How frequently is a prime the smallest divisor of all integers

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Abstract

In this paper I will cover how to calculate how frequently a prime is the smallest divisor of all positive integers . I will use that information to approximate how likely specific integers near some value C_n are to be prime . Finally, I will construct a proof utilizing these approximations.

Objective

The motivation for these results is to have a counting method that includes all positive integers only once. The chosen approach requires counting relative to the primes which leads to unique factorization. Given the fundamental theorem of arithmetic we know that all positive integers can be written as a unique product of primes. Each positive integer will have only one prime as its smallest divisor. This will be the premise for unique factorization. We will calculate how many times some prime divides all positive integers as the smallest factor. We will express this count in terms of n using an approximation from Mertens third theorem. We will give its count relative to some interval to determine a relative frequency. Finally, we will analyze sums of the first n terms and sums of the trailing terms, to determine growth and decay behavior of these prime frequencies.

Main Results

Let γ, e be constants. Let P_n be a function of p_n , and let ϕ be some function .
Prime Frequency Theorem: Assume p_n is the n-th prime , then $F_n = \frac{\phi(P_{n-1})}{P_n}$ yields the average likelihood that p_n is the smallest divisor of all integers.
Result 2: Assume $F_n = \frac{\phi(P_{n-1})}{P_n}$, then $F_n \approx \frac{e^{-\gamma}}{n(\ln n)^2}$
Divisibility Decay Theorem Assume $E_n = \sum_{m>n} \frac{e^{-\gamma}}{m(\ln m)^2}$ is the tail sum of F_n, then $E_n > E_{n+1}$.
Result 4: Assume we have $P_n \pm 1, P_{n+1} \pm 1$, then the likelihood that $P_n \pm 1$ is prime, is less than the likelihood that $P_{n+1} \pm 1$ is prime.

1 Prime Frequency Theorem

Assume p_n is the n-th prime , then $\frac{\phi(P_{n-1})}{P_n}$ yields the average likelihood that p_n is the smallest divisor of all integers .

Determine the number of times a Prime is the Smallest Factor Over a Fixed Interval

Primorial : The primorial of p_n, denoted P_n, is the product of the first n primes :

$$P_n = \prod_{p \leq p_n} p.$$

$$P_4 = \prod_{p \leq 7} p = 7 \cdot 5 \cdot 3 \cdot 2 = 210.$$

$P_n = a \cdot p_n$

Consider the product $a \cdot p_n$, where $a = p_1 \cdot p_2 \cdot \cdots \cdot p_{n-1}$ is the product of the first $n-1$ primes , and p_n is the n-th prime . We wish to determine the number of times p_n is the smallest factor of integers in the interval $[1, a \cdot p_n]$.
Let $n = 3$ then $p_3 = 5$. a is the product of all primes less than $p_3 = 5$.

$$a = 2 \cdot 3 = 6$$

$$P_n = a \cdot p_n = 6 \cdot 5 = 30$$

Integers Factorable by p_n

All integers in $[1, a \cdot p_n]$ that are divisible by p_n take the form:

$$p_n \cdot 1,\ p_n \cdot 2,\ p_n \cdot 3,\ \ldots,\ p_n \cdot a.$$

This set contains a elements, as a represents the range of possible values for the multiplier m.

Integers divisible by no smaller primes than p_n

Coprime (Relatively Prime) Integers: Two positive integers a and b are coprime if their greatest common divisor is 1:

$$\gcd(a, b) = 1, \gcd(7, 3) = 1.$$

Totient Function : The Euler totient function $\phi(a)$ is calculated using the prime factorization of a. If a has the prime factorization:

$$a = p_1^{e_1} p_2^{e_2} \cdots p_b^{e_b},$$

then the totient function is given by:

$$\phi(a) = a\prod_{i=1}^{b}\left(1-\frac{1}{p_i}\right),$$

where $p_1, p_2, \ldots, p_b$ are the distinct prime factors of a. This formula represents the count of integers $x < a$ that do not share any factors with a .

$$\phi(6) = \phi(2\cdot 3) = 6\cdot(1-\frac{1}{2})\cdot(1-\frac{1}{3}) = 6\cdot\frac{1}{2}\cdot\frac{2}{3} = 2$$

For p_n to be the smallest factor of an integer $p_n \cdot m$, m must not share any factors with a, if m shares a factor with a , then m is divisible by a prime less than p_n , since a is composed of all the primes less than p_n. The number of integers j in $[1, a]$ that are coprime to a is given by $\phi(a)$. Thus, there are $j = \phi(a)$ integers in $[1, a\cdot p_n]$ which take the form $p_n \cdot m$, where p_n is the smallest factor . The totient function $\phi(a)$ allows us to determine the exact number of times p_n is the smallest factor of integers in the interval $[1, a\cdot p_n]$. This is precisely $\phi(a)$, as it counts the values of m coprime to a .
Use the totient function to exclude any m divisible by a factor of a , when $a = 6$ and $p_n = 5$.

