Twin Prime Likelihood

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Abstract

In this paper I will cover how to calculate how frequently a prime is the smallest factor over an interval. I will use this to estimate the likelihood that an integer is prime . I will then determine the likelihood that the product of the first n primes is a twin prime center. I will show that this likelihood increases as more primes are included in the product , and define specific growth and decay rates for approximating this increasing likelihood .

Key Definitions

1. **Coprime (Relatively Prime) Integers:** Two integers a and b are coprime if their greatest common divisor is 1:

$$\gcd(a, b) = 1.$$

2. **Primorial :** The primorial of p_n, denoted P_n, is the product of the first n primes :

$$P_n = \prod_{p \le p_n} p.$$

3. **Totient Function :** The Euler totient function $\phi(a)$ is calculated using the prime factorization of a. If a has the prime factorization:

$$a = p_1^{e_1} p_2^{e_2} \cdots p_b^{e_b},$$

then the totient function is given by:

$$\phi(a) = a \prod_{i=1}^{b} \left(1 - \frac{1}{p_i}\right),$$

where $p_1, p_2, \ldots, p_k$ are the distinct prime factors of a. This formula represents the count of integers x that do not share any factors with a .

Using the Totient Function to Determine the Frequency of a Prime as the Smallest Factor Over a Fixed Interval

$P_n = a \cdot p_n$

Consider the product $a \cdot p_n$, where $a = p_1 \cdot p_2 \cdot \dots \cdot p_{n-1}$ is the product of the first $n-1$ primes , and p_n is the n-th prime . We wish to determine the number of times p_n is the smallest factor of integers in the interval $[1, a \cdot p_n]$.

Integers Factorable by p_n

All integers in $[1, a \cdot p_n]$ that are divisible by p_n take the form:

$$p_n \cdot 1,\, p_n \cdot 2,\, p_n \cdot 3,\, \dots,\, p_n \cdot a.$$

This set contains a elements, as a represents the range of possible values for the multiplier m.

Integers whose smallest factor is p_n

For p_n to be the smallest factor of an integer $p_n \cdot m$, m must not share any factors with a, ensuring that p_n is the smallest prime dividing $p_n \cdot m$. The number of integers x in $[1, a]$ that are coprime to a is given by $\phi(a)$. Thus, there are $x = \phi(a)$ integers in $[1, a \cdot p_n]$ for which $p_n \cdot m$ is the smallest factor .

Frequency of p_n as the Smallest Factor

To compute the relative frequency in $[1, a \cdot p_n]$ where p_n is the smallest factor:

$$\text{Frequency} = F_n = \frac{\phi(a)}{a \cdot p_n} = \frac{\phi(P_{n-1})}{P_n}$$

The totient function $\phi(a)$ allows us to determine the exact number of times p_n is the smallest factor of numbers in the interval $[1, a \cdot p_n]$. This is precisely $\phi(a)$, as it counts the values of m coprime to a, which ensures p_n is the smallest prime dividing $p_n \cdot m$. The relative frequency of these numbers decreases as p_n grows .

Sum of frequencies

The total sum of frequencies S_n of numbers in $[1, P_n]$ that are divisible by some prime p_i up to p_n, for which p_i is the smallest factor, is given by:

$$S_n = \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \frac{\phi(P_{i-1})}{P_{i-1} \cdot p_i} = \sum_{i=1}^{n} \frac{\phi(P_{n-1})}{P_n}.$$

S_n is the measure of the likelihood of an integer to share a factor with a prime less than or equal to p_n .

The likelihood that a number is coprime to S_n is $1 - S_n$

Approximating the decay rate of F_n

We start from

$$F_n = \frac{\phi(P_{n-1})}{P_n}.$$

$$F_n = \frac{\phi(P_{n-1})}{P_n} = \frac{P_{n-1}\prod_{i=1}^{n-1}\left(1-\frac{1}{p_i}\right)}{P_{n-1}\cdot p_n}.$$

$$F_n = \frac{\cancel{P_{n-1}}\prod_{i=1}^{n-1}\left(1-\frac{1}{p_i}\right)}{\cancel{P_{n-1}}\cdot p_n} = \frac{\prod_{i=1}^{n-1}\left(1-\frac{1}{p_i}\right)}{p_n}.$$

We can use Mertens' third theorem aka Mertens' product formula , to approximate the numerator asymptotically , we have

$$\prod_{i=1}^{n-1}\left(1-\frac{1}{p_i}\right) \sim \frac{e^{-\gamma}}{\log p_{n-1}}.$$

where γ is the Euler–Mascheroni constant. This provides us with an asymptotic approximation for the numerator .

Since $p_{n-1} \approx p_n$ for large n, this gives

$$\prod_{i=1}^{n-1}\left(1-\frac{1}{p_i}\right) \approx \frac{e^{-\gamma}}{\log p_n}.$$

To complete the original definition we must factor in the denominator .

$$F_n = \frac{1}{p_n}\prod_{i=1}^{n-1}\left(1-\frac{1}{p_i}\right) \approx \frac{1}{p_n}\cdot\frac{e^{-\gamma}}{\log p_n}.$$

Given

$$F_n \approx \frac{e^{-\gamma}}{p_n \log p_n},$$

and $p_n \sim n\log n$, we have

$$F_n \approx \frac{e^{-\gamma}}{n\log n\cdot\log(n\log n)}$$

$$F_n \approx \frac{e^{-\gamma}}{n\log n\cdot(\log n + \cancel{\log\log n}))}$$

For large n F_n will grow as $\log n$ as opposed to $\log\log n$, since $\log n \gg \log\log n$ for large n .

$$F_n \approx \frac{e^{-\gamma}}{n(\log n)^2}$$

Thus:

$$S_n \sim \sum_{h=1}^{n} \frac{1}{h(\log h)^2}$$

As $n \to \infty$, $F_n \to 0$.

Coprimality to S_n

Let P_n be the product of the first n primes $G = \{2, 3, 5, ..., p_n\}$. This means P_n is factorable by all of these primes . It can be shown that $P_n \pm 2$ is factorable by 2 , $P_n \pm 3$ is factorable by 3 etc . It can be deduced that $P_n \pm 1$ is not factorable by any of the elements in G . $P_n \pm 1$ is coprime to G .

Error Terms

There are primes $p_{n+1}, p_{n+2}, ...$ in the interval $[p_n, P_n]$ whose frequencies are not accounted for and which may be a factor of $P_n \pm 1$, we will call the sum of the frequencies of these terms E_n. If $P_n \pm 1$ is coprime to the primes used to calculate S_n and coprime to the primes used to approximate E_n then $P_n \pm 1$ is prime .

Quantifying the Error Term E_n

We have:

$$E_n = \sum_{m>n}^{\sqrt{P_n+1}} F_m,$$

To estimate E_n, consider the sum:

$$E_n = \sum_{m>n} \frac{e^{-\gamma}}{m(\log m)^2}.$$

For large n, we can approximate sums by integrals. Specifically:

$$\sum_{m>n} \frac{1}{m(\log m)^2} \approx \int_n^{\infty} \frac{dx}{x(\log x)^2}.$$

Simplify and Evaluate the Integral

$$u = \log x$$

$$\frac{du}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x} \quad \Rightarrow \quad du = \frac{1}{x}dx \quad \Rightarrow \quad dx = x\,du$$

$$\frac{1}{x(\log x)^2} = \frac{1}{xu^2}$$

$$I = \int_n^\infty \frac{dx}{xu^2} = \int_{\log n}^\infty \frac{x\,du}{xu^2} = \int_{\log n}^\infty \frac{du}{u^2}$$

$$\int \frac{du}{u^2} = \frac{1}{u^2} = u^{-2} = \frac{u^{-2+1}}{-2+1} = \frac{u^{-1}}{-1} = -\frac{1}{u} + C$$

$$I = \left[-\frac{1}{u}\right]_{\log n}^\infty = \left(-\frac{1}{\infty}\right) - \left(-\frac{1}{\log n}\right) = 0 - \left(-\frac{1}{\log n}\right) = \frac{1}{\log n}$$

Thus:

$$\int_n^\infty \frac{dx}{x(\log x)^2} = \frac{1}{\log n}.$$

Incorporating the constant $e^{-\gamma}$:

$$E_n \approx e^{-\gamma} \cdot \frac{1}{\log n} \quad \text{as } n \to \infty.$$

E_n is on the order of $\frac{1}{\log n}$.

If the sum defining E_n is not from n to ∞ but to some finite upper limit M we can write:

$$\int_n^M \frac{dx}{x(\log x)^2} = \frac{1}{\log n} - \frac{1}{\log M}.$$

If M is extremely large the dominant term is $1/\log n$. , ensuring it remains on the order of $1/\log n$ for large n.

$$E_n \approx \frac{e^{-\gamma}}{\log n}.$$

Approximating primality of $P_n \pm 1$

If $P_n \pm 1$ is coprime to the primes used to calculate E_n then it is prime . The likelihood that either $P_n \pm 1$ are coprime to the primes of E_n is $1 - E_n$.

Evaluating Individual Cases

$E_n = F_{m_1} + F_{m_2} + F_{m_3} .....F_{m_m}$, all likelihoods contributed by individual F_m are with respect to primes p_m . Where $p_n \leq p_m \leq \sqrt{P_n + 1}$, which are all larger than 2 , since 2 is always in G . Since $(P_n + 1) - (P_n - 1) = 2$, we can determine that if any prime p_m is a factor of $P_n - 1$ then it cannot be a factor of $P_n + 1$.

Dependent Case

Assume $P_n - 1$ is composite and factorable by one or more primes p_{m_1}, p_{m_2} . Let us call the sum of the frequencies of these primes ϵ , $\epsilon = F_{m_1} + F_{m_2}$. If $P_n - 1$ has a factorable likelihood of E_n , and $P_n + 1$ cannot be factorable by p_{m_1}, p_{m_2} , it stands that the factorable likelihood of $P_n + 1$ is $E_n - \epsilon$.

Independent Case

Assume $P_n - 1$ is prime . Therefore it has no relation to any primes p_m . The error for $P_n - 1$ is E_n . Since there are no p_m, F_m , $\epsilon = 0$. The error of $P_n + 1$ is $E_n - \epsilon = E_n$ since $\epsilon = 0$.

The likelihood that $P_n - 1$ is prime is approximately $(1 - E_n)$. The likelihood that $P_n - 1$ is prime goes to 1 as E_n goes to 0 . Therefore, as n increases, the likelihood that $P_n - 1$ is prime increases.

Proof

Let A denote the infinite ordered set of all prime numbers. We construct a subset $B \subseteq A$ consisting of primorials that serve as twin prime centers. The construction proceeds iteratively as follows:

1. For each positive integer n, define the primorial P_n as the product of the first n primes:
$$P_n = \prod_{i=1}^{n} p_i,$$
where p_i is the i-th prime number.

2. Define B as the set of primorials P_n for which both $P_n - 1$ and $P_n + 1$ are prime. Formally,
$$B = \left\{ P_n \in A \;\middle|\; P_n - 1 \text{ and } P_n + 1 \text{ are both prime} \right\}.$$

3. Analyze the likelihood that P_n is a twin prime center as n increases.

4. Define an error term E_n representing the sum of frequencies of primes beyond p_n that may divide $P_n \pm 1$:
$$E_n = \sum_{m>n}^{\sqrt{P_n+1}} F_m \approx \frac{e^{-\gamma}}{\log n}.$$
This approximation indicates that E_n diminishes as n increases.

5. Consider the likelihood that $P_n - 1$ is coprime to the set of primes accounted for in E_n. This is approximately $1 - E_n$. As $E_n \to 0$ when $n \to \infty$, this likelihood approaches 1. Thus $P_n - 1$ is increasingly likely to be prime .

6. We now have that $P_n - 1$ is increasingly likely to be prime for large n. When $P_n - 1$ is prime we can independently calculate the likelihood that $P_n + 1$ is also prime as $(1 - E_n)$. The probability that both are coprime to E_n is approximately
$$(1 - E_n)^2 .$$

7. Therefore, as n increases, the likelihood that P_n serves as a twin prime center increases. Consequently, the set B is expected to contain more elements as n grows.

8. Since B is constructed by iteratively including primorials that satisfy the twin prime center condition, and the likelihood of inclusion increases with n, we infer that B is an infinite subset of A.

9. Finally, assuming that there are infinitely many primes and that the likelihood of primorials being twin prime centers tends to 1, it follows that there are infinitely many twin prime centers.