Exotic Options – Creating a Structured Product

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What are Structured Products?

A structured product is a pre-packaged finance investment strategy based on a single security, a basket of securities, options, indices, commodities, debt issuance or foreign currency, and derivatives. This type of product is designed for investors who are prepared to invest for a fixed period, and who want some degree of protection over their invested capital.
We can treat a structured product as a portfolio. This is a tailor-made product, which aims to provide the best solution for the investors that purchase it.

Our Structured product

We are going to implement the following strategy, which consists of three instruments:

- 1 zero-coupon bond,
- 1 American Up & Out Call option (Long),
- 1 American Up & Out Call option (Short).

A 3-Year US note to be issued on 07/08/2021 consists of our safe instrument. Therefore, the redemption date of such bond is 07/07/2024, which becomes the maturity for our structured product. The structured product is going to be issued the same day as the zero-coupon bond, providing us with more than 3 weeks of time to prepare this ad-hoc product.
The bond has a face value of 1,000 USD, and the current YTM is 0.31% (see treasury.gov). Thanks to this data, we are able to calculate the expected price of the bond; expected because we need the price for the issue date. Assuming that the YTM will not change at issue date, we can calculate the price of the bond by discounting the future payoff (face value):

$$P_{bond} \approx \frac{1000}{(1+0.0031)^3} = 990.76\ USD$$

Now, we need to price the options. They are risky instruments, where the payoff depends on the price of the underlying asset, which consists in 1 stock of McCormick & Company Inc. (Ticker: MKC). We don't know how this price will look like in the future.
For both the option which we sell and the one which we purchase, we can assume the following:

- MKC's price (S) is, as of 06/11/2021, equal to 88.85 USD,
- Volatility (σ) is 19.75%, calculated basing on the last 252 trading days' returns,
- The risk-free rate (r) is equal to 0.24%, from the 1-year LIBOR interest rate,

- Dividend yield (q) is equal to 1.5%, from the last dividend released by the company, which happened on 04/09/2021,
- Time to maturity (T) is the same as for our bond, 3 years.

In addition, for the American Up & Out Call option which we purchase, we assume:
- A barrier (H_1) of 110 USD,
- The option is in-the-money, with a strike price (K_1) of 70 USD.

While for the American Up & Out Call option which we short, we assume:
- A barrier (H_2) of 100 USD,
- The option is in-the-money, with a strike price (K_2) of 80 USD.

Since the options have the "up" feature, the barrier is higher than the underlying asset price at the moment of writing the options.

Using the pricing formulas for an American Up & Out Call option, we can find the price of the two options:

$$c_1 = 3.49\ USD$$
$$c_2 = 0.36\ USD$$

Where c_1 is the price of the American Up & Out Call which we purchase, and c_2 is the price of the American Up & Out Call which we short.

Now we can calculate the margin for our structured product. This would be:

$$Margin = 1000 - 990.76 - 3.49 + 0.36 = 6.11\ USD$$

We can imagine that we represent an investment bank, and we created a portfolio which we sell to our client for 1,000 USD, keeping a 6.11 USD margin. This is a safe product, because at the end the client will receive for sure the face value of the bond (1,000 USD), and the potential payoff from the options, which will be 0 USD only in the worst case.

Payoff functions

The payoff formula from our product from the point of view of the customer can be presented. Such payoff is the face value of the 3-Year US note, which is known and sure. In addition to that, there will be the 2 payoffs given by the options, which depend on the behavior of MKC stock. Therefore:

$$Payoff = 1000 + \begin{cases} \max(S_t - K_1; 0) & if\ S_t \leq H_1 \\ 0 & if\ S_t > H_1 \end{cases} - \begin{cases} \max(S_t - K_2; 0) & if\ S_t \leq H_2 \\ 0 & if\ S_t > H_2 \end{cases}$$

Where S_t is the future and unknown MKC price, K_1 and K_2 are the strike prices which were assumed to be 70 USD and 80 USD, and H_1 and H_2 are the barriers which were assumed to be equal to 110 USD and 100 USD. Knowing these values, we can substitute in the above formula, obtaining:

$$Payoff = 1000 + \begin{cases} \max(S_t - 70; 0) & if\ S_t \leq 110 \\ 0 & if\ S_t > 110 \end{cases} - \begin{cases} \max(S_t - 80; 0) & if\ S_t \leq 100 \\ 0 & if\ S_t > 100 \end{cases}$$

Hence, from the point of view of the buyer of our structured product, this is the payoff function, the outcome of which depends only and exclusively on the value of the underlying asset price.