$$5\cdot 1, \cancel{5\cdot 2}, \cancel{5\cdot 3}, \cancel{5\cdot 4}, 5\cdot 5, \cancel{5\cdot 6},$$

$$\phi(6) = 6\cdot(1-\frac{1}{2})\cdot(1-\frac{1}{3}) = 6\cdot\frac{1}{2}\cdot\frac{2}{3} = 2$$

Frequency of p_n as the Smallest Factor

To compute the relative frequency in $[1, a\cdot p_n]$ where p_n is the smallest factor , we will use the ratio of the number of times that p_n is the smallest factor , in the interval of its primorial P_n:

$$\text{Frequency} = F_n = \frac{\phi(a)}{a\cdot p_n} = \frac{\phi(P_{n-1})}{P_n}$$

For $p_3 = 5$ and $a = 6$

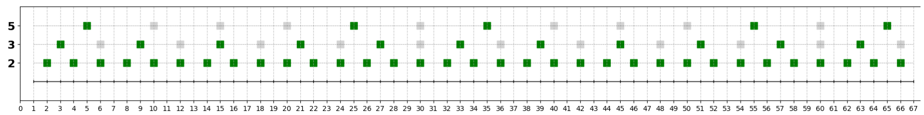
$$F_3 = \frac{\phi(a)}{a\cdot p_n} = \frac{\phi(6)}{6\cdot 5} = \frac{\phi(6)}{30} = \frac{2}{30}$$

We can say that $p_3 = 5$ is the smallest divisor of $\frac{2}{30}$ of all integers .

Case for F_1 , $p_1 = 2$

F_1 is a special case since there are no primes smaller than 2 . So the frequency for 2 will be calulated heuristically . Since 2 divides $\frac{1}{2}$ of all integers and there are no primes smaller than 2 we will say that F_1 is $\frac{1}{2}$. 2 is the smallest divisor of $\frac{1}{2}$ of all integers .

Visual Proof that the frequency repeats



Green ticks are when p_n is the smallest factor of an integer . Grey ticks are for when p_n divides an integer but there are smaller primes which also divide that integer . We can see that the frequency of p_n to divide an integer as the smallest factor, repeats over the interval of its primorial . This is because frequency is measured relative to smaller primes and its primorial is the least common factor $a \cdot p_n$.

Conclusion

If p_n is the n-th prime, then $F_n = \frac{\phi(P_{n-1})}{P_n}$. This is useful in problem contexts where divisibilty with respect to certain primes are necessary . Existing methods only utilize simple heuristics to determine the frequency that a prime is a factor of all integers ie: $\frac{1}{p_n}$, which is limited in its usefulness depending on the problem setup .

2 Approximating the behavior of F_n asymtotically for large n

We start from

$$F_n = \frac{\phi(P_{n-1})}{P_n}.$$

$$F_n = \frac{\phi(P_{n-1})}{P_n} = \frac{P_{n-1} \prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right)}{P_{n-1} \cdot p_n}.$$

$$F_n = \frac{\cancel{P_{n-1}} \prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right)}{\cancel{P_{n-1}} \cdot p_n} = \frac{\prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right)}{p_n}.$$

We can use Mertens' third theorem aka Mertens' product formula , to approximate the numerator asymptotically , we have

$$\prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right) \sim \frac{e^{-\gamma}}{\ln p_{n-1}}.$$

where $\gamma \approx .577$ is the Euler–Mascheroni constant. This provides us with an asymptotic approximation for the numerator .
Since $p_{n-1} \approx p_n$ for large n, this gives

$$\prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right) \approx \frac{e^{-\gamma}}{\ln p_n}.$$

To complete the original definition we must factor in the denominator .

$$F_n = \frac{1}{p_n} \prod_{i=1}^{n-1} \left(1 - \frac{1}{p_i}\right) \approx \frac{1}{p_n} \cdot \frac{e^{-\gamma}}{\ln p_n}.$$

Given

$$F_n \approx \frac{e^{-\gamma}}{p_n \ln p_n},$$

and $p_n \sim n \ln n$, we have

$$F_n \approx \frac{e^{-\gamma}}{n \ln n \cdot \ln(n \ln n)}$$

$$F_n \approx \frac{e^{-\gamma}}{n \ln n \cdot (\ln n + \cancel{\ln \ln n}))}$$

For large n , F_n will grow as $\ln n$ as opposed to $\ln \ln n$, since $\ln n \gg \ln \ln n$ for large n .

