Laboratory Exercise 1: Pendulum Acceleration Measurement and Prediction

Laboratory Handout

AME 20213: Fundamentals of Measurements and Data Analysis

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Date exercises to be performed: 4 – 15 September, 2006
Due date: 21 September, 2006

Overview:

In this experiment, the motion of a pendulum will be measured by a microcontroller based data acquisition system using an on-board accelerometer. The data acquisition card is mounted directly to the pendulum, measuring the acceleration of the pendulum system as well as a component of gravity. From the data, the period and the initial angle of the pendulum release can be determined. Data will be collected for both a small angle and a large angle assigned to your lab group. Comparison of the small angle case and the large angle case will be made. Differences between the theoretical and experimental results will be examined. Uncertainty in the results will also be computed.

Deliverables:

There are six deliverables for this lab:

1) Derive, in symbolic form, the theoretical period for:
   a. Pendulum 1
      i. Point Mass, Small Angle approximation
      ii. Distributed Mass, Large Angle approximation
   b. Pendulum L
      i. Point Mass, Small Angle approximation
      ii. Distributed Mass, Large Angle approximation

   Leave all solutions in terms of system variables, i.e., length, mass, etc., and do not substitute any numeric values.

2) Compute the period values for the experimental cases you ran in lab, using the equations derived from Deliverable (1):
   a. Pendulum 1
      i. Small Angle
      ii. Large Angle
   b. Pendulum L
      i. Large Angle
   c. Pendulum S
      i. Large Angle

   Results should be given in seconds, with appropriate significant digits.

3) Determine the first and last period for all the cases listed in Deliverable (2). Compare these values with those you found in Deliverable (2). How is the period affected by a large or small release angle? What can you say about the decay of the period in time? Note, that for the small angle approximation, the period is independent of the release angle.

4) Determine the maximum apex angle for the first and last period of data for all cases listed in Deliverable (2). Compare these results with the release angles recorded in lab. To find the maximum apex angle for the first and last periods of the ten second data sets, use the maxima and minima of the data. (All data points for a given case make up a 'data set' or 'data series'.) How does the apex angle change from the beginning to the end of the ten second cycle? Is there a great effect on the large angle case compared to the small angle case?

5) Compare Experimental Data with Theoretical Results.
a. How well does theory predict the experimental results?
b. Determine the percent error between experimental and theoretical results.
c. How do the large angle and small angle data differ?
d. Discuss your findings.

6) Compute the uncertainty in acceleration for the accelerometer and explain each uncertainty in detail, making sure to describe each type of uncertainty error in your own words.

Introduction:

Galileo Galilei[1] was the first person to discover that pendulums keep good time. As a medical student in Pisa, he noticed that lamps swinging in the cathedral oscillated with a period that remained fairly constant. After Galileo, Christiaan Huygens experimented with pendulums and built on the work the former scientist began. Huygens was able to make pendulum clocks that kept time accurate to better than one minute a day. The work that Galileo initially did led to the creation of clocks and eventually changed how the world thought about the order of the universe.

For small angle deflections, Galileo's idea that the period of a pendulum is constant was and still is true today. Understanding the centuries old experiment of an oscillating pendulum is the focus of this laboratory experiment. Thus, the equation of motion for a simple pendulum point mass system undergoing small angle oscillations is presented[2]. One of the experimental pendulums used in this laboratory experiment is constructed of a thin metal rod connected to a thin disk on which the accelerometer is mounted. Figure (1) shows the forces that act on a pendulum at some angle θ. The mass moment of inertia is counteracted by gravity, which tries to restore the pendulum to an angle of zero.

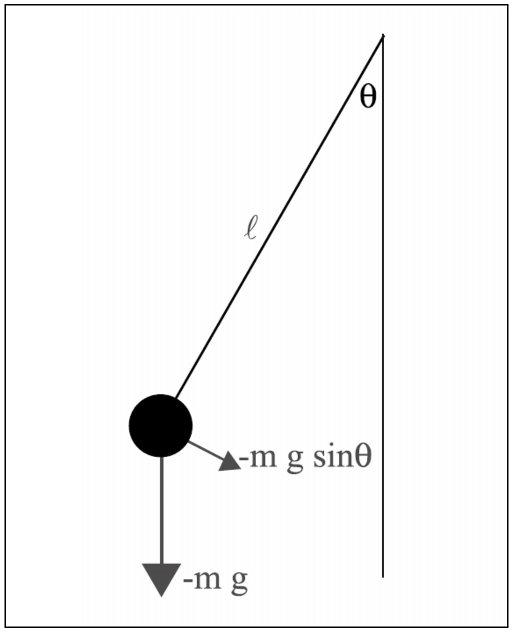


Figure (1): Forces Acting on Pendulum.