Scenario analysis

Now that we know payoff formulas, let us consider scenario analysis for different market prices. We can try to imagine how would the payoff look like depending on the value of our underlying asset price. In *scenario 1*, we imagine having an S_t which is higher than both barriers, for instance 120 USD. In such situation, the payoff would be:

$$Payoff = 1000 + \begin{cases} \max\ (120 - 70; 0\ ) & if\ S_t \leq 110 \\ 0 & if\ S_t > 110 \end{cases} - \begin{cases} \max(120 - 80; 0) & if\ S_t \leq 100 \\ 0 & if\ S_t > 100 \end{cases}$$

Hence:

$$Payoff = 1000 + 0 - 0 = 1000\ USD$$

As we can see, breaking both barriers would lead to a payoff that is equal to the face value of our coupon bond only, not less than that.

Let's imagine now that the S_t will be included between the two barriers, for instance 105. This would be our *scenario 2*:

$$Payoff = 1000 + \begin{cases} \max(105 - 70; 0) & if\ S_t \leq 110 \\ 0 & if\ S_t > 110 \end{cases} - \begin{cases} \max(105 - 80; 0) & if\ S_t \leq 100 \\ 0 & if\ S_t > 100 \end{cases}$$

This scenario would result in the following payoff:

$$Payoff = 1000 + 35 - 0 = 1035\ USD$$

As we can see, such a scenario would lead to a positive payoff, greater than the value invested by our customer.

Finally, let's verify the payoff in the case that the underlying asset price was lower than both barriers, for instance with a value of 90 USD. This would be our *scenario 3*:

$$Payoff = 1000 + \begin{cases} \max(90 - 70; 0) & if\ S_t \leq 110 \\ 0 & if\ S_t > 110 \end{cases} - \begin{cases} \max(90 - 80; 0) & if\ S_t \leq 100 \\ 0 & if\ S_t > 100 \end{cases}$$

Hence:

$$Payoff = 1000 + 20 - 10 = 1010\ USD$$

As in the previous scenario, we can see that the payoff from the point of view of our client would be greater than the value invested, even though smaller than in *scenario 2*. In fact, the best scenario for the customer is when the underlying asset price is included between the two barriers, as in that case only the Up & Out Call option which we purchase would be exercised. This is shown as well in Figure 1.