$$F_n \approx \frac{e^{-\gamma}}{n (\ln n)^2}$$

$$\begin{aligned} F_{11} &\approx \frac{e^{-\gamma}}{11 \left(\ln 11\right)^2} \\ &\approx \frac{0.561459}{63.265169} \\ &\approx 0.008878 \end{aligned}$$

Conclusion

Assume $F_n = \frac{\phi(P_{n-1})}{P_n}$, then $F_n \approx \frac{e^{-\gamma}}{n(\ln n)^2}$. Above I have provided a means for approximating $F_n = \frac{\phi(P_{n-1})}{P_n}$ asymtotically. This is useful so that sums of F_n, may be approximated in terms of some large $n|x$ by integration.

Sum of F_n

To determine the likelihood that an integer q is composite , we must consider the likelihood that q is divisible by any smaller prime. This can be done using a sum with respect to the frequencies of all smaller primes . The total sum of frequencies S_n of integers in $[1, P_n]$ that are divisible by some prime p_i up to p_n, for which p_i is the smallest factor, is given by:

$$S_n = \sum_{i=1}^{n} \frac{\phi(P_{i-1})}{P_{i-1} \cdot p_i} = \sum_{i=1}^{n} \frac{\phi(P_{n-1})}{P_n} = \sum_{i=1}^{n} F_i \approx \sum_{i=1}^{n} \frac{e^{-\gamma}}{i(\ln i)^2}.$$

S_n is the measure of the likelihood of an integer q to share a factor with a prime less than or equal to p_n . Let $G = \{p_i \leq p_n | p_i\}$, then the likelihood that an integer is divisble by G is S_n . S_n strictly increases for subsequent n , because all terms are positive .

Divisibility by primes not in G

Let $K = \{p_i > p_n | p_i\}$. If q is not divisble by an element in G, then it must be divisble by an element in K , since G and K inlcude all primes, and via the fundamental theorem of arithmetic at least one of these primes must factor q. The likelihood that an integer is not factorable by G is equal to the tail sum of S_n , we will call this tail/end sum E_n . E_n is the likelihood that an element of K divides some q. In practice K is not the entire tail, however it is significantly many terms.

3 Divisibilty Decay Theorem

Assume we have E_n, E_{n+1}, then $E_n > E_{n+1}$.

Quantifying the Tail Sum of S_n as E_n

We have:

$$E_n = \sum_{m>n} F_m,$$

To estimate E_n, consider the sum:

$$E_n = \sum_{m>n} \frac{e^{-\gamma}}{m(\ln m)^2}.$$

For large n, we can approximate sums by integrals. Specifically:

$$\sum_{m>n} \frac{1}{m(\ln m)^2} \approx \int_n^\infty \frac{dx}{x(\ln x)^2}.$$

Simplify and Evaluate the Integral

$$u = \ln x$$

$$\frac{du}{dx} = \frac{d}{dx}(\ln x) = \frac{1}{x} \quad \Rightarrow \quad du = \frac{1}{x}dx \quad \Rightarrow \quad dx = x\,du$$

$$\frac{1}{x(\ln x)^2} = \frac{1}{xu^2}$$

$$I = \int_n^\infty \frac{dx}{xu^2} = \int_{\ln n}^\infty \frac{x\,du}{xu^2} = \int_{\log n}^\infty \frac{du}{u^2}$$

$$\int \frac{du}{u^2} = \frac{1}{u^2} = u^{-2} = \frac{u^{-2+1}}{-2+1} = \frac{u^{-1}}{-1} = -\frac{1}{u} + C$$

$$I = \left[-\frac{1}{u}\right]_{\ln n}^{\infty} = \left(-\frac{1}{\infty}\right) - \left(-\frac{1}{\ln n}\right) = 0 - \left(-\frac{1}{\ln n}\right) = \frac{1}{\ln n}$$

Thus:

$$\int_n^{\infty} \frac{dx}{x(\ln x)^2} = \frac{1}{\ln n}.$$

Incorporating the constant $e^{-\gamma}$:

$$E_n \approx e^{-\gamma} \cdot \frac{1}{\ln n} \quad \text{as } n \to \infty.$$

E_n is on the order of $\frac{1}{\ln n}$. If the sum defining E_n is not from n to ∞ but to some finite upper limit M we can write:

$$\int_n^{M} \frac{dx}{x(\ln x)^2} = \left(-\frac{1}{\ln M}\right) - \left(-\frac{1}{\ln n}\right).$$

