Geometric Modeling

Assignment 4: Mesh Parametrization

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Acknowledgements: Olga Diamanti

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Administrative Details

- HW4 extended 1 week:
  - Due 4/18/2017, 11:59PM
  - Demo session still 4/26/2017, 2-3PM
- HW5 out next week

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Assignment 4: Mesh Parametrization

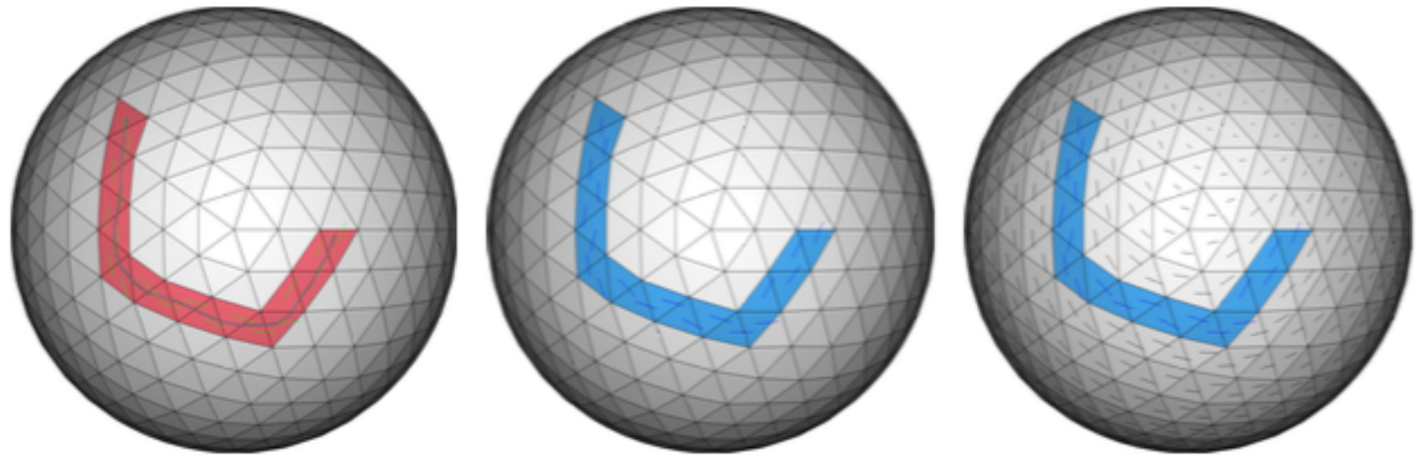
Handout date: 03/27/2017
Submission deadline: 4/11/2017, 11:59PM
Demo session: 4/26/2017, 2-3PM

In this exercise you will

- Familiarize yourself with vector field design on surfaces.
- Create scalar fields whose gradients align with given vector fields as closely as possible.
- Experiment with the `libigl` implementation of harmonic and least-squares conformal parameterizations.

1. Tangent vector fields for scalar field design

Our first task is to design smooth tangent vector fields on a surface; these will be "integrated" later to define a scalar field. A (piecewise constant) vector field on a triangle mesh is defined as an assignment of a single vector to each triangle such that each vector lies in the tangent plane containing the triangle. We will design fields to follow a set of alignment constraints provided by the user: the user specifies the field vectors at a subset of the mesh triangles (the constraints) and those constraints are interpolated smoothly throughout the surface to define a field.



(A) Selection (B) Constraints (C) Vector Field

Figure 1. Selecting faces and assigning vector constraints via the UI; interpolated vector field.

1.1. **Creating vector constraints.** The provided assignment code already implements a minimal, stroke-based interface for assigning vector constraints at faces. To use it, enable `selection_mode` from the Nanogui and drag with your mouse to draw a stroke over the surface. The faces below the

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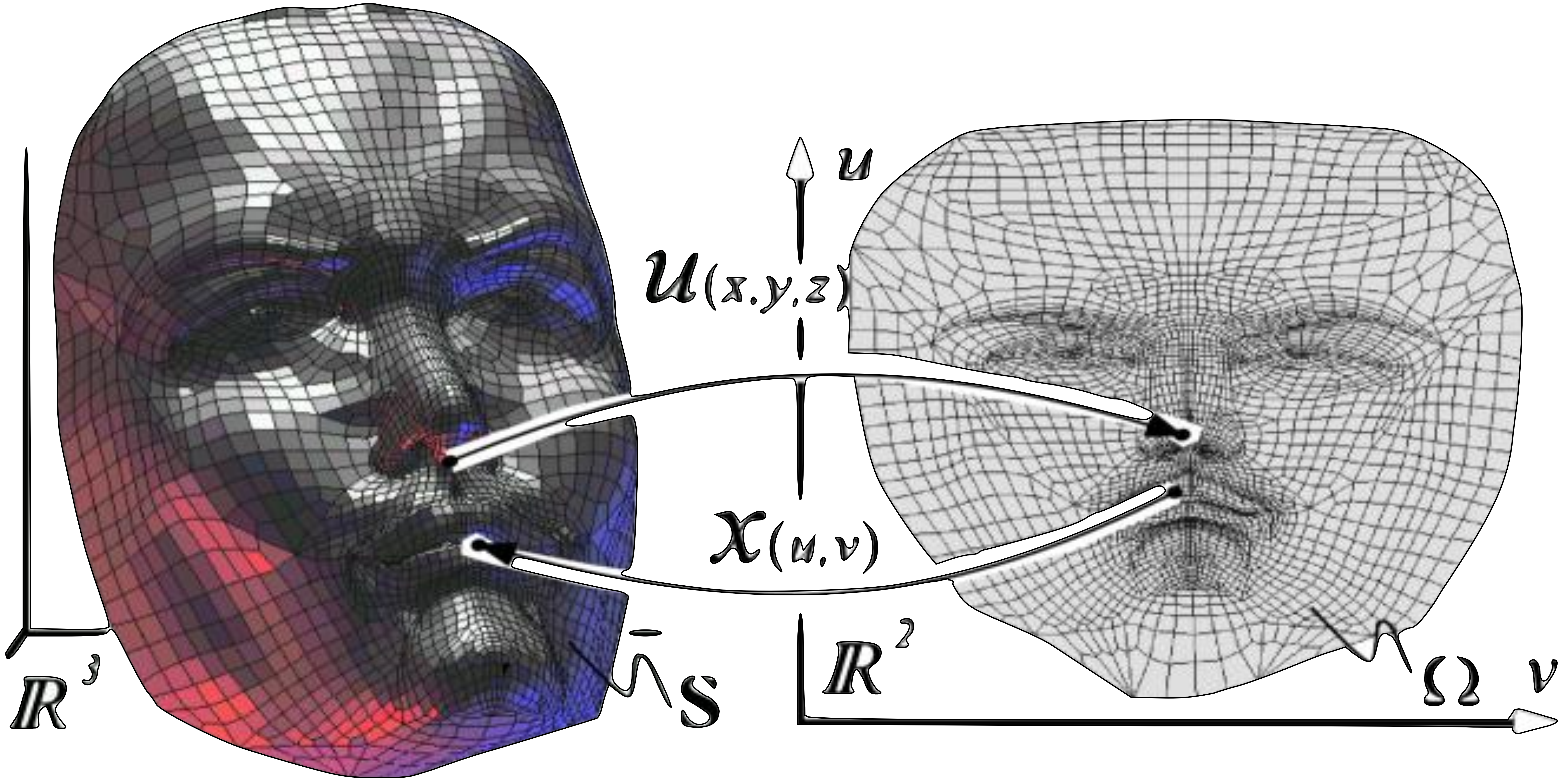
Mesh Parametrization



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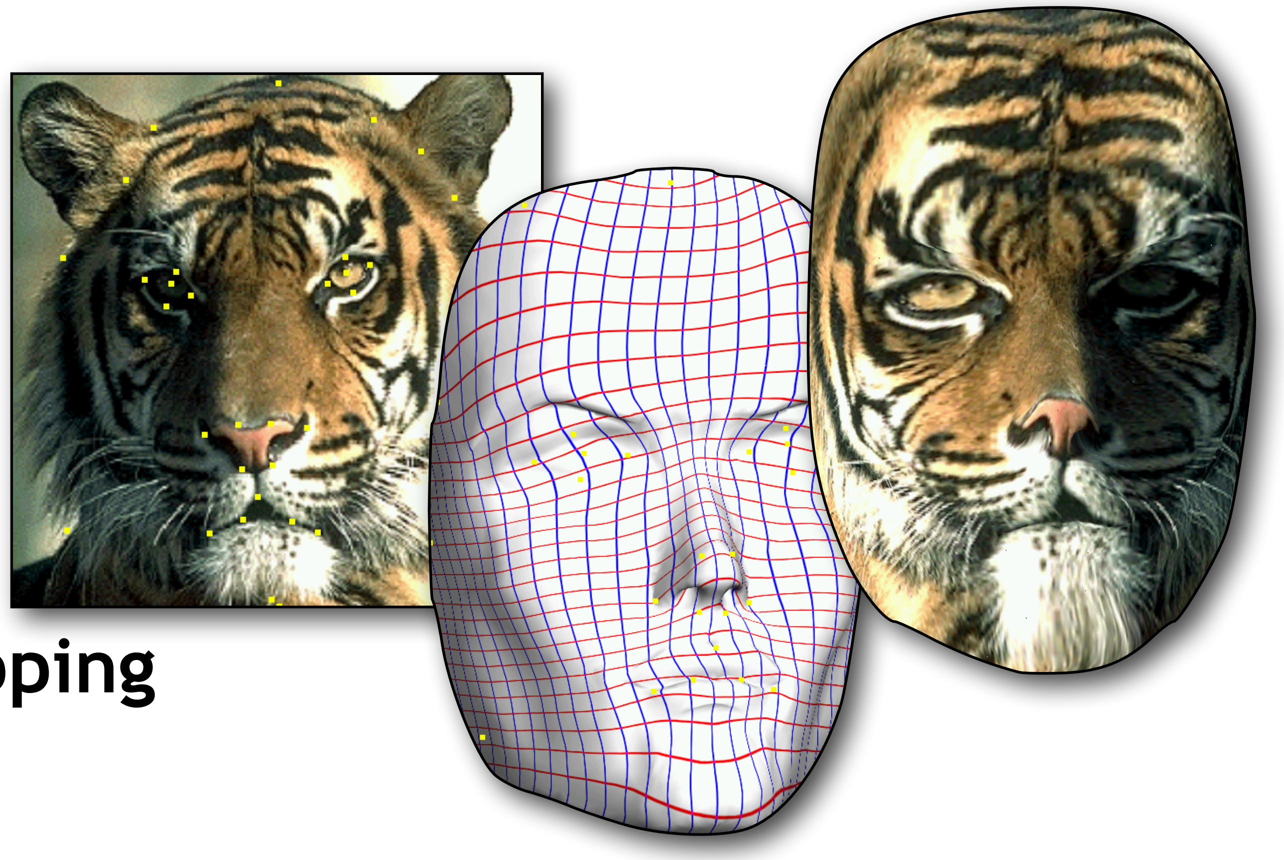
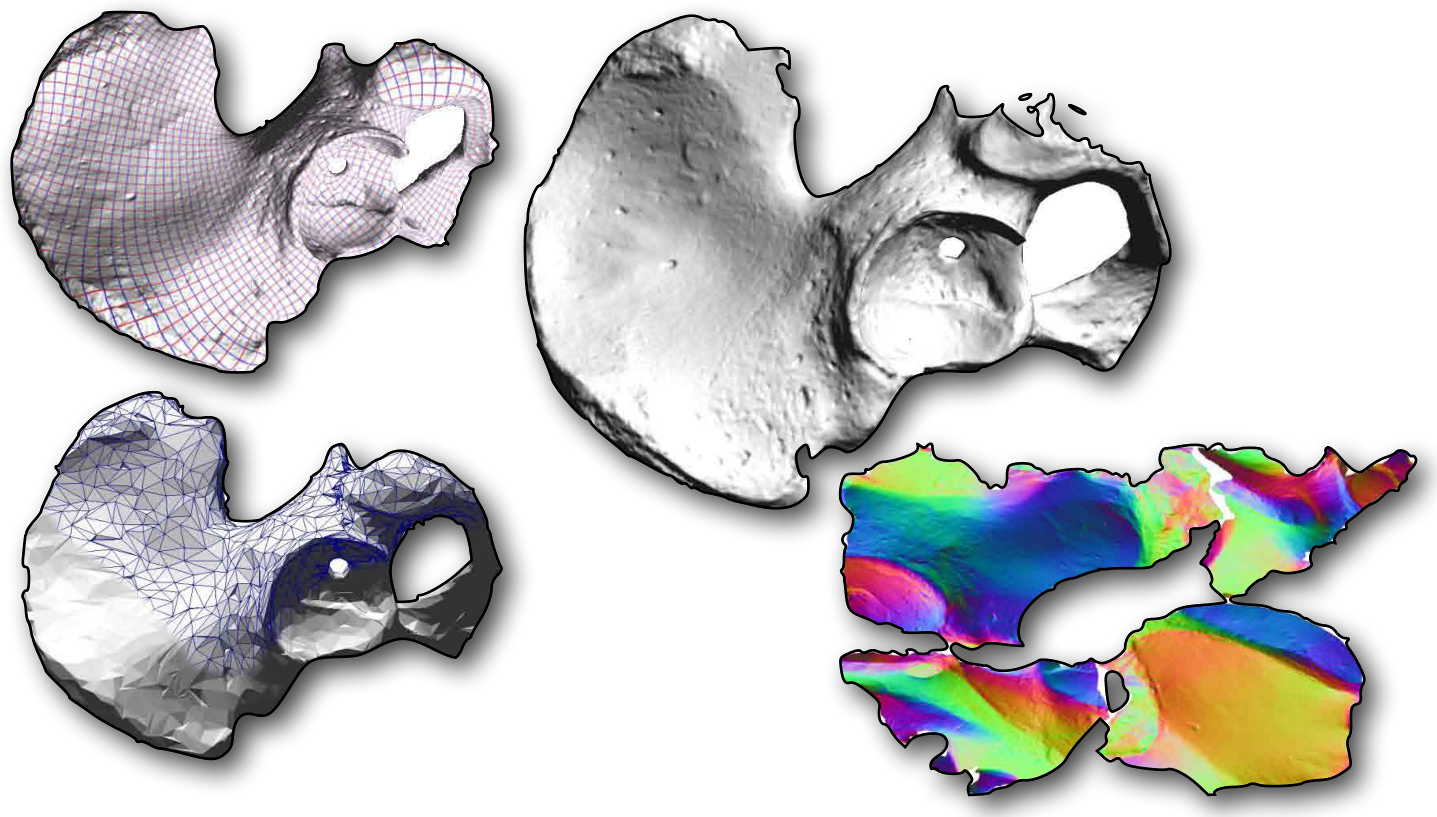
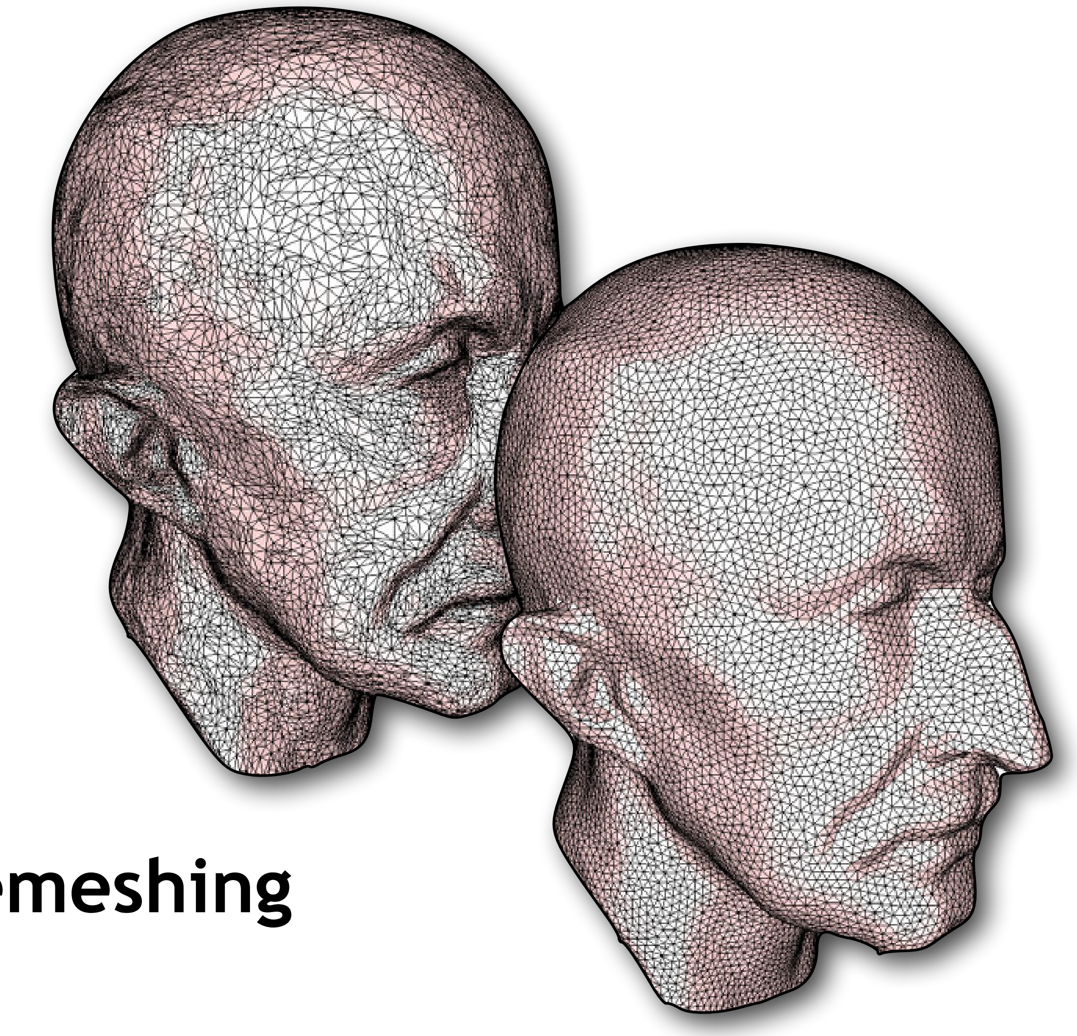
Parametrization Applications



