Fixed strike lookback option

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What are Lookback options?

Lookback options, in the terminology of finance, are a type of exotic option where the payoff depends on the optimal (maximum or minimum) underlying asset's price occurring over the life of the option. The option allows the holder to "look back" over time to determine when to exercise their option and so as to get the best payoff. This type of option reduces uncertainties associated with the timing of market entry and reduces the chances the option will expire worthlessly. There exist two kinds of lookback options: with floating strike and with fixed strike.

Lookback Option with floating strike

As the name suggests, the option's strike price is floating and determined at maturity. The floating strike is the optimal value of the underlying asset's price during the option life. The payoff is the maximum difference between the market asset's price at maturity and the floating strike.

For the call, the strike price is fixed at the asset's lowest price during the option's life, and, for the put, it is fixed at the asset's highest price thus maximizing the payoff. This option is never out-of-the-money, which makes it more expensive than a standard option.

The payoff functions for the lookback call and the lookback put, respectively, are given by:

$$LC_{float} = \max(S_T - S_{min}, 0) = S_T - S_{min}$$

$$LP_{float} = \max(S_{max} - S_T, 0) = S_{max} - S_T$$

where S_{max} is the asset's maximum price during the life of the option, S_{min} is the asset's minimum price during the life of the option, and S_T is the underlying asset's price at maturity T.

Lookback Option with fixed strike

Here the option's strike price is fixed. The option is not exercised at the price at maturity: the payoff is the maximum difference between the optimal underlying asset price and the strike. So, unlike the floating strike rate lookback option where strike price was not fixed, here the exercise price is not fixed. For example, for the call option, the holder chooses to exercise at the point when the underlying asset price is at its highest level and for the put option, the holder chooses to exercise at the underlying asset's lowest price, thus maximizing the payoff.

The **payoff functions** for the lookback call and the lookback put, respectively, are given by:

$$LC_{fixed} = \max\,(S_{max} - K, 0)$$

$$LP_{fixed} = \max\,(K - S_{min}, 0)$$

where S_{max} is the asset's maximum price during the life of the option, S_{min} is the asset's minimum price during the life of the option, and K is the strike price, without considering the option premium paid.

Scenario analysis

Now that we know payoff formulas, let us consider scenario analysis for different market prices:

K= 100, Maturity = T

Here we are considering a situation for an asset which reached maximum market price of S_{max} and minimum of S_{min} during the life of a lookback option with maturity T. Strike price was fixed at 100 and P is the premium paid to acquire the option.
Let us consider how and when the Fixed strike Lookback Call will be exercised:

- $S_{max} = 50,\ K = 100$

$$LC_{payoff} = max(S_{max} - K, 0) - P$$

Payoff = - P, as the call option won't be exercised as it would bring no benefit.

- $S_{max} = 150,\ K = 100$

$$LC_{payoff} = max(S_{max} - K, 0) - P$$

Payoff = (150-100) -(P) = 50 - P

- $S_{max} = 100,\ K = 100$

$$LC_{payoff} = max(S_{max} - K, 0) - P$$

Payoff = (100-100) - P = - P, however, here it wouldn't make sense to exercise the option.

Let us consider how and when the Fixed strike Lookback Put will be exercised.

- $S_{min} = 50,\ K = 100$

$$LP_{payoff} = max(K - S_{min}, 0) - P$$

Payoff = (100-50) - P = 50 – P

- $S_{min} = 150,\ K = 100$

$$LP_{payoff} = max(K - S_{min}, 0) - P$$

Payoff = - P as the option will not be exercised, because it would bring no benefit.

- $S_{min} = 100, \ K = 100$

$$LP_{payoff} = max(K - S_{min}, 0) - P$$

Payoff = (100-100) - P = - P, even here it wouldn't make sense to exercise the option.

