

# How frequently is a prime the smallest divisor of all integers

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## Abstract

In this paper I will cover how to calculate how frequently a prime is the smallest divisor of all positive integers . I will use that information to approximate how likely specific integers near some value  $P_n$  are to be prime . Finally, I will construct a proof utilizing these approximations.

## Objective

The motivation for these results is to have a counting method that includes all positive integers only once. The chosen approach requires counting relative to the primes which leads to unique factorization. Given the fundamental theorem of arithmetic we know that all positive integers can be written as a unique product of primes. Each positive integer will have only one prime as its smallest divisor. This will be the premise for unique factorization. We will calculate how many times some prime divides all positive integers as the smallest factor. We will express this count in terms of  $n$  using an approximation from Mertens third theorem. We will give its count relative to some interval to determine a relative frequency. Finally, we will analyze sums of the first  $n$  terms and sums of the trailing terms, to determine growth and decay behavior of these prime divisibility frequencies.

$p_n$  : Let  $p_n$  be the  $n$ -th prime.  $n = 1, p_1 = 2$

**Primorial** : The primorial of  $p_n$ , denoted  $P_n$ , is the product of the first  $n$  primes :

$$P_n = \prod_{p \leq p_n} p.$$

$$P_4 = \prod_{p \leq 7} p = 7 \cdot 5 \cdot 3 \cdot 2 = 210.$$

**Totient Function** : The Euler totient function  $\phi(a)$  is calculated using the prime factorization of  $a$ . If  $a$  has the prime factorization:

$$a = p_1^{e_1} p_2^{e_2} \cdots p_b^{e_b},$$

then the totient function is given by:

$$\phi(a) = a \prod_{i=1}^b \left(1 - \frac{1}{p_i}\right),$$

where  $p_1, p_2, \dots, p_b$  are the distinct prime factors of  $a$ . This formula represents the count of integers  $x < a$  that do not share any factors with  $a$ .

$$\phi(6) = \phi(2 \cdot 3) = 6 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) = 6 \cdot \frac{1}{2} \cdot \frac{2}{3} = 2$$

Let  $e$  be Eulers constant  $e \approx 2.718$

Let  $\gamma$  be the Euler-Mascheroni constant  $\gamma \approx 0.577$

Let  $F_n$  be the likelihood that some prime  $p_n$  is the smallest divisor of all integers.  $p_1 = 2$  divides  $\frac{1}{2}$  of all integers. There are no smaller primes than 2, so there are no smaller primes to exclude, so any time 2 divides some integer it will be the smallest prime dividing that integer. 2 is the smallest divisor of  $\frac{1}{2}$  of all integers.  $F_1 = \frac{1}{2}$

## Main Results

**1. Prime Divisibility Theorem** Assume  $p_n$ , then  $F_n = \frac{\phi(P_{n-1})}{P_n}$  yields the average likelihood that  $p_n$  is the smallest divisor of all integers.

**2. Approximation** Assume  $F_n = \frac{\phi(P_{n-1})}{P_n}$ , then  $F_n \approx \frac{e^{-\gamma}}{n(\ln n)^2}$

**4. Divisibility Decay Theorem** Assume  $E_n = \sum_{m>n} \frac{e^{-\gamma}}{m(\ln m)^2}$  is the tail sum of  $F_n$ , then  $E_n > E_{n+1}$ .

**Section 6.** Assume we have  $P_n \pm 1, P_{n+1} \pm 1$ , then the likelihood that  $P_n \pm 1$  is prime, is less than the likelihood that  $P_{n+1} \pm 1$  is prime.

## 1 Prime Divisibility Theorem

Assume  $p_n$ , then  $\frac{\phi(P_{n-1})}{P_n}$  yields the average likelihood that  $p_n$  is the smallest divisor of all integers.

### 1.1 Determine the number of times a Prime is the Smallest Factor Over a Fixed Interval

$$P_n = a \cdot p_n$$

Consider the product  $a \cdot p_n$ , where  $a = p_1 \cdot p_2 \cdot \dots \cdot p_{n-1}$  is the product of the first  $n-1$  primes, and  $p_n$  is the  $n$ -th prime. We wish to determine the number of times  $p_n$  is the smallest factor of integers in the interval  $[1, a \cdot p_n]$ .