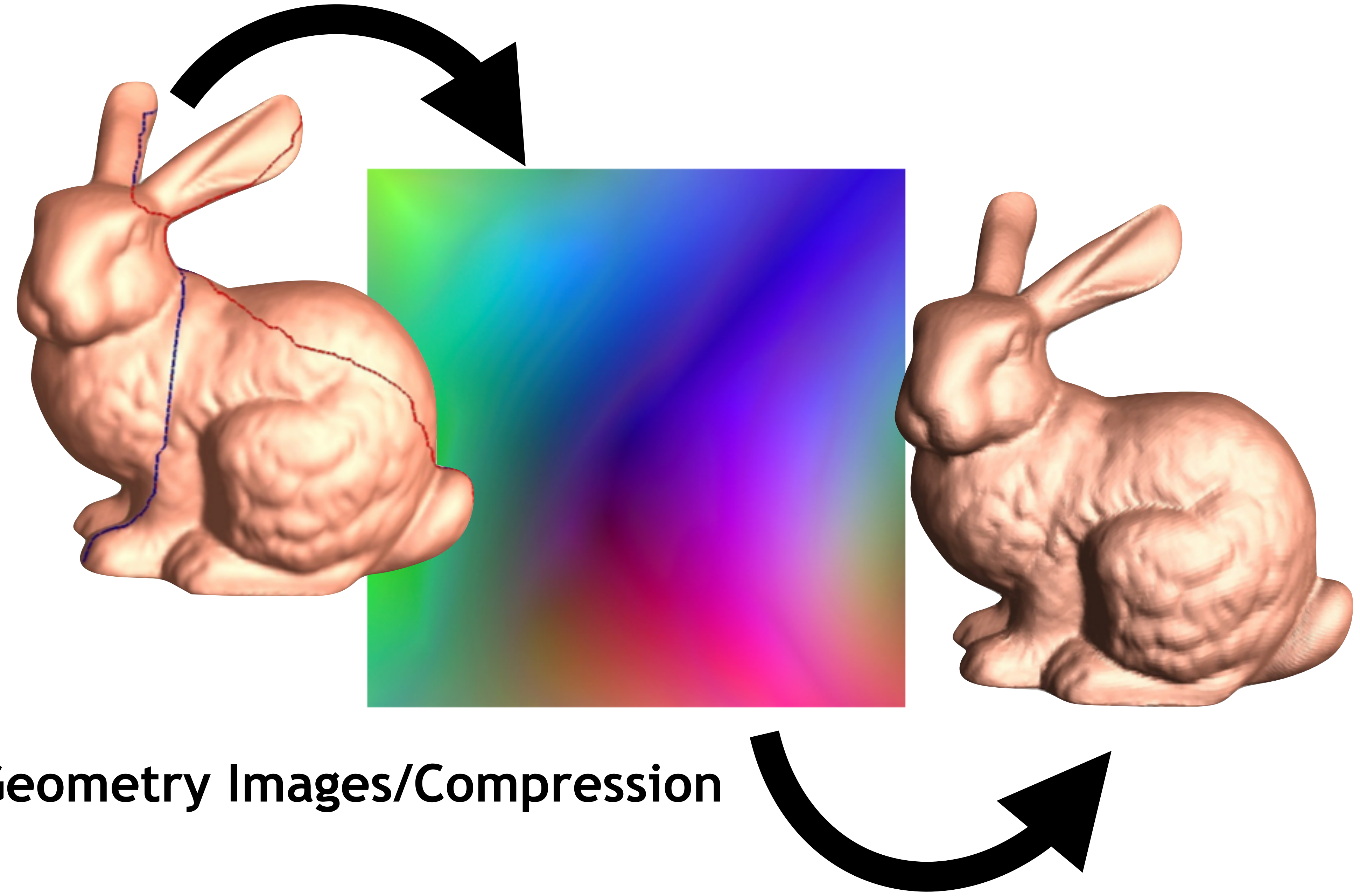
Texture Mapping



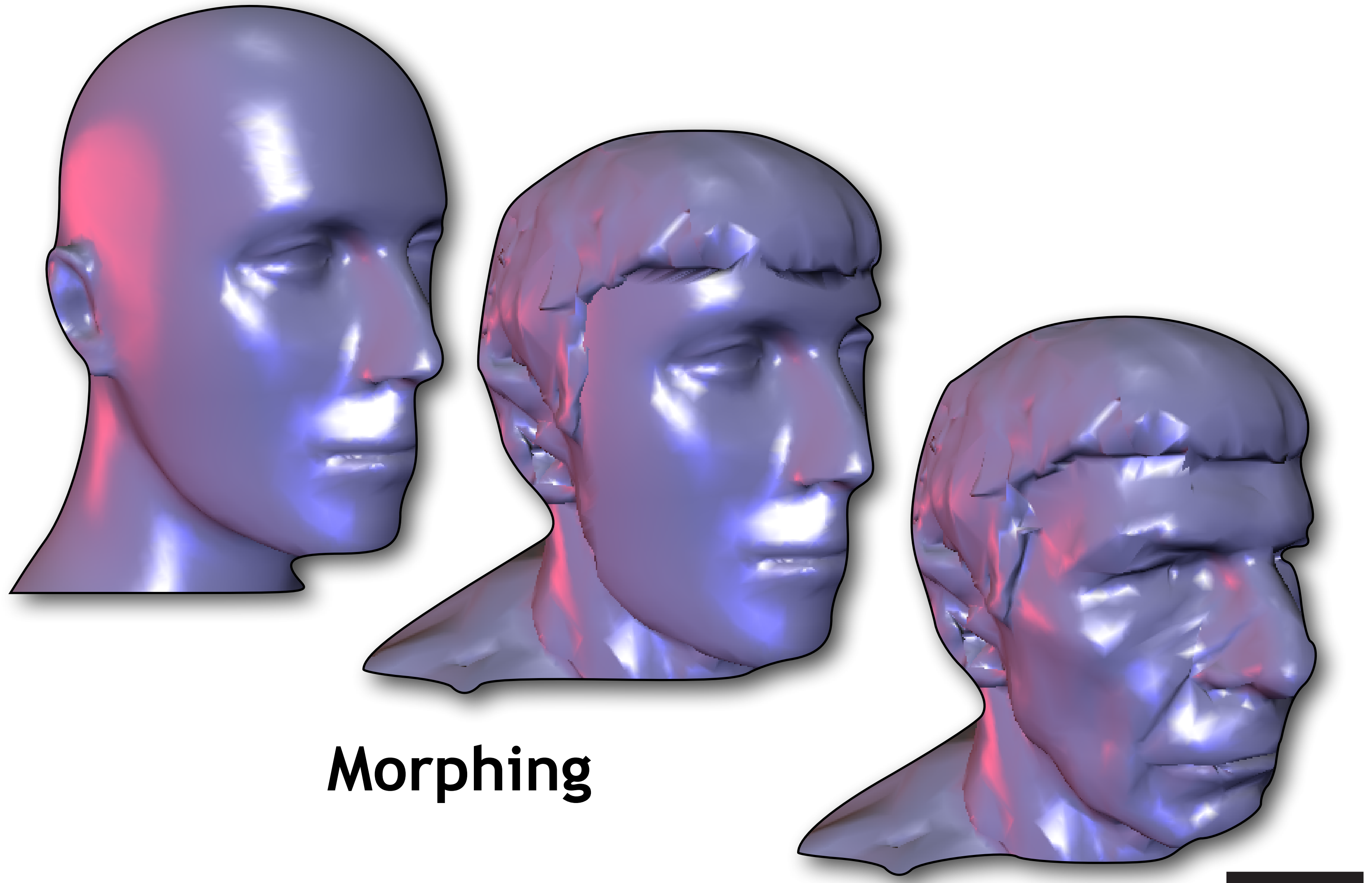
Normal/Bump Mapping



Remeshing



Geometry Images/Compression



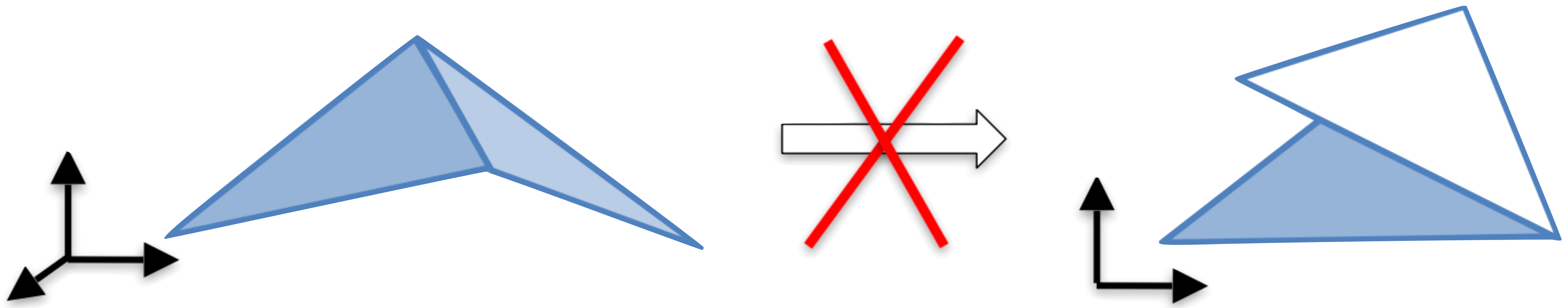
Morphing

- Detail Transfer
- Mesh Completion
- Editing
- Surface Fitting

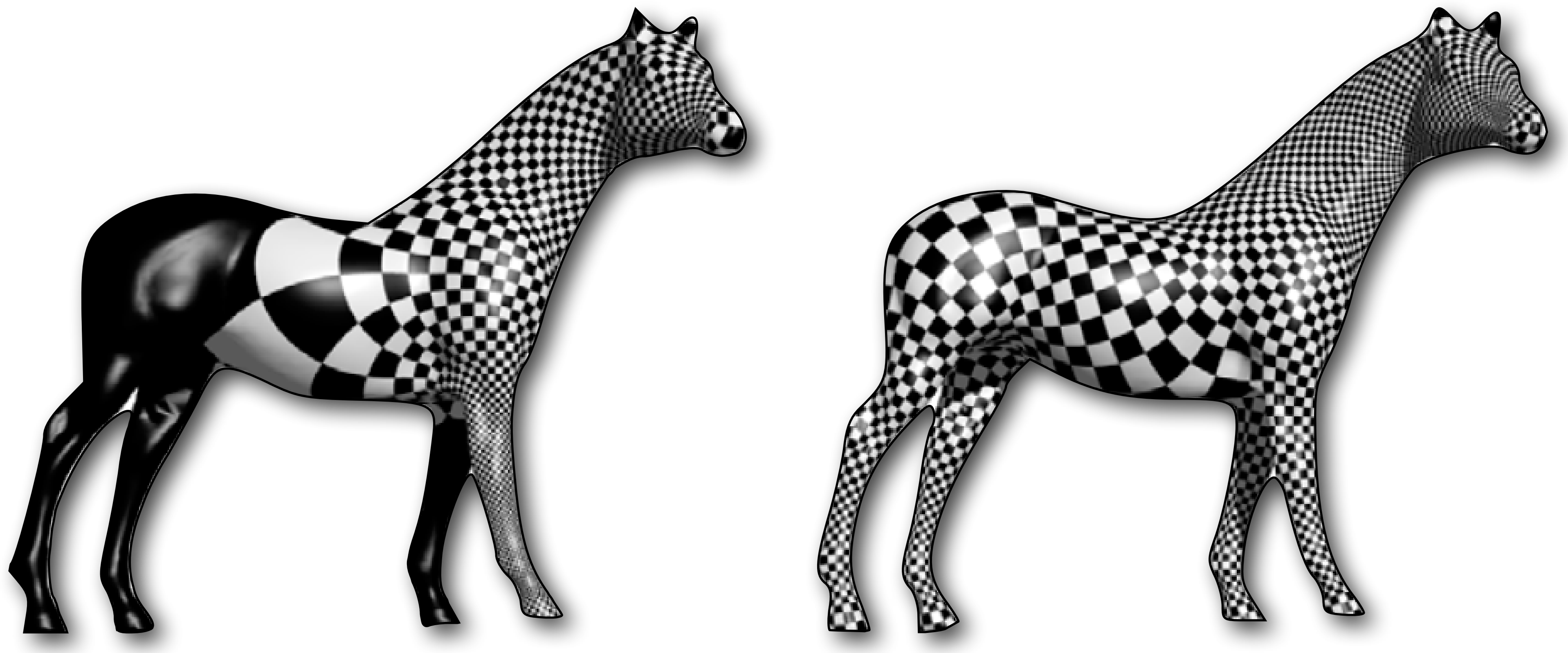
...

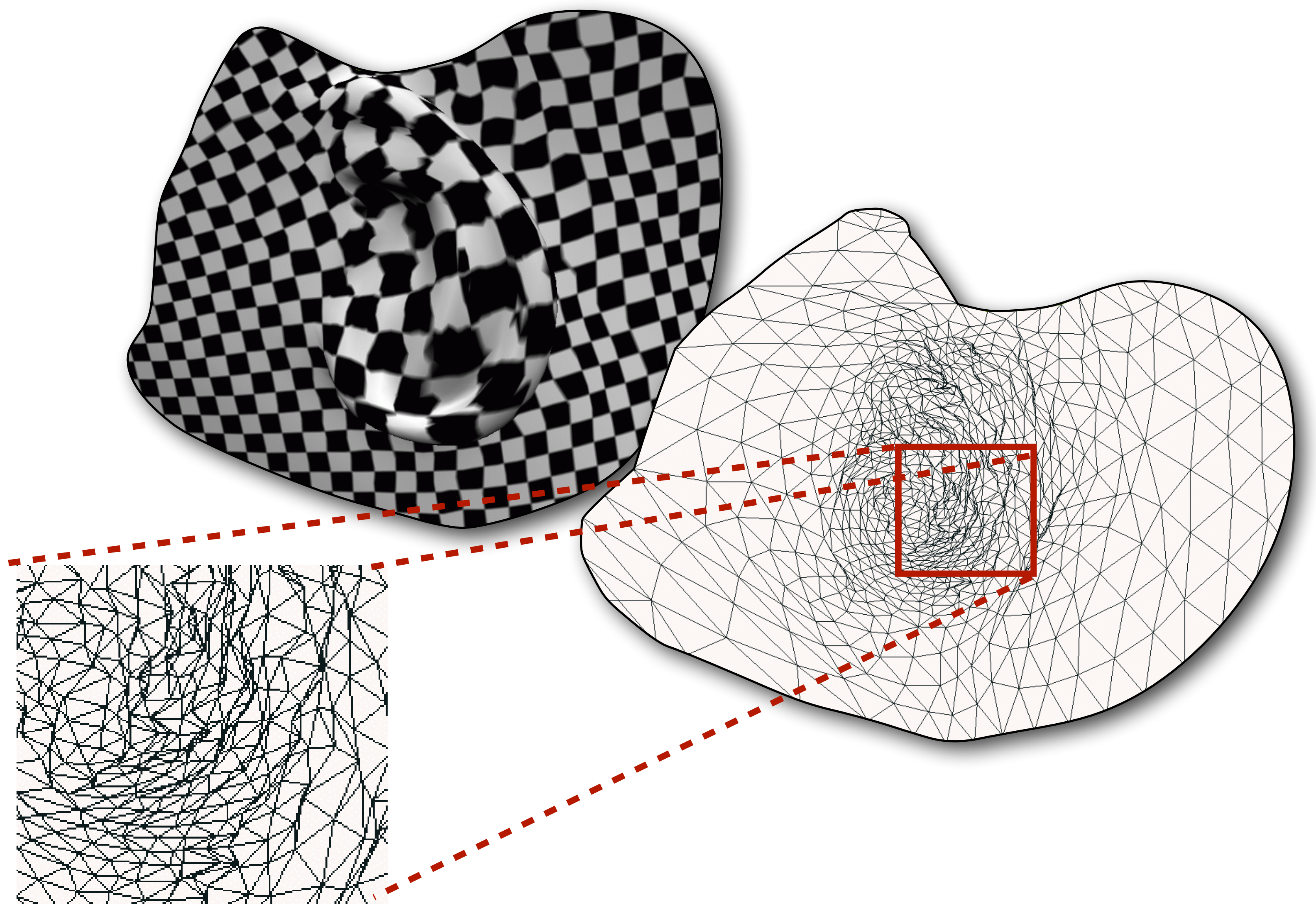
Desirable Properties

- Bijective (invertible): avoid flips/overlaps



- Minimal distortion
  - –Area
  - –Angle





- Efficiently computable

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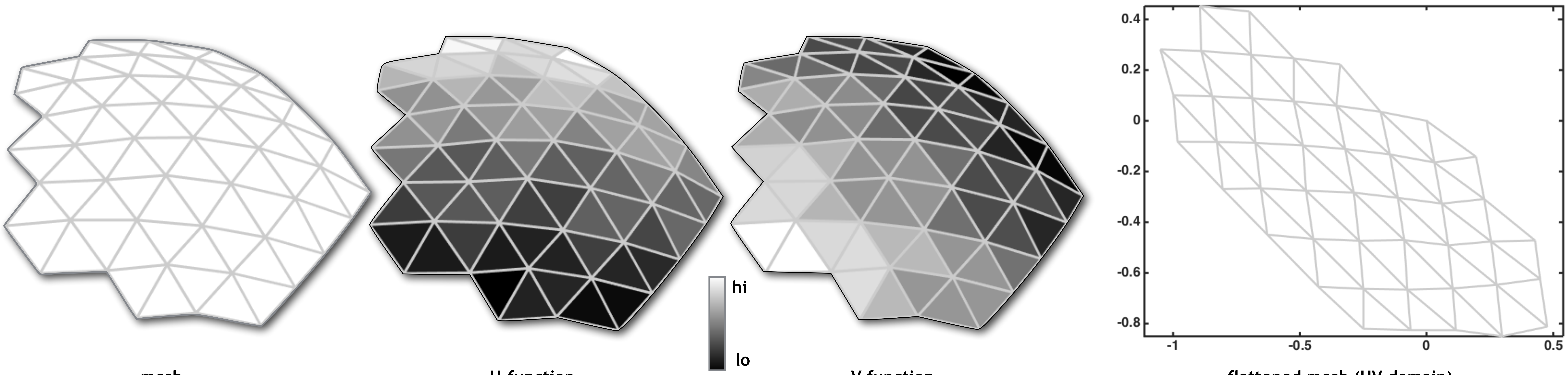
Parametrization Approaches

