Understanding And Generalising The Convolution

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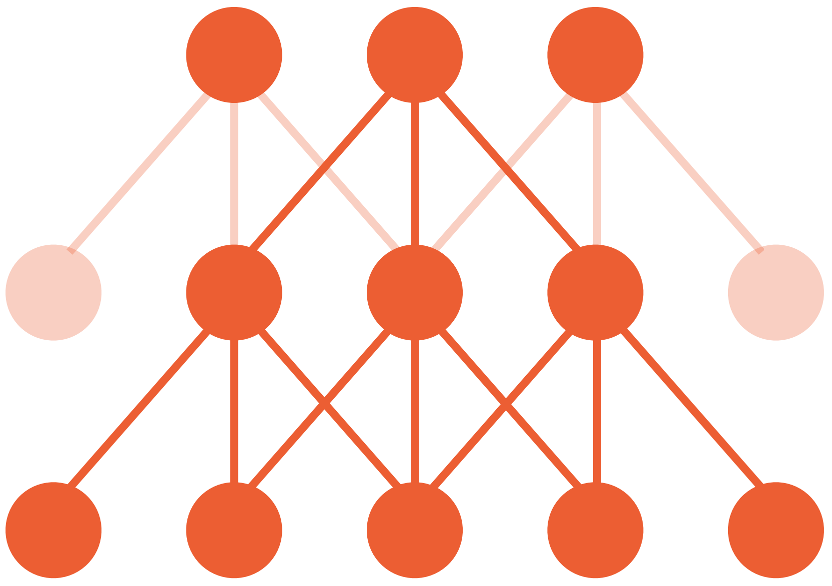
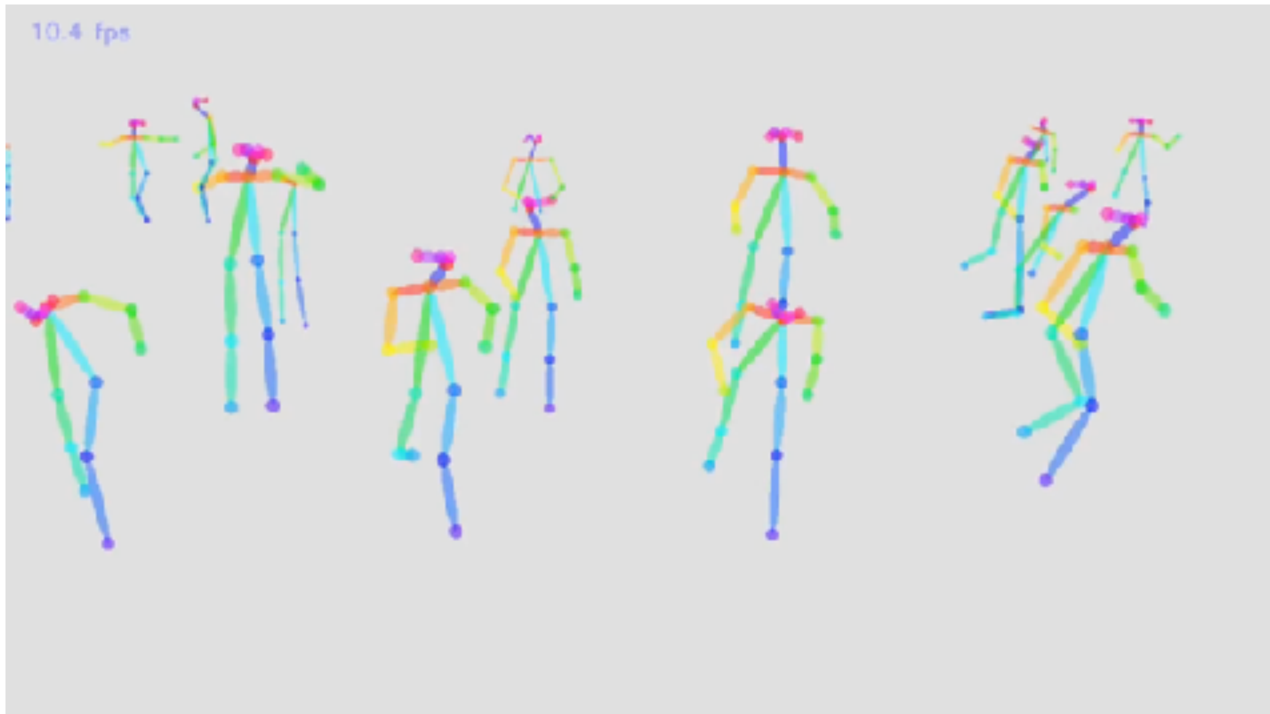