Let  $n = 3$  then  $p_3 = 5$ .  $a$  is the product of all primes less than  $p_3 = 5$ .

$$a = 2 \cdot 3 = 6$$

$$P_n = a \cdot p_n = 6 \cdot 5 = 30$$

## 1.2 Integers Factorable by $p_n$

All integers in  $[1, a \cdot p_n]$  that are divisible by  $p_n$  take the form:

$$p_n \cdot 1, p_n \cdot 2, p_n \cdot 3, \dots, p_n \cdot a.$$

This set contains  $a$  elements, as  $a$  represents the range of possible values for the multiplier  $m$ .

## 1.3 Integers divisible by no smaller primes than $p_n$

**Coprime (Relatively Prime) Integers:** Two positive integers  $a$  and  $b$  are coprime if their greatest common divisor is 1:

$$\gcd(a, b) = 1, \gcd(7, 3) = 1.$$

For  $p_n$  to be the smallest factor of an integer  $p_n \cdot m$ ,  $m$  must not share any factors with  $a$ , if  $m$  shares a factor with  $a$ , then  $m$  is divisible by a prime less than  $p_n$ , since  $a$  is composed of all the primes less than  $p_n$ . The number of integers  $j$  in  $[1, a]$  that are coprime to  $a$  is given by  $\phi(a)$ . Thus, there are  $j = \phi(a)$  integers in  $[1, a \cdot p_n]$  which take the form  $p_n \cdot m$ , where  $p_n$  is the smallest factor. The totient function  $\phi(a)$  allows us to determine the exact number of times  $p_n$  is the smallest factor of integers in the interval  $[1, a \cdot p_n]$ . This is precisely  $\phi(a)$ , as it counts the values of  $m$  coprime to  $a$ .

Use the totient function to exclude any  $m$  divisible by a factor of  $a$ , when  $a = 6$  and  $p_n = 5$ .

$$5 \cdot 1, \cancel{5 \cdot 2}, \cancel{5 \cdot 3}, \cancel{5 \cdot 4}, 5 \cdot 5, \cancel{5 \cdot 6},$$

$$\phi(6) = 6 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) = 6 \cdot \frac{1}{2} \cdot \frac{2}{3} = 2$$

5 is the smallest divisor 2 times in the interval from 1 to 30.

## 1.4 Frequency of $p_n$ as the Smallest Factor

To compute the relative frequency in  $[1, a \cdot p_n]$  where  $p_n$  is the smallest factor, we will use the ratio of the number of times that  $p_n$  is the smallest factor, in the interval of its primorial  $P_n$ :

$$\text{Frequency} = F_n = \frac{\phi(a)}{a \cdot p_n} = \frac{\phi(P_{n-1})}{P_n}$$

For  $p_3 = 5$  and  $a = 6$

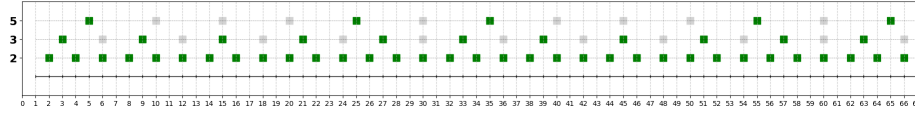
$$F_3 = \frac{\phi(a)}{a \cdot p_n} = \frac{\phi(6)}{6 \cdot 5} = \frac{\phi(6)}{30} = \frac{2}{30}$$

We can say that  $p_3 = 5$  is the smallest divisor of  $\frac{2}{30}$  of all integers.

### 1.5 Case for $F_1$ , $p_1 = 2$

$F_1$  is a special case since there are no primes smaller than 2 . So the frequency for 2 will be calculated heuristically . Since 2 divides  $\frac{1}{2}$  of all integers and there are no primes smaller than 2 we will say that  $F_1$  is  $\frac{1}{2}$  . 2 is the smallest divisor of  $\frac{1}{2}$  of all integers .

### 1.6 Visual Proof that the frequency repeats



Green ticks are when  $p_n$  is the smallest factor of an integer . Grey ticks are for when  $p_n$  divides an integer but there are smaller primes which also divide that integer . We can see that the frequency of  $p_n$  to divide an integer as the smallest factor, repeats over the interval of its primorial . This is because frequency is measured relative to smaller primes and its primorial is the least common factor  $a \cdot p_n$  .