- This assignment will consider 3 approaches
  - Least-Squares Conformal (LSCM)
  - Harmonic
  - Field-Guided
- LSCM, Harmonic implemented in libigl

Formulation: Design Scalar Fields

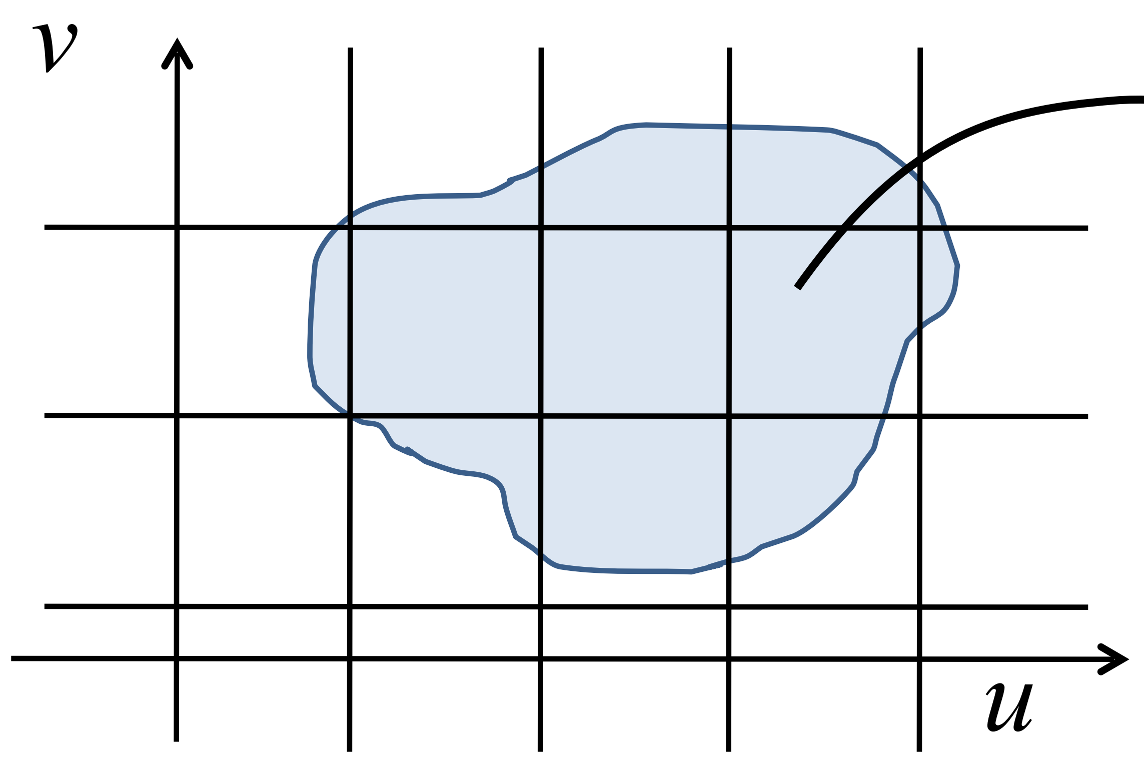
- Compute two scalar fields (u, v) giving the plane coordinates of each mesh point

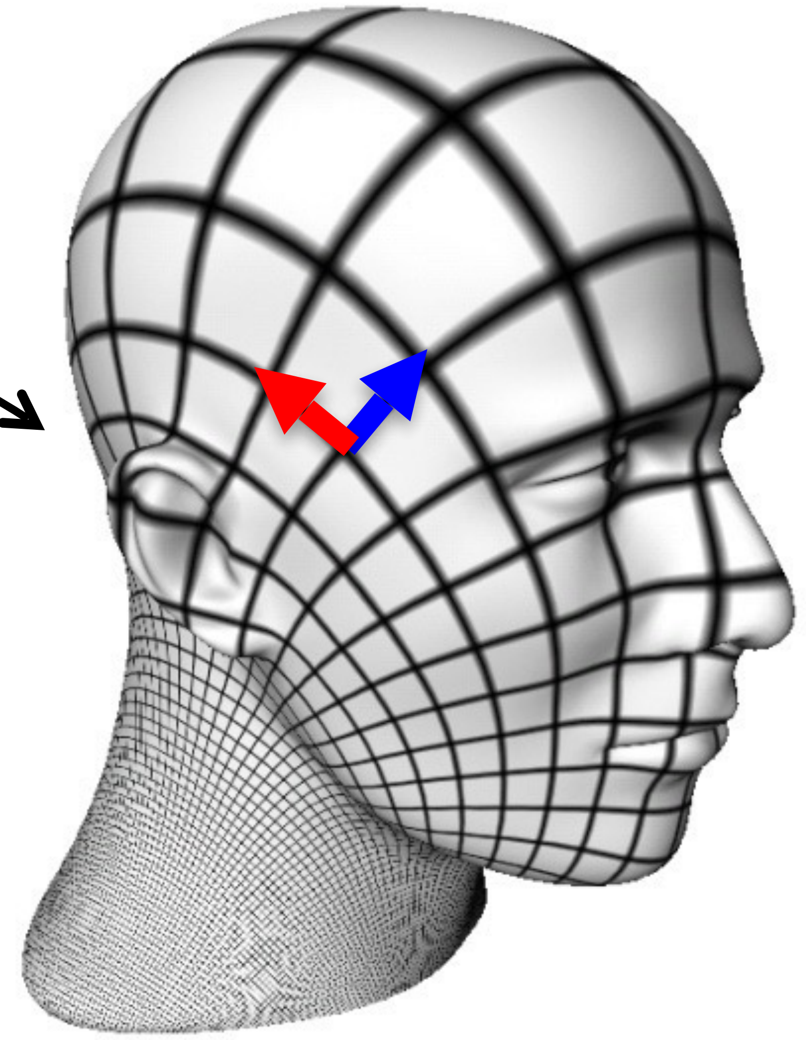


mesh U-function V-function flattened mesh (UV-domain)

Least-Squares Conformal

- **Conformal** maps have **zero angle distortion** (preserve 90° angles)





- Least-Squares Conformal: minimize angle distortion

$$\min_{u,v} \int_M \|\nabla u^{\perp} - \nabla v\|^2 \, \mathrm{d}V$$

- Encourage perpendicular, equally scaled coordinate vectors

Least-Squares Conformal

$$\min_{u,v} \frac{1}{2}\int_M \|\nabla u^\perp - \nabla v\|^2 \, \mathrm{d}V$$

$$= \min_{u,v} \frac{1}{2}\int_M \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)^2 \, \mathrm{d}V$$

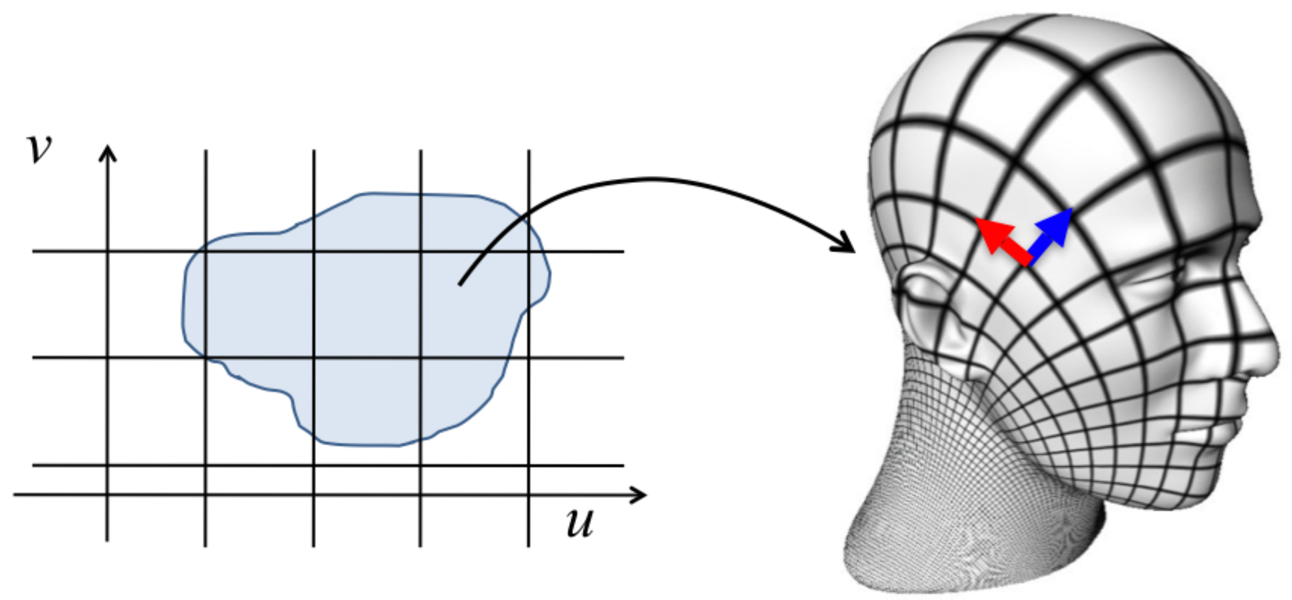
$$= \min_{u,v} \frac{1}{2}\int_M \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 - 2\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \mathrm{d}V$$

$$= \min_{u,v} \frac{1}{2}\int_M \|\nabla u\|^2 + \|\nabla v\|^2 - 2\left|\frac{\partial(u,v)}{\partial(x,y)}\right| \mathrm{d}V$$

Jacobian Determinant

$$= \min_{u,v} \underline{\frac{1}{2}\int_M \|\nabla u\|^2 \, \mathrm{d}V + \frac{1}{2}\int_M \|\nabla v\|^2 \, \mathrm{d}V - A(u,v)}$$

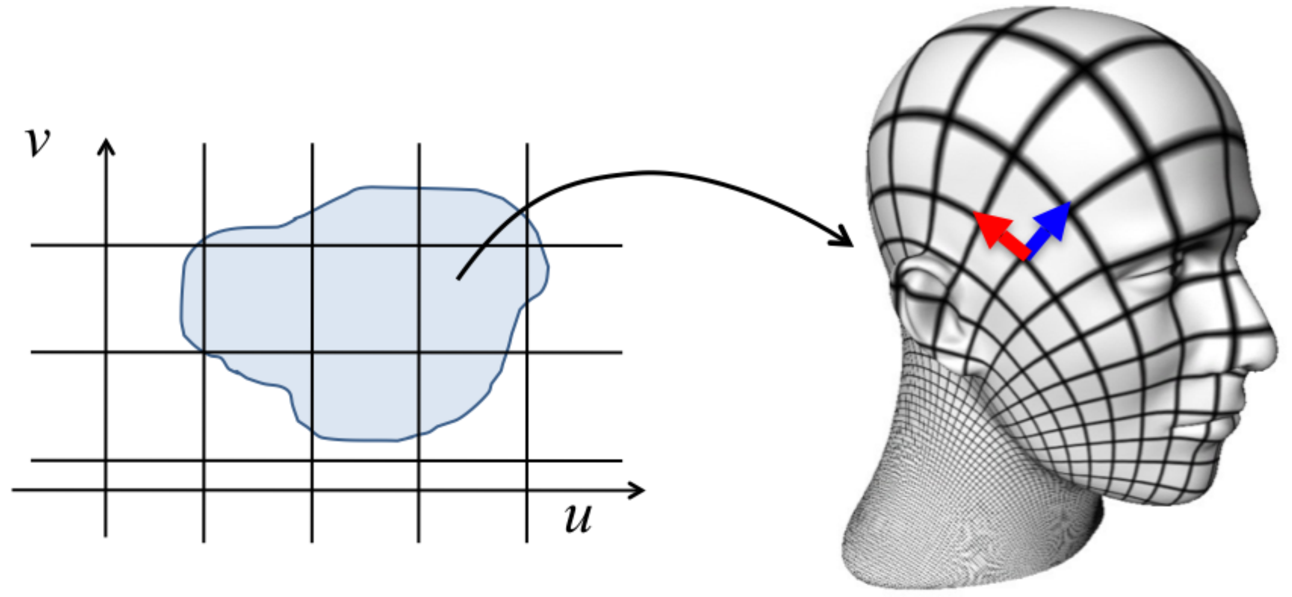
Area in Plane!



LSCM ==> Harmonic

- We showed LSCM decomposes into:

$$\frac{1}{2}\int_M \|\nabla u^\perp - \nabla v\|^2 \,\mathrm{d}V = \underbrace{\frac{1}{2}\int_M \|\nabla u\|^2 \,\mathrm{d}V + \frac{1}{2}\int_M \|\nabla v\|^2 \,\mathrm{d}V}_{\text{Smoothness of } u \text{ and } v \text{ ("Dirichlet Energy")}} - \underbrace{A(u,v)}_{\text{Area in plane}}$$



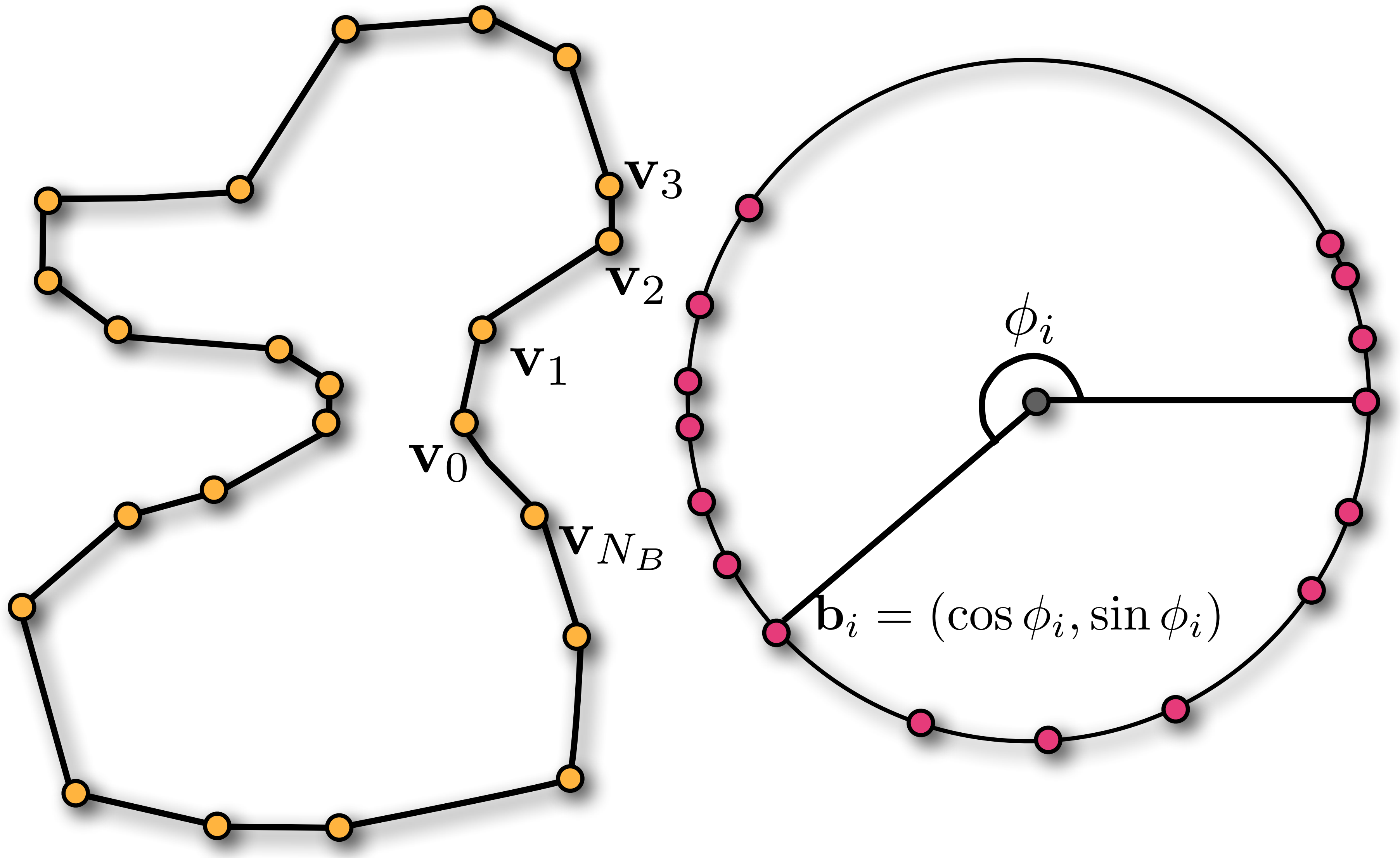
Smoothness of u and v
("Dirichlet Energy")

Area in plane

- If we **fix the flattened boundary** (area), LSCM is equivalent to designing **smooth scalar fields**

Harmonic Parametrization

- Map mesh boundary to a convex polygon in plane



- Solve for smoothly interpolating (u, v):

$$\min_{u=u_b \text{ on } \partial M} \frac{1}{2}\int_M \|\nabla u\|^2 \, \mathrm{d}V \quad \Longrightarrow \quad \begin{cases} \Delta u = 0 & \text{in } M \\ u = u_b & \text{on } \partial M \end{cases}$$

Just solve Laplace equation with fixed boundary values!

