How frequently is a prime the smallest divisor of all integers , and very prime integers

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Abstract

In this paper I will cover how to calculate how frequently a prime is the smallest divisor over an interval. I will then extend that definition to show how frequently that prime is the smallest divisor of all integers. Lastly , I will use this information to discuss very composite integers and very prime integers .

Key Definitions

1. **Coprime (Relatively Prime) Integers:** Two integers a and b are coprime if their greatest common divisor is 1:

$$\gcd(a, b) = 1.$$

2. **Primorial :** The primorial of p_n, denoted P_n, is the product of the first n primes :

$$P_n = \prod_{p \leq p_n} p.$$

3. **Totient Function :** The Euler totient function $\phi(a)$ is calculated using the prime factorization of a. If a has the prime factorization:

$$a = p_1^{e_1} p_2^{e_2} \cdots p_b^{e_b},$$

then the totient function is given by:

$$\phi(a) = a \prod_{i=1}^{b} \left(1 - \frac{1}{p_i}\right),$$

where $p_1, p_2, \ldots, p_b$ are the distinct prime factors of a. This formula represents the count of integers $x < a$ that do not share any factors with a .

Using the Totient Function to Determine the Frequency of a Prime as the Smallest Factor Over a Fixed Interval

$P_n = a \cdot p_n$

Consider the product $a \cdot p_n$, where $a = p_1 \cdot p_2 \cdot \dots \cdot p_{n-1}$ is the product of the first $n-1$ primes , and p_n is the n-th prime . We wish to determine the number of times p_n is the smallest factor of integers in the interval $[1, a \cdot p_n]$.

Integers Factorable by p_n

All integers in $[1, a \cdot p_n]$ that are divisible by p_n take the form:

$$p_n \cdot 1,\ p_n \cdot 2,\ p_n \cdot 3,\ \ldots,\ p_n \cdot a.$$

This set contains a elements, as a represents the range of possible values for the multiplier m.

Integers divisible by no smaller primes than p_n

For p_n to be the smallest factor of an integer $p_n \cdot m$, m must not share any factors with a, if m shares a factor with a , then m is divisible by a prime less than p_n , since a is composed of all the primes less than p_n. The number of integers j in $[1, a]$ that are coprime to a is given by $\phi(a)$. Thus, there are $j = \phi(a)$ integers in $[1, a \cdot p_n]$ which take the form $p_n \cdot m$, where p_n is the smallest factor . The totient function $\phi(a)$ allows us to determine the exact number of times p_n is the smallest factor of integers in the interval $[1, a \cdot p_n]$. This is precisely $\phi(a)$, as it counts the values of m coprime to a .

Frequency of p_n as the Smallest Factor

To compute the relative frequency in $[1, a \cdot p_n]$ where p_n is the smallest factor:

$$\text{Frequency} = F_n = \frac{\phi(a)}{a \cdot p_n} = \frac{\phi(P_{n-1})}{P_n}$$

Example 1

Let $n = 2$ then $p_2 = 3$. Let a be the product of all primes less than $p_2 = 3$. $a = 2$

$$\phi(a) = \phi(2) = 2 \cdot (1 - \frac{1}{2}) = 2 \cdot \frac{1}{2} = 1$$

$$a \cdot p_n = 2 \cdot 3 = 6$$

$$F_2 = \frac{\phi(a)}{a \cdot p_n} = \frac{1}{6}$$

We can say that $p_2 = 3$ is the smallest divisor of $\frac{1}{6}$ of all integers . By use of the totient function we may exclude m where m is a factor of a .

$$3 \cdot 1, \cancel{3 \cdot 2},$$

Example 2

Let $n = 3$ then $p_3 = 5$. Let a be the product of all primes less than $p_3 = 5$. $a = 2 \cdot 3 = 6$

$$\phi(a) = \phi(6) = 6 \cdot (1 - \frac{1}{2}) \cdot (1 - \frac{1}{3}) = 6 \cdot \frac{1}{2} \cdot \frac{2}{3} = 2$$

$$a \cdot p_n = 6 \cdot 5 = 30$$

$$F_3 = \frac{\phi(a)}{a \cdot p_n} = \frac{2}{30}$$

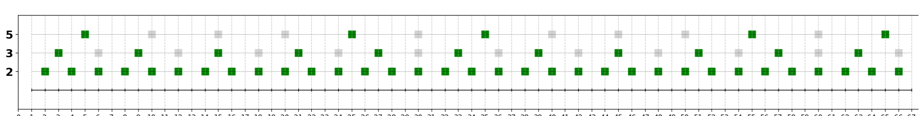
We can say that $p_3 = 5$ is the smallest divisor of $\frac{2}{30}$ of all integers .By use of the totient function we may exclude m where m is a factor of a .

$$5 \cdot 1, \cancel{5 \cdot 2}, \cancel{5 \cdot 3}, \cancel{5 \cdot 4}, 5 \cdot 5, \cancel{5 \cdot 6},$$

Case for F_1 , $p_1 = 2$

F_1 is a special case since there are no primes smaller than 2 . So the frequency for 2 will be calulated heuristically . Since 2 divides $\frac{1}{2}$ of all integers and there are no primes smaller than 2 we will say that F_1 is $\frac{1}{2}$. 2 is the smallest divisor of $\frac{1}{2}$ of all integers .