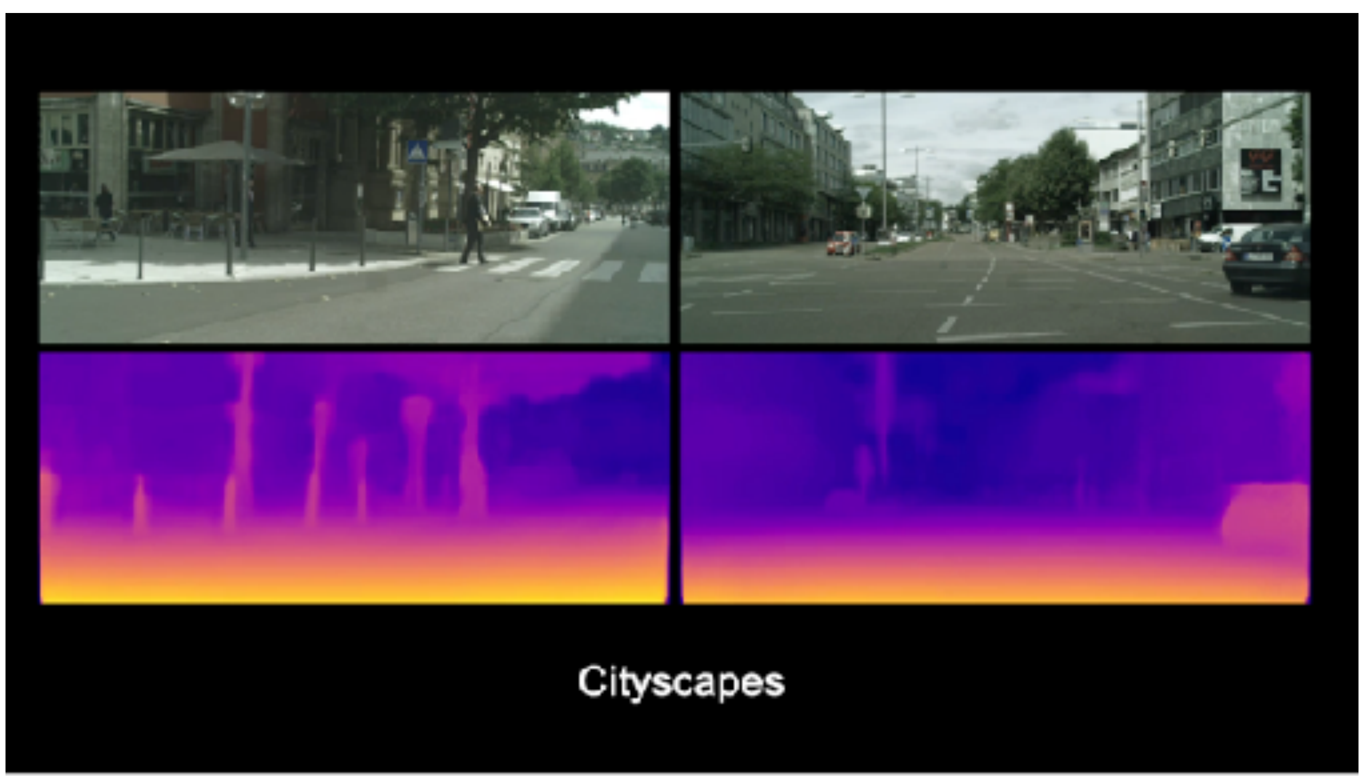
What makes CNNs special?