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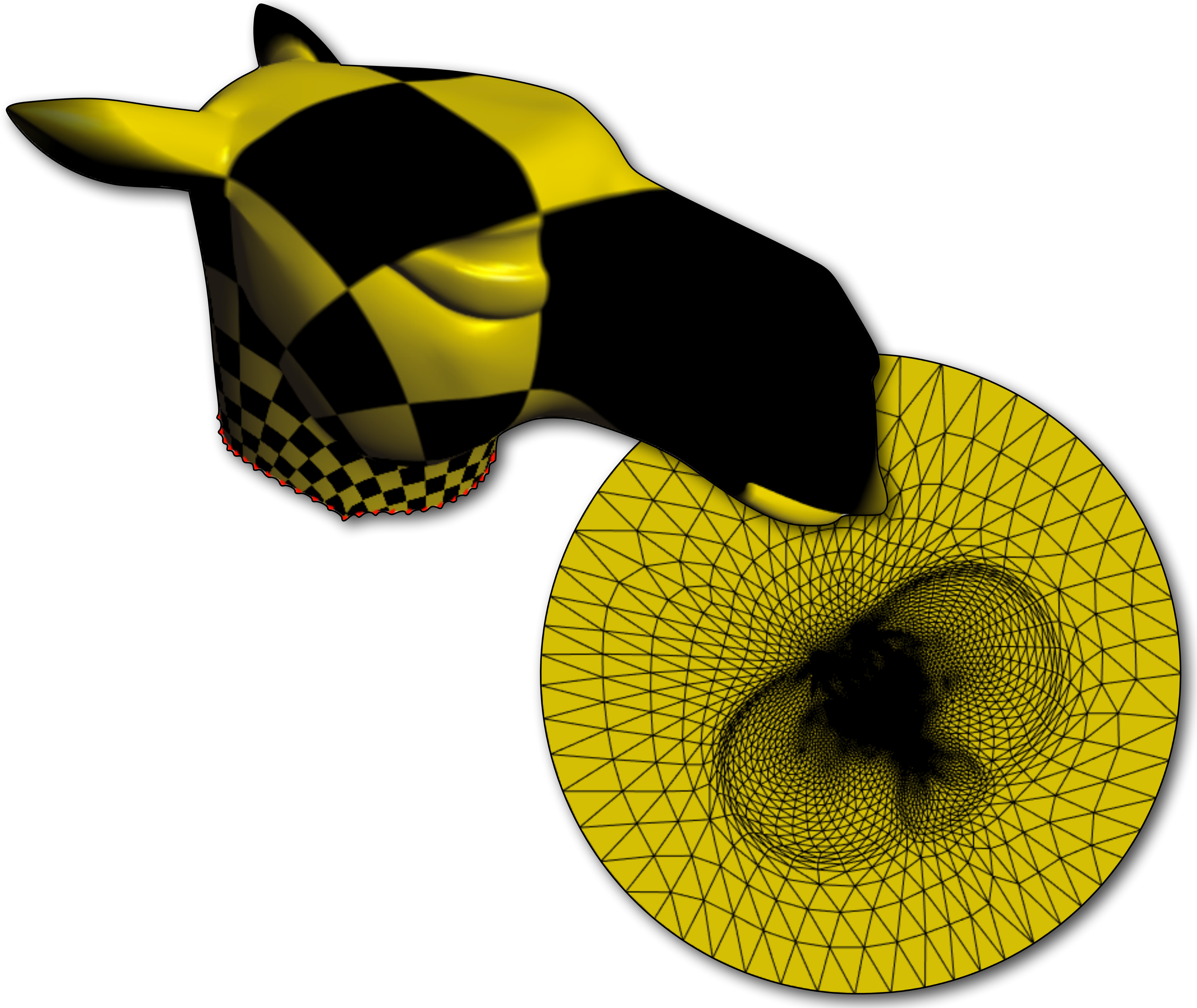
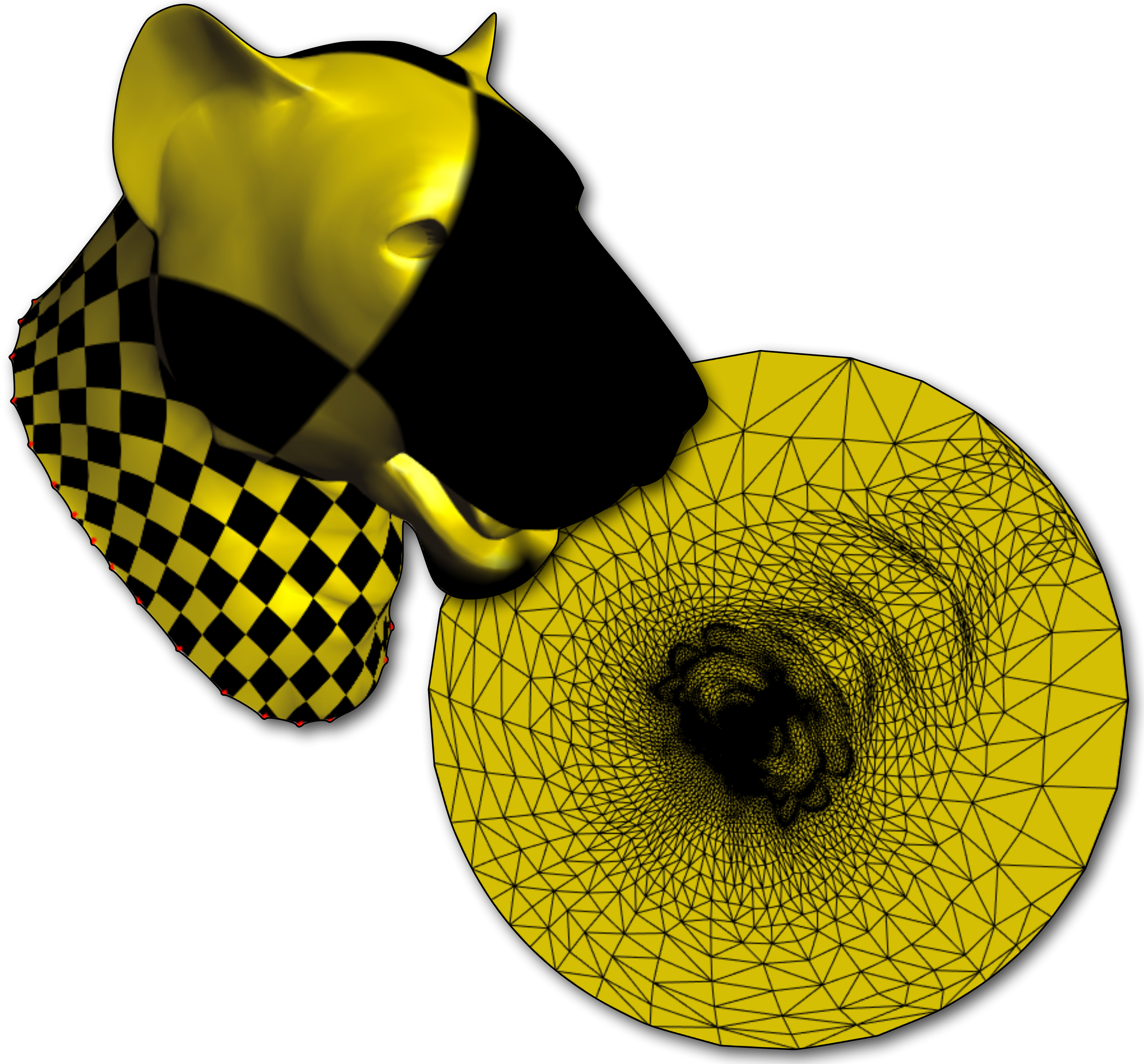
Harmonic Parametrization





Already implemented in libigl!
See tutorial 501.

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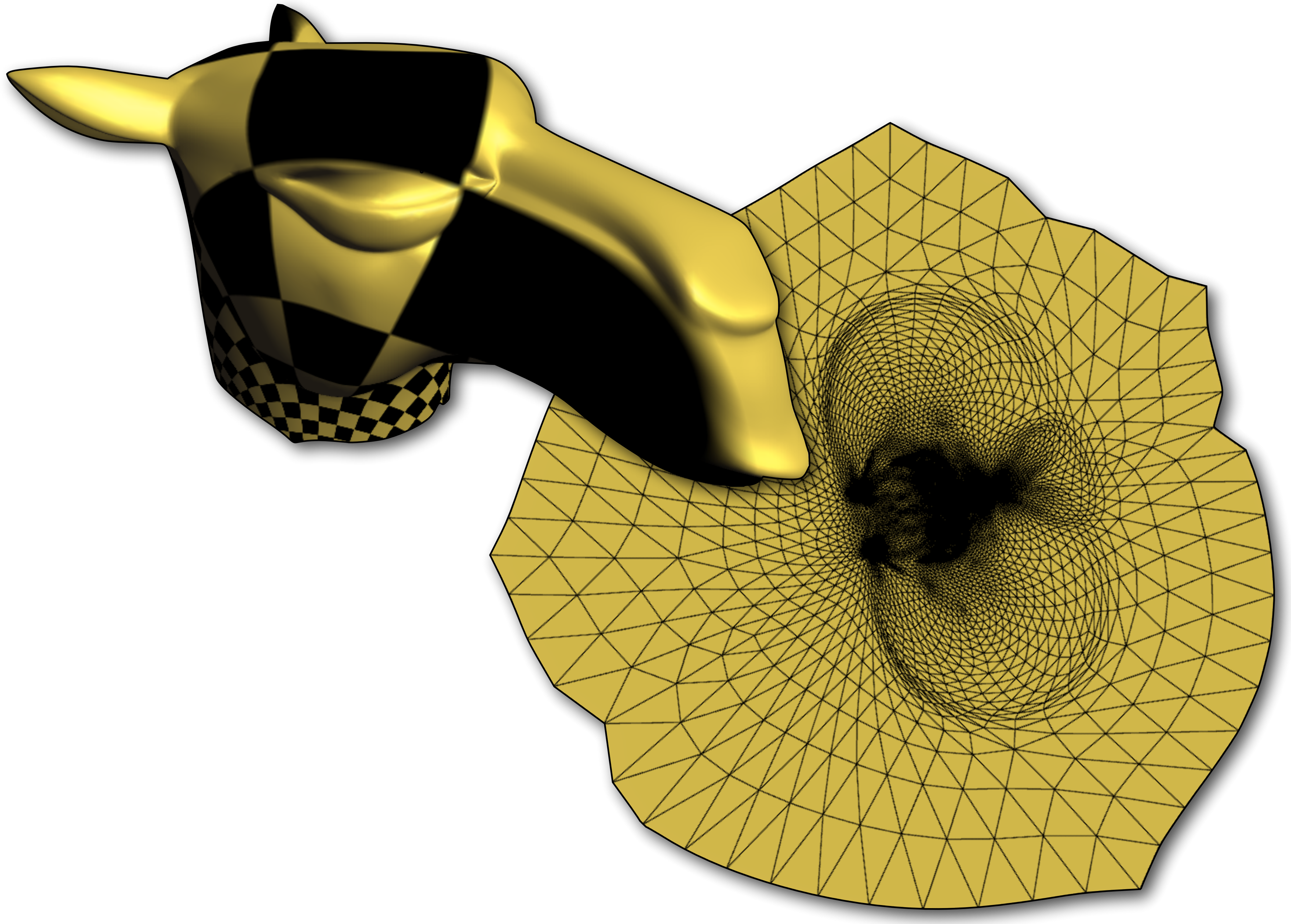
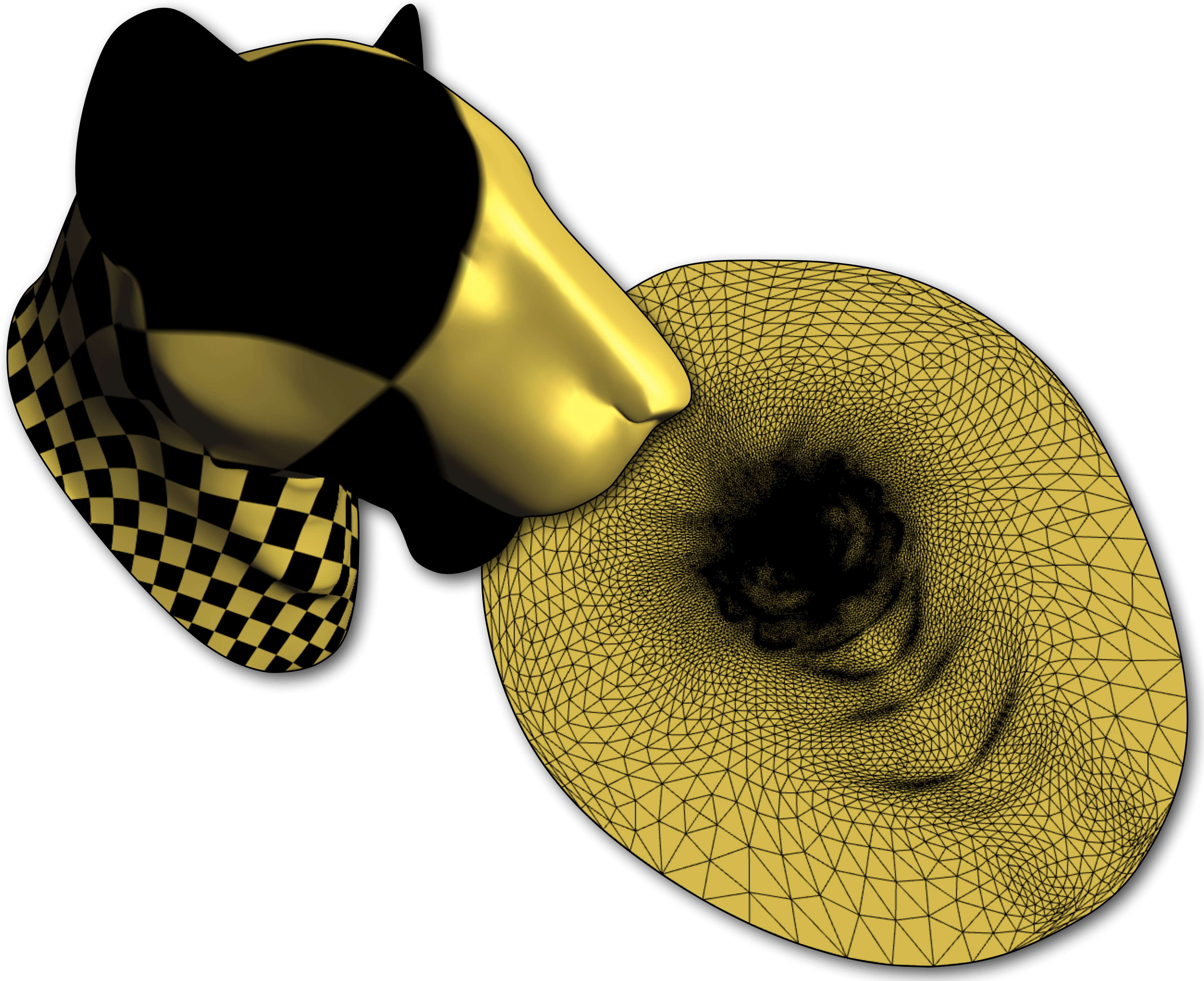
LSCM Parametrization

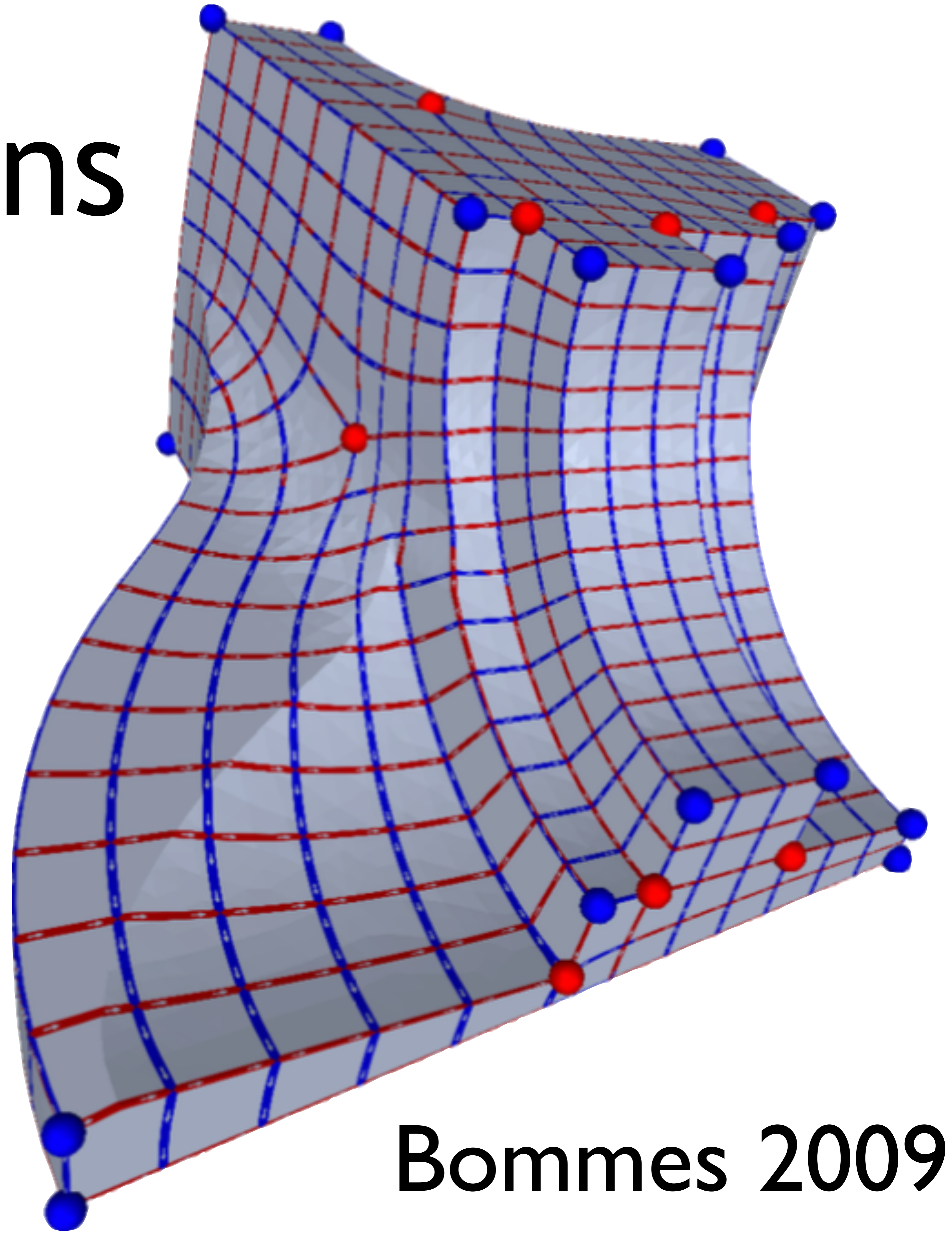




Already implemented in libigl!
See tutorial 502.