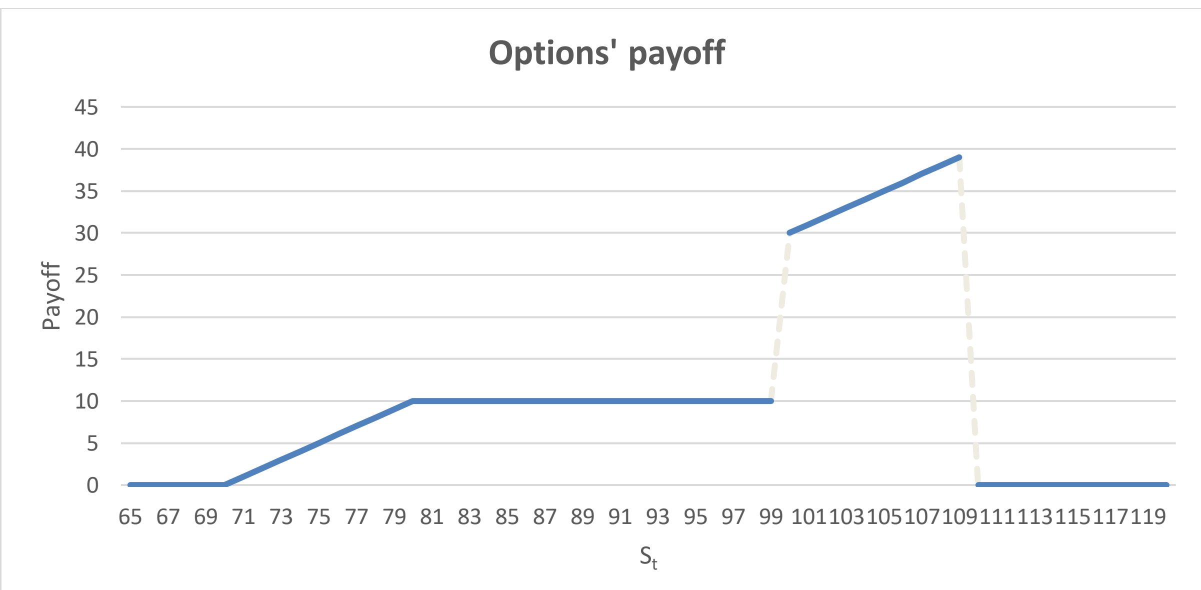

Figure 1. Total options' payoff from the point of view of the client.

The partial payoffs given by the options separately are presented below (Fig. 2 and Fig. 3). Looking at these graphs, we can see how the long option allows us to hedge from the risk of obtaining a negative payoff in case the short option was exercised.

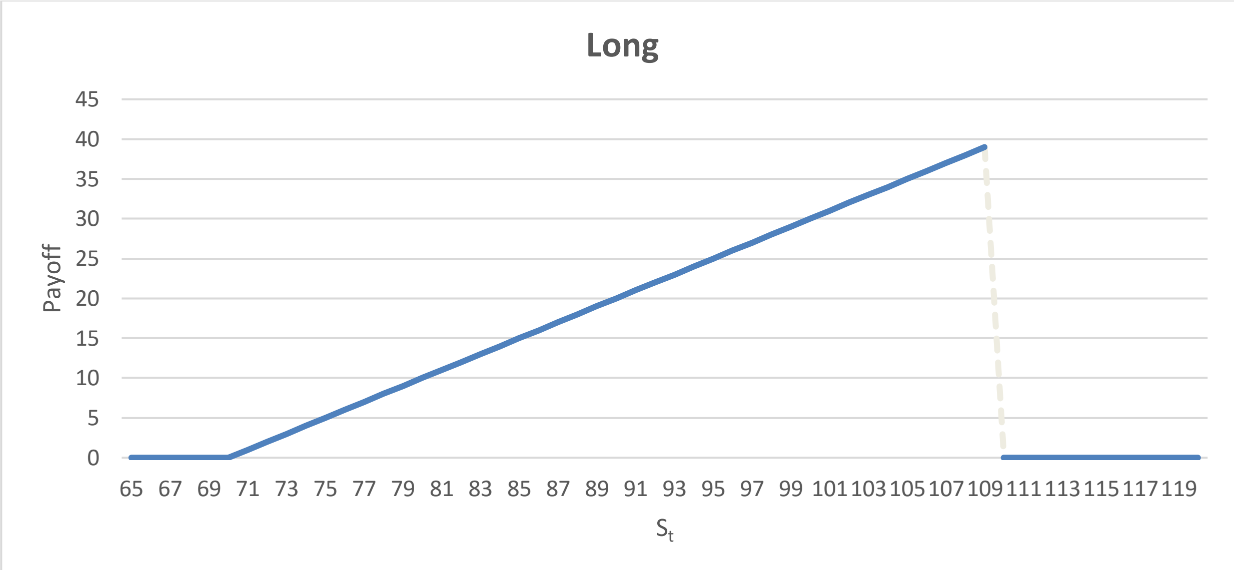

Figure 2. Payoff from the Long American Up & Out Call option.