### 1.7 Conclusion

Assume  $p_n$  , then  $F_n = \frac{\phi(P_{n-1})}{P_n}$ . This is useful in problem contexts where divisibility with respect to certain primes are necessary . Existing methods only utilize simple heuristics to determine the frequency that a prime is a factor of all integers ie:  $\frac{1}{p_n}$  , which is limited in its usefulness depending on the problem setup.

## 2 Approximating the behavior of $F_n$ asymptotically for large $n$

We start from

$$F_n = \frac{\phi(P_{n-1})}{P_n}.$$

$$F_n = \frac{\phi(P_{n-1})}{P_n} = \frac{P_{n-1} \prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right)}{P_{n-1} \cdot p_n}.$$

$$F_n = \frac{\cancel{P_{n-1}} \prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right)}{\cancel{P_{n-1}} \cdot p_n} = \frac{\prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right)}{p_n}.$$

We can use Mertens' third theorem aka Mertens' product formula , to approximate the numerator asymptotically , we have

$$\prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right) \sim \frac{e^{-\gamma}}{\ln p_{n-1}}.$$

where  $\gamma \approx .577$  is the Euler–Mascheroni constant. This provides us with an asymptotic approximation for the numerator .

Since  $p_{n-1} \approx p_n$  for large  $n$ , this gives

$$\prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right) \approx \frac{e^{-\gamma}}{\ln p_n}.$$

To complete the original definition we must factor in the denominator .

$$F_n = \frac{1}{p_n} \prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right) \approx \frac{1}{p_n} \cdot \frac{e^{-\gamma}}{\ln p_n}.$$

Given

$$F_n \approx \frac{e^{-\gamma}}{p_n \ln p_n},$$

and  $p_n \sim n \ln n$ , we have

$$F_n \approx \frac{e^{-\gamma}}{n \ln n \cdot \ln(n \ln n)}$$

$$F_n \approx \frac{e^{-\gamma}}{n \ln n \cdot (\ln n + \ln \ln n)}$$

For large  $n$  ,  $F_n$  will grow as  $\ln n$  as opposed to  $\ln \ln n$  , since  $\ln n \gg \ln \ln n$  for large  $n$  .

$$F_n \approx \frac{e^{-\gamma}}{n(\ln n)^2}$$

$$\begin{aligned} F_{11} &\approx \frac{e^{-\gamma}}{11 (\ln 11)^2} \\ &\approx \frac{0.561459}{63.265169} \\ &\approx 0.008878 \end{aligned}$$

## 2.1 Conclusion

Assume  $F_n = \frac{\phi(P_{n-1})}{P_n}$  , then  $F_n \approx \frac{e^{-\gamma}}{n(\ln n)^2}$ . Above I have provided a means for approximating  $F_n = \frac{\phi(P_{n-1})}{P_n}$  asymptotically. This is useful so that sums of  $F_n$ , may be approximated in terms of some large  $n|x$  by integration.

## 3 Sum of $F_n$

To determine the likelihood that an integer  $q$  is composite , we must consider the likelihood that  $q$  is divisible by any smaller prime. This can be done using a sum with respect to the frequencies of all smaller primes . The total sum of

frequencies  $S_n$  of integers in  $[1, P_n]$  that are divisible by some prime  $p_i$  up to  $p_n$ , for which  $p_i$  is the smallest factor, is given by:

$$S_n = \sum_{i=1}^n \frac{\phi(P_{i-1})}{P_{i-1} \cdot p_i} = \sum_{i=1}^n \frac{\phi(P_{n-1})}{P_n} = \sum_{i=1}^n F_i \approx \sum_{i=1}^n \frac{e^{-\gamma}}{i(\ln i)^2}.$$

$S_n$  is the measure of the likelihood of an integer  $q$  to share a factor with a prime less than or equal to  $p_n$ . Let  $G = \{p_i \leq p_n | p_i\}$ , then the likelihood that an integer is divisible by  $G$  is  $S_n$ .  $S_n$  strictly increases for subsequent  $n$ , because all terms are positive.