If M is extremely large the dominant term is $1/\ln n$, ensuring it remains on the order of $1/\ln n$ for large n. As $n \to \infty$, $E_n \to 0$.

$$E_n \approx \frac{e^{-\gamma}}{\ln n}.$$

$$\begin{aligned} E_{15} &\approx \frac{e^{-\gamma}}{\ln 15} \\ &\approx \frac{0.561459}{2.708050} \\ &\approx 0.2073 \end{aligned}$$

$$\begin{aligned} E_{16} &\approx \frac{e^{-\gamma}}{\ln 16} \\ &\approx \frac{0.561459}{2.772589} \\ &\approx 0.2026 \end{aligned}$$

$$E_{15} > E_{16}$$

Conclusion

If we have E_n, E_{n+1}, then $E_n > E_{n+1}$. This is useful because it allows us to see that if some q is not divisible by G then the likelihood that it is divisible by K goes to 0, for large n.

Adjacent Coprime Divisibility

Assume P_n is the the product of the first n primes, then $P_n \pm 1$ is only divisible by K.
Let P_n be the product of the first n primes $G = \{2, 3, 5, ..., p_n\}$. This means P_n is factorable by all elements in G . It can be shown that $P_n \pm 2$ is factorable by 2 , $P_n \pm 3$ is factorable by 3 ... $P_n \pm p_n$ is factorable by p_n. It can be deduced that $P_n \pm 1$ is not factorable by any of the elements in G. $P_n \pm 1$ is coprime to G. There are primes $p_{n+1}, p_{n+2}, ...$ in the interval $[p_n, P_n + 1]$, which may be a factor of $P_n \pm 1$. This is defined as $E_n = \sum_n^M F_m$, where $M = P_n + 1$. Let $K = \{P_n + 1 > p_m > p_n | p_m\}$ which is also the set of primes accounted for in E_n.
Assume P_n is the the product of the first n primes, then $P_n \pm 1$ is only divisible by K. This is useful because it constrains the likelihood that $P_n \pm 1$ is prime or composite, relative to only a single set of primes K, whose behavior of F_n and E_n for large n is already known.
If $P_n \pm 1$ is coprime to G and coprime to K then $P_n \pm 1$ is prime.

4 Quantifying the likelihood that either $P_n - 1$ or $P_n + 1$ are prime for subsequent n

Assume we have $P_n \pm 1, P_{n+1} \pm 1$, then the likelihood that $P_n \pm 1$ is prime, is less than the likelihood that $P_{n+1} \pm 1$ is prime.
As n increases , the likelihood of $P_n \pm 1$ being divisible by K decreases via the Divisibility Decay Theorem . Given that E_n is the likelihood of K to divide some positive integer and $E_n > E_{n+1}$, the likelihood that K is composite to $P_n \pm 1$ decreases for large n. The likelihood that either $P_n + 1$ or $P_n - 1$ is composite is approximately E_n for each , so the combined likelihood that either is composite is at most $\approx E_n + E_n = 2 \cdot E_n$. Therefore, as n increases, the likelihood that either $P_n \pm 1$ are composite decreases as $2 \cdot E_n$. It follows that as $n \to \infty$, the likelihood that both $P_n - 1$ and $P_n + 1$ are both not composite is at least $(1 - 2 \cdot E_n)$, which increases for large n . Thus, the likelihood that $P_n \pm 1$ are both prime increases for larger n as $(1 - 2 \cdot E_n)$.

Conclusion

Assume we have $P_n \pm 1, P_{n+1} \pm 1$, then the likelihood that $P_n \pm 1$ is prime, is less than the likelihood that $P_{n+1} \pm 1$ is prime. This is useful in locating large prime numbers with a decreased error rate.

Infinite Twin Primes Proof

Twin primes are two primes seperated by a distance of 2 .
Let A denote the infinite ordered set of all prime numbers. We construct a subset $B \subseteq A$ consisting of P_n where $P_n \pm 1$ are both prime. The construction proceeds iteratively as follows:

1. For each positive integer n, define the primorial P_n as the product of the first n primes:

$$P_n = \prod_{i=1}^{n} p_i,$$

2. Define B as the set of primorials P_n for which both $P_n - 1$ and $P_n + 1$ are prime. Formally,

$$B = \left\{ P_n \in A \;\middle|\; P_n - 1 \text{ and } P_n + 1 \text{ are both prime} \right\}.$$

3. The likelihood that $P_n \pm 1$ are both prime increases as n increases, by result 4 .

4. The set B is expected to contain more elements as n grows.

5. Since B is constructed by iteratively including primorials where $P_n \pm 1$ are both prime, and the likelihood of inclusion increases with n, we infer that B is an infinite subset of A.

6. There are an infinite number of twin primes.