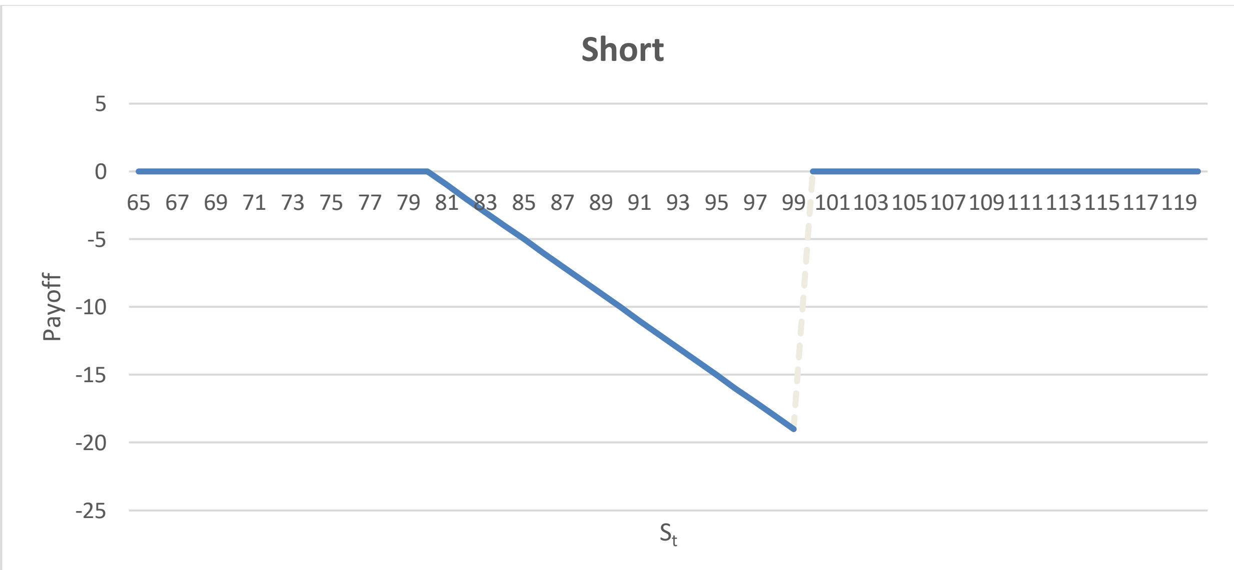

Figure 3. Payoff from the Short American Up & Out Call option.

In fact, it is easy to notice how the potential negative payoffs given by the option which we sell when the S_t is included between 80 and 99 is always offset by the greater positive payoffs which would be granted by the option which we purchased along the same interval. Moreover, these plots allow us to confirm that the best scenario would be the one in which the S_t is included between 100 and 109; in fact, only the call option which we purchased would be used providing us with a great payoff.

Valuation

Now comes the valuation of our structured product. The price of the product is the sum of the bond's price and the option price which we purchase, minus the price of the option which we short:

$$Price = \frac{N}{(1 + YTM_{bond})^T} + Up\&Out\ Call_{price}(long) - Up\&Out\ Call_{price}(Short)$$

Where:

- N is the notional of the bond,
- YTM_{bond} is the rate of return of the considered bond,
- T is the time to maturity,
- $Up\&Out\ Call_{price}(long)$ is the price of the American Up & Out Call option which we purchase,
- $Up\&Out\ Call_{price}(Short)$ is the price of the American Up & Out Call option which we short.

The American Up & Out options are priced according to the formulas developed by Merton, Reiner and Rubinstein, which use the following set of factors:

$$A = \phi \cdot S \cdot e^{-qT} N(\phi \cdot x_1) - \phi \cdot K \cdot e^{-rT} N(\phi \cdot x_1 - \phi \cdot \sigma\sqrt{T})$$
$$B = \phi \cdot S \cdot e^{-qT} N(\phi \cdot x_2) - \phi \cdot K \cdot e^{-rT} N(\phi \cdot x_2 - \phi \cdot \sigma\sqrt{T})$$
$$C = \phi \cdot S \cdot e^{-qT} \left(\frac{H}{S}\right)^{2(\mu+1)} N(\eta \cdot y_1) - \phi \cdot K \cdot e^{-rT} \left(\frac{H}{S}\right)^{2\mu} N(\eta \cdot y_1 - \eta \cdot \sigma\sqrt{T})$$
$$D = \phi \cdot S \cdot e^{-qT} \left(\frac{H}{S}\right)^{2(\mu+1)} N(\eta \cdot y_2) - \phi \cdot K \cdot e^{-rT} \left(\frac{H}{S}\right)^{2\mu} N(\eta \cdot y_2 - \eta \cdot \sigma\sqrt{T})$$

Where:

$$x_1 = \frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{T}} + (1 + \mu)\sigma\sqrt{T}, \qquad x_2 = \frac{\ln\left(\frac{S}{H}\right)}{\sigma\sqrt{T}} + (1 + \mu)\sigma\sqrt{T}$$
$$y_1 = \frac{\ln\left(\frac{H^2}{S \cdot K}\right)}{\sigma\sqrt{T}} + (1 + \mu)\sigma\sqrt{T}, \qquad y_2 = \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T}} + (1 + \mu)\sigma\sqrt{T}$$
$$\mu = \frac{r - q - \frac{\sigma^2}{2}}{\sigma^2}$$

It is important to include the frequency of observation of the MKC's price, since we are dealing with American options. In order to take into account this feature, we adjust the pricing formulas according to the approach suggested by Broadie, Glasserman and Kou, who proposed the following substitution:

$$H^* = He^{0.5826\sigma\sqrt{\frac{T}{m}}}$$

Where H^* is the adjusted barrier, m is the frequency of observation of S_t. On a time-horizon of 3 years, we set it to be 756, which means that the value of the underlying asset price would be observed with daily frequency. Therefore, we obtain the following adjusted barriers:

$$H_1^* = 110.80\,USD$$
$$H_2^* = 100.73\,USD$$

Where H_1^* is the adjusted barrier for the option which we purchase, and H_2^* is the adjusted barrier for the option which we sell.

Finally, with the inclusion of all the parameters presented above and the newly introduced adjusted barriers, both American Up & Out Call options can be priced by selecting the following parameters:

$$\eta = -1, \qquad \phi = 1$$

If $K > H$: $Up\&Out\ Call_{price} = 0$
If $K < H$: $Up\&Out\ Call_{price} = A - B + C - D$

Substituting the values, which we already know, in the formula:

$$Price = \frac{N}{(1 + YTM_{bond})^T} + Up\&Out\ Call_{price}(long) - Up\&Out\ Call_{price}(Short)$$

We obtain the following price of our product:

$$Price = 990.76 + 3.49 - 0.36 = 993.89\,USD$$

Sensitivity analysis

It is worth mentioning that the price calculated above is subject to fluctuations, as we are preparing this structured product about 25 days before its issuance. Therefore, we should test the sensitivity of our options' prices to the following market risk factors:

- Underlying asset price S_t,
- Volatility σ,
- Risk-free interest rate r,
- Cash flows (dividend yields etc…) q.

We assume other parameters to be the same as specified at the beginning of this project, therefore:

- The barriers H_1 and H_2 of the options are 110 USD and 100 USD for the long and short option respectively,
- The strike prices K_1 and K_2 of the options are 70 USD and 80 USD for the long and short option respectively,
- Time to maturity T equals 3 years.

Below are the plots which show the sensitivity of the American Up & Out Call option which we purchase to the above-mentioned risk factors:

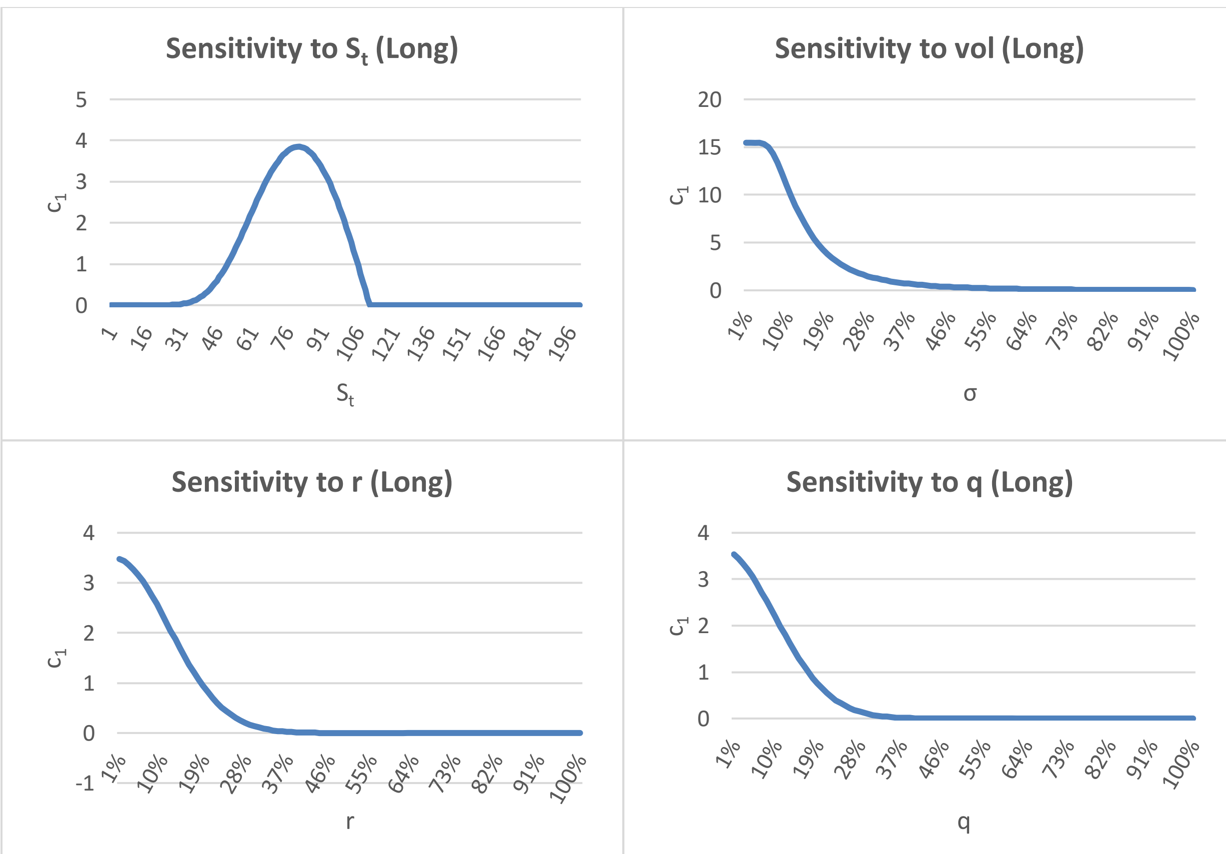

Figure 4. Sensitivity analysis for the Long American Up & Out Call option.

As we can see, the sensitivity analysis to the spot price presents a concave shape, with the price of the option increasing until the value of the underlying asset reaches approximately 80 USD, and then starting to decrease until the value of 110 USD; afterwards, the price of this option keeps being 0, as the barrier was hit.
A low volatility indicates initial impressive prices of our option, which soon start to decade, approximating 0 towards the highest values of volatility.
The lowest the r, the higher the value of the option, the same can be said relating to q.