Visual Proof



Green ticks are when p_n is the smallest factor of an integer . Grey ticks are for when p_n divides an integer but there are smaller primes which also divide that integer . We can see that the frequency of p_n to divide an integer as the smallest factor, repeats over the interval of its primorial . This is because frequency is measured relative to smaller primes and its primorial is the least common factor $a \cdot p_n$.

Sum of frequencies

The total sum of frequencies S_n of integers in $[1, P_n]$ that are divisible by some prime p_i up to p_n, for which p_i is the smallest factor, is given by:

$$S_n = \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \frac{\phi(P_{i-1})}{P_{i-1} \cdot p_i} = \sum_{i=1}^{n} \frac{\phi(P_{n-1})}{P_n}.$$

S_n is the measure of the likelihood of an integer to share a factor with a prime less than or equal to p_n . The likelihood that an integer is coprime to S_n is $1 - S_n$

Approximating the behavior of F_n for large n

We start from

$$F_n = \frac{\phi(P_{n-1})}{P_n}.$$

$$F_n = \frac{\phi(P_{n-1})}{P_n} = \frac{P_{n-1} \prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right)}{P_{n-1} \cdot p_n}.$$

$$F_n = \frac{\cancel{P_{n-1}} \prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right)}{\cancel{P_{n-1}} \cdot p_n} = \frac{\prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right)}{p_n}.$$

We can use Mertens' third theorem aka Mertens' product formula , to approximate the numerator asymptotically , we have

$$\prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right) \sim \frac{e^{-\gamma}}{\ln p_{n-1}}.$$

where $\gamma \approx .577$ is the Euler–Mascheroni constant. This provides us with an asymptotic approximation for the numerator .
Since $p_{n-1} \approx p_n$ for large n, this gives

$$\prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right) \approx \frac{e^{-\gamma}}{\ln p_n}.$$

To complete the original definition we must factor in the denominator .

$$F_n = \frac{1}{p_n} \prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right) \approx \frac{1}{p_n} \cdot \frac{e^{-\gamma}}{\ln p_n}.$$

Given

$$F_n \approx \frac{e^{-\gamma}}{p_n \ln p_n},$$

and $p_n \sim n \ln n$, we have

$$F_n \approx \frac{e^{-\gamma}}{n \ln n \cdot \ln(n \ln n)}$$

$$F_n \approx \frac{e^{-\gamma}}{n \ln n \cdot (\ln n + \cancel{\ln \ln n}))}$$

For large n F_n will grow as $\ln n$ as opposed to $\ln \ln n$, since $\ln n \gg \ln \ln n$ for large n .

$$F_n \approx \frac{e^{-\gamma}}{n(\ln n)^2}$$

Thus:

$$S_n \sim \sum_{h=1}^{n} \frac{e^{-\gamma}}{h(\ln h)^2}$$

As $n \to \infty$, $F_n \to 0$.

Extremely composite integers and extremely coprime integers

Let P_n be the product of the first n primes $G = \{2, 3, 5, ..., p_n\}$. This means P_n is factorable by all of these primes . Thus P_n is composite with respect to S_n. It can be shown that $P_n \pm 2$ is factorable by 2 , $P_n \pm 3$ is factorable by 3 , $P_n \pm 5$ is factorable by 5 etc . It can be deduced that $P_n \pm 1$ is not factorable by any of the elements in G . $P_n \pm 1$ is coprime to G .

Error Terms

There are primes $p_{n+1}, p_{n+2}, ...$ in the interval $[p_n, P_n + 1]$ whose frequencies are not accounted for and which may be a factor of $P_n \pm 1$, we will call the sum of the frequencies of these primes E_n. Let K be the set of primes accounted for in E_n. If $P_n \pm 1$ is coprime to the primes used to calculate S_n, G and coprime to the primes used to approximate E_n, K then $P_n \pm 1$ is prime .

Quantifying the Error Term E_n

We have:

$$E_n = \sum_{m>n}^{P_n+1} F_m,$$

To estimate E_n, consider the sum:

$$E_n = \sum_{m>n}^{P_n+1} \frac{e^{-\gamma}}{m(\ln m)^2}.$$

For large n, we can approximate sums by integrals. Specifically:

$$\sum_{m>n} \frac{1}{m(\ln m)^2} \approx \int_n^\infty \frac{dx}{x(\ln x)^2}.$$

Simplify and Evaluate the Integral

$$u = \ln x$$

$$\frac{du}{dx} = \frac{d}{dx}(\ln x) = \frac{1}{x} \quad \Rightarrow \quad du = \frac{1}{x}dx \quad \Rightarrow \quad dx = x\,du$$

$$\frac{1}{x(\ln x)^2} = \frac{1}{xu^2}$$

$$I = \int_n^\infty \frac{dx}{xu^2} = \int_{\ln n}^\infty \frac{x\,du}{xu^2} = \int_{\log n}^\infty \frac{du}{u^2}$$

$$\int \frac{du}{u^2} = \frac{1}{u^2} = u^{-2} = \frac{u^{-2+1}}{-2+1} = \frac{u^{-1}}{-1} = -\frac{1}{u} + C$$

$$I = \left[-\frac{1}{u}\right]_{\ln n}^\infty = \left(-\frac{1}{\infty}\right) - \left(-\frac{1}{\ln n}\right) = 0 - \left(-\frac{1}{\ln n}\right) = \frac{1}{\ln n}$$

