = u(sa) < g_W = sa \quad \Rightarrow \quad u < 1 \quad \Rightarrow \quad \dot{g}_K < 0$$

## The equilibrium ('warranted') rate of growth

- Warranted vs natural growth rate:

$$g_Y = sa \neq n$$

- Dynamic instability:

$$g_K = u(sa) > g_W = sa \quad \Rightarrow \quad u > 1 \quad \Rightarrow \quad \dot{g}_K > 0$$

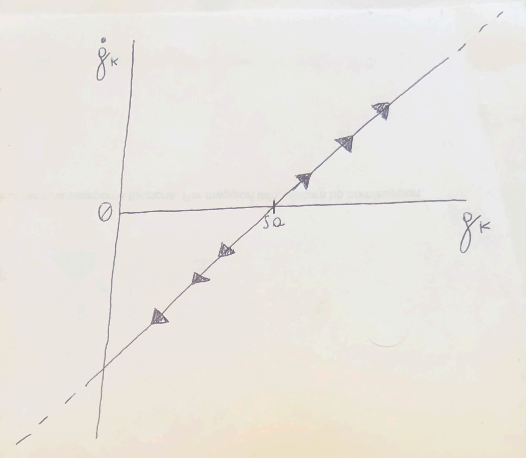
$$g_K = u(sa) < g_W = sa \quad \Rightarrow \quad u < 1 \quad \Rightarrow \quad \dot{g}_K < 0$$

- More formally (by plugging I=S condition into investment function):

$$\dot{g}_K = \alpha \left[ \frac{g_K}{g_W} - 1 \right]$$

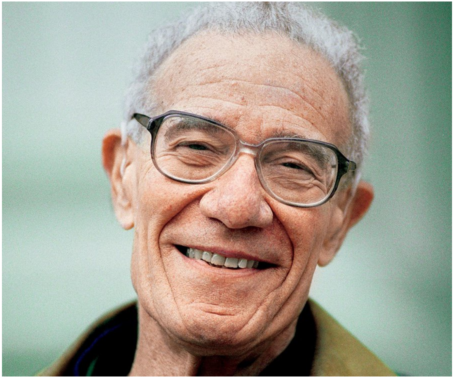
# Harrod

$$\dot{g}_K = \alpha \left[ \frac{g_K}{g_W} - 1 \right] \quad \text{with } \alpha > 0$$




- Phase diagram.
- positive slope $\rightarrow$ instability

# The Solow model



# Solow growth model

## Key premises:

- *neoclassical* production function
- Say's law: full employment at all times.

## Main implications:

- stable steady-state with $g_Y = n + g$
- saving rate determines output level but not growth rate
- K accumulation cannot explain long-run growth or cross-country income differences.

# Production function

- One-good economy
- 4 variables: $Y$, $K$, $L$, $A$.
- Say's law: full employment of $L$ & $K$ at each $t$.
- Neoclassical aggregate production function

$$Y(t) = F[K(t), A(t)L(t)]$$

  - AL: labor-augmenting technological progress.

# Constant returns to scale (CRS)

$$Y = F[K, AL]$$
$$F(cK, cAL) = cF(K, AL)$$

# Constant returns to scale (CRS)

$$Y = F[K, AL]$$

$$F(cK, cAL) = cF(K, AL)$$

- *Intensive form* of the production function:

$$\frac{Y}{AL} = F(\frac{K}{AL}, \frac{AL}{AL}) = F(\frac{K}{AL}, 1)$$

$$\Downarrow$$

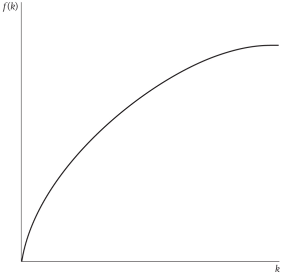
$$y = f(k)$$

with $k = \frac{K}{AL}$ , $y = \frac{Y}{AL}$ and $f(k) = F(k, 1)$

## Other assumptions about the production function

$$f(0) = 0, \qquad f'(k) > 0, \qquad f''(k) < 0$$

$$lim_{k \to 0} f'(k) = \infty, \qquad lim_{k \to \infty} f'(k) = 0$$



## Evolution of production inputs

$$\dot{L}(t) = nL(t) \rightarrow g_L = n$$

$$\dot{A}(t) = gA(t) \rightarrow g_A = g$$

$$\dot{K}(t) = sY(t) - \delta K(t), \qquad 0 < s \leq 1$$

# The dynamics of the model

- Strategy: focus on $k = \frac{K}{AL}$
- Take the derivative of $k$ wrt time

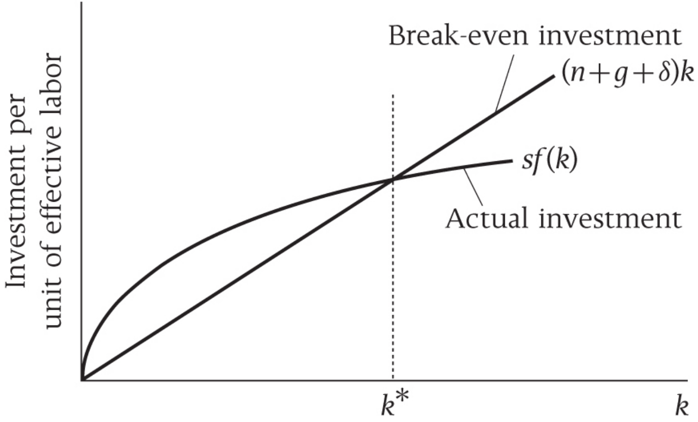
$$\dot{k}(t) = \frac{d(K/AL)}{dt} = \frac{\dot{K}}{AL} - \frac{K}{(AL)^2}(A\dot{L} + \dot{A}L) = \frac{\dot{K}}{AL} - \frac{K}{AL}\frac{\dot{L}}{L} - \frac{K}{AL}\frac{\dot{A}}{A}$$

- using $k = \frac{A}{AL}$, $y = \frac{Y}{AL}$ & the assumptions about inputs:

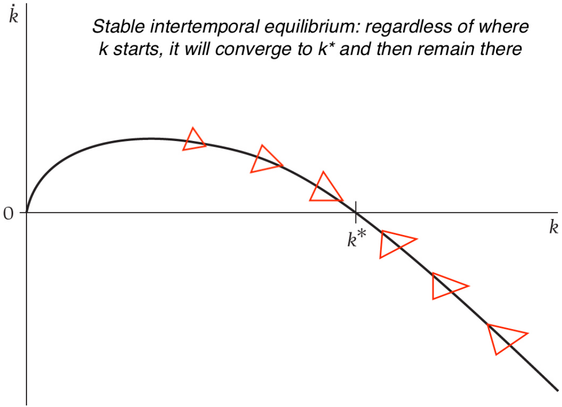
$$\dot{k}(t) = sf[k(t)] - (n + g + \delta)k(t)$$

change in $k$ = investment − breakeven investment

# Actual vs. break-even investment



# Phase diagram



# The steady state

In the intertemporal equilibrium...

- by assumption, $g_L = n$ and $g_A = g$;
- $K = ALk \to g_K = n + g$
- $Y = ALf(k) \to g_Y = n + g$
- $\frac{K}{L} = Ak \to g_{\frac{K}{L}} = g$
- $\frac{Y}{L} = Af(k) \to g_{\frac{Y}{L}} = g$

balanced growth path: all variables grow at constant rates.

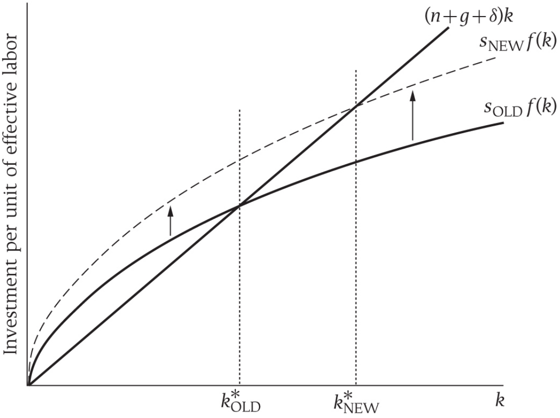
## Other things we want to know:

1. Qualitative effect of an increase in $s$ (*direction*)
2. What level of $k$ maximizes consumption (*golden-rule* $k^*$)
3. Size of the effect of an increase in $s$ (*how big*)
4. Speed of convergence: *how long* does transition take?

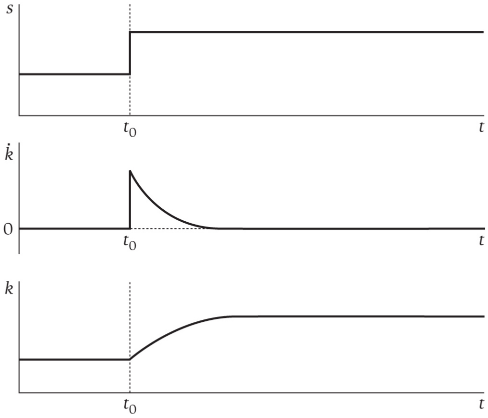
# Other things we want to know:

1. Qualitative effect of an increase in $s$ (*direction*)
2. What level of $k$ maximizes consumption (*golden-rule* $k^*$)
3. Size of the effect of an increase in $s$ (*how big*)
4. Speed of convergence: *how long* does transition take?

## An increase in the saving rate



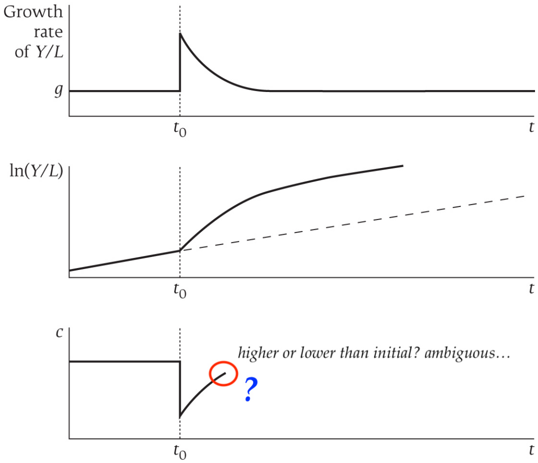
# An increase in the saving rate



## An increase in the saving rate

$Y/L = Af(k)$

$c = f(k)(1-s)$



## Other things we want to know:

1. Qualitative effect of an increase in $s$ (*direction*)
2. What level of $k$ maximizes consumption (*golden-rule* $k^*$)
3. Size of the effect of an increase in $s$ (*how big*)
4. Speed of convergence: *how long* does transition take?

# Consumption and the golden-rule

- $c^* = f(k^*) - (n + g + \delta)k^*$
- $k^* = k^*(s, n, g, \delta)$

*What value of $s$ maximizes $c^*$?*

# Consumption and the golden-rule

- $c^* = f(k^*) - (n+g+\delta)k^*$
- $k^* = k^*(s, n, g, \delta)$

*What value of $s$ maximizes $c^*$?*

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - (n+g+\delta)]\frac{\partial k^*}{\partial s} = 0$$

$$\Downarrow$$

$$f'(k^*) = (n+g+\delta)$$

# Consumption and the golden-rule

- $c^* = f(k^*) - (n+g+\delta)k^*$
- $k^* = k^*(s, n, g, \delta)$

*What value of s maximizes $c^*$?*

$$\frac{\partial c^*}{\partial s} = [f'(k^*) - (n+g+\delta)]\frac{\partial k^*}{\partial s} = 0$$

$$\Downarrow$$

$$f'(k^*) = (n+g+\delta)$$

- golden-rule $k^*$
- characterized by $MPK = (n+g+\delta)$.
- but no reason for $s$ to be exactly at the level which implies the golden-rule $k^*$!

## Other things we want to know:

1. Qualitative effect of an increase in $s$ (*direction*)
2. What level of $k$ maximizes consumption (*golden-rule* $k^*$)
3. Size of the effect of an increase in $s$ (*how big*)
4. Speed of convergence: *how long* does transition take?