Similarly, here are the plots which show the sensitivity of the American Up & Out Call option which we short, to the above-mentioned risk factors:

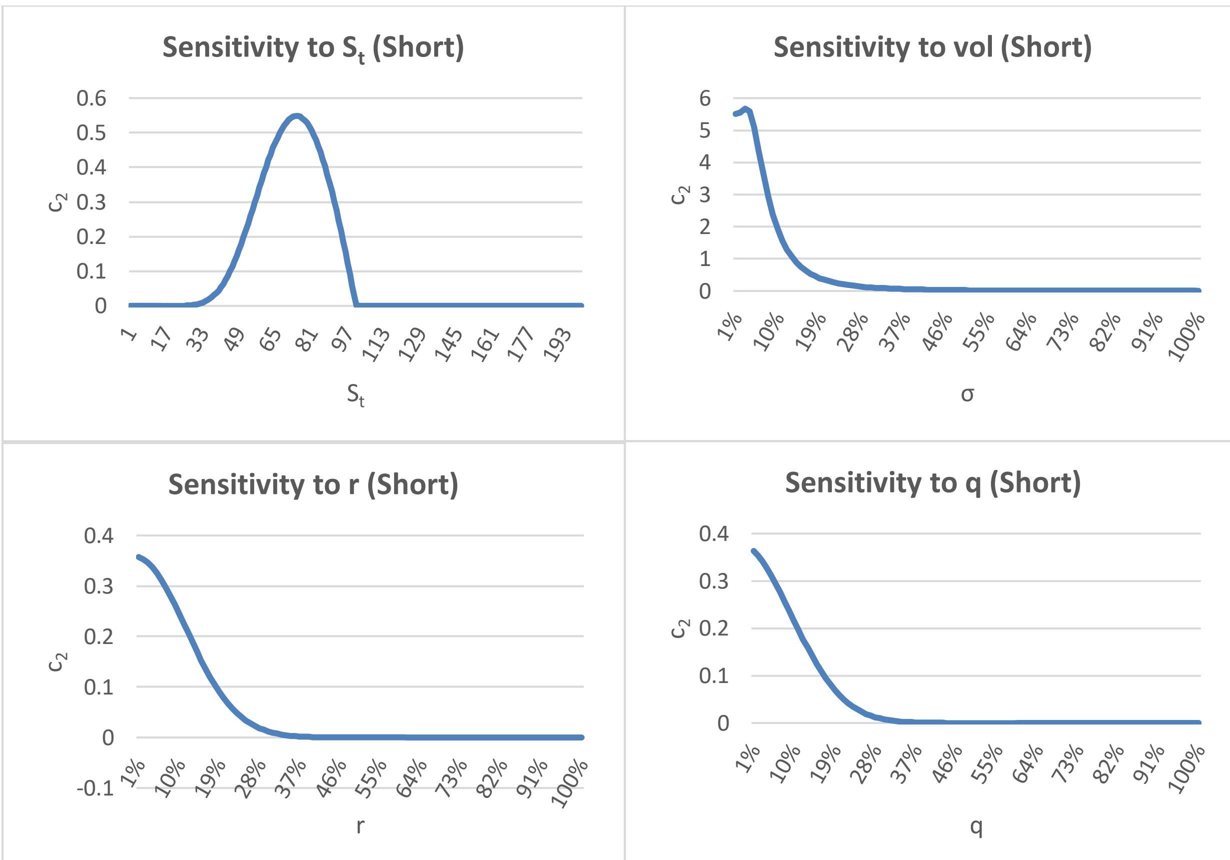

Figure 5.Sensitivity analysis for the Short American Up & Out Call option.

We can easily notice that the behavior followed by the price of this option depending on the value of the market risk factors resembles the one followed by the purchased Up & Out Call option. Perhaps, the only difference which is easily visible is the small spike in the price when volatility approximates 5%.

Finally, here is the change in price of our structured product depending on the changes in underlying asset price, volatility, risk-free rate of interest, and dividend yield, plotted taking into account the price of the bond as fixed:

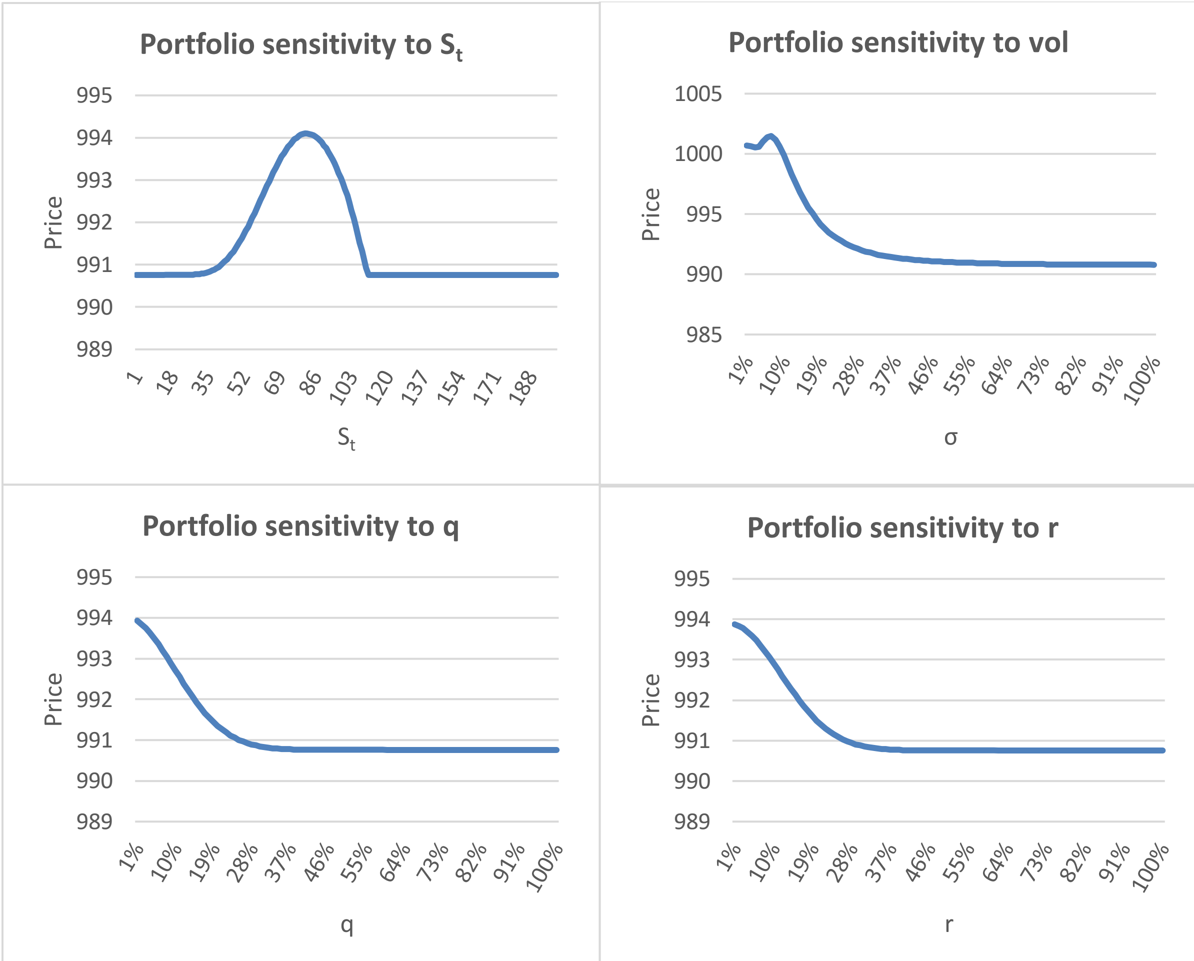

Figure 6. Sensitivity analysis for the price of the structured product.

As we can see, our product was in built in such way, that no matter what change may occur to underlying asset price, volatility, risk-free interest rate or dividend yield, the price will never be higher than 1,000 USD, apart from the case in which volatility ranges between 1% and 9%. Currently, MKC's volatility is estimated above 19%, therefore, it's unlikely that it will drop so drastically within the next 25 days. Also, the current market condition forces us to spend 993.89 USD on the structured product, however, there are market conditions that could be more favorable in the future.

For example, a volatility, risk-free rate, or dividend yield greater than approximately 30% would allow us to spend the least on the product, approximately 990.76 USD. However, it would be almost impossible to meet such market conditions, especially in terms of risk-free interest rate and dividend yield.