Thus:

$$\int_n^\infty \frac{dx}{x(\ln x)^2} = \frac{1}{\ln n}.$$

Incorporating the constant $e^{-\gamma}$:

$$E_n \approx e^{-\gamma} \cdot \frac{1}{\ln n} \quad \text{as } n \to \infty.$$

E_n is on the order of $\frac{1}{\ln n}$. If the sum defining E_n is not from n to ∞ but to some finite upper limit M we can write:

$$\int_n^M \frac{dx}{x(\ln x)^2} = \left(-\frac{1}{\ln M}\right) - \left(-\frac{1}{\ln n}\right).$$

If $M = P_n + 1$ is extremely large the dominant term is $1/\ln n$, ensuring it remains on the order of $1/\ln n$ for large n.

$$E_n \approx \frac{e^{-\gamma}}{\ln n}.$$

$$E_n \to 0 \text{ as } n \to \infty$$

Extremely prime integers

Let P_n be the product of the first n primes $G = \{2, 3, 5, ..., p_n\}$. $P_n \pm 1$ is not factorable by any of the elements in G . $P_n \pm 1$ is coprime to G . Let K be the primes accounted for in E_n. As n increases , the likelihood of $P_n \pm 1$ being divisible by K decreases as $n \to \infty$ since as $n \to \infty, E_n \to 0$. It follows that as $n \to \infty, P_n \pm 1$ has an increasing likelihood of being prime . The likelihood that either $P_n \pm 1$ is prime is approximately $(1 - E_n)$. So the likelihood that both are prime is approximately $(1 - E_n)^2$. Therefore, as n increases, the likelihood that $P_n \pm 1$ are prime increases.

Values for different n

$$n = 1, p_1 = 2, F_1 = \frac{1}{2}, S_1 = \frac{1}{2}, G = \{2\}, K = \{3\}$$

$$n = 2, p_2 = 3, F_2 = \frac{1}{6}, S_2 = \frac{1}{2} + \frac{1}{6}, G = \{2, 3\}, K = \{5, 7\}$$

$$n = 3, p_3 = 5, F_3 = \frac{2}{30}, S_2 = \frac{1}{2} + \frac{1}{6} + \frac{2}{30}, G = \{2, 3, 5\}, K = \{7, 11, 13, 17, 19, 23\}$$

$n = 4, p_4 = 7, F_4 = \frac{8}{210}, S_2 = \frac{1}{2} + \frac{1}{6} + \frac{2}{30} + \frac{8}{210}, G = \{2, 3, 5, 7\},$
$K = \{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$

$n = 5, p_5 = 11, F_5 = \frac{48}{2310}, S_2 = \frac{1}{2} + \frac{1}{6} + \frac{2}{30} + \frac{8}{210} + \frac{48}{2310}, G = \{2, 3, 5, 7, 11\},$
$K = \{13, 17, 19, 23, 29, 31, 37, 41, 43, 47, ..., 107, 109, 113\}$

Iterative recursive running total for F_n as a square free integer

Totient Function for Square-Free Integers The Euler totient function $\phi(a)$ counts the number of positive integers less than a that are coprime to a. For a square-free integer a, which has no repeated prime factors, for a

$$a = p_1 p_2 \cdots p_b,$$

where $p_1, p_2, \ldots, p_b$ are distinct prime factors of a. Then,

$$\phi(a) = (p_1 - 1)(p_2 - 1) \cdots (p_b - 1).$$

This formula represents the count of integers $x < a$ that are coprime to a.

Iterative F_n

F_n is defined for the F_n nth prime as

$$F_n = \frac{N_{n-1} \cdot \phi(p_{n-1})}{D_{n-1} \cdot p_n}$$

$$F_n = \frac{N_{n-1} \cdot (p_{n-1} - 1)}{D_{n-1} \cdot p_n}$$

$$n = 1, p_1 = 2, F_1 = \frac{1}{2} = \frac{N_1}{D_1}.$$

$$n = 2, p_2 = 3, p_1 = 2, F_1 = \frac{1}{2}, F_2 = \frac{1 \cdot (2-1)}{2 \cdot 3} = \frac{1}{6}.$$

$$n = 3, p_3 = 5, p_2 = 3, F_2 = \frac{1}{6}, F_3 = \frac{1 \cdot (3-1)}{6 \cdot 5} = \frac{2}{30}.$$

$$n = 4, p_4 = 7, p_3 = 5, F_3 = \frac{2}{30}, F_4 = \frac{2 \cdot (5-1)}{30 \cdot 7} = \frac{8}{210}.$$