Field-Guided Parametrization

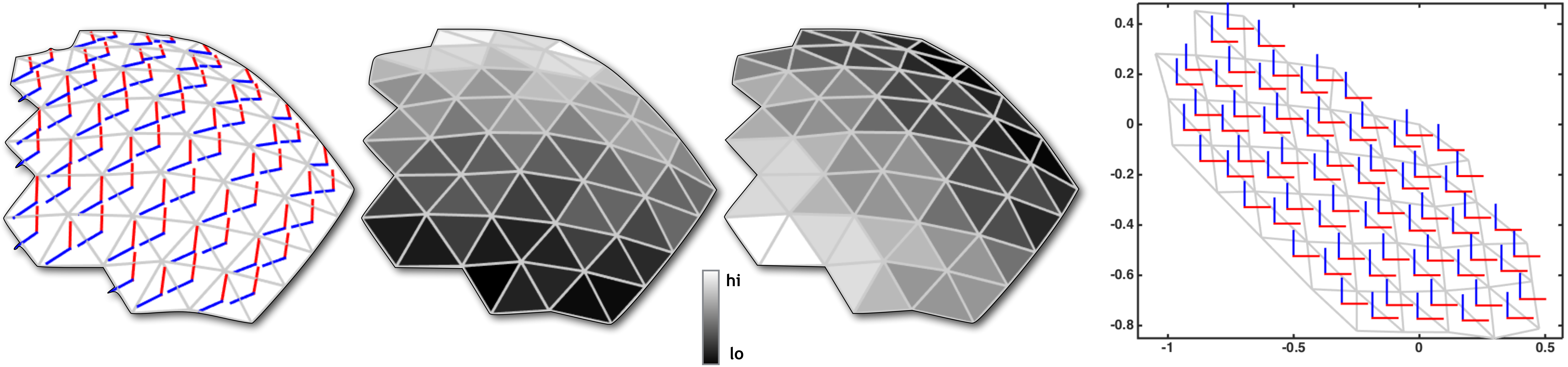
- Often we want coordinate directions to **align** with mesh features.
- We can do this by **specifying vector fields** and **fitting (u, v) gradients** (coordinate directions) to them.
- Orthonormal vectors ==> minimize **angle and area distortion**



Bommes 2009

Field-Guided Parametrization

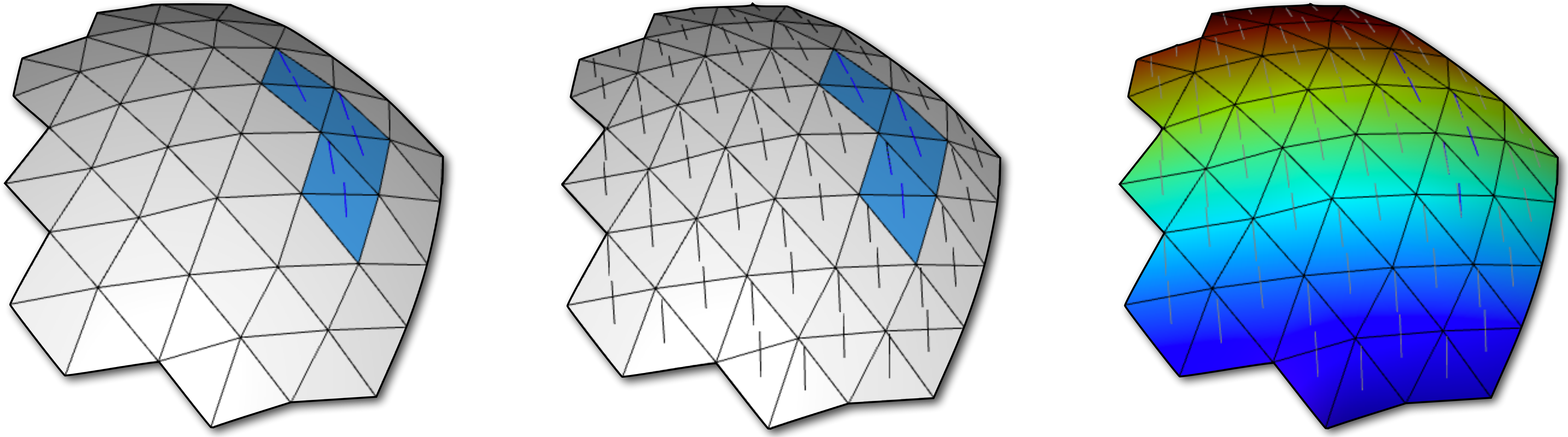
- We'll consider piecewise linear (per-vertex) scalar fields, so gradients and vector fields are constant per triangle:



mesh U-function V-function flattened mesh (UV-domain)

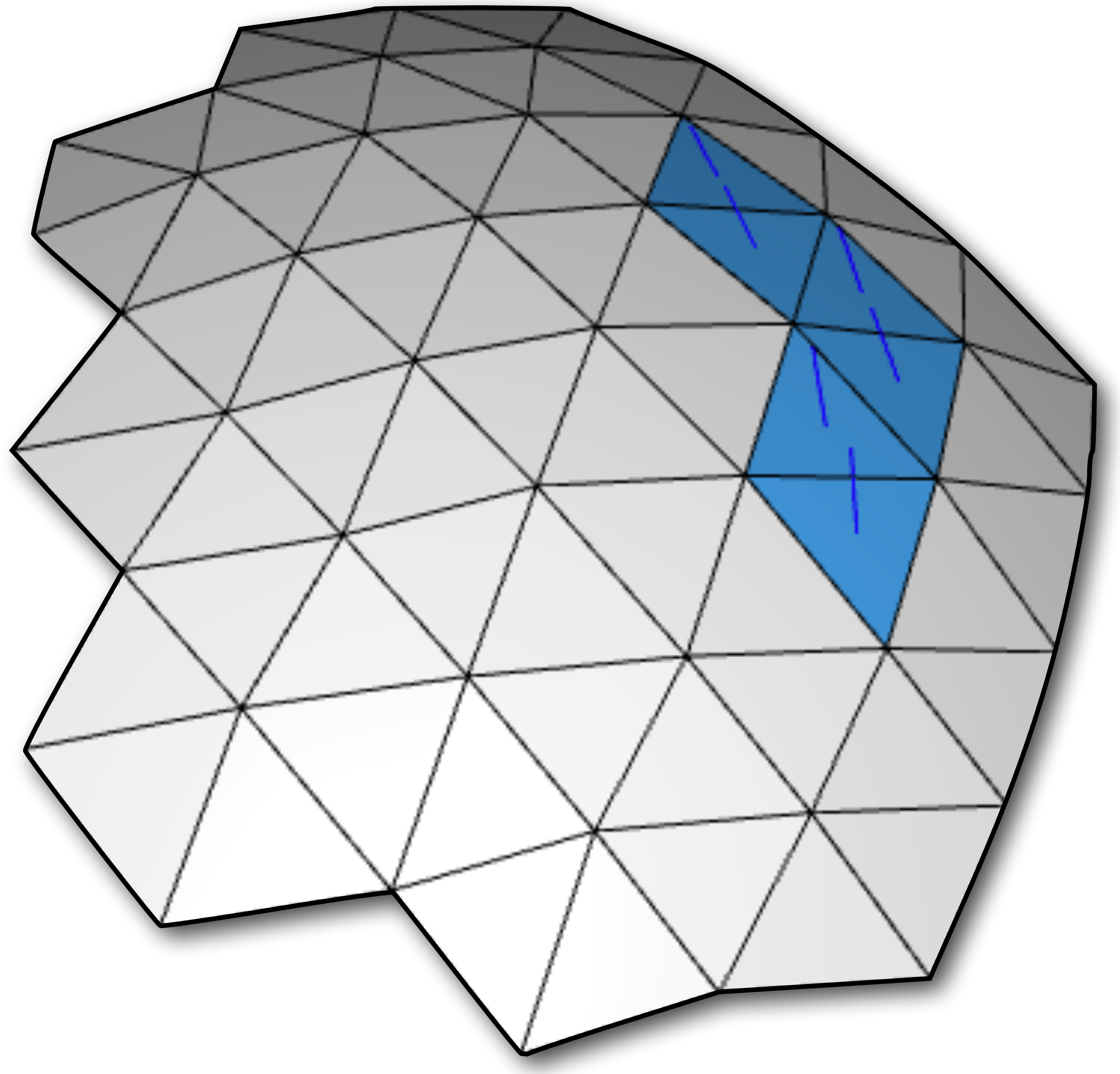
Vector-Guided Scalar Fields

- Main HW4 task: design smooth vector fields and compute a scalar field aligned to them.

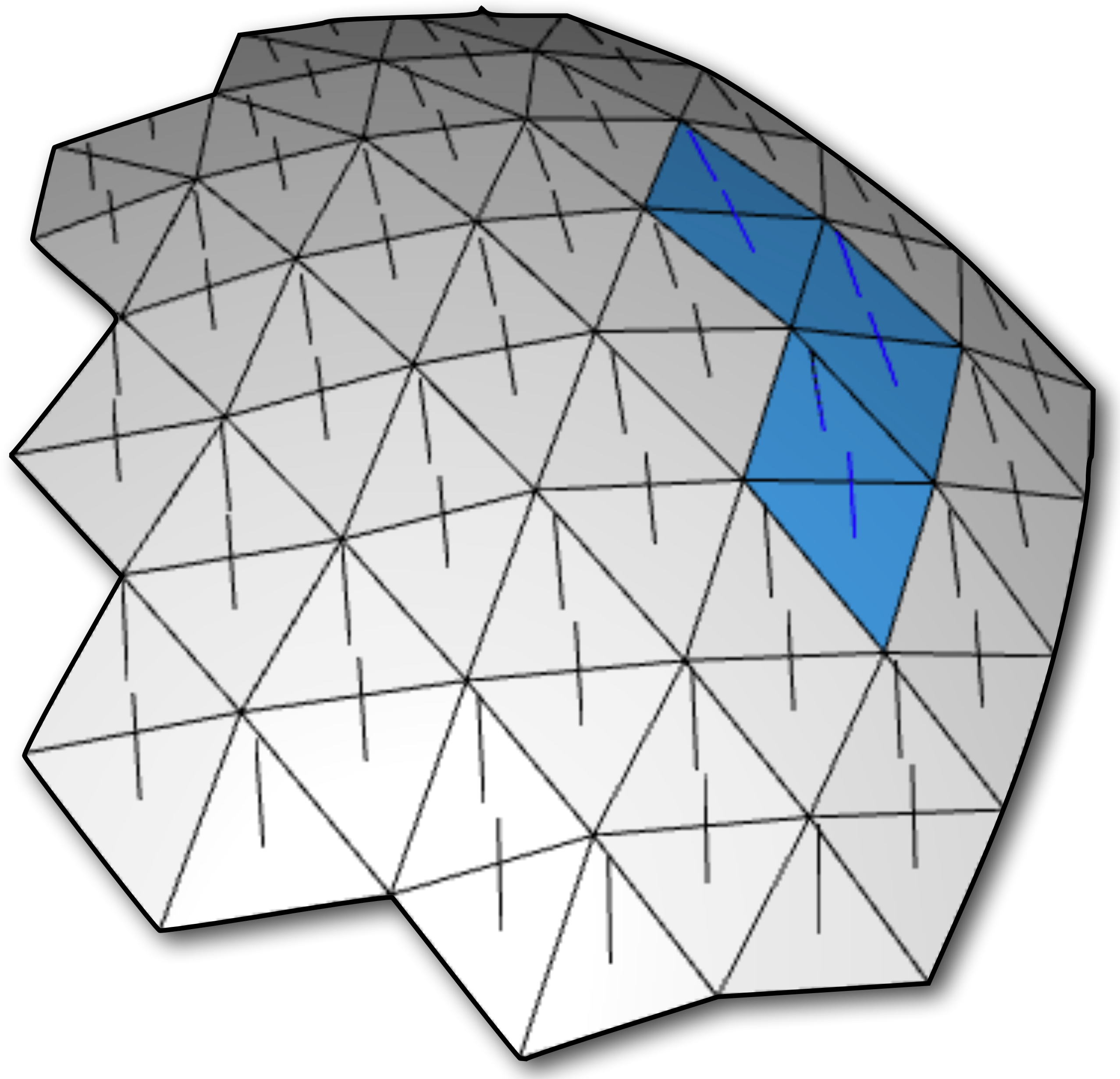


vector constraints | smooth interpolated field | reconstructed scalar function and gradient

Step 1: Vector Field Design



Paint constraint vectors on some faces



Smoothly interpolated field

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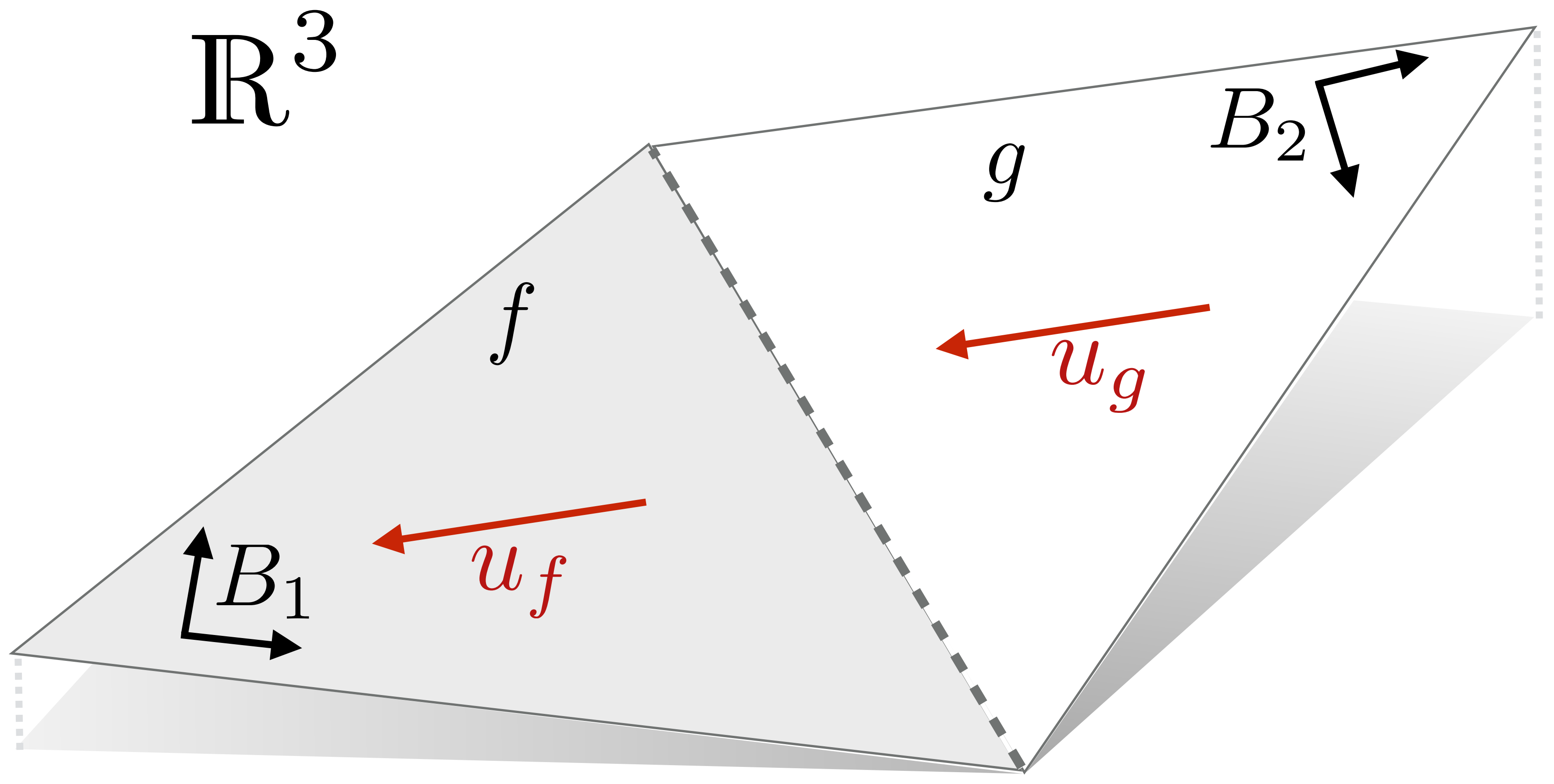
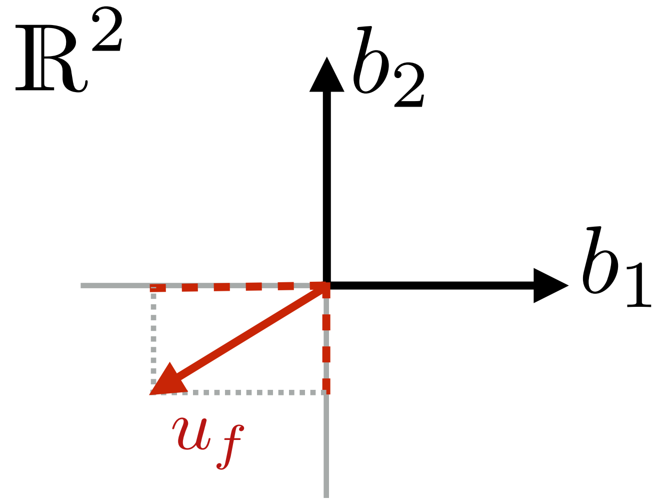
Vector Field Representation