CNNs Are Responsible For Many Successes

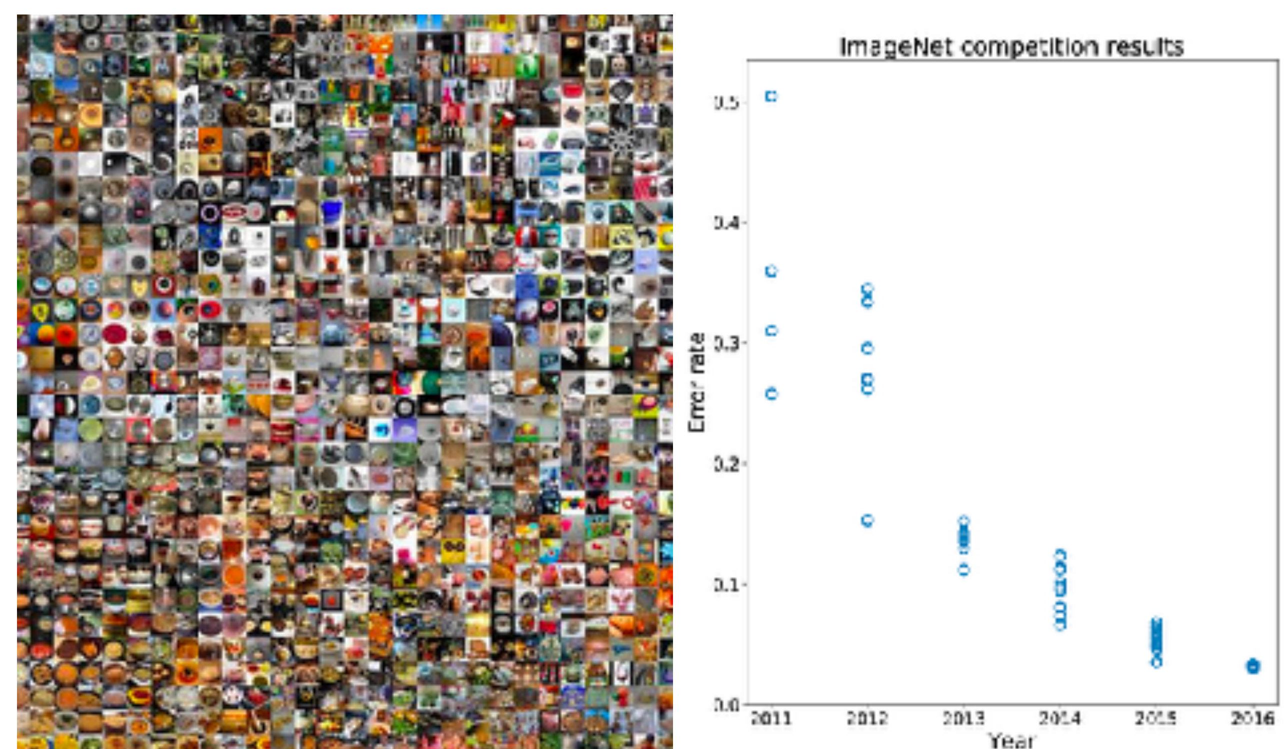


Cao et al. (2017)

Karras et al. (2017)



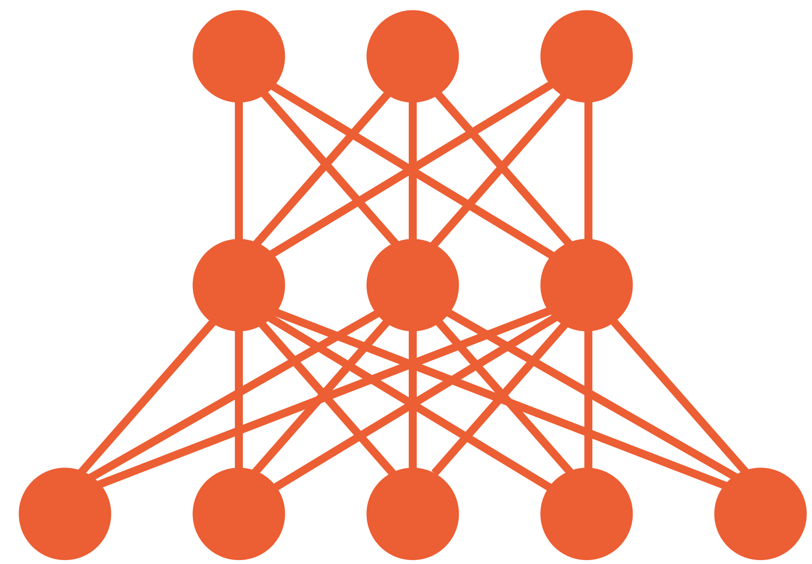
Godard et al. (2017)