## How big is the effect of the saving rate on output?

$$\frac{\partial y^*}{\partial s} = \frac{\partial f(k^*)}{\partial s} = f'(k^*)\frac{\partial k^*(s,n,g,\delta)}{\partial s}$$

## How big is the effect of the saving rate on output?

$$\frac{\partial y^*}{\partial s} = \frac{\partial f(k^*)}{\partial s} = f'(k^*)\frac{\partial k^*(s,n,g,\delta)}{\partial s}$$

- from implicit differentiation $\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n+g+\delta)-sf'(k^*)}$

## How big is the effect of the saving rate on output?

$$\frac{\partial y^*}{\partial s} = \frac{\partial f(k^*)}{\partial s} = f'(k^*)\frac{\partial k^*(s,n,g,\delta)}{\partial s}$$

- from implicit differentiation $\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n+g+\delta)-sf'(k^*)}$

$$\Rightarrow \frac{\partial y^*}{\partial s} = \frac{f'(k^*)f(k^*)}{(n+g+\delta)-sf'(k^*)}$$

## How big is the effect of the saving rate on output?

$$\frac{\partial y^*}{\partial s} = \frac{\partial f(k^*)}{\partial s} = f'(k^*)\frac{\partial k^*(s,n,g,\delta)}{\partial s}$$

- from implicit differentiation $\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n+g+\delta)-sf'(k^*)}$

$$\Rightarrow \frac{\partial y^*}{\partial s} = \frac{f'(k^*)f(k^*)}{(n+g+\delta)-sf'(k^*)}$$

- convert this to an elasticity, use $sf(k^*) = (n+g+\delta)k^*$ and rearrange to get:

$$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_k(k^*)}{1-\alpha_k(k^*)}$$

where $\alpha_K(k^*) = \frac{k^*}{f(k^*)}f'(k^*)$ is the elasticity of $y$ w.r.t. $k$.

# Effect of the saving rate on output

$$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_k(k^*)}{1-\alpha_k(k^*)}$$

$$\text{Assume} \quad \alpha \approx 1/3 \quad \Rightarrow \quad \frac{s}{y^*}\frac{\delta y^*}{\delta s} \approx \frac{1}{2} \tag{1}$$

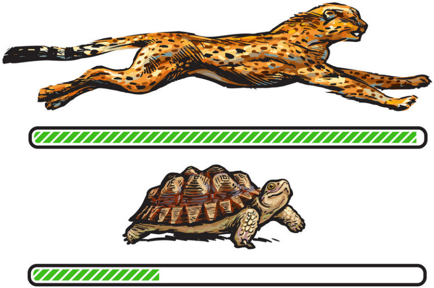
- A 1 percent increase in $s$ increases $y^*$ by 0.5 percent;
- ex: raising $s$ from 0.20 to 0.22 (+10%) increases $y^*$ by 5%;
- significant but not very big.

## Other things we want to know:

1. Qualitative effect of an increase in $s$ (*direction*)
2. What level of $k$ maximizes consumption (*golden-rule* $k^*$)
3. Size of the effect of an increase in $s$ (*how big*)
4. Speed of convergence: *how long* does transition take?

## Speed of convergence

- *How fast* will $k$ reach $k^*$ when starting out of equilibrium?



## Refresher: Taylor approximations

- *Taylor's theorem:* any (continuously differentiable) function $\phi(x)$ can be approximated, around a point $x_0$, by a n-th degree polynomial.
- n-th degree Taylor approximation around $x_0$:

$$\phi(x) = [\frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)}{1!}(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{\phi^{(n)}(x_0)}{n!}(x - x_0)^n] + R_n$$

($R_n$ = remainder)

- linear approximation around $x_0$:

$$\phi(x) \approx \phi(x_0) + \phi'(x_0)(x - x_0)$$

## Speed of convergence

- $\dot{k} = \dot{k}(k)$
- linear approximation around $k^*$:

$$\dot{k} \approx \left[ \frac{\partial \dot{k}(k)}{\partial k} \bigg|_{k=k^*} \right] (k - k^*) \quad \Rightarrow \quad \dot{k} \approx -\lambda (k - k^*) \quad \Rightarrow \quad \dot{k} + \lambda k = \lambda k^*$$

## Speed of convergence

- $\dot{k} = \dot{k}(k)$
- linear approximation around $k^*$:

$$\dot{k} \approx \left[ \frac{\partial \dot{k}(k)}{\partial k} \bigg|_{k=k^*} \right] (k - k^*) \quad \Rightarrow \quad \dot{k} \approx -\lambda (k - k^*) \quad \Rightarrow \quad \dot{k} + \lambda k = \lambda k^*$$

- $k(t) \approx k^* + e^{-\lambda t}(k(0) - k^*)$

## Speed of convergence

- $\dot{k} = \dot{k}(k)$
- linear approximation around $k^*$:

$$\dot{k} \approx \left[ \frac{\partial \dot{k}(k)}{\partial k} \bigg|_{k=k^*} \right] (k - k^*) \quad \Rightarrow \qquad \dot{k} \approx -\lambda(k - k^*) \quad \Rightarrow \quad \dot{k} + \lambda k = \lambda k^*$$

- $k(t) \approx k^* + e^{-\lambda t}(k(0) - k^*)$
- $\lambda = -\frac{\delta \dot{k}}{\delta k}|_{k=k^*} = (n + g + \delta)(1 - \alpha_K)$
- Bottom line: for plausible parameters, convergence is not fast.
- eg: $(n + g + \delta) = 6\%$ and $\alpha_K = 1/3 \rightarrow \lambda = 0.04$

## Solow model: Takeaways

1. Convergence to a balanced growth path, where $g_Y = n + g$.
   - $g_Y > n + g$ only during convergence.

## Solow model: Takeaways

1. Convergence to a balanced growth path, where $g_Y = n + g$.
   - $g_Y > n + g$ only during convergence.

2. $s$ affects $Y$ but not $g_Y$ (in equilibrium).

## Solow model: Takeaways

1. Convergence to a balanced growth path, where $g_Y = n + g$.
   - $g_Y > n + g$ only during convergence.

2. $s$ affects $Y$ but not $g_Y$ (in equilibrium).