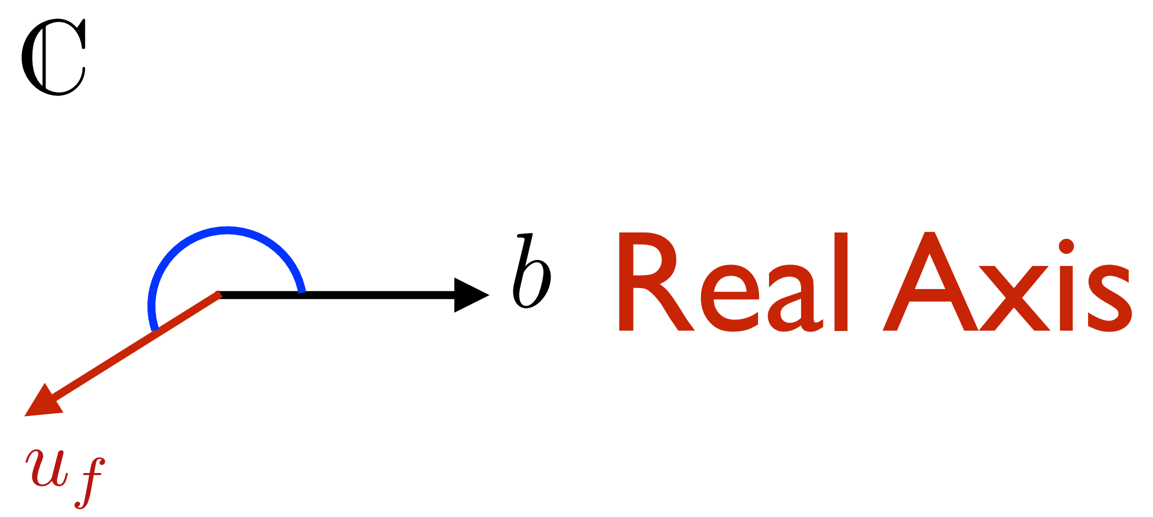
- Tangent vector per triangle:
  represent in an arbitrary local basis for each triangle:





$$u_f = (u_f^x, u_f^y) \in \mathbb{R}^2$$

- (Optional) notation simplification:
  Vector = complex number
  (Tangent plane = complex plane)



$$u_f = u_f^x + i u_f^y \in \mathbb{C}$$

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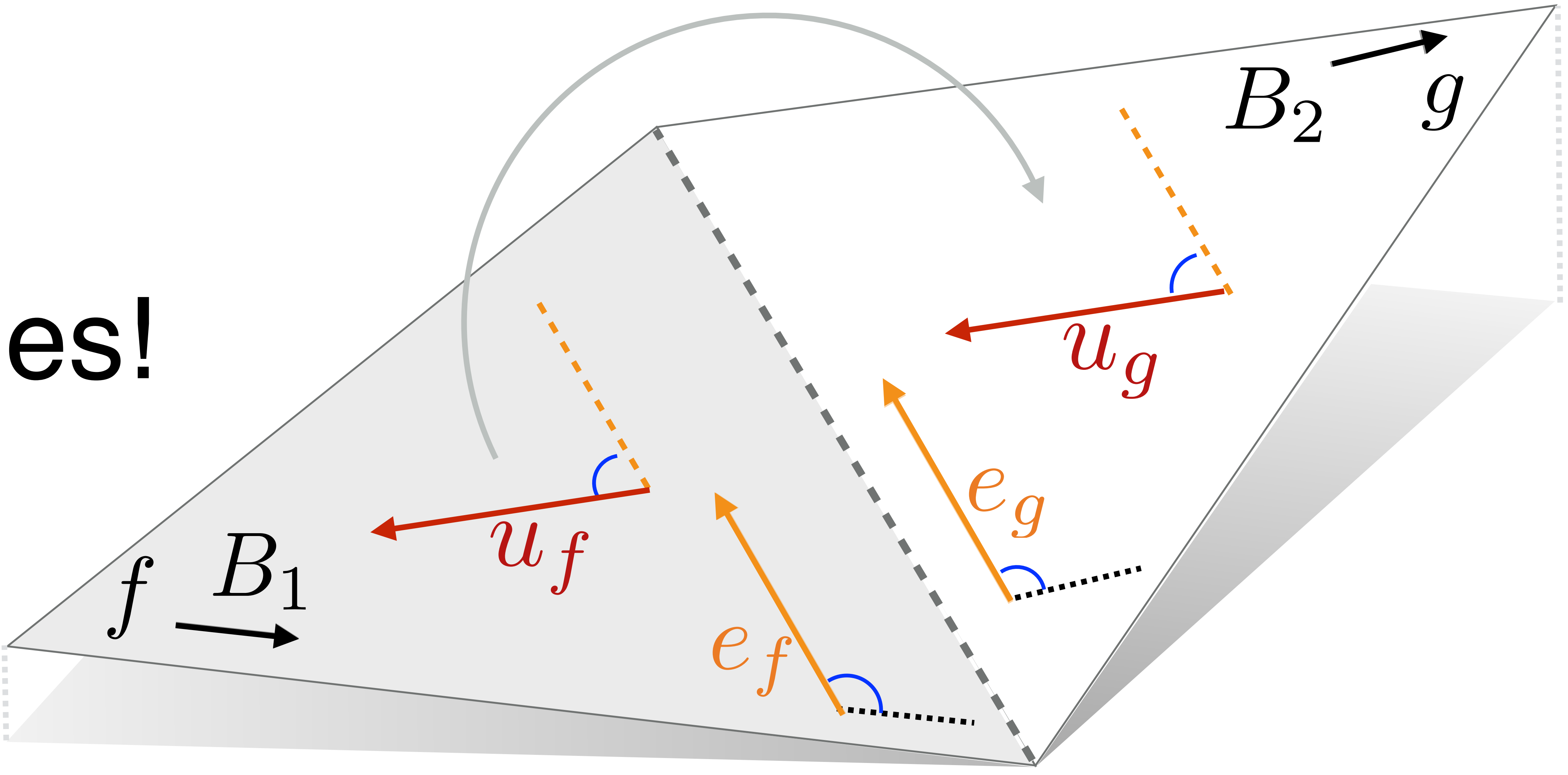
Vector Field Smoothness

- Try to keep vectors constant
- Vectors lie in different tangent planes! Must transport to compare.
- Idea: unfold triangle pair along shared edge, compare vectors
- Equivalent: represent vectors in orthonormal basis determined by shared edge.
  (Rotate so shared edge is real axis)



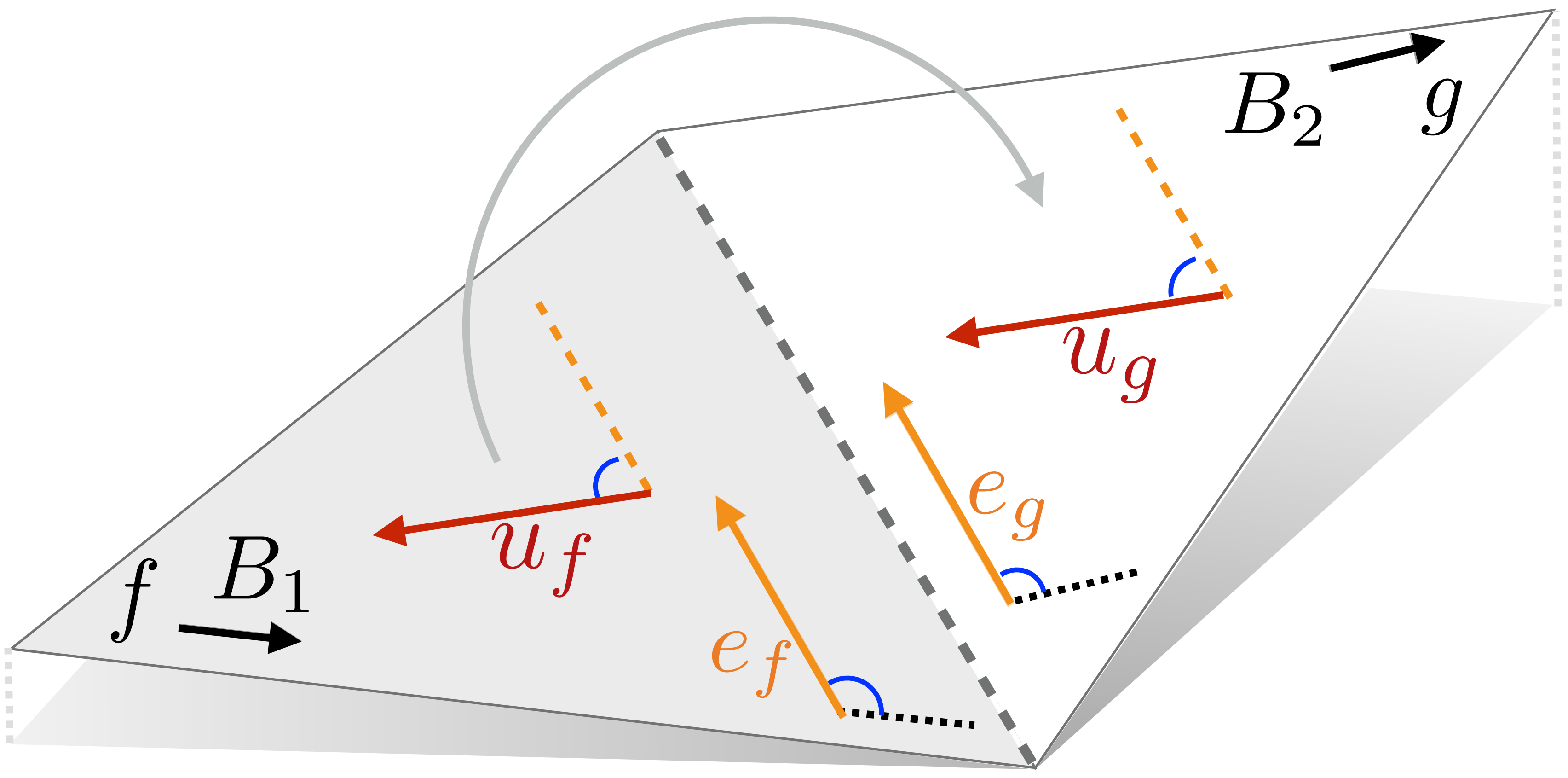
Vectors in common basis:

$$\tilde{u}_f = u_f \bar{e}_f$$
$$\tilde{u}_g = u_g \bar{e}_g$$
$$|\tilde{u}_f - \tilde{u}_g|^2$$

Since for unit complex numbers:

$$e_f^{-1} = \overline{e_f}$$

Vector Field Smoothness



- Minimize non-smoothness across all edges:

$$\sum_{\text{edge }(f,g)} \left|u_f\overline{e_f} - u_g\overline{e_g}\right|^2 = \sum_{\text{edge }(f,g)} \begin{bmatrix}\overline{u_f} & \overline{u_g}\end{bmatrix} Q_{fg} \begin{bmatrix}u_f \\ u_g\end{bmatrix}$$

- Can be re-written as one large complex quadratic form:

$$\min_{\mathbf{u}|_{cf}=\vec{c}} \mathbf{u}^* Q \mathbf{u} \quad \Longleftrightarrow \quad \frac{\partial}{\partial \overline{\mathbf{u}}} \mathbf{u}^* Q \mathbf{u} = Q\mathbf{u} = 0$$

Complex equiv to gradient

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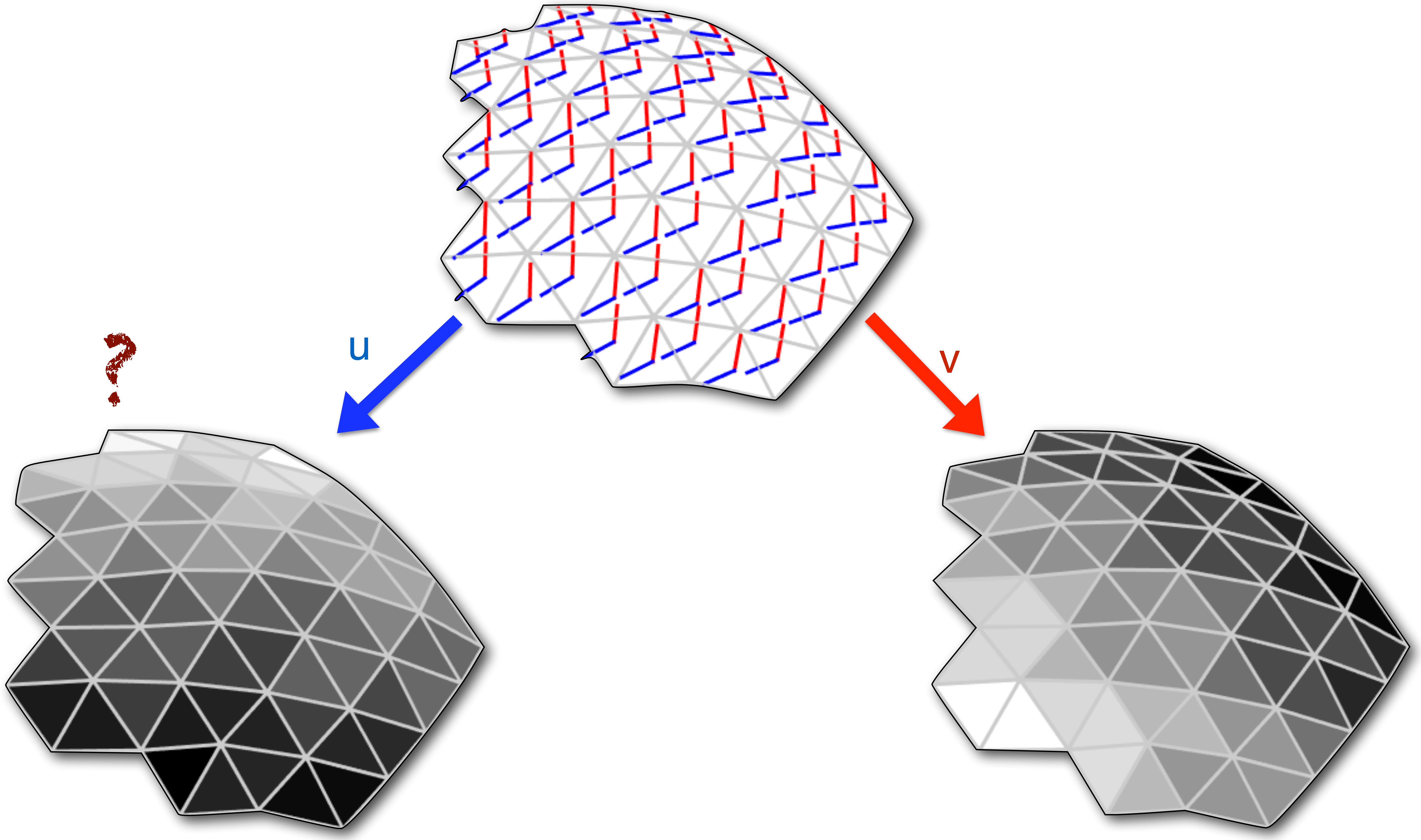
Step 2: Scalar Field Design

- Find a scalar field whose gradients match the vector field:



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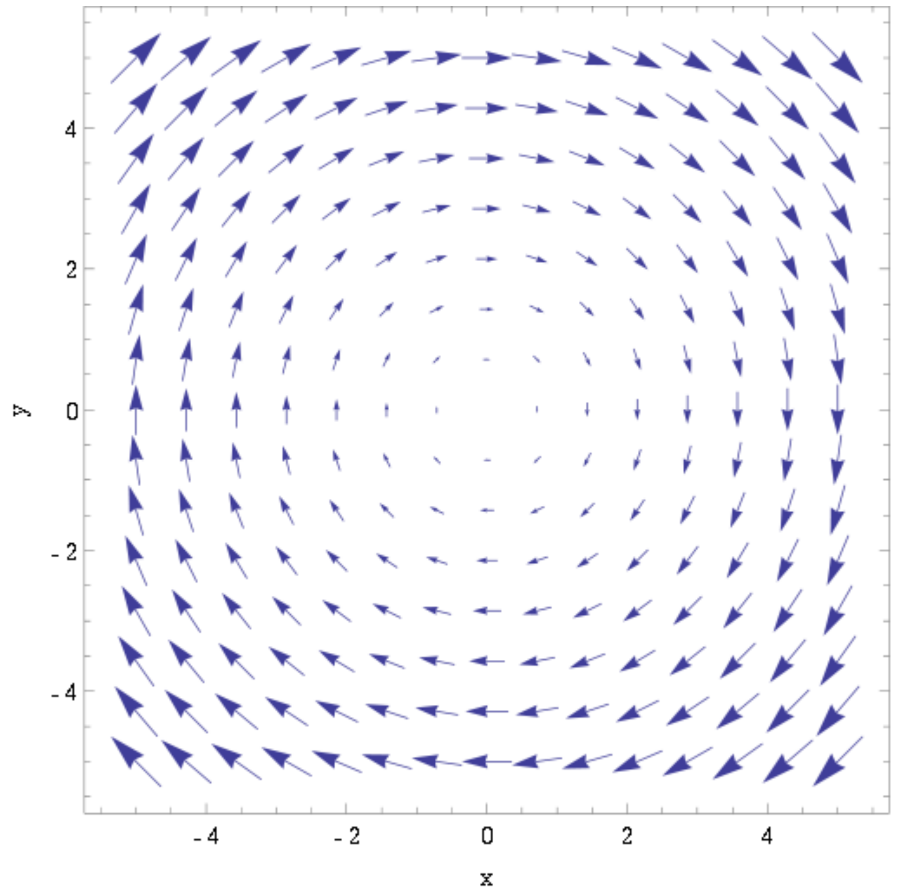
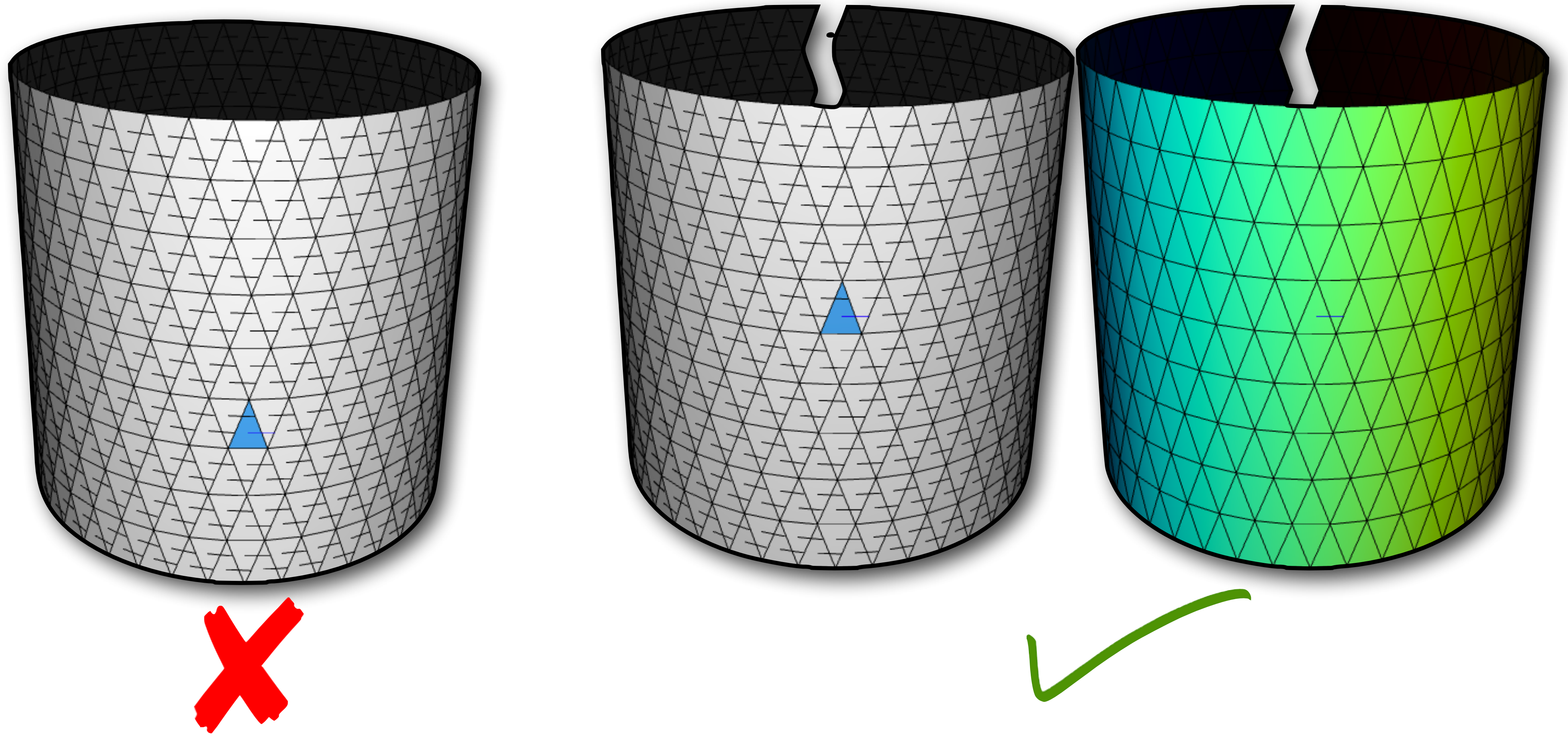
Integrating the Vector Field

- Why not just assign zero to some vertex and "integrate" the vector field outward to get all other vertex values?
- Problem: most vector fields are not actually integrable (i.e. not the gradient of a scalar field)
- This can happen for local or global reasons:



Local: nonzero curl



Global: non-simply-connected domains

Integrating the Vector Field

- **The fix:** integrate in the “least-squares sense”

$$\min \sum_{\text{face } t} w_t \|\mathbf{g}_t - \mathbf{u}_t\|^2 \qquad\qquad \mathbf{g}_t = \nabla f|_t$$

- This turns out to be equivalent to the Poisson equation:

$$\begin{aligned} \nabla \cdot \nabla f &= \nabla \cdot u \quad \text{in } M \\ n \cdot \nabla f &= n \cdot u \quad \text{on } \partial M \end{aligned}$$

- Can also be written as a quadratic form:

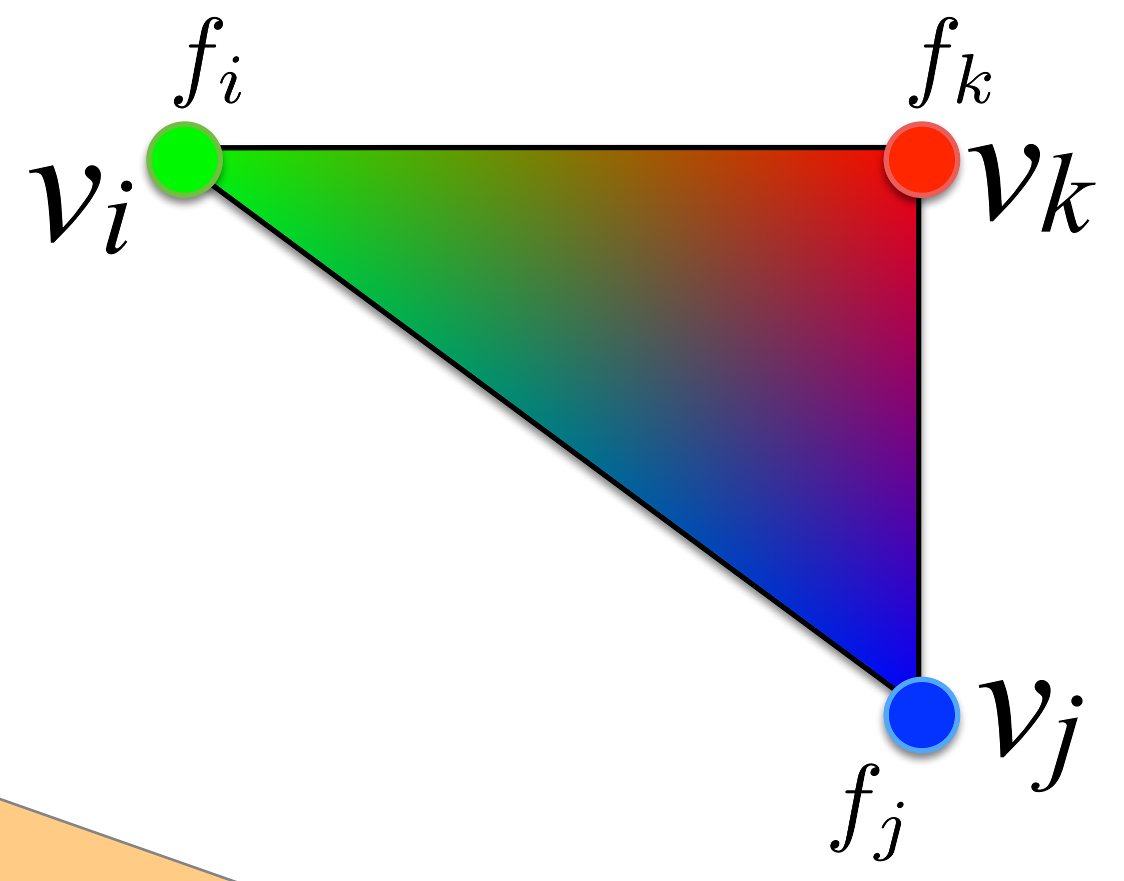
$$\frac{1}{2}\mathbf{f}^T K \mathbf{f} + \mathbf{f}^T \mathbf{b} + c$$

- Only determined up to constant: fix one vertex to zero!

Expressing as Quadratic Form

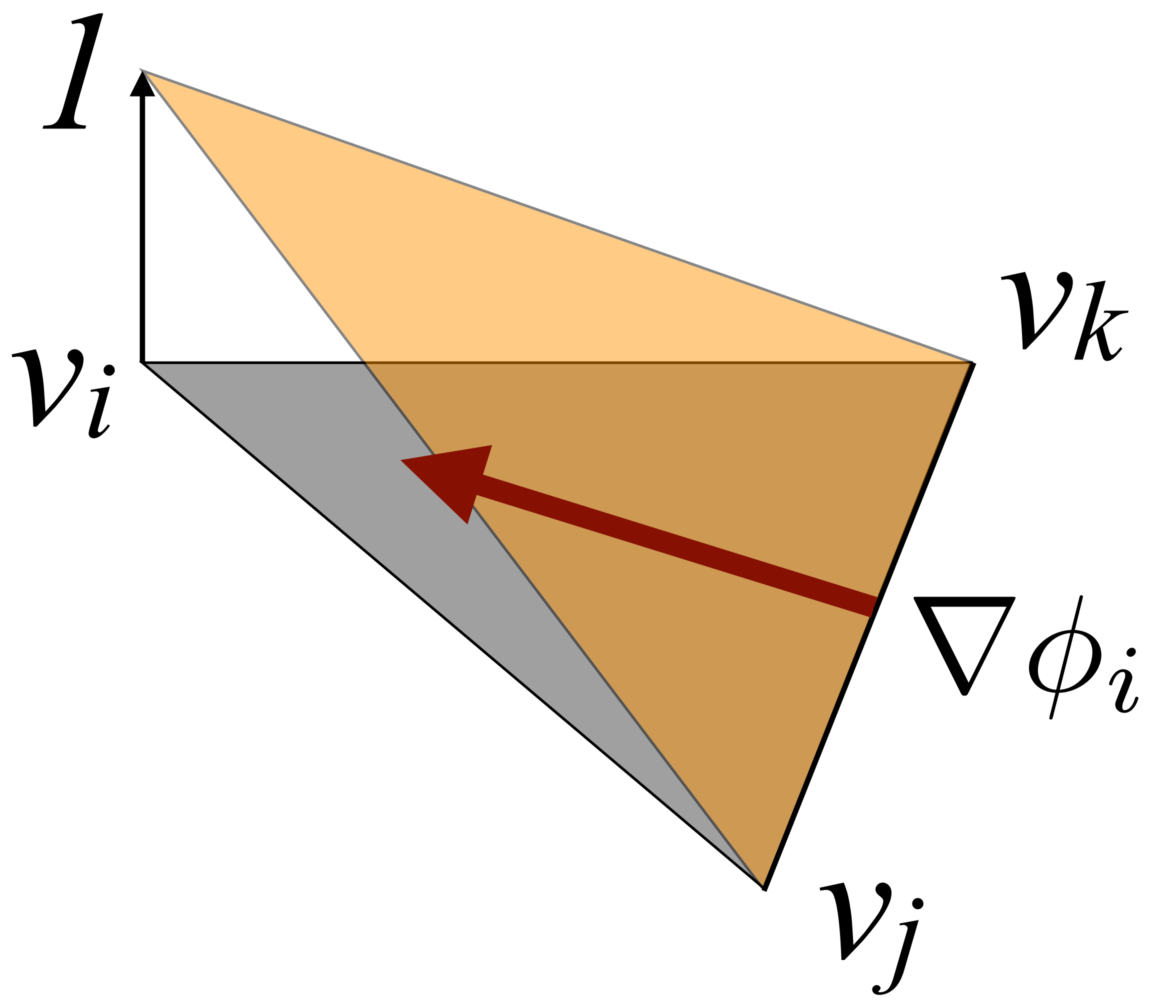
$$\min \sum_{\text{face } t} w_t \|\mathbf{g}_t - \mathbf{u}_t\|^2 \qquad \mathbf{g}_t = \nabla f|_t$$

- Recall, piecewise linear f can be written as a sum of “hat” basis functions:



$$f(x) = f_i\phi_i(x) + f_j\phi_j(x) + f_k\phi_k(x)$$

$$\nabla f(x) = f_i\nabla\phi_i(x) + f_j\nabla\phi_j(x) + f_k\nabla\phi_k(x)$$



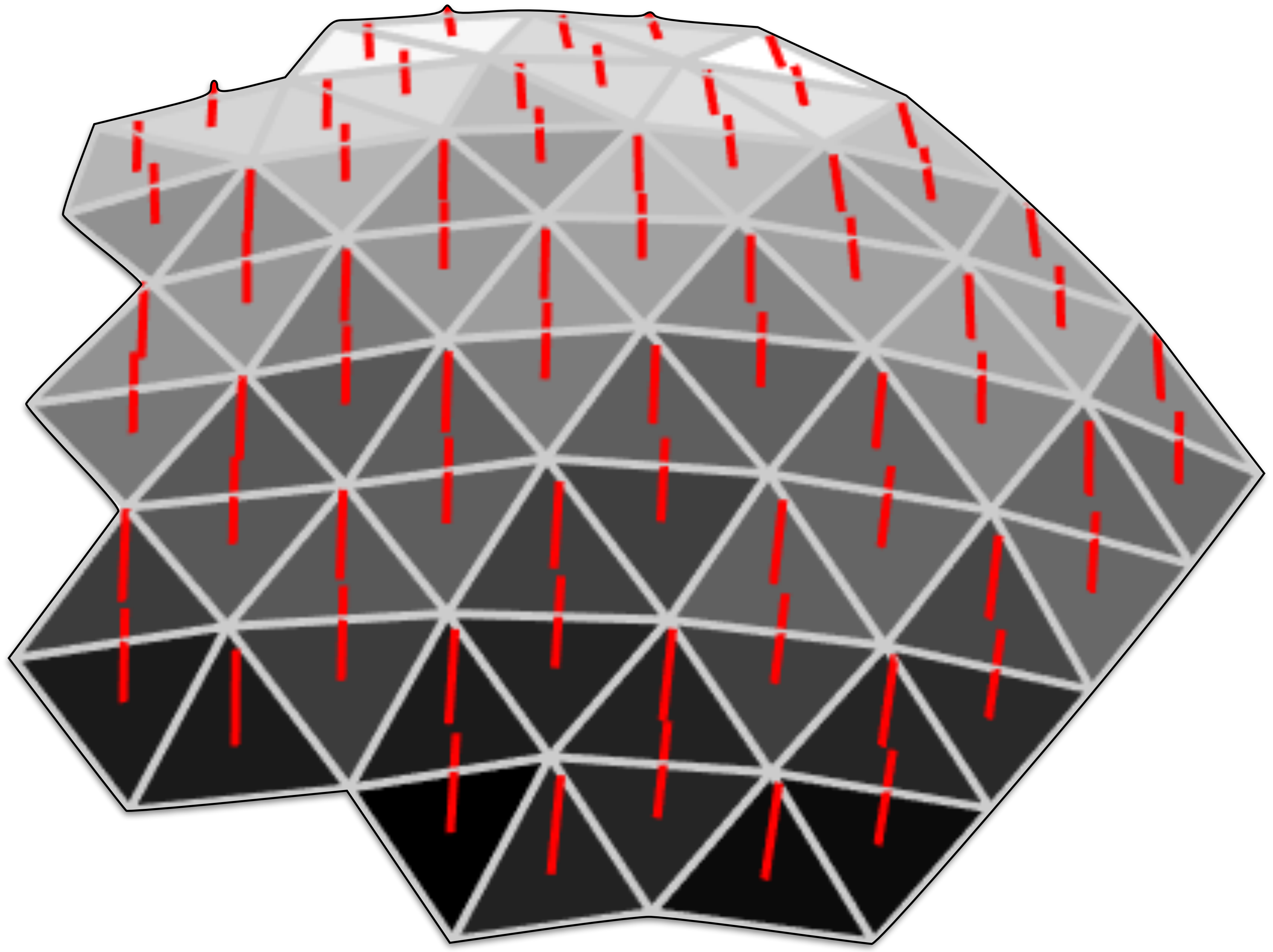
The "Gradient Matrix"

- So the gradient is just a linear combination of vertex scalar field values

$$\underbrace{\begin{bmatrix} \left.\frac{\partial f}{\partial x}\right|_{t_1} \\ \vdots \\ \left.\frac{\partial f}{\partial x}\right|_{t_{\#F}} \\ \left.\frac{\partial f}{\partial y}\right|_{t_1} \\ \vdots \\ \left.\frac{\partial f}{\partial z}\right|_{t_{\#F}} \end{bmatrix}}_{3\#F \times 1} = G \underbrace{\begin{bmatrix} f|_{v_1} \\ \vdots \\ f|_{v_{\#V}} \end{bmatrix}}_{\#V \times 1}$$