Pricing formulas

Taking into account that M is the realized maximum of the asset from the start of the sampling period $t = 0$ until T, the formula for **pricing fixed strike lookback call options** is the following:

If $K > M$:

$$Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) + Se^{-r(T-t)}\frac{\sigma^2}{2(r-q)} \times$$
$$\times \left(-\left(\frac{S}{K}\right)^{-\left(\frac{2(r-q)}{\sigma^2}\right)} N\left(d_1 - \frac{2(r-q)\sqrt{T-t}}{\sigma}\right) + e^{(r-q)(T-t)}N(d_1) \right)$$

Where:

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

And:

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

If $K < M$:

$$(M-K)e^{-r(T-t)} + Se^{-q(T-t)}N(d_1) - Me^{-r(T-t)}N(d_2) + Se^{-r(T-t)}\frac{\sigma^2}{2(r-q)} \times$$
$$\times \left(-\left(\frac{S}{M}\right)^{-\left(\frac{2(r-q)}{\sigma^2}\right)} N\left(d_1 - \frac{2(r-q)\sqrt{T-t}}{\sigma}\right) + e^{(r-q)(T-t)}N(d_1) \right)$$

Where:

$$d_1 = \frac{\log\left(\frac{S}{M}\right) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

And:

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

And now, taking into account that M is the realized minimum of the asset from the start of the sampling period $t = 0$ until T, the formula for **pricing fixed strike lookback put options** is the following:

<u>If $K < M$</u>:

$$Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1) + Se^{-r(T-t)}\frac{\sigma^2}{2(r-q)} \times$$
$$\times\left(\left(\frac{S}{K}\right)^{-\left(\frac{2(r-q)}{\sigma^2}\right)} N\left(-d_1 + \frac{2(r-q)\sqrt{T-t}}{\sigma}\right) - e^{(r-q)(T-t)}N(-d_1)\right)$$

Where:

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

And:

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

<u>If $K > M$</u>:

$$(K-M)e^{-r(T-t)} - Se^{-q(T-t)}N(-d_1) + Me^{-r(T-t)}N(-d_2) + Se^{-r(T-t)}\frac{\sigma^2}{2(r-q)} \times$$
$$\times\left(\left(\frac{S}{M}\right)^{-\left(\frac{2(r-q)}{\sigma^2}\right)} N\left(-d_1 + \frac{2(r-q)\sqrt{T-t}}{\sigma}\right) - e^{(r-q)(T-t)}N(-d_1)\right)$$

Where:

$$d_1 = \frac{\log\left(\frac{S}{M}\right) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

And:

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

The implementation of such formulas in VBA looks in the following way:

```
Function FX_call(S, K, r, q, vol, T, Max)

If K > Max Then
    d1 = (Log(S / K) + (r - q + 0.5 * vol ^ 2) * T) / (vol * Sqr(T))
    d2 = d1 - vol * Sqr(T)
    d3 = d1 - ((2 * (r - q) * Sqr(T)) / vol)
    Nd1 = Application.WorksheetFunction.Norm_S_Dist(d1, 1)
    Nd2 = Application.WorksheetFunction.Norm_S_Dist(d2, 1)
    Nd3 = Application.WorksheetFunction.Norm_S_Dist(d3, 1)
    FX_call = S * Exp(-q * T) * Nd1 - K * Exp(-r * T) * Nd2 + S * Exp(-r * T) * ((vol ^ 2) /
(2 * (r - q))) * ((-(S / K) ^ -(2 * (r - q) / vol ^ 2)) * Nd3 + Exp((r - q) * T) * Nd1)
Else
    d1 = (Log(S / Max) + (r - q + 0.5 * vol ^ 2) * T) / (vol * Sqr(T))
    d2 = d1 - vol * Sqr(T)
    d3 = d1 - ((2 * (r - q) * Sqr(T)) / vol)
    Nd1 = Application.WorksheetFunction.Norm_S_Dist(d1, 1)
    Nd2 = Application.WorksheetFunction.Norm_S_Dist(d2, 1)
    Nd3 = Application.WorksheetFunction.Norm_S_Dist(d3, 1)
    FX_call = (Max - K) * Exp(-r * T) + S * Exp(-q * T) * Nd1 - Max * Exp(-r * T) * Nd2 + S *
Exp(-r * T) * ((vol ^ 2) / (2 * (r - q))) * ((-(S / Max) ^ -(2 * (r - q) / vol ^ 2)) * Nd3 +
Exp((r - q) * T) * Nd1)
End If
End Function

Function FX_put(S, K, r, q, vol, T, Min)

If K < Min Then
    d1 = (Log(S / K) + (r - q + 0.5 * vol ^ 2) * T) / (vol * Sqr(T))
    d2 = d1 - vol * Sqr(T)
    d3 = d1 - ((2 * (r - q) * Sqr(T)) / vol)
    Nd1 = Application.WorksheetFunction.Norm_S_Dist(-d1, 1)
    Nd2 = Application.WorksheetFunction.Norm_S_Dist(-d2, 1)
    Nd3 = Application.WorksheetFunction.Norm_S_Dist(-d3, 1)
    FX_put = K * Exp(-r * T) * Nd2 - S * Exp(-q * T) * Nd1 + S * Exp(-r * T) * ((vol ^ 2) / (2
* (r - q))) * (((S / K) ^ -(2 * (r - q) / vol ^ 2)) * Nd3 - Exp((r - q) * T) * Nd1)
Else
    d1 = (Log(S / Min) + (r - q + 0.5 * vol ^ 2) * T) / (vol * Sqr(T))
    d2 = d1 - vol * Sqr(T)
    d3 = d1 - ((2 * (r - q) * Sqr(T)) / vol)
    Nd1 = Application.WorksheetFunction.Norm_S_Dist(-d1, 1)
    Nd2 = Application.WorksheetFunction.Norm_S_Dist(-d2, 1)
    Nd3 = Application.WorksheetFunction.Norm_S_Dist(-d3, 1)
    FX_put = (K - Min) * Exp(-r * T) - S * Exp(-q * T) * Nd1 + Min * Exp(-r * T) * Nd2 + S *
Exp(-r * T) * ((vol ^ 2) / (2 * (r - q))) * (((S / Min) ^ -(2 * (r - q) / vol ^ 2)) * Nd3 -
Exp((r - q) * T) * Nd1)
End If
End Function
```

Sensitivity analysis

We can test the sensitivity of call and put fixed strike lookback options prices to the following parameters:

- Underlying asset price S;

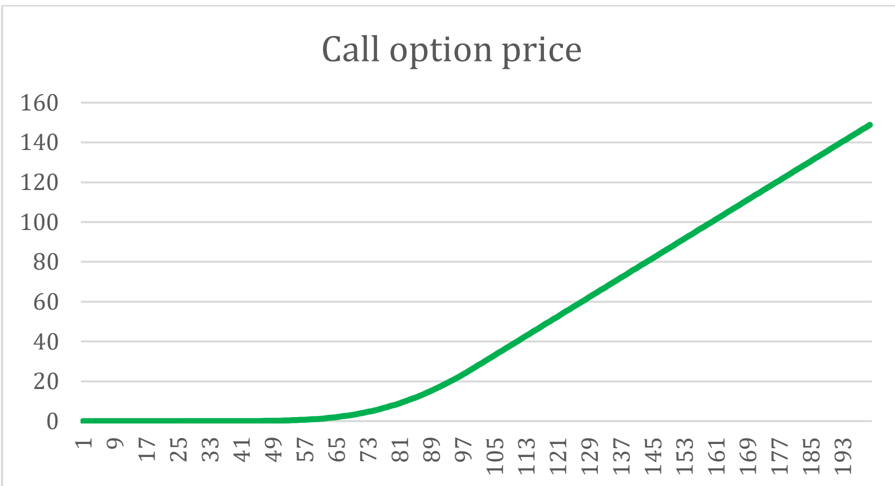
- Realized maximum Max;
- Realized minimum Min;
- Strike price K;
- Time to maturity T;
- Volatility;
- Risk-free interest rate r;
- Cash flows (dividend yields etc…) q.