### 3.1 Divisibility by primes not in $G$

Let  $K = \{p_i > p_n | p_i\}$ . If  $q$  is not divisible by an element in  $G$ , then it must be divisible by an element in  $K$ , since  $G$  and  $K$  include all primes, and via the fundamental theorem of arithmetic at least one of these primes must factor  $q$ . The likelihood that an integer is not factorable by  $G$  is equal to the tail sum of  $S_n$ , we will call this tail/end sum  $E_n$ .  $E_n$  is the likelihood that an element of  $K$  divides some  $q$ . In practice  $K$  is not the entire tail, however it is significantly many terms.

## 4 Divisibility Decay Theorem

Assume we have  $E_n, E_{n+1}$ , then  $E_n > E_{n+1}$ .

### 4.1 Quantifying the Tail Sum of $S_n$ as $E_n$

We have:

$$E_n = \sum_{m>n} F_m,$$

To estimate  $E_n$ , consider the sum:

$$E_n = \sum_{m>n} \frac{e^{-\gamma}}{m(\ln m)^2}.$$

For large  $n$ , we can approximate sums by integrals. Specifically:

$$\sum_{m>n} \frac{1}{m(\ln m)^2} \approx \int_n^\infty \frac{dx}{x(\ln x)^2}.$$

### 4.2 Simplify and Evaluate the Integral

$$u = \ln x$$

$$\frac{du}{dx} = \frac{d}{dx}(\ln x) = \frac{1}{x} \Rightarrow du = \frac{1}{x} dx \Rightarrow dx = x du$$

$$\begin{aligned}
\frac{1}{x(\ln x)^2} &= \frac{1}{xu^2} \\
I &= \int_n^\infty \frac{dx}{xu^2} = \int_{\ln n}^\infty \frac{x du}{xu^2} = \int_{\log n}^\infty \frac{du}{u^2} \\
\int \frac{du}{u^2} &= \frac{1}{u^2} = u^{-2} = \frac{u^{-2+1}}{-2+1} = \frac{u^{-1}}{-1} = -\frac{1}{u} + C \\
I &= \left[ -\frac{1}{u} \right]_{\ln n}^\infty = \left( -\frac{1}{\infty} \right) - \left( -\frac{1}{\ln n} \right) = 0 - \left( -\frac{1}{\ln n} \right) = \frac{1}{\ln n}
\end{aligned}$$

Thus:

$$\int_n^\infty \frac{dx}{x(\ln x)^2} = \frac{1}{\ln n}.$$

Incorporating the constant  $e^{-\gamma}$ :

$$E_n \approx e^{-\gamma} \cdot \frac{1}{\ln n} \quad \text{as } n \rightarrow \infty.$$

$E_n$  is on the order of  $\frac{1}{\ln n}$ . If the sum defining  $E_n$  is not from  $n$  to  $\infty$  but to some finite upper limit  $M$  we can write:

$$\int_n^M \frac{dx}{x(\ln x)^2} = \left( -\frac{1}{\ln M} \right) - \left( -\frac{1}{\ln n} \right).$$

If  $M$  is extremely large the dominant term is  $1/\ln n$ , ensuring it remains on the order of  $1/\ln n$  for large  $n$ . As  $n \rightarrow \infty$ ,  $E_n \rightarrow 0$ .

$$\begin{aligned}
E_n &\approx \frac{e^{-\gamma}}{\ln n} \\
E_{15} &\approx \frac{e^{-\gamma}}{\ln 15} \\
&\approx \frac{0.561459}{2.708050} \\
&\approx 0.2073 \\
E_{16} &\approx \frac{e^{-\gamma}}{\ln 16} \\
&\approx \frac{0.561459}{2.772589} \\
&\approx 0.2026 \\
E_{15} &> E_{16}
\end{aligned}$$

### 4.3 Conclusion

If we have  $E_n, E_{n+1}$ , then  $E_n > E_{n+1}$ . This is useful because it allows us to see that if some  $q$  is not divisible by  $G$  then the likelihood that it is divisible by  $K$  goes to 0, for large  $n$ .

## 5 Adjacent Coprime Divisibility

Assume  $P_n$  is the the product of the first  $n$  primes, then  $P_n \pm 1$  is only divisible by  $K$ .

Let  $P_n$  be the product of the first  $n$  primes  $G = \{2, 3, 5, \dots, p_n\}$ . This means  $P_n$  is factorable by all elements in  $G$ . It can be shown that  $P_n \pm 2$  is factorable by 2,  $P_n \pm 3$  is factorable by 3 ...  $P_n \pm p_n$  is factorable by  $p_n$ . It can be deduced that  $P_n \pm 1$  is not factorable by any of the elements in  $G$ .  $P_n \pm 1$  is coprime to  $G$ . There are primes  $p_{n+1}, p_{n+2}, \dots$  in the interval  $[p_n, P_n + 1]$ , which may be a factor of  $P_n \pm 1$ . This is defined as  $E_n = \sum_n^M F_m$ , where  $M = P_n + 1$ . Let  $K = \{P_n + 1 > p_m > p_n | p_m\}$  which is also the set of primes accounted for in  $E_n$ .

Assume  $P_n$  is the the product of the first  $n$  primes, then  $P_n \pm 1$  is only divisible by  $K$ . This is useful because it constrains the likelihood that  $P_n \pm 1$  is prime or composite, relative to only a single set of primes  $K$ , whose behavior of  $F_n$  and  $E_n$  for large  $n$  and  $M$  is already known.

If  $P_n \pm 1$  is coprime to  $G$  and coprime to  $K$  then  $P_n \pm 1$  is prime.

## 6 Quantifying the likelihood that either $P_n - 1$ or $P_n + 1$ are prime for subsequent $n$

Assume we have  $P_n \pm 1, P_{n+1} \pm 1$ , then the likelihood that  $P_n \pm 1$  is prime, is less than the likelihood that  $P_{n+1} \pm 1$  is prime.

As  $n$  increases, the likelihood of  $P_n \pm 1$  being divisible by  $K$  decreases via the Divisibility Decay Theorem. Given that  $E_n$  is the likelihood of  $K$  to divide some positive integer and  $E_n > E_{n+1}$ , the likelihood that  $K$  is composite to  $P_n \pm 1$  decreases for large  $n$ . The likelihood that either  $P_n + 1$  or  $P_n - 1$  is composite is approximately  $E_n$  for each, so the combined likelihood that either is composite is at most  $\approx E_n + E_n = 2 \cdot E_n$ . Therefore, as  $n$  increases, the likelihood that either  $P_n \pm 1$  are composite decreases as  $2 \cdot E_n$ . It follows that as  $n \rightarrow \infty$ , the likelihood that both  $P_n - 1$  and  $P_n + 1$  are both not composite is at least  $(1 - 2 \cdot E_n)$ , which increases for large  $n$ . Thus, the likelihood that  $P_n \pm 1$  are both prime increases for larger  $n$  as  $(1 - 2 \cdot E_n)$ .

### 6.1 Conclusion

Assume we have  $P_n \pm 1, P_{n+1} \pm 1$ , then the likelihood that  $P_n \pm 1$  is prime, is less than the likelihood that  $P_{n+1} \pm 1$  is prime. This is useful in locating large prime numbers with a decreased error rate.



## 7 Infinite Twin Primes Proof

Twin primes are two primes separated by a distance of 2.

Ex: [17,19].  $19 - 17 = 2$ . 19 and 17 are twin primes.

Let  $A$  denote the infinite ordered set of all prime numbers. We construct a subset  $B \subseteq A$  consisting of  $P_n$  where  $P_n \pm 1$  are both prime. The construction proceeds iteratively as follows:

1. For each positive integer  $n$ , define the primorial  $P_n$  as the product of the first  $n$  primes:

$$P_n = \prod_{i=1}^n p_i,$$

2. Define  $B$  as the set of primorials  $P_n$  for which both  $P_n - 1$  and  $P_n + 1$  are prime. Formally,

$$B = \left\{ P_n \in A \mid P_n - 1 \text{ and } P_n + 1 \text{ are both prime} \right\}.$$

3. The likelihood that  $P_n \pm 1$  are both prime increases as  $n$  increases, by section 6 .
4. The set  $B$  is expected to contain more elements as  $n$  grows.
5. Since  $B$  is constructed by iteratively including primorials where  $P_n \pm 1$  are both prime, and the likelihood of inclusion increases with  $n$ , we infer that  $B$  is an infinite subset of  $A$ .
6. There are an infinite number of twin primes.