The pendulum is acted on by a component of gravity, the moment of which is defined as the cross product of the distance the mass is from the point of rotation with the gravitational force acting on the mass:

$$\tau = \vec{l} \times m\vec{g}$$

$$\tau = -lm\vec{g}\sin(\theta)$$

where l is the length of the rod from the axis of rotation to the center of the disk, m is the mass of the disk, g is the gravitational acceleration due to gravity, and θ is the angle of the pendulum with respect to the vertical axis. From Newton's Second Law, the only other force acting on the pendulum is that caused by the system's inertia. The moment created from the inertial force is defined as

$$\tau = I\ddot{\theta}.$$

Here, I is the mass moment of inertia of the disk and $\ddot{\theta}$ is the angular acceleration of the pendulum, or simply the second derivative of acceleration with respect to time. For a simple pendulum in which the mass of the rod is neglected, the moment becomes

$$\tau = ml^2\ddot{\theta}.$$

The sum of these moments represent all the forces that act on the system, so adding them together will yield the following governing equation for this system

$$ml^2\ddot{\theta} + m\lg\sin(\theta) = 0.$$

This equation is nonlinear in θ, so the solution is not straightforward. Often, to simplify the solution, the following small angle approximation is made

$$\sin(\theta) \approx \theta.$$

The resulting governing equation is a formula of simple harmonic motion,

$$\ddot{\theta} + p^2\theta = 0,$$

where p is the circular frequency, defined by

$$p = \sqrt{\frac{g}{l}}\ .$$

There are four solutions for this differential equation:

$$\begin{aligned} \theta &= C\sin\sqrt{\frac{g}{l}}t \\ \theta &= -C\sin\sqrt{\frac{g}{l}}t \\ \theta &= C\cos\sqrt{\frac{g}{l}}t \\ \theta &= -C\cos\sqrt{\frac{g}{l}}t \end{aligned},$$

with C being a constant that corresponds to the initial release angle. Based on the initial conditions of the pendulum, the value of C can be computed by differentiating these solutions twice and plugging back into the original differential equation. During movement through one period, the argument of the sine or cosine moves from zero through 2π . So the period for this solution can be found by setting

$$\sqrt{\frac{g}{l}}T = 2\pi\ .$$

Thus the period for a simple pendulum undergoing small angle oscillations is

$$T = 2\pi\sqrt{\frac{l}{g}}\ .$$

This result is the same that Galileo found, that for small angle oscillations the period is not dependent on the initial release angle of the pendulum or on the mass of the pendulum. For a distributed mass pendulum or one which experiences oscillations where the small angle approximation is not valid, the period will differ from that stated above.

For this laboratory experiment, you will be required to derive the period of the pendulum for both a point mass and a distributed mass using the small angle approximation and large angle perturbation solution.

Accelerometer and Data Acquisition Equipment:

The accelerometer[3] used in this experiment, an Analog Devices model ADXL105 single axis unit, is a spring mass system that is sensitive to accelerations along a specific axis of movement. This sensor is described in the accompanying literature in the following manner:

> *"The sensor is a surface micromachined polysilicon structure built on top of the silicon wafer. Polysilicon springs suspend the structure over the surface of the wafer and provide a resistance against acceleration-induced forces. Deflection of the structure is measured with a differential capacitor structure that consists of two independent fixed plates and a central plate attached to the moving mass. A 180 degrees out-of-phase square wave drives the fixed plates. An acceleration*

causing the beam to deflect will unbalance the differential capacitor resulting in an output square wave whose amplitude is proportional to acceleration. Phase sensitive demodulation techniques are then used to rectify the signal and determine the direction of the acceleration."

The accelerometer outputs a voltage that is proportional to the force acting on the sensing element. The accelerometer output voltage is an analog signal that is digitized by an analog-to-digital (A/D) converter and stored in binary form. The output voltage is sampled and stored by the on-board data acquisition system, which in this lab is integrated into the microcontroller processing unit (MPU), allowing for remote storage of data. The MPU samples the accelerometer voltage for a finite number of points at a given sampling frequency over a specified time interval. Here, the sampling frequency is 100 Hz, and 1000 data points are acquired over a 10 second time period. This set of data points, commonly called a 'data set' or 'data series', is stored on-board the MPU that can then be downloaded later to a computer. The results in this laboratory experiment are converted to acceleration using LabVIEW with a calibration previously conducted by the teaching assistant. The calibration consists of placing the MPU with the accelerometer in a centrifuge and subjecting the unit to a range of known accelerations. The recorded voltage of the accelerometer can be plotted versus the known acceleration to obtain a linear calibration equation that relates output voltage to acceleration values. Thus, the accelerations felt under unknown forcing can be reconstructed from acquired voltages stored remotely on the MPU.