3. Two sources of cross-country variation in $Y/L$: $s$ and $A$.
   - BUT implausibly huge differences in $s$ would be needed to produce sizable differences in $Y/L$.

## Solow model: Takeaways

1. Convergence to a balanced growth path, where $g_Y = n + g$.
   - $g_Y > n + g$ only during convergence.

2. $s$ affects $Y$ but not $g_Y$ (in equilibrium).

3. Two sources of cross-country variation in $Y/L$: $s$ and $A$.
   - BUT implausibly huge differences in $s$ would be needed to produce sizable differences in $Y/L$.

4. Technology ($A$) is the only possible explanation of vast cross-country differences in $Y/L$.

## Solow model: the conventional criticisms

- Main driver of growth ($A$) falls from the sky and has a vague definition;

## Solow model: the conventional criticisms

- Main driver of growth ($A$) falls from the sky and has a vague definition;
- No microfoundations (not robust to Lucas critique);

## Solow model: the conventional criticisms

- Main driver of growth ($A$) falls from the sky and has a vague definition;
- No microfoundations (not robust to Lucas critique);
- Can’t say anything about welfare;

## Solow model: the conventional criticisms

- Main driver of growth ($A$) falls from the sky and has a vague definition;
- No microfoundations (not robust to Lucas critique);
- Can't say anything about welfare;
- Subsequent developments of neoclassical growth theory address these issues.

## Solow model: more radical criticisms

- Overcomes Harrodian instability by assuming it away
  - continuous full employment by assumption.
  - $g = g_W$ is assumed, not demonstrated.

## Solow model: more radical criticisms

- Overcomes Harrodian instability by assuming it away
  - continuous full employment by assumption.
  - $g = g_W$ is assumed, not demonstrated.
- Relies on a fictional aggregate production function $Y(K, AL)$.
  - In reality, many production processes and many types of inputs
  - They do not add up to an aggregate production function with the properties assumed by Solow model (Cambridge capital controversy)
  - One-good economy: *'Venerable Solow may make peculiar assumptions, but he never makes a mistake'* (A. Sen, 1974)
  - Recent discussion of the problem: Baqaee & Fahri (2019).
- Subsequent developments of neoclassical growth theory do *not* address these issues.

## Growth accounting

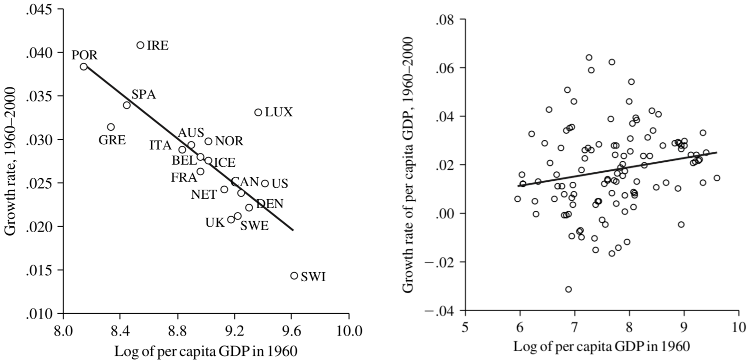
Given the production function, we can decompose $g_{Y/L}(t)$ into:

$$g_{Y/L} = \alpha_K g_{K/L} + R$$

- $\alpha_K$ estimated using $P/Y$;
- residual $R$ interpreted as the contribution of unobservable technological progress.
- But it could also be every other thing that the model leaves out!

## Convergence regressions

- Do poor countries catch up?
- Solow model suggests convergence, if $A$ non-excludable.
- Empirical test:
$$\Delta ln(Y/L)_{i,1} = \alpha + \beta ln(Y/N)_{i,0} + \epsilon_i$$
- with $t = 0$ and $t = 1$ usually quite apart in time (40/50 years).
- $\beta = -1$ = perfect convergence
- Evidence from 1960-2000: some convergence among core-OECD countries ($\beta \approx -1$), but little or no convergence overall ($\beta \approx 0$).

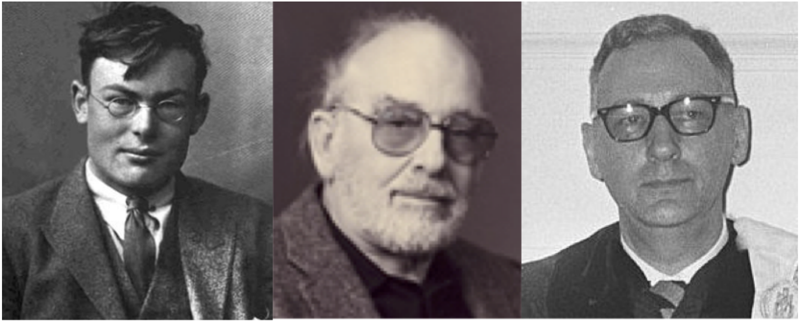



(a) 18 original OECD members

(b) world (114 countries)

Figure: International evidence on convergence: 1960 income and subsequent growth

## Ramsey-Cass-Koopmans model



Frank Ramsey David Cass Tjalling Koopmans

## Assumptions about production

- Production technology and inputs evolution exactly the same as in Solow, but $\delta = 0$ for simplicity.
- $Y = F(K, AL)$; CRS; $f(0) = 0$; $f'(k) > 0$; $f''(k) < 0$;

  $lim_{k\to 0} f'(k) = \infty$; $lim_{k\to\infty} f'(k) = 0$.
- $\frac{\dot{A}}{A} = g_A = g$; $\frac{\dot{L}}{L} = g = n$
- $\dot{K}(t) = Y(t) - \zeta(t)$ where $\zeta$ is total consumption.
- *Representative firm* assumption.

## Assumptions about households

Large but fixed number of identical households:

- each grows at rate $n$;
- household members are infinitely lived, forward-looking *and* have perfect foresight into the infinite future;
- they supply 1 unit of L at each point in time and earn wages;
- own $K$, that they rent to firms, earning $K$ income;
- divide their income between $C$ and $I$ in such a way as to maximize utility over their (infinite) lifetime.
- *representative household* assumption.

## The Euler equation

- Assumed households' behavior implies the **Euler equation** – the fundamental driver of this model:

$$g_C = \frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta}$$

- Higher interest rate induces to postpone consumption, so it contributes *positively* to its growth in time.
- Higher discount rate induces to anticipate consumption, so it contributes *negatively* to its growth in time.
- Let's now study formally how this equation follows from the assumptions of the model...

## Household's utility function

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt$$

- gives PV of total utility enjoyed by household members over their lifetime.
- $C$ = consumption per person.
- $L/H$ = number of household members.
- $\rho$ = discount rate.
- instantaneous utility $u()$:

$$u[C(t)] = \frac{C(t)^{1-\theta}}{1-\theta} \qquad \theta > 0; \qquad \rho - n - (1-\theta)g > 0$$

## Firms & factors' prices

Perfectly competitive firms in single-good economy, therefore

- interest rate: $r(t) = f'[k(t)]$
- wage per unit of eff. labor: $w(t) = \frac{W(t)}{A} = [f(k) - kf'(k)]$

## No-Ponzi condition

- Household consumption is constrained by the PV of their wealth:

$$\lim_{s\to\infty} e^{-R(s)}K(s) \geq 0$$

- Or, in ‘intensive form’ (scaled by AL):

$$\lim_{s\to\infty} e^{-R(s)}e^{(n+g)s}k(s) \geq 0$$

- *No-Ponzi condition:* household’s asset holdings cannot be negative in the limit.
- Will be satisfied with equality.

## Households' dynamic optimization problem

- Household maximizes PV of lifetime utility (intensive form):

$$\text{Max } U = B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt$$

with $B = A(0)^{1-\theta} \frac{L(0)}{H}$ and $\beta = \rho - n - (1-\theta)g$

- subject to the state equation:

$$\dot{k}(t) = (r - n - g)k(t) + w(t) - c(t)$$

- and the transversality (no-Ponzi w/ equality) condition:

$$\lim_{s \to \infty} e^{-R(s)} e^{(n+g)s} k(s) = 0$$

## The Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

- This is the **Euler equation** we have seen before, but scaled by $AL$ (intensive form). Same meaning and interpretation.
- $r > \rho \rightarrow$ households postpone consumption $\rightarrow c(t)$ increases in time.
- $r < \rho \rightarrow$ households anticipate consumption $\rightarrow c(t)$ decreases in time.

## The dynamics of the economy: c & k

- Dynamics of $c$ (Euler equation):

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

- Dynamics of $k$ (like in Solow but w/o depreciation):

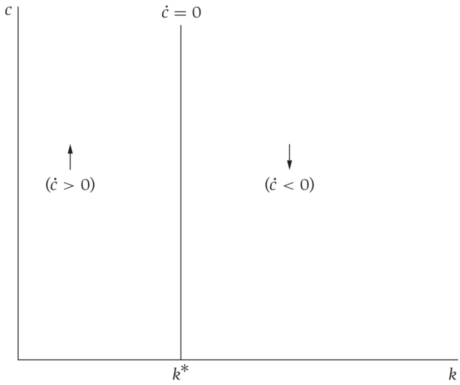
$$\dot{k} = f(k(t)) - c(t) - (n+g)k(t)$$

- intertemporal equilibrium: $\dot{c} = 0$ and $\dot{k} = 0$;

- two variables phase diagram

## The dynamics of the economy: consumption

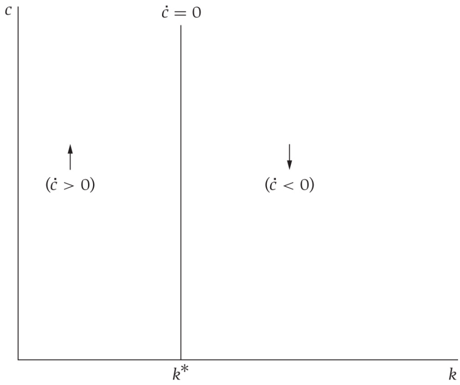
$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$




- what's going on in this graph?

## The dynamics of the economy: consumption

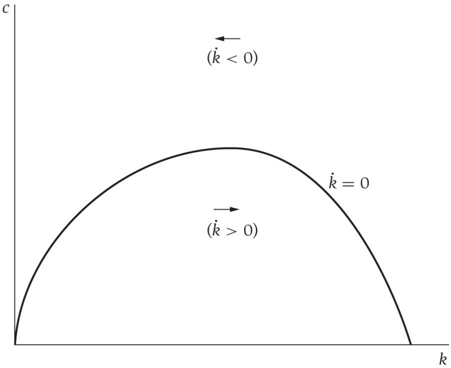
$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$




- what's going on in this graph?
- $\dot{c} = 0 \quad \rightarrow \quad f'(k) = \rho + \theta g$
- implicitly defines $k^*$.
- $k^*$ is unique & independent of $c$ (vertical line).
- $k > k^* \rightarrow f'(k) < \rho + \theta g \rightarrow \dot{c} < 0$
- $k < k^* \rightarrow f'(k) > \rho + \theta g \rightarrow \dot{c} > 0$

## The dynamics of the economy: capital stock

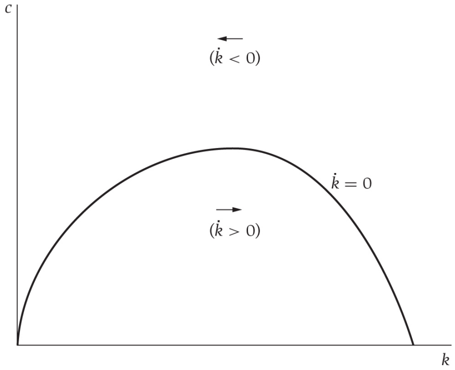
$$\dot{k} = f(k(t)) - c(t) - (n+g)k(t)$$




- what's going on in this graph?

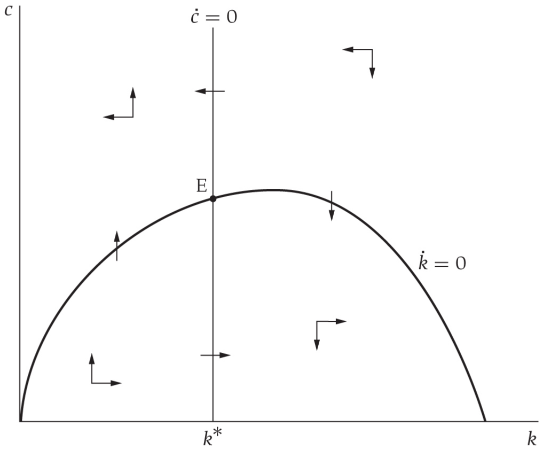
## The dynamics of the economy: capital stock

$$\dot{k} = f(k(t)) - c(t) - (n+g)k(t)$$

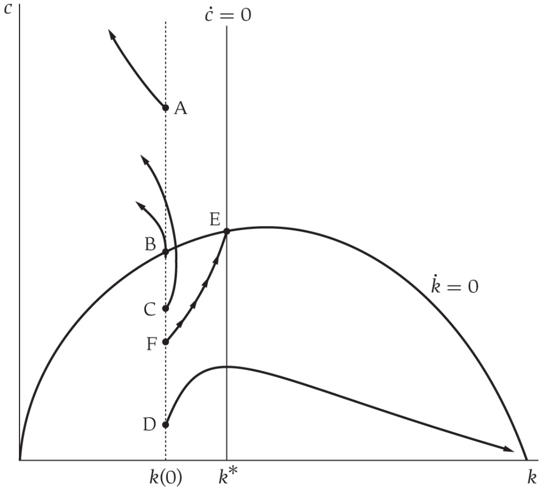



- what's going on in this graph?
- $\dot{k} = 0 \quad \rightarrow \quad c^* = f(k) - (n+g)k$
- $c^*$ U-shaped: increasing in $c$ as long as $f'(k) > (n+g)$.
- $c > c^* \rightarrow$, investment lower than break-even $\rightarrow \dot{k} < 0$.
- $c < c^* \rightarrow$, investment higher than break-even $\rightarrow \dot{k} > 0$.

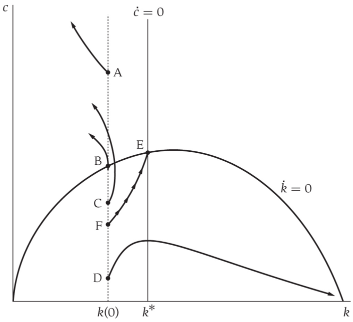
## The dynamics of the economy: phase diagram



## The dynamics of the economy: phase diagram



## The dynamics of the economy: phase diagram




- E = intertemporal equilibrium ($\dot{k} = \dot{c} = 0$);
- given $k(0)$, only $c(0) = F$ is on the 'stable branch' that leads to E;
- $c(0) < F$ leads to zero $c$ and infinite $k$: not utility-maximizing!
- $c(0) > F$ leads to negative $k$ but positive $c$: not feasible!
- $c(0) = F$ is the only $c(0)$ that implies

$$\lim_{s\to\infty} e^{-R(s)} e^{(n+g)s} k(s) = 0$$

so it is the only feasible and utility-maximizing one.

## The saddle path

- For any possible $k(0)$, there is a unique $c(0)$ that satisfies

$$\lim_{s\to\infty} e^{-R(s)} e^{(n+g)s} k(s) = 0$$

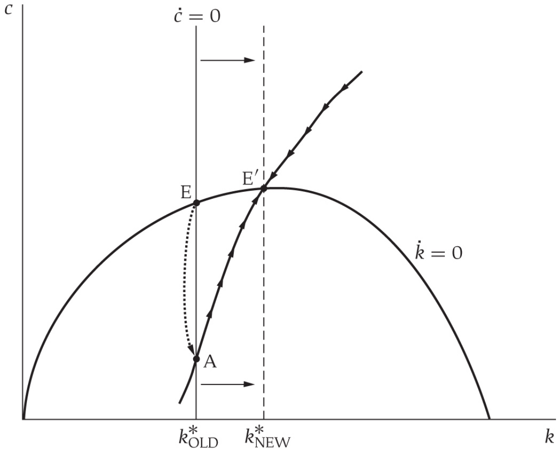
- this $c(0)$ is the one on the 'saddle path'[1] towards steady state.
- all other $c(0)$'s are on unstable trajectories, but are ruled out by the no-Ponzi condition or by intertemporal optimization;
- *saddle-path stable* equilibrium.
- (quite obviously) it is Pareto-efficient.

[1] The reason for this name is the analogy with a marble left on top of a saddle. There is one point on the saddle where, if left there, the marble does not move. This point corresponds to the steady state. There is a trajectory on the saddle with the property that if the marble is left at any point on that trajectory, it rolls toward the steady state. But if the marble is left at any other point, the marble falls to the ground. (Barro & Sala i-Martin, *Economic growth*, 1990).

## The balanced growth path

- $\dot{k} = \dot{y} = \dot{c} = 0$
- $g_Y = g_K = g_C = n + g$
- $g_{Y/L} = g_{K/L} = g_{C/L} = g$
- exactly as in Solow model!
- here, however, $k^* < k_{GR}$
  - because $\rho + \theta g > n + g$ (one of the assumptions of RCK model).
  - households don't maximize $c$ (as in the *golden rule*), but $PV(c)$.
  - $\rho > 0$ creates a bias towards the present.

## A fall in the discount rate




- $f'(k) = \rho + \theta g$
- $c^* = f(k) - (n+g)k$

## Diamond (1965): The Overlapping Generations (OLG) model

## OLG model: assumptions about households

- Time is discrete ($t = 0, 1, 2, \ldots$);
- each individual lives for two periods;
- $L_t = (1+n)L_{t-1}$ individuals born at time $t$;
- young (1st period):
  - no $K$
  - supplies 1 unit of $L$;
  - divides resulting wage between $C$ and $S$;
- old (2nd period):
  - rents her $K$ (=1st period savings)
  - then consumes $(1+r)K$

## OLG model: assumptions about production

- At each $t$, old people's $K$ and young people's $L$ are combined to produce $Y$;
- $Y = F(K_t, A_t L_t)$
- CRS and Inada conditions (as in Solow & Ramsey);
- $\delta = 0$ for simplicity;
- $A_t = (1+g)A_{t-1}$;
- Competitive markets
  - $r_t = f'(k_t)$
  - $w_t = f(k_t) - k_t f'(k_t)$

## OLG model: The Plan

- Focus on $k = \frac{K}{AL}$.
- Intertemporal equilibrium: $k_{t+1} = k_t$

## OLG model: The Plan

- Focus on $k = \frac{K}{AL}$.
- Intertemporal equilibrium: $k_{t+1} = k_t$

Our strategy:

1. $U$ maximization $\rightarrow$ $C$ dynamics (Euler equation);
2. $C$ dynamics $\rightarrow$ dynamics of $K$ & $k$ $\quad \Rightarrow \quad$ $k_{t+1}$ as a function of $k_t$;
3. set $k_{t+1} = k_t$ to study intertemporal equilibrium & stability.

## OLG model: The Plan

- Focus on $k = \frac{K}{AL}$.
- Intertemporal equilibrium: $k_{t+1} = k_t$

Our strategy:

1. $U$ maximization $\rightarrow$ $C$ dynamics (Euler equation);
2. $C$ dynamics $\rightarrow$ dynamics of $K$ & $k$ $\quad\Rightarrow\quad$ $k_{t+1}$ as a function of $k_t$;
3. set $k_{t+1} = k_t$ to study intertemporal equilibrium & stability.

- 1. using relatively general production and utility functions;
- stronger functional form assumptions needed to do 2. & 3.

## Consumption dynamics

- Maximization of lifetime-utility...

$$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta} \quad \text{with} \quad \theta > 0, \quad \rho > -1$$

- ...subject to the budget constraint

$$C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1} = A_t w_t$$

## Consumption dynamics

- Maximization of lifetime-utility...

$$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta} \quad \text{with} \quad \theta > 0, \quad \rho > -1$$

- ...subject to the budget constraint

$$C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1} = A_t w_t$$

$\Downarrow$

- f.o.c. imply the Euler Equation:

$$\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}$$

## Discrete-time Euler equation: intuitive derivation

- At the optimal point, a marginal reallocation of $C$ from 1st to 2nd period does not affect utility:

$$C_{1t}^{-\theta}\Delta C = \frac{1}{1+\rho} C_{2t+1}^{-\theta}(1+r_{t+1})\Delta C \tag{2}$$

(marginal effect of change in $C_1$ = marginal effect of change in $C_2$)

- rearrange as

$$\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}$$

## Discrete-time Euler equation: 'systematic' derivation

- Set the Lagrangian for the utility-maximization problem

$$\mathcal{L} = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta} + \lambda[A_t w_t - (C_{1t} + \frac{1}{1+r_{t+1}}C_{2t+1})]$$

- f.o.c. for $C_{1t}$ and $C_{2t}$:

$$C_{1t}^{-\theta} = \lambda; \qquad \frac{1}{1+\rho}C_{2t+1}^{-\theta} = \frac{1}{1+r_{t+1}}\lambda$$

- Substitute for $\lambda$ and rearrange:

$$\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta}$$

## Consumption dynamics

- Substitute Euler Equation into budget constraint to get

$$C_{1t} = \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} A_t w_t$$

- or more simply:

$$C_{1t} = [1 - s(r)] A_t w_t$$

$$\text{with} \quad s(r) = 1 - \frac{C_{1t}}{A_t w_t} = \frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}$$

- Implication: 1st period saving increasing in $r$ if $\theta < 1$; decreasing if $\theta > 1$; $s$ independent of $r$ with logarithmic utility ($\theta = 1$)

## Consumption dynamics

- Substitute Euler Equation into budget constraint to get

$$C_{1t} = \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} A_t w_t$$

- or more simply:

$$C_{1t} = [1 - s(r)]A_t w_t$$

$$\text{with} \quad s(r) = 1 - \frac{C_{1t}}{A_t w_t} = \frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}$$

- Implication: 1st period saving increasing in $r$ if $\theta < 1$; decreasing if $\theta > 1$; $s$ independent of $r$ with logarithmic utility ($\theta = 1$)
- $r$ has both an *income* and a *substitution* effect.

## The dynamics of the OLG economy

- Capital accumulation in a given period is:

$$K_{t+1} = s(r_{t+1})A_t w_t L_t$$

## The dynamics of the OLG economy

- Capital accumulation in a given period is:

$$K_{t+1} = s(r_{t+1})A_t w_t L_t$$

- Focus on the intensive form (divided by $A_{t+1}L_{t+1}$):

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r_{t+1}) w_t$$

## The dynamics of the OLG economy

- Capital accumulation in a given period is:

$$K_{t+1} = s(r_{t+1})A_t w_t L_t$$

- Focus on the intensive form (divided by $A_{t+1}L_{t+1}$):

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r_{t+1})w_t$$

- Given one-good competitive economy, we can substitute for factors' prices:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(f'(k_{t+1}))[f(k_t) - k_t f'(k_t)]$$

## The dynamics of the OLG economy

- We now have $k_{t+1}$ as a (implicit) function of $k_t$:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(f'(k_{t+1}))[f(k_t) - k_t f'(k_t)] \tag{3}$$

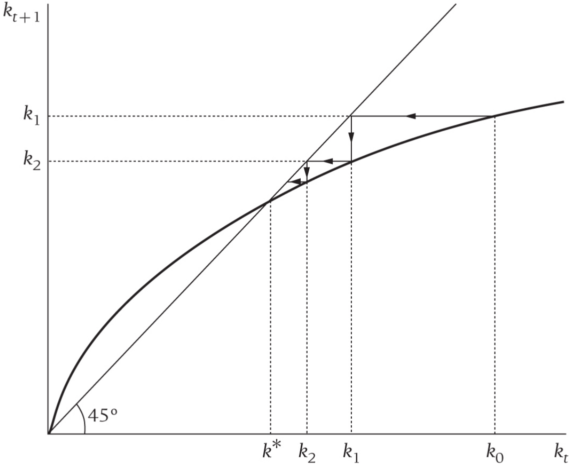
- Assume $f(k) = k^\alpha$ & $\theta = 1$:

$$f(k) = k^\alpha; \quad f'(k) = \alpha k^{\alpha-1}; \quad s = 1/(2+\rho)$$

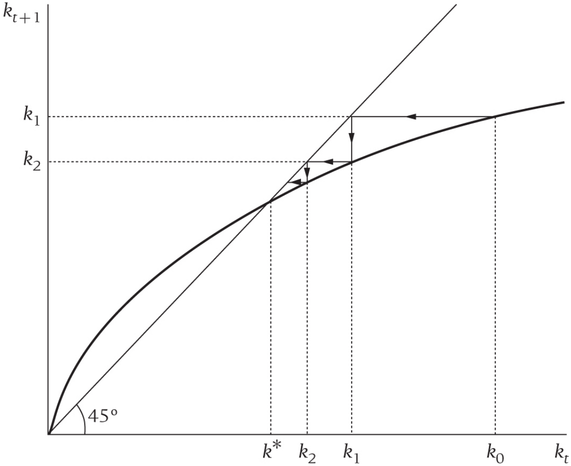
- So the equation of motion for $k$ is:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho} (1-\alpha) k_t^\alpha = \beta k_t^\alpha \quad \text{with } 0 < \alpha < 1$$

## The dynamics of the OLG economy



## The dynamics of the OLG economy




- Equilibrium: $k_{t+1} = k_t = k^*$;
- decreasing MPK & Inada conditions ensure existence & uniqueness (except $k = 0$);
- dynamically stable;
- steady state à la Solow-Ramsey: constant $s$ and $k$; $Y/L$ grows at rate $g$.

## The dynamics of the OLG economy

- In intertemporal equilibrium, $k_t = k_{t+1} = k^*$

$$k^* = \frac{1}{(1+n)(1+g)(2+\rho)}(1-\alpha){k^*}^{\alpha}$$

- Solving for $k^*$

$$k^* = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{1}{1-\alpha}}$$

How fast is convergence in the OLG economy?

- *Linear approximation* around steady state

$$k_{t+1} - k^* \approx \lambda(k_t - k^*) \qquad \text{with } \lambda = \left.\frac{dk_{t+1}}{dk_t}\right|_{k_t = k^*}$$

- Solving the (1st order linear) difference equation:

$$k_t - k^* \approx \lambda^t (k_0 - k^*)$$

## How fast is convergence in the OLG economy?

- *Linear approximation* around steady state

$$k_{t+1} - k^* \approx \lambda(k_t - k^*) \qquad \text{with } \lambda = \left.\frac{dk_{t+1}}{dk_t}\right|_{k_t = k^*}$$

- Solving the (1st order linear) difference equation:

$$k_t - k^* \approx \lambda^t (k_0 - k^*)$$

- w/ log $U$ and Cobb-Douglas production: $0 < \lambda = \alpha < 1$;
- With $\alpha = 1/3$, two-thirds of the ‘gap’ removed in one period (=half a lifetime).

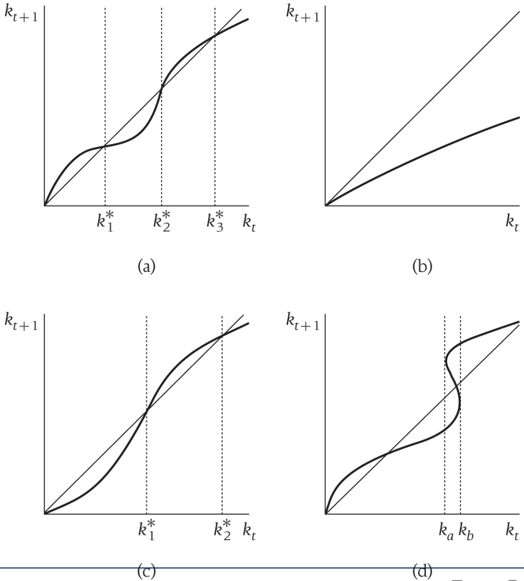
## The general case

- With more general utility and production functions?
- Equation of motion for $k$:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(r) \frac{f(k_t) - k_t f'(k_t)}{f(k_t)} f(k_t)$$

4 components: [$AL_t/AL_{t+1}$] [saving rate] [wage share] [Y/AL]

- $k_{t+1}$ depends on $k_t$ through three channels;
- (almost) anything goes.




- (a) and (c): multiple equilibria ($s$ or $W/Y$ increasing);
- (b): stable zero-output equilibrium (either $s$ or $W/Y$ approach 0 when $k = 0$);
- (d): indeterminacy ($s$ ‘very increasing’ in $k_t$).

## Welfare & dynamic inefficiency

- OLG equilibrium can be Pareto-inefficient ($k^* > k^{GR}$);

## Welfare & dynamic inefficiency

- OLG equilibrium can be Pareto-inefficient ($k^* > k^{GR}$);
- Assume $g = 0$, Cobb-Douglass production and log utility:
  - $k^{GR}$ implies $f'(k) = n$.
  - $f'(k^*) = \frac{\alpha}{1-\alpha}(1+n)(2+\rho)$
  - $\alpha$ small $\rightarrow f'(k^*) < n \rightarrow k^* > k^{GR}$
  - This possible *dynamic inefficiency* arises from relaxing the assumption of a finite number of agents.

## OLG model: Takeaways

- Same conclusions as Solow/Ramsey on sources of long-run growth;
- but possibility of multiple equilibria and dynamic inefficiency (overaccumulation)

## OLG model: Takeaways

- Same conclusions as Solow/Ramsey on sources of long-run growth;
- but possibility of multiple equilibria and dynamic inefficiency (overaccumulation)
- Is neoclassical growth theory fragile even within its 'one-good economy with perfect markets' assumptions?
  - However, Barro (1974) shows that a bequest motive (intergenerational altruism) may make OLG practically equivalent to Ramsey (no inefficiencies).
  - Moreover, OLG dynamic inefficiency (over-accumulation) does not seem empirically relevant: too much accumulation??