Sensitivity to S

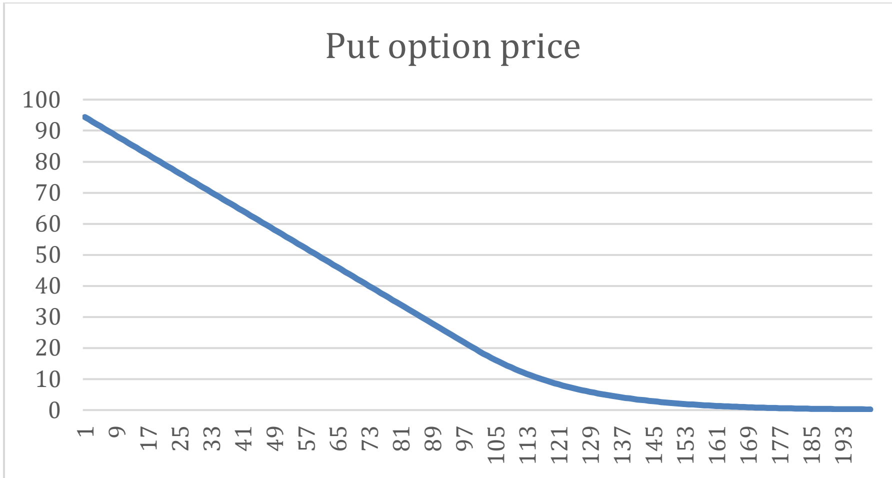
It is a little bit complex to estimate the price of calls and puts for fixed strike lookback options. We decided to assume that the underlying asset's price S at time T equals the realized maximum of the asset from the start of the sampling period $t = 0$ until T in the case of calls, and the realized minimum of the asset from the start of the sampling period $t = 0$ until T in the case of puts. Moreover:

- $K = 100$;
- $r = 5\%$;
- $q = 2\%$;
- $vol = 30\%$;
- $T = 1$.

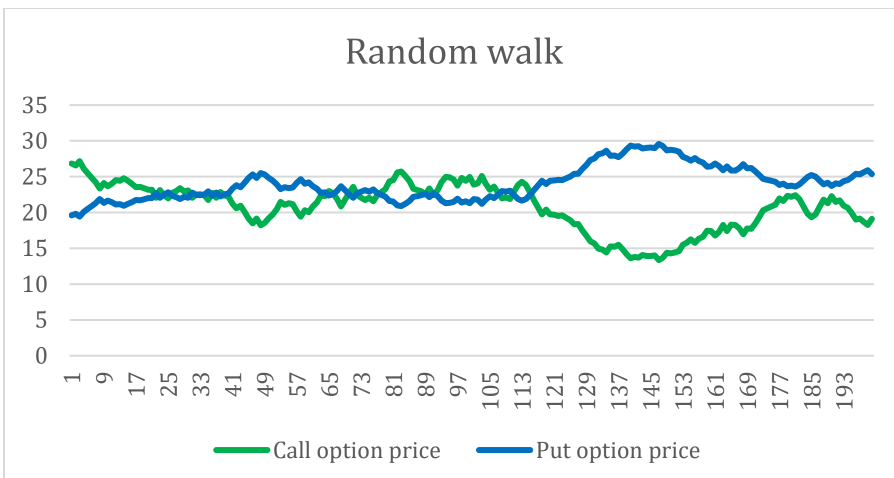
Hence:



As we can see in the case of a call option, as the underlying asset price rises the price of the option rises as well.



In the case of a put option, as the underlying asset price rises, the price of the option decreases. If we create a random walk for the path of the option price, and we calculate the prices basing on such random walk:



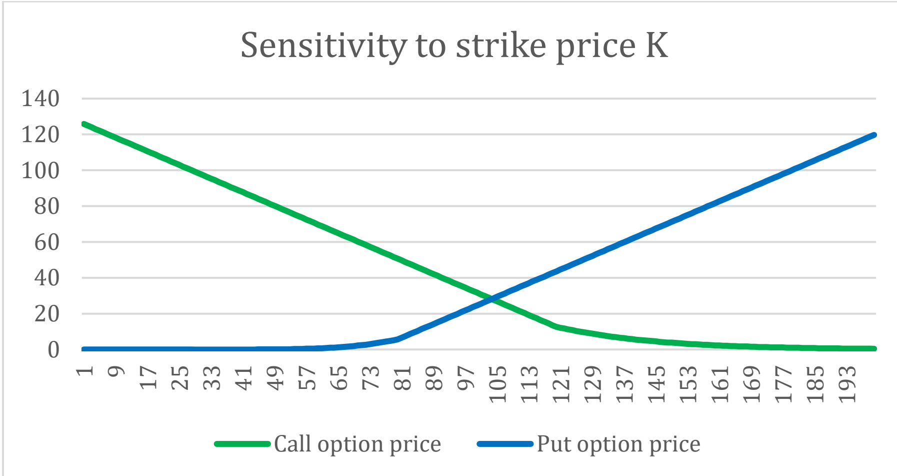
We see that there is a sort of "mirror effect", which can be explained simply by the fact that the higher the underlying asset price, the higher the price of the call option and the lower the price of the put option. The lower the stock price, the lower the price of the call option and the higher the price of the put option.

<u>Sensitivity to K</u>

With a K ranging from 1 to 200 and the other parameters set in the following way:

- $S = 100$;
- $r = 5\%$;
- $q = 2\%$;
- vol =30%;
- $T = 1$;
- $Max = 120$;
- $Min = 80$.

The sensitivity of call and put options looks in the following way:



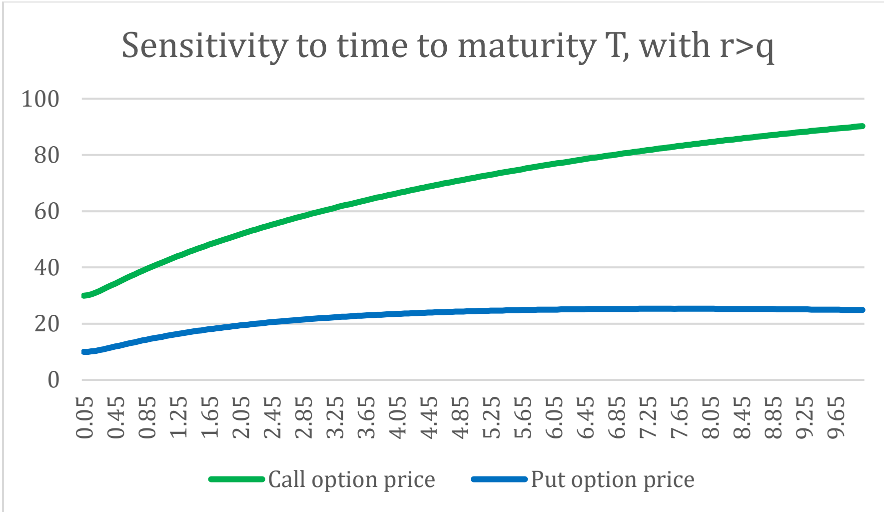
The higher the strike price, the lower the value of the call option and the higher the value of the put option and vice versa. We can notice that the increase of price of the put option becomes linear after hitting a strike price of 80, which corresponds to the set realized minimum. In the case of the call option, the decrease of the value of the option is linear until K equals 120, which is the set realized maximum.