In this experiment, the accelerometer is mounted so that it records accelerations perpendicular to the long axis, or radial component, of the pendulum rod in the plane of motion. Therefore, when the pendulum is moving, the accelerations recorded always include centripetal acceleration and a component of gravity, written as

$$a = r\dot{\theta}^2 + g\cos(\theta).$$

To decipher the contributions of both accelerations to the resultant acceleration recorded by the data acquisition system, the acceleration of the pendulum at its apex as well as at an angle of zero degrees can be used. At its apex angle where the inertia and gravitational forces bring the pendulum to a momentary stop, the centripetal acceleration is zero and the accelerometer measures only the component of gravity. This appears in the set of data points as a minimum. As the pendulum passes through zero degrees, both accelerations are included, which yields a maximum in the data.

As the pendulum begins a period at its apex angle, it then swings through zero degrees, or a quarter cycle in the period (T/4) up to the other side, where it momentarily comes to rest (T/2). It then reverses direction and swings back through zero degrees (3T/4) up to another apex angle (T), completing one full period. From the maximum acceleration, the apex angle is found by:

$$\theta_m = \arccos\left(1 - \left[\frac{a_{\max} - g}{2g}\right]\right) \; [\mathrm{rad}]$$

where g is the gravitational constant, and $a_{\max}$ is the maximum acceleration being considered. From the minimum acceleration, the apex angle is:

$$\theta_m = \arccos\left(\frac{a_{\min}}{g}\right) \; [\text{rad}]$$

where $a_{\min}$ is the minimum acceleration being looked at. Referring to Figure (2) below, local minima will occur three times in one period. This can also be applied to the maximum acceleration values as well.

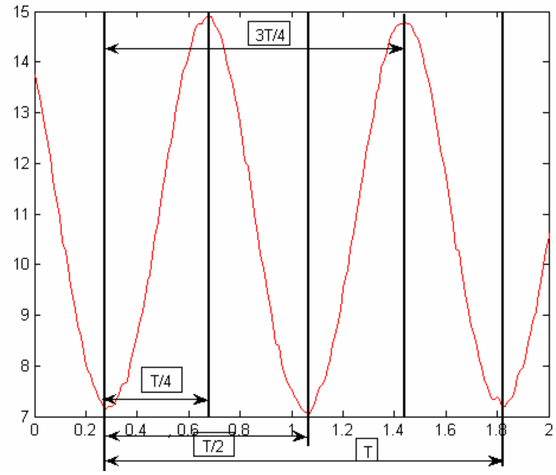


Figure (2): Period calculation

To determine the first and last periods of a given data series, take the difference in time between the first three maxima apart, or first three minima apart as in Figure (2), and the difference in time between the last three maxima or minima, respectively.

Experimental Setup Physical Properties:

The properties of the three pendulum apparatus are listed below. These will be needed to complete the deliverables required for the Technical Memo. Remember to include the accelerometer/MPU assembly and digital protractor when computing the moment of inertia for pendulum L and pendulum S. For pendulum 1, the accelerometer assembly is already included as part of the 'thin disk'. Also be sure to use the parallel axis theorem for each component.

Pendulum 1:

Thin Disk properties: (This includes all supports, mounts, data acquisition equipment, and battery.)

Mass: 0.6673 kg
Diameter: 0.30 m

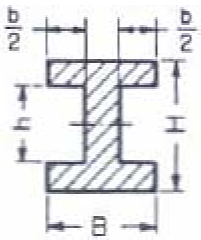
Note: Assume the center of mass of the disk is located at the center point of the disk.

Slender Rod properties:

Mass: 0.4970 kg
Length: 0.8954 kg
Diameter: 0.0093 m

Pendulum S:
I-beam support properties:

Cross section:



$$I_{COM} = \frac{BH^3 - bh^3}{12}$$

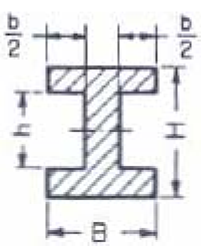
b: 0.025 m
h: 0.059 m
B: 0.028 m
H: 0.064 m

Overall length: 1.22 m
Bearing location offset from center of pendulum: 0.29 m
Mass: 1.03 kg

Note: Pendulum S has no added brass masses.

Pendulum L:
I-beam support properties:

Cross section:



$$I_{COM} = \frac{BH^3 - bh^3}{12}$$

b: 0.025 m
h: 0.059 m
B: 0.028 m
H: 0.064 m

Overall length: 1.22 m
Bearing location offset from center of pendulum: 0.57 m
Mass: 1.03 kg

Note: Pendulum L has the following added brass masses.

Brass masses:
Hollow Cylinder properties:

Mass: 0.200 kg each
Outer diameter: 0.05 m
Inner diameter: 6.731 mm
Thickness: 12.014 mm

Note: Each position holds 2 washers.