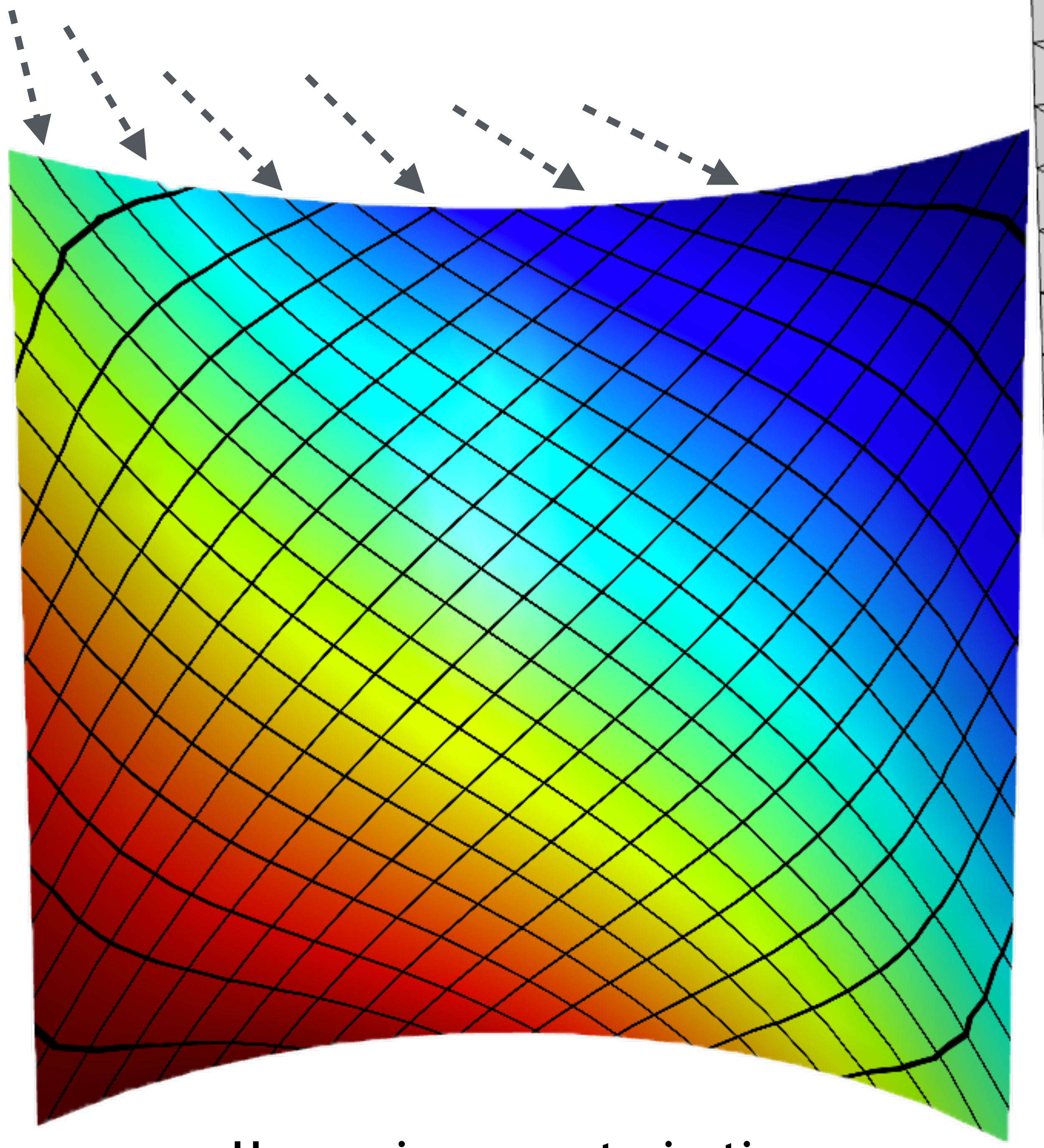
$$\underset{3\#F \times \#V}{G}$$

Maps per-vertex scalars to per-triangle gradient vectors

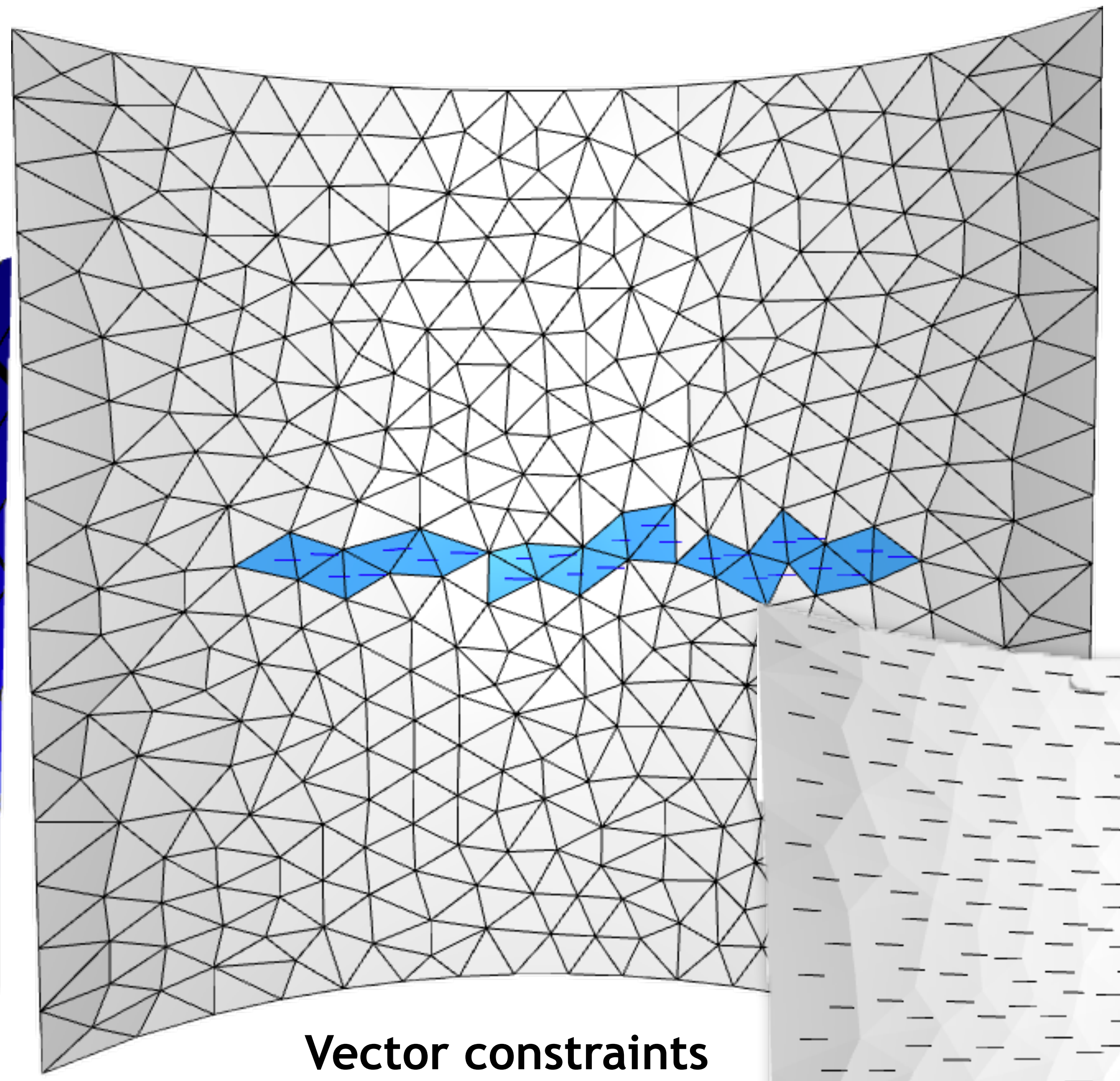


```
void igl::grad(const Eigen::MatrixXd &V,
               const Eigen::MatrixXi &F,
               Eigen::SparseMatrix<double> &G);
```

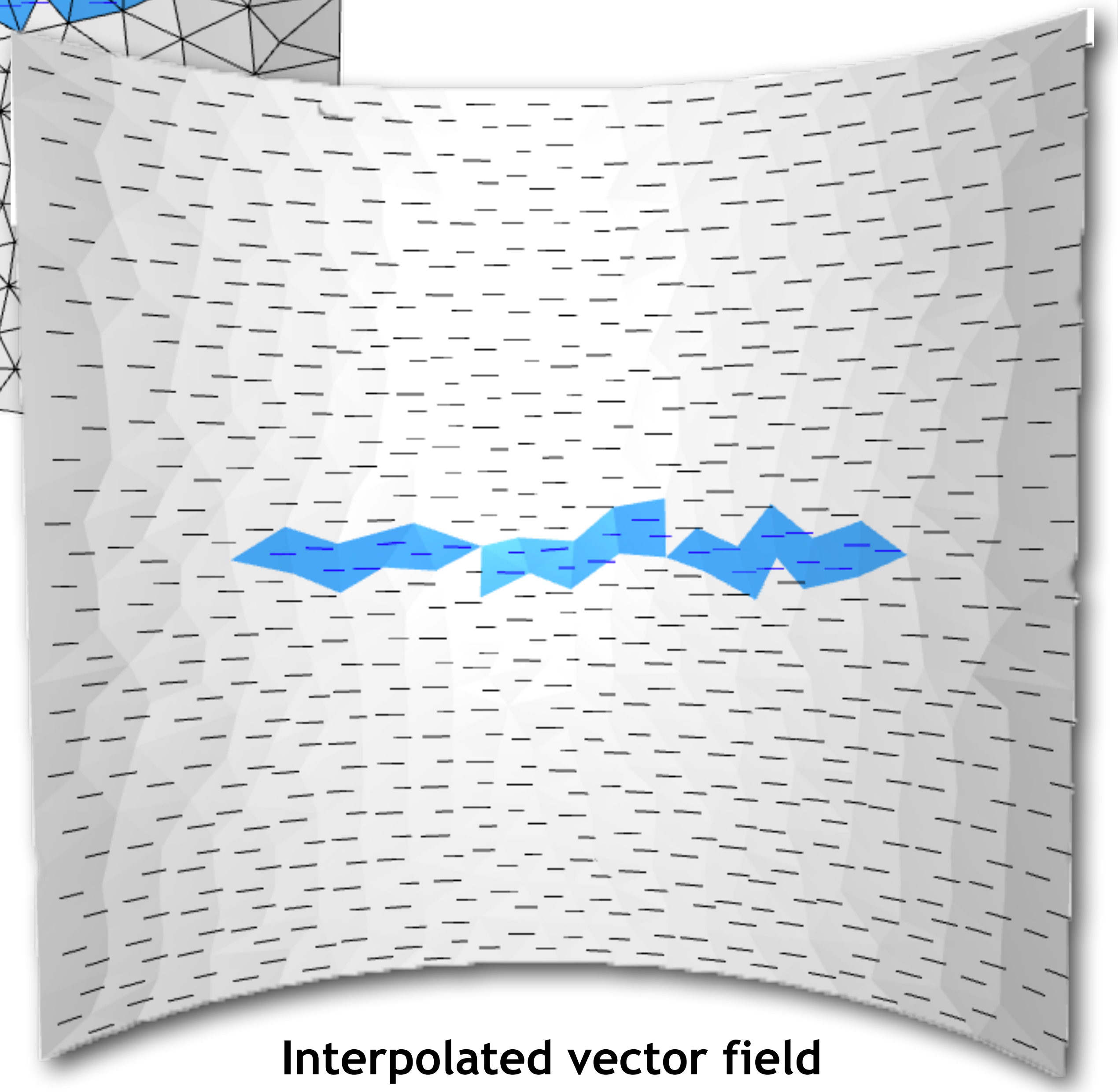
Application: Edit Parameterizations



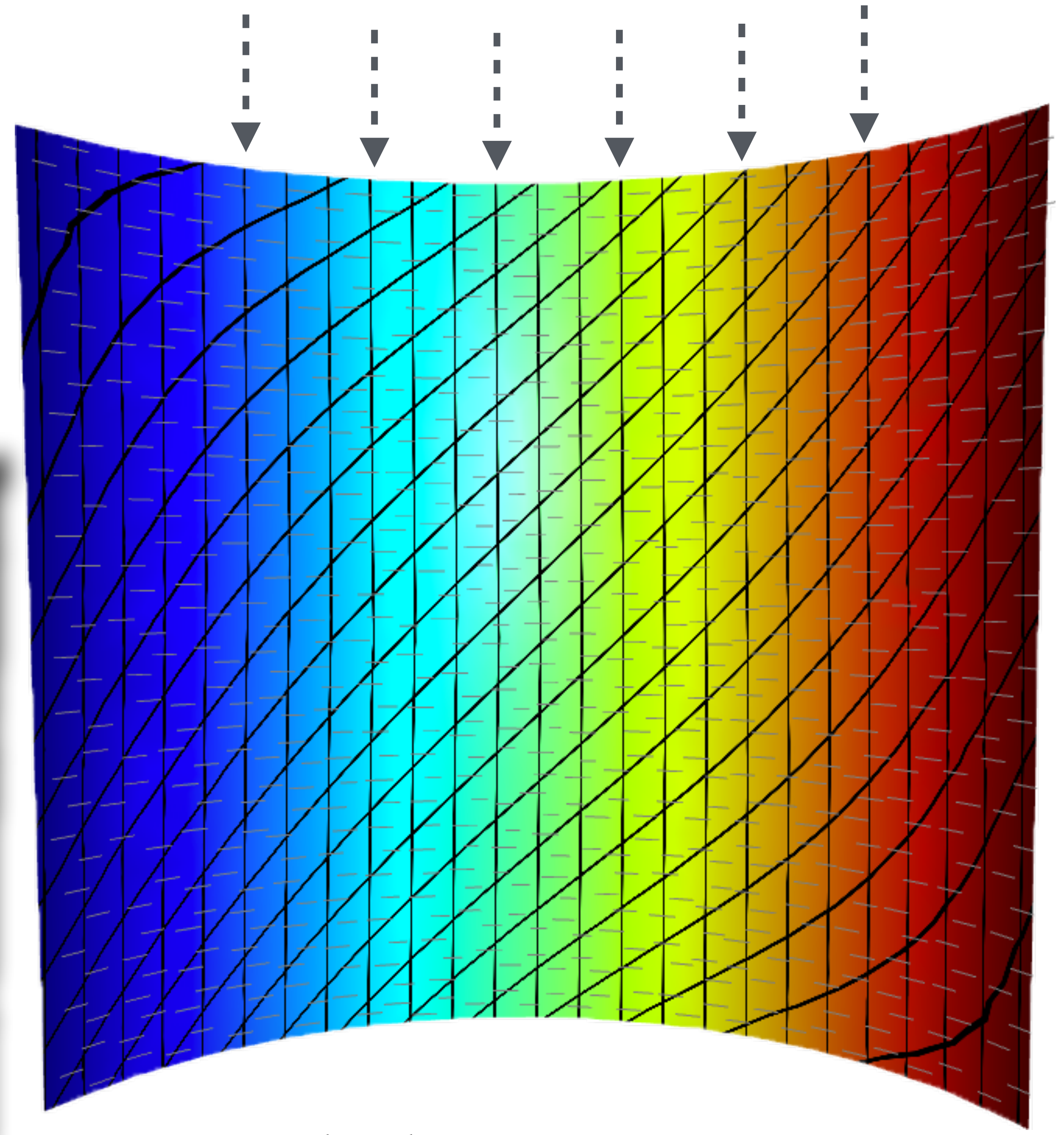
Harmonic parameterization
(initial u function)



Vector constraints
for u’s gradient



Interpolated vector field
(approximate u gradient)



Edited parameterization
(new u function)

Equality Constraints

- Both steps involve equality constraints
  - –Painted vectors on constrained faces
  - –Scalar field value on single vertex fixed to zero
- Two approaches:
  - –Lagrange multipliers (standard approach)
  - –Variable elimination (much more efficient here)

Variable Elimination via Row/Col Removal

- I'll derive this since it's less discussed than Lagrange multipliers
- Most efficient way to fix variables to constants:

$$\min_{\mathbf{x}} \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x} + c \quad \text{s.t.} \quad \mathbf{x}_c = \mathbf{d}$$

$$\left(\mathbf{x} = \begin{bmatrix} \mathbf{x}_f \\ \mathbf{x}_c \end{bmatrix} \right) \begin{array}{l} \leftarrow \text{Free Variables} \\ \leftarrow \text{Constrained Variables} \end{array}$$

- For derivation, assume vars ordered like this; Trick works for any ordering

Row/Col Removal

$$
\begin{aligned}
&\frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x} + c \\
=&\frac{1}{2}\begin{bmatrix}\mathbf{x}_f^T & \mathbf{x}_c^T\end{bmatrix}\begin{bmatrix}A_{ff} & A_{fc} \\ A_{cf} & A_{cc}\end{bmatrix}\begin{bmatrix}\mathbf{x}_f \\ \mathbf{x}_c\end{bmatrix} - \begin{bmatrix}\mathbf{b}_f^T & \mathbf{b}_c^T\end{bmatrix}\begin{bmatrix}\mathbf{x}_f \\ \mathbf{x}_c\end{bmatrix} + c \\
=&\frac{1}{2}\left(\mathbf{x}_f^T A_{ff}\mathbf{x}_f + 2\mathbf{x}_f^T A_{fc}\mathbf{x}_c + \underbrace{\mathbf{x}_c^T A_{cc}\mathbf{x}_c}_{\text{const}}\right) - \mathbf{b}_f^T\mathbf{x}_f - \underbrace{\mathbf{b}_c^T\mathbf{x}_c}_{\text{const}} + c
\end{aligned}
$$

$$
\min_{\mathbf{x}_f} \implies 0 = A_{ff}\mathbf{x}_f + A_{fc}\mathbf{x}_c - \mathbf{b}_f
$$

$$
\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}
$$

$$
\tilde{A} = A_{ff} \quad \tilde{\mathbf{x}} = \mathbf{x}_f \quad \tilde{\mathbf{b}} = \mathbf{b}_f - A_{fc}\mathbf{x}_c
$$

Rows/cols corresponding to fixed variables removed!

Columns corresponding to fixed vars contribute to RHS

Eigen Complex Sparse Matrices

```
Eigen::SparseMatrix<std::complex<double> > Q;


Eigen::SparseLU< Eigen::SparseMatrix<std::complex<double> > > solver;
```

Questions?