<u>Sensitivity to T</u>

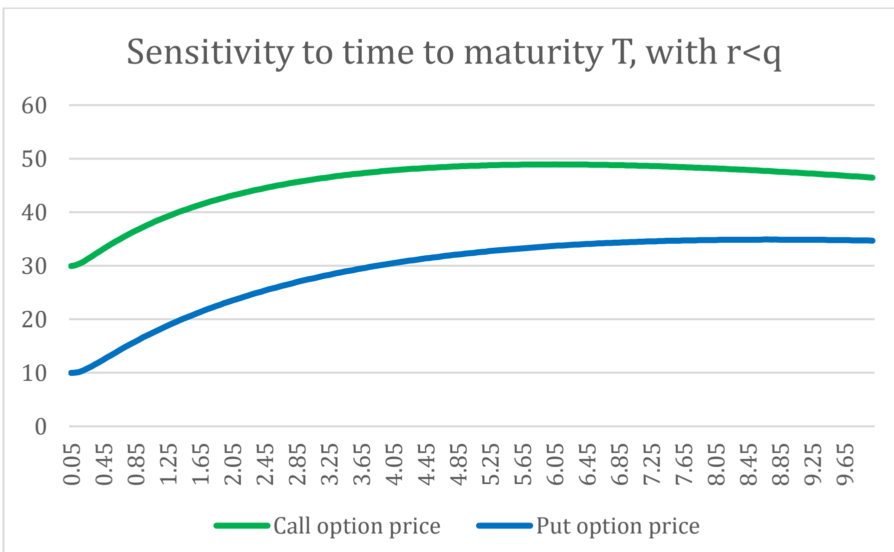
With a time to maturity ranging from 0.05 years to 10 years and the other parameters set in the following way:

- $K = 90;$
- $S = 100;$
- $r = 5\%;$
- $q = 2\%;$
- $vol = 30\%;$
- $Max = 120;$
- $Min = 80.$

The sensitivity of call and put options to T looks in the following way:



The higher the time to maturity, the higher the value of the call option and the higher the value of the put option, up to certain extent. However, this is affected by the relationship between q and r. In fact, the graph represented above stands for the case in which $r > q$. In the opposite case, $r < q$:



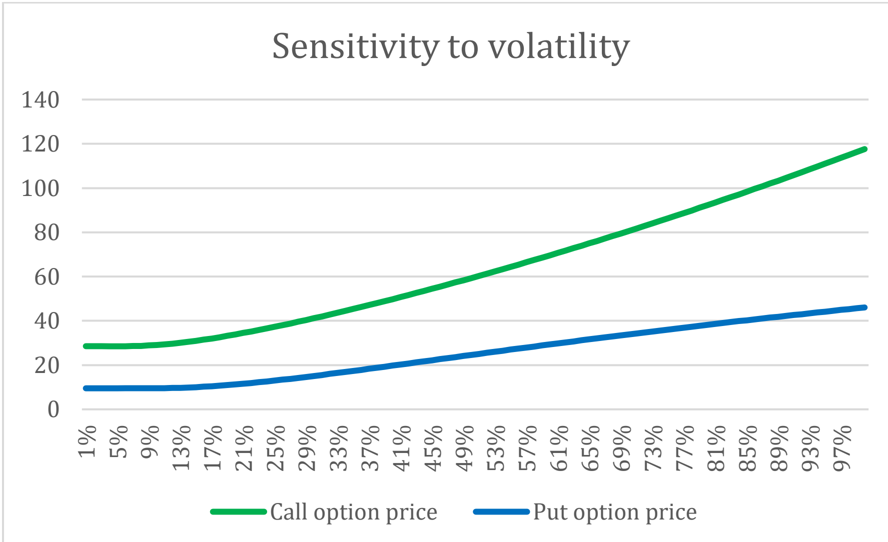
We can see that the options' prices tend to increase initially, but after a while they start to decrease gently.

<u>Sensitivity to volatility</u>
With a volatility ranging from 1% to 100% and the other parameters set in the following way:

- $K = 90$;
- $S = 100$;
- $r = 5\%$;
- $q = 2\%$;
- $T = 1$;

- $Max = 120$;
- $Min = 80$.