Mass locations from pendulum L bearing location:
W2: 0.17 m
W3: 0.23 m
W5: 0.51 m
W6 0.63 m
W7: 0.88 m

Digital Protractor:
Rectangular properties:

Mass: 0.23 kg
Width: 0.0358 m
Depth: 0.169 m
Offset from bearing location on Pendulum L: 0.76 m
Offset from bearing location on Pendulum S: 0.48 m

Accelerometer/MPU assembly:

Mass: 0.20 kg
Offset from bearing location on Pendulum L: 1.15 m
Offset from bearing location on Pendulum S: 0.87 m

Notre: Assume assembly is a point mass for computing the moment of inertia.

Uncertainty:

To understand the results obtained from an experiment, the error inherent in the results must be obtained. The uncertainty in a measured result must be determined by performing an uncertainty analysis[4] before the validity of the result can be quantified. Since uncertainty analysis has not yet been taught in this class, the major errors associated with this experiment will be explained and given to you. However, an uncertainty analysis must be completed individually in the future.

The overall error, or uncertainty, is a combination of systematic (bias) and random (precision) errors. Systematic errors are errors between true and measured quantities, whereas precision errors are due to statistical fluctuations in a measured value. Systematic errors include errors associated with the data acquisition equipment and errors that propagate through equations.

In this laboratory experiment, the errors which are of most importance are Random error, Quantization error, Misalignment error, and Temperature dependence error. The last three errors listed here are bias errors. Random error (e_R) is the statistical fluctuation in a measured quantity under fixed conditions over a repeated number of acquisitions.

Quantization error (e_Q) is the result of digitizing an analog signal. It occurs in measuring physical quantities, such as the initial release angle, the length of the pendulum, and the masses of the disk, and the length of the rod. It is also found in the data recorded by the accelerometer, since the data acquisition system converts continuous data into a set of discrete points, which is limited by the resolution of the analog to digital converter. The resolution of the device used to measure the signal causes to error in the final measured result. The resolution of an instrument is the smallest physically indicated division that the instrument displays or is marked. Normally, the quantization error is defined as one-half the resolution of an instrument. For example, if a thermometer is only marked in one-degree increments, the resolution is one degree and quantization error in reading this is one-half of a degree. Misalignment error (e_M) is due to the accelerometer sensitivity direction being misaligned with the radial direction of the pendulum. Due to manufacturing tolerances, the angle at which the sensitive direction of the accelerometer is mounted may deviate by a small angle. This will result in an incorrect acceleration being measured. Temperature Dependence error (e_T) is a result of the accelerometer response being a function of ambient temperature. As the room temperature fluctuates during an experiment, the response will be affected by this fluctuation.

For this experiment, which contains a 12-bit analog to digital converter with a 2.43 V full scale range, 1 degree maximum misalignment, and a temperature dependence of 0.50% of full scale, the errors are computed to be:

$$e_R = 0.3197 \text{ m/s}^2$$
$$e_Q = 0.0229 \text{ m/s}^2$$
$$e_M = 0.2618 \text{ m/s}^2$$
$$e_T = 0.0750 \text{ m/s}^2$$

All elemental errors such as those above are combined to form the total uncertainty, e, using the method of Kline and McClintock[5]:

$$e = \sqrt{e_R^2 + e_Q^2 + e_M^2 + e_T^2} \; .$$

Thus for this experiment the uncertainty in the acceleration is 0.421 m/s^2. Other elemental errors that are important but which are not included in this simple error analysis include: Hysteresis, Sensitivity, Zero-shift, Repeatability, and Stability. For a more detailed error analysis in the future, you could refer to the specifications sheets for the accelerometer[6] and data acquisition card[7] that list all errors associated with these devices. For more information see Reference [4].

References:

[1] Galileo, Huygens, http://www.museum.vic.gov.au/scidiscovery/gravity/pendulums.asp, Online, Museum Victoria Australia, January 18, 2005

[2] Hibbeler, R. C., *Engineering Mechanics: Dynamics*, Eighth edition, Prentice-Hall, Inc., Upper Saddle River, NJ, ©1998

[3] Szarek, T., *AME250 Laboratory Exercise 3A: Microprocessor-Based Data Acquisition on a Pendulum*, University of Notre Dame, ©2004

[4] Dunn, P. F., *Measurement and Data Analysis for Engineering and Science*, First Edition, McGraw-Hill, New York, NY, ©2005

[5] Kline, S.J. and McClintock, F.A., *Describing Uncertainties in Single-Sample Experiments*, Mechanical Engineering, pp. 3-9, © January 1953.

[6] Analog Devices Single Axis Accelerometer, http://www.nd.edu/~dvanness/ame250/ADXL105.pdf, Online, Analog Devices, January 18, 2005

[7] Silicon Laboratories Microprocessor Chip Unit, http://www.nd.edu/~dvanness/ame250/C8051F020_Short.pdf, Online, Silicon Laboratories, January 18, 2005