4.5 SAMPLING DISTRIBUTION OF THE OLS ESTIMATOR

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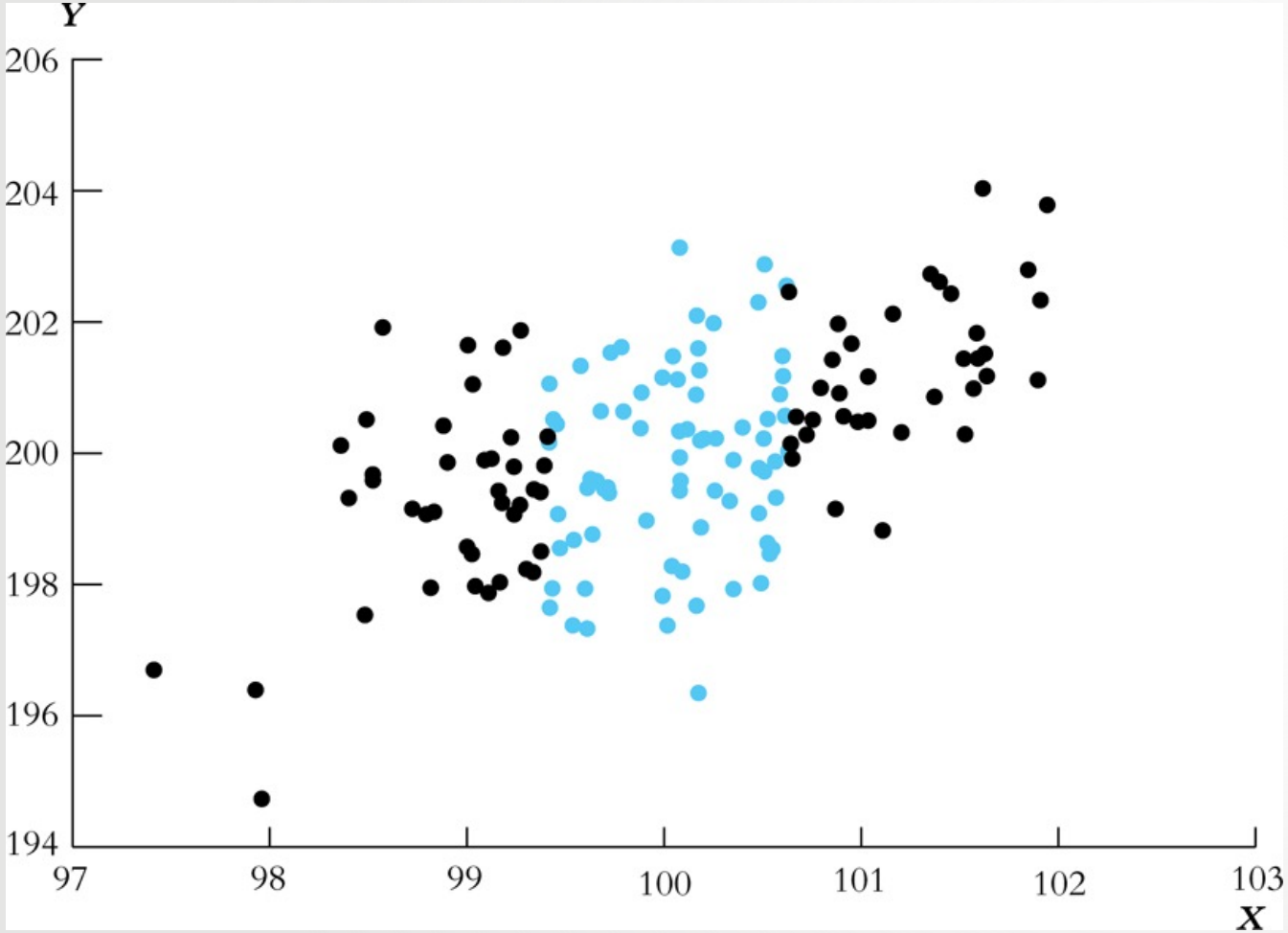
SAMPLING DISTRIBUTION OF $\widehat{\beta}_0$ AND $\widehat{\beta}_1$

- $\widehat{\beta}_0$ and $\widehat{\beta}_1$ = random variables (*why?)*
- $\mathrm{E}(\widehat{\beta}_0) = \beta_0$ and $\mathrm{E}(\widehat{\beta}_1) = \beta_1$
- $\widehat{\beta}_0$ and $\widehat{\beta}_1$ tend to be normally distributed in large samples.
- $\widehat{\beta}_0 \sim N(\beta_0, \sigma^2_{\widehat{\beta}_0})$ and $\widehat{\beta}_1 \sim N(\beta_1, \sigma^2_{\widehat{\beta}_1})$
- What determines $\sigma^2_{\widehat{\beta}_0}$ and $\sigma^2_{\widehat{\beta}_1}$?