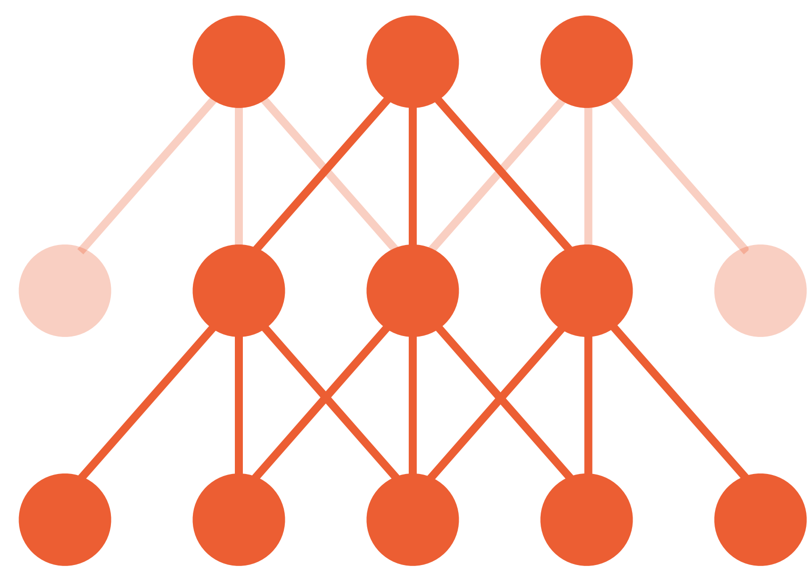
Ongoing [figure from Wikipedia]

WHAT MAKES CNNS SPECIAL?

- **Multilayer perceptrons (MLPs)**
  1. Global connectivity
  2. Seldom used in computer vision



- **Convolutional neural networks (CNNs)**
  1. Local "receptive field" connectivity
  2. Translational weight tying ***of convolution***
     - Basis of ***many*** state-of-the-art models



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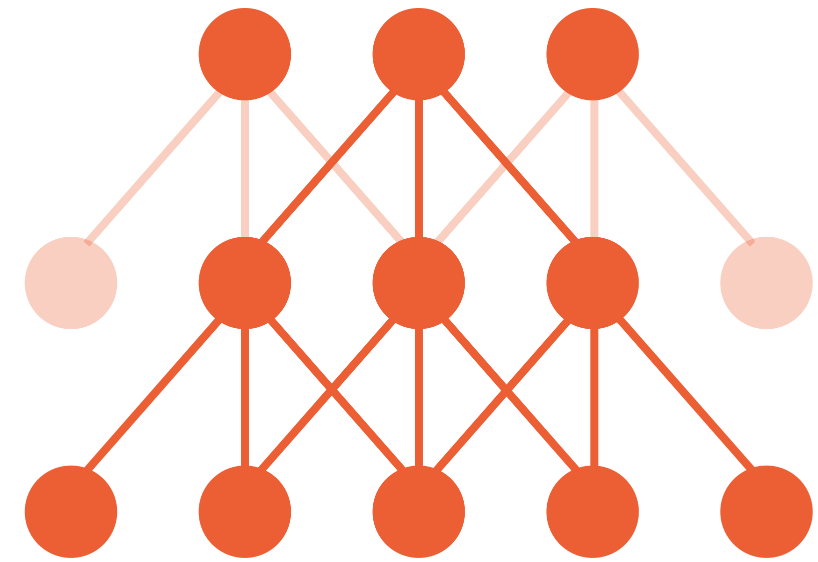