The sensitivity of call and put options to volatility looks in the following way:

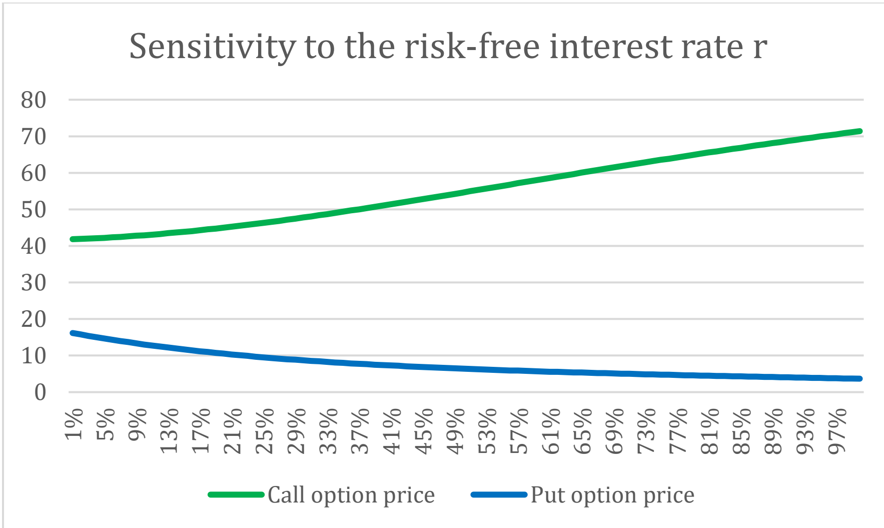


Namely, in the case of both options an increase in volatility leads to an increase in price. Fixed strike lookback options are practically riskless options which never expire out of the money. So, with more volatile markets, the lower bound stays fixed (in practice), while the possibility of higher returns increases.

<u>Sensitivity to r</u>
With risk-free interest rate ranging from 1% to 100% and the other parameters set in the following way:

- $K = 90$;
- $S = 100$;
- $vol = 30\%$;
- $q = 0\%$;
- $T = 1$;
- $Max = 120$;
- $Min = 80$.

The sensitivity of call and put options to r looks in the following way:



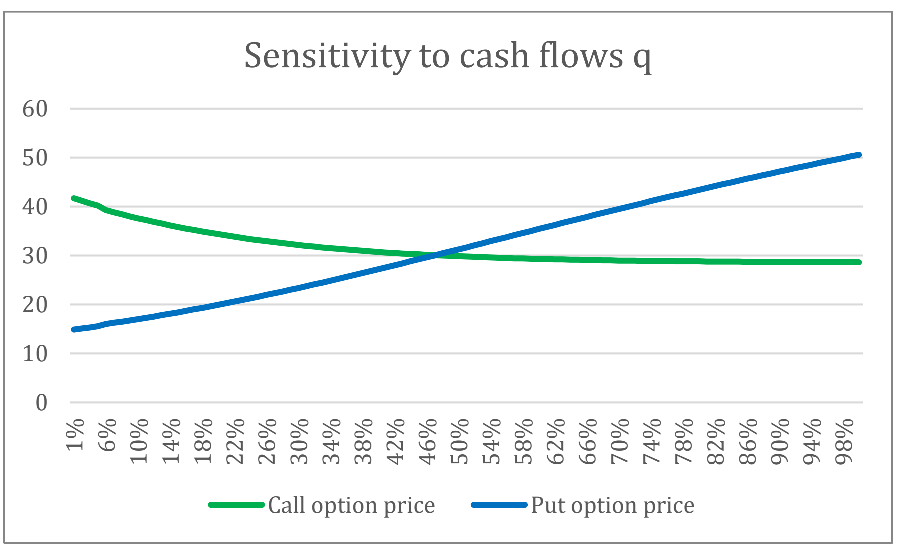
So, the variation of r influences lightly the value of the options. But generally, the higher the risk-free interest rate, the higher the value of the call option and the lower the value of the put option.

Sensitivity to q

With q ranging from 1% to 100% and the other parameters set in the following way:

- $K = 90$;
- $S = 100$;
- $vol = 30\%$;
- $r = 5\%$;
- $T = 1$;
- $Max = 120$;
- $Min = 80$.

The sensitivity of call and put options to r looks in the following way:



The higher q, the higher the value of the put option and the lower the value of the call option (to some extent).

Sensitivity analysis (summary)

Variable (increase)	Fixed strike lookback call	Fixed strike lookback put
S	+	-
K	-	+
T (r>q)	+	?
T (r<q)	?	?
vol	+	+
r	+	-
q	-	+