THE VARIANCE OF THE OLS ESTIMATOR

$$\text{var}(\hat{\beta}_1) = \frac{1}{n} \times \frac{\text{var}[(X_i - \mu_x)u_i]}{(\sigma_X^2)^2}.$$

THE VARIANCE OF THE OLS ESTIMATOR



- The number of black and blue dots is the same and they come from the same joint distribution.
- Using which would you get a more accurate regression line?
- Increasing the spread of X decreases $var(\hat{\beta}_1)$

4.6 HYPOTHESIS TESTS ABOUT REGRESSION COEFFICIENTS

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HYPOTHESIS TESTS

- Null and alternative hypotheses:

$$H_0: \beta_1 = \beta_{1,0} \text{ vs. } H_1: \beta_1 \neq \beta_{1,0}$$

- Three steps for testing H_0 :

1. Compute $\hat{\beta}_1$ and $SE(\hat{\beta}_1)$ using sample data.
2. Compute the t-statistics
3. Compute the p-value

HYPOTHESIS TESTS

- Null and alternative hypotheses:

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- Three steps for testing H_0 :

1. Compute $\hat{\beta}_1$ and $SE(\hat{\beta}_1)$ using sample data.
2. Compute the t-statistics
3. Compute the p-value

THE STANDARD ERROR OF $\hat{\beta}_1$

- $SE(\hat{\beta}_1)$ is an estimator of $\sigma_{\hat{\beta}_1}$.
- $SE(\hat{\beta}_1) = \sqrt{\hat{\sigma}^2_{\hat{\beta}_1}} = \frac{1}{n} \times \frac{\frac{1}{n-2}\sum_{i=1}^{n}(X_i - \bar{X})^2 \hat{u}_i^2}{\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right]^2}$
- (complicated, but STATA will do it for you)
- Also called *robust* standard error.

```
regress testscr str, robust
Regression with robust standard errors                Number of obs =     420
                                                      F(  1,   418) =   19.26
                                                      Prob > F      =  0.0000
                                                      R-squared     =  0.0512
                                                      Root MSE      =  18.581
------------------------------------------------------------------------------
             |               Robust
    testscr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         str |  -2.279808   .5194892    -4.39   0.000    -3.300945   -1.258671
       _cons |    698.933   10.36436    67.44   0.000     678.5602    719.3057
------------------------------------------------------------------------------
```

- $\hat{\beta}_1 = -2.28$ and $SE(\hat{\beta}_1) = 0.52$
- $\hat{\beta}_0 = 698.9$ and $SE(\hat{\beta}_1) = 10.36$

HYPOTHESIS TESTS

- Null and alternative hypotheses:

$$H_0: \beta_1 = \beta_{1,0} \text{ vs. } H_1: \beta_1 \neq \beta_{1,0}$$

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1. Compute $\hat{\beta}_1$ and $SE(\hat{\beta}_1)$ using sample data.
2. Compute the t-statistics
3. Compute the p-value

T-STATISTICS FOR OLS PARAMETERS

$$t = \frac{estimator - hypothesized\ value}{standard\ error\ of\ estimator}$$

$$t = \frac{\hat{\beta}_1 - \hat{\beta}_{1,0}}{SE(\hat{\beta}_1)}$$

- t has a *standard normal distribution* in large samples
- $t \sim N(0,1)$

```
regress testscr str, robust
Regression with robust standard errors                Number of obs =     420
                                                      F(  1,   418) =   19.26
                                                      Prob > F      =  0.0000
                                                      R-squared     =  0.0512
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         str |  -2.279808   .5194892    -4.39   0.000    -3.300945   -1.258671
       _cons |    698.933   10.36436    67.44   0.000     678.5602    719.3057
------------------------------------------------------------------------------
```

- $\hat{\beta}_1 = -2.28$ and $SE(\hat{\beta}_1) = 0.52$ and $t = -4.39$
- $\hat{\beta}_0 = 698.9$ and $SE(\hat{\beta}_1) = 10.36$ and $t = 67.44$

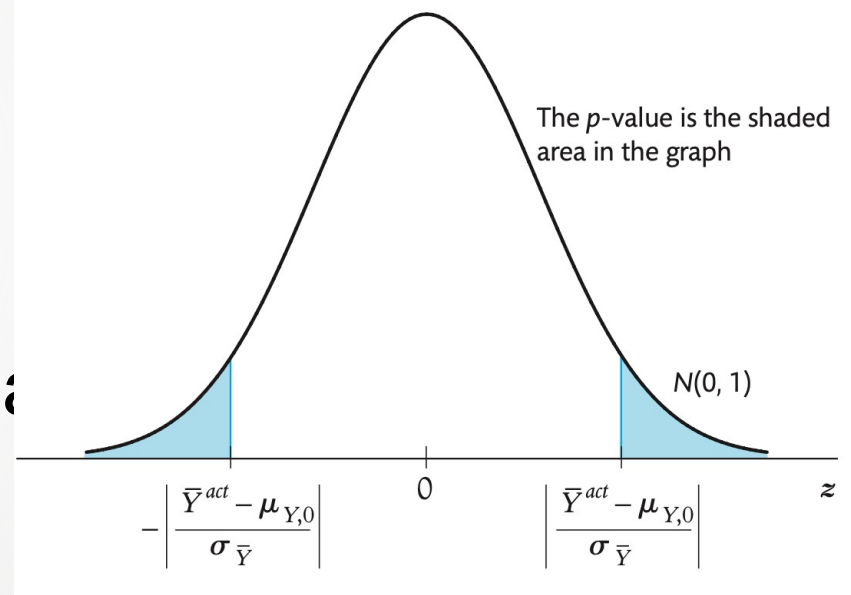
HYPOTHESIS TESTS

- Null and alternative hypotheses:

$$H_0: \beta_1 = \beta_{1,0} \text{ vs. } H_1: \beta_1 \neq$$

- Three steps for testing H_0 :

1. Compute $\hat{\beta}_1$ and $SE(\hat{\beta}_1)$ using sample da
2. Compute the t-statistics
3. Compute the p-value



COMPUTING THE P-VALUE

- p-value $= Pr_{H_0}[|\hat{\beta}_1 - \beta_{1,0}| > |\hat{\beta}_1^{act} - \hat{\beta}_{1,0}|$

$$= Pr_{H_0}\left[\left|\frac{\hat{\beta}_1 - \beta_{1,0}}{SE(\hat{\beta}_1)}\right| > \left|\frac{\hat{\beta}_1^{act} - \hat{\beta}_{1,0}}{SE(\hat{\beta}_1)}\right|\right]$$

$$= Pr_{H_0}(|t| > |t^{act}|)$$

$$= Pr_{H_0}(|Z| > |t^{act}|)$$

$$= 2\phi(-|t^{act}|)$$

```
regress testscr str, robust
Regression with robust standard errors                Number of obs =     420
                                                      F(  1,   418) =   19.26
                                                      Prob > F      =  0.0000
                                                      R-squared     =  0.0512
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    testscr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
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         str |  -2.279808   .5194892    -4.39   0.000    -3.300945   -1.258671
       _cons |    698.933   10.36436    67.44   0.000     678.5602    719.3057
------------------------------------------------------------------------------
```

- $\hat{\beta}_1 = -2.28$ and $SE(\hat{\beta}_1) = 0.52$ and $t = -4.39$ and $p < 0.001$
- $\hat{\beta}_0 = 698.9$ and $SE(\hat{\beta}_1) = 10.36$ and $t = 67.44$ and $p < 0.001$

```
regress testscr str, robust
Regression with robust standard errors                Number of obs =     420
                                                      F(  1,   418) =   19.26
                                                      Prob > F      =  0.0000
                                                      R-squared     =  0.0512
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------------------------------------------------------------------------------
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    testscr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         str |  -2.279808   .5194892    -4.39   0.000    -3.300945   -1.258671
       _cons |   698.933   10.36436     67.44   0.000     678.5602    719.3057
------------------------------------------------------------------------------
```

1. Compute $SE(\beta_1) = 0.52$

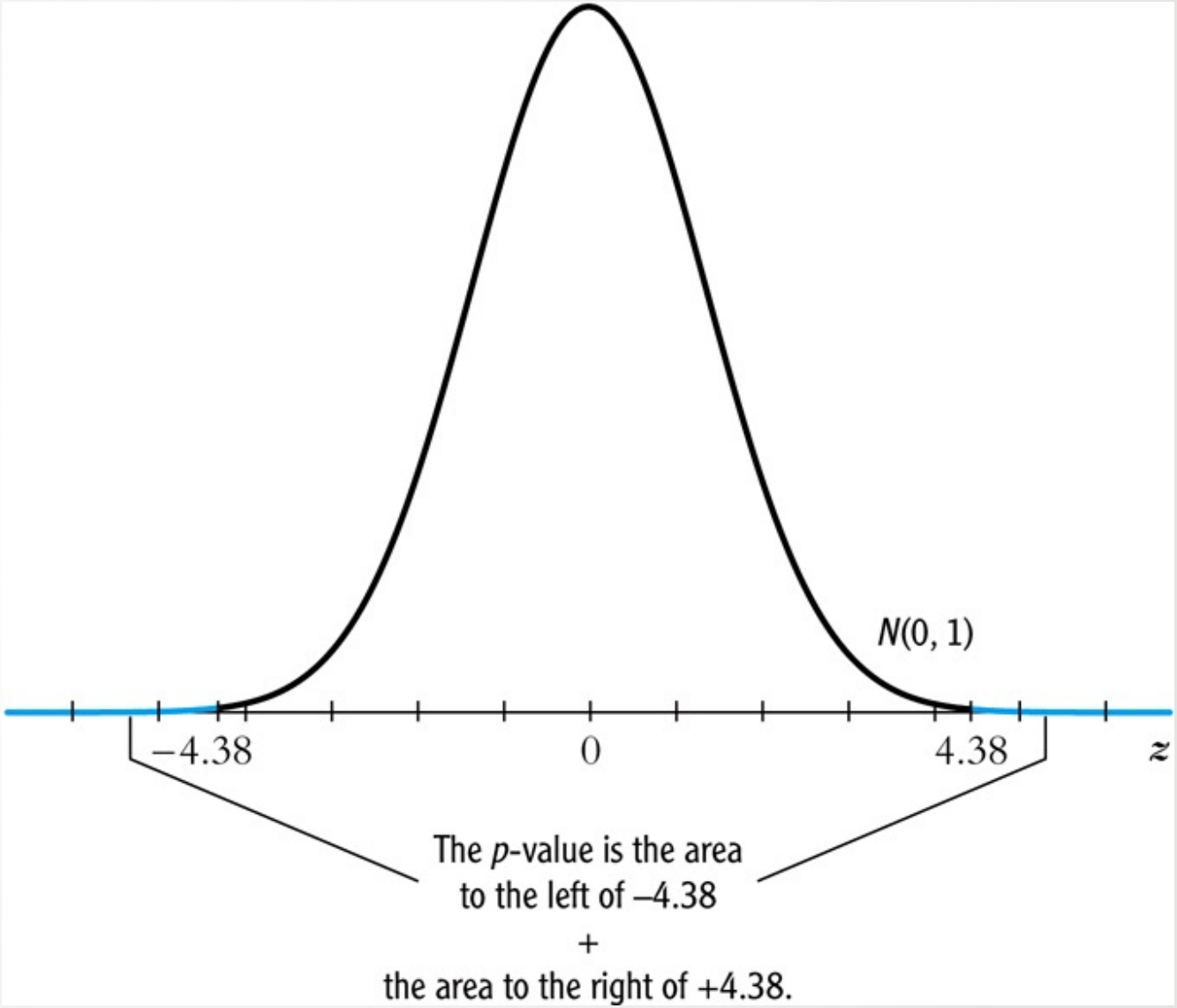
2. Compute the t-statistic:

$$t = \frac{\hat{\beta}_1 - \beta_{1,0}}{SE(\hat{\beta}_1)} = \frac{-2.28 - 0}{0.52} = -4.39$$

3. Compute the p-value: $2\Phi(-|t|) = 2\Phi(-4.39) = 0.00001$

***p*-value =0.00001 (or 10^{-5})**

We can reject the null hypothesis: smaller classes do have higher test scores.



4.7 CONFIDENCE INTERVALS FOR REGRESSION COEFFICIENTS

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CONFIDENCE INTERVAL FOR β_1

- **95% confidence interval:** a range of values that is 95% likely to include the "true" population coefficient $\boldsymbol{\beta_1}$.
- Includes all β_1 values that we *cannot* reject at the 5% significance level.
- 95% confidence interval for β_1:

$$\hat{\beta}_1 - 1.96 * SE(\hat{\beta}_1) \leq \beta_1 \leq \hat{\beta}_1 + 1.96 * SE(\hat{\beta}_1)$$

CONFIDENCE INTERVAL FOR β_1

- 95% confidence interval for the effect of test scores.
- We estimated $\hat{\beta}_1 = -2.28$ and $SE(\hat{\beta}_1) = 0.52$
- So the true β_1 is 95% likely to be between:
  - $-2.28 - (1.96 \times 0.52) = -3.30$
  - $-2.28 + (1.96 \times 0.52) = -1.26$

```
regress testscr str, robust
Regression with robust standard errors                Number of obs =     420
                                                      F(  1,   418) =   19.26
                                                      Prob > F      =  0.0000
                                                      R-squared     =  0.0512
                                                      Root MSE      =  18.581
------------------------------------------------------------------------------
             |               Robust
    testscr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         str |  -2.279808   .5194892    -4.39   0.000    -3.300945   -1.258671
       _cons |    698.933   10.36436    67.44   0.000     678.5602    719.3057
------------------------------------------------------------------------------
```

- Confidence interval for β_1: [-3.30 $\leq \beta_1 \leq$ -1.26]

Confidence interval for β_1(coefficient of STR):

$$[-3.30 \leq \beta_1 \leq -1.26]$$

YOUR TURN:

Can you compute a confidence interval for the average effect of a 3.5 increase in STR?

Confidence interval for β_1(coefficient of STR):

$$[-3.30 \leq \beta_1 \leq -1.26]$$

Lower bound: -3.30 * 3.5 = -11.55

Upper bound: -1.26 * 3.5 = -4.41

Increasing STR by 3 students will decrease test scores by between 4.41 and 11.55 points.

CONFIDENCE INTERVAL FOR PREDICTED EFFECTS

- Confidence interval for the effect of a Δx change in X:

$$\left[(\hat{\beta}_1 lower\ bound) \times \Delta x \ ; \ (\hat{\beta}_1 lower\ bound) \times \Delta x \right]$$

$$\left[(\hat{\beta}_1 - 1.96 * SE(\hat{\beta}_1)) \times \Delta x \ ; \ \hat{\beta}_1 + 1.96 * SE(\hat{\beta}_1) \times \Delta x \right]$$

4.8 REGRESSION WHEN X IS A BINARY VARIABLE

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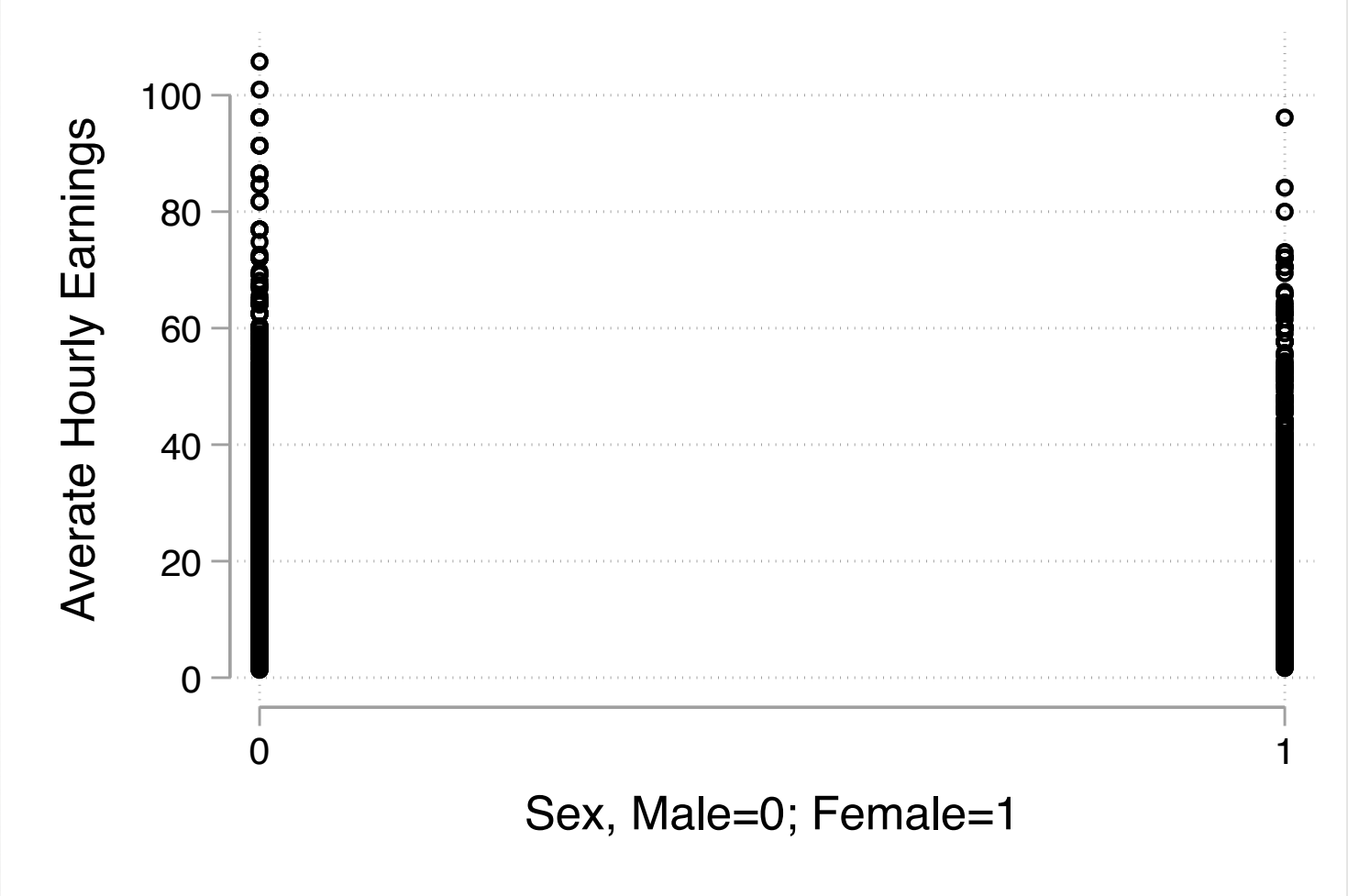
REGRESSION WHEN X IS BINARY

- **Binary** (or *indicator* or *dummy*) **variables**
  - Sex at birth (1 = female; 0 = male)
  - Urban or rural (1 = urban; 0 = rural)
  - Treatment or placebo (1 = treatment; 0 = placebo)
  - ….

CPS 2015 data

scatter ahe female



REGRESSION WHEN X IS BINARY

$$E(Y|D) = \beta_0 + \beta_1 D$$

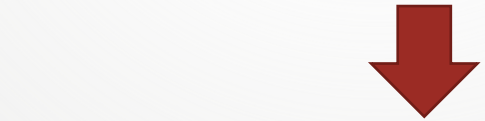
$$Y_i = \beta_0 + \beta_1 D_i + u_i$$

When $D_i = 0$:

$$E(Y|D = 0) = \beta_0 + \beta_1 \times 0 = \beta_0$$

When $D_i = 1$:

$$E(Y|D = 1) = \beta_0 + \beta_1 \times 1 = \beta_0 + \beta_1$$



$$\beta_1 = E(Y|D = 1) - E(Y|D = 0)$$

EXAMPLE: GENDER GAP IN EARNINGS

```
. reg ahe female, robust

Linear regression                               Number of obs     =     13,201
                                                F(1, 13199)       =     184.93
                                                Prob > F          =     0.0000
                                                R-squared         =     0.0131
                                                Root MSE          =     10.695

------------------------------------------------------------------------------
             |               Robust
         ahe |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      female |  -2.495648   .1835205   -13.60   0.000    -2.855375   -2.135922
       _cons |   18.32845   .1300679   140.91   0.000     18.0735    18.5834
------------------------------------------------------------------------------
```

- AHE for men:
  $\beta_0 = 18.33$
- Difference between women and men:
  $\beta_1 = -2.50$
- AHE for women:
  $\beta_0 + \beta_1 =$
  $= 18.32 - 2.50$
  $= 15.83$

EXAMPLE: GENDER GAP IN EARNINGS

```
. ttest ahe, by(female) unequal unpaired

Two-sample t test with unequal variances

   Group |     Obs        Mean    Std. err.   Std. dev.   [95% conf. interval]
---------+--------------------------------------------------------------------
       0 |   7,694    18.32845    .1300665    11.40884    18.07348    18.58341
       1 |   5,507     15.8328    .1294705      9.6079    15.57899    16.08661
---------+--------------------------------------------------------------------
Combined |  13,201    17.28735    .0936909    10.76467     17.1037     17.471
---------+--------------------------------------------------------------------
    diff |             2.495648    .1835209                2.13592    2.855377
------------------------------------------------------------------------------
    diff = mean(0) - mean(1)                                      t =  13.5987
H0: diff = 0                     Satterthwaite's degrees of freedom =  12855.9
```

- T-test for difference in means (see “review of statistics”).
- Regression does exactly the same thing!

REGRESSION WHEN X IS BINARY: SUMMING UP

$$Y_i = \beta_0 + \beta_1 D_i + u_i$$

- $\beta_0 =$ mean of Y when X=0
- $\beta_0 + \beta_1 =$ mean of Y when X=1
- β_1= difference in group means: $E(Y|X = 1) - E(Y|X = 0)$
- T-stats, p-value, confidence intervals calculated as usual.
- Will give the same result as a t-test for difference in means.

4.9 HETEROSKEDASTICITY AND HOMOSKEDASTICITY

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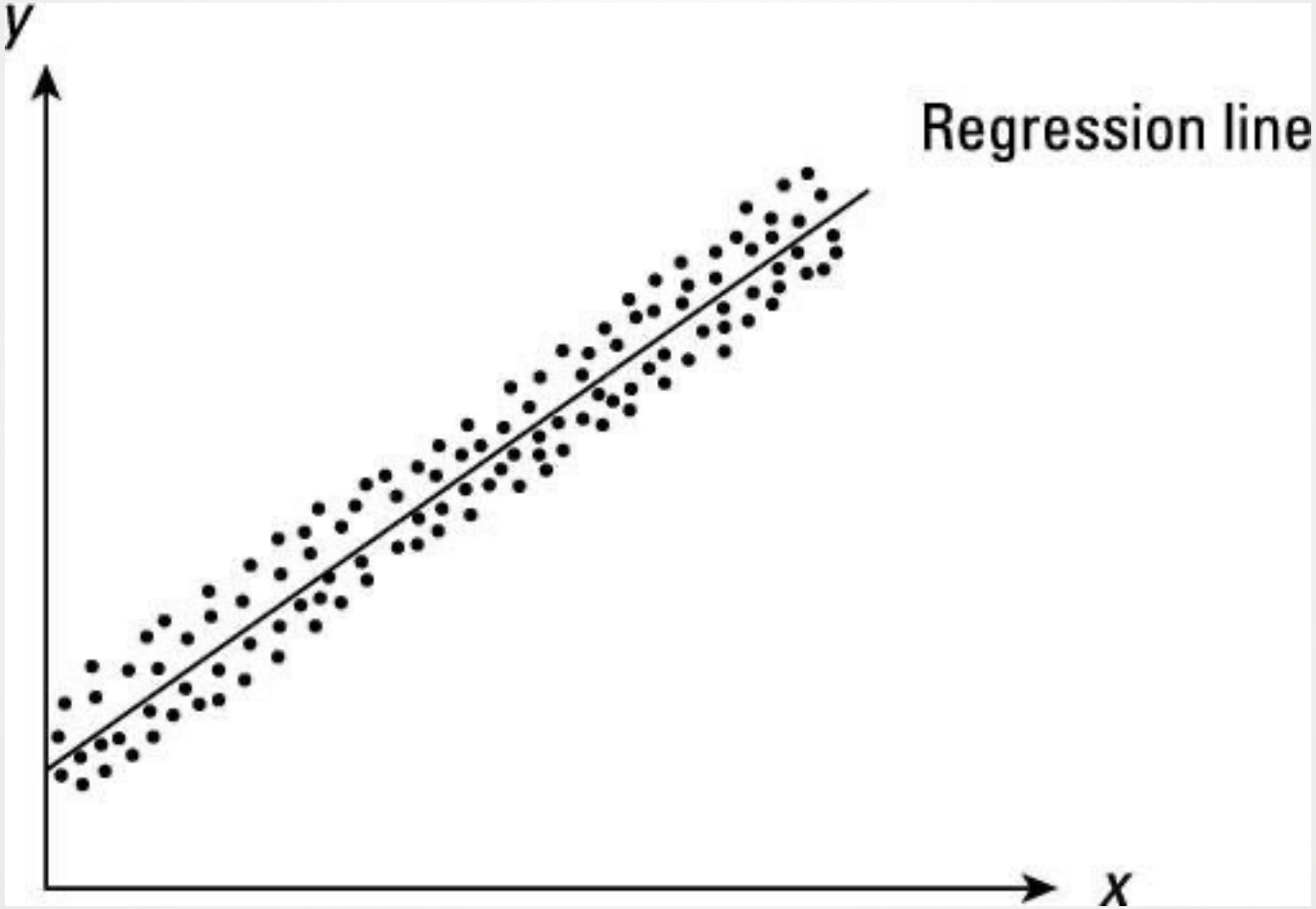
HETEROSKEDASTICITY & HOMOSKEDASTICITY

- Concerns the *conditional variance* of the error term:

$$var(u|X = x)$$

- Homoskedasticity: $var(u|X = x)$ is constant (does not depend on X).
- Heteroskedasticity: $var(u|X = x)$ varies with X.

HOMOSKEDASTICITY IN A PICTURE:



VARIANCE OF β_1 UNDER HOMOSKEDASTICITY

- In general the variance of $\hat{\beta}_1$ is

$$var(\hat{\beta}_1) = \frac{1}{n} \times \frac{var[(X_i - \mu_x)]u_i]}{[var(X_i)]^2}$$

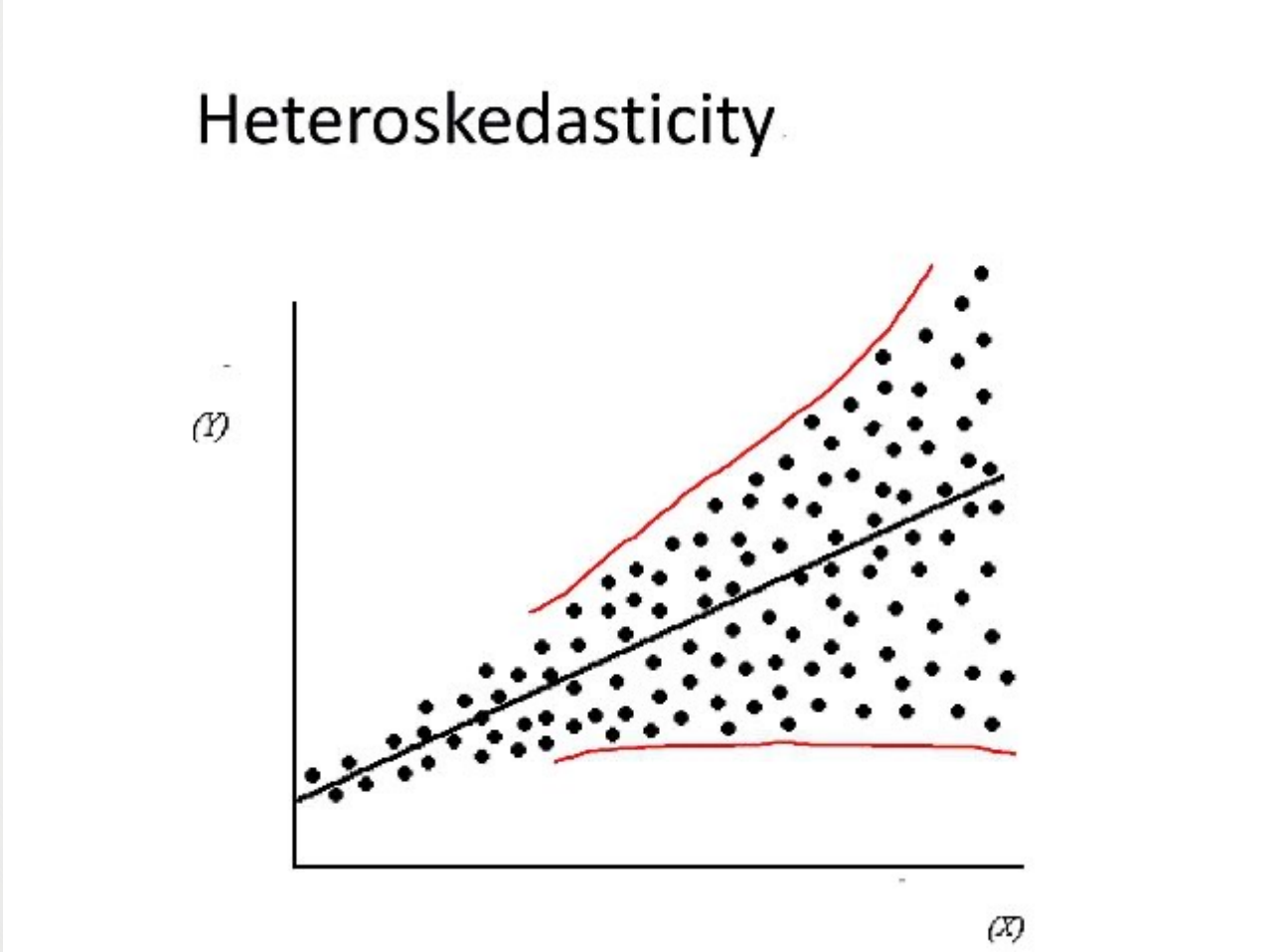
- If u_i is homoscedastic we get a simpler formula:

$$var(\hat{\beta}_1) = \frac{1}{n} \times \frac{var[u_i]}{[var(X_i)]^2}$$

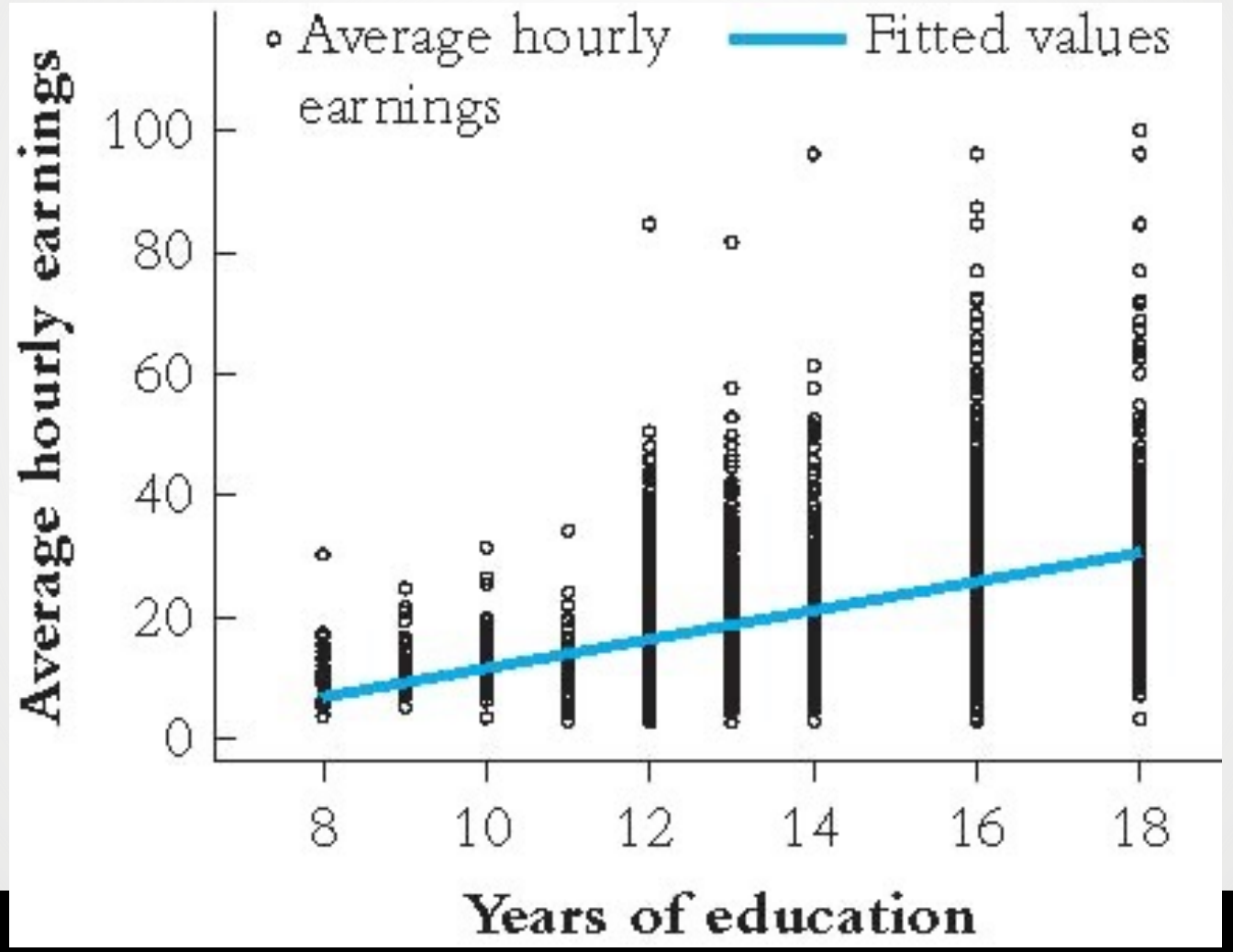
- So the homoskedasticity-only standard error of $\hat{\beta}_1$ is also simpler:

$$SE(\hat{\beta}_1) = \sqrt{\hat{\sigma}^2_{\hat{\beta}_1}} = \sqrt{\frac{\sigma^2_{\hat{u}}}{\sum_{i=1}^{n}(X_i - \bar{X})^2}}$$

HETEROSKEDASTICITY IN A PICTURE:



REAL-DATA EXAMPLE OF HETEROSKEDASTICITY (CPS DATA)



SO, WHEN SHOULD YOU USE HOMOSKEDASTICITY-ONLY STANDARD ERRORS?

- Never.
- Our usual (heteroskedasticity-robust) SEs are always fine.
- Always use the *robust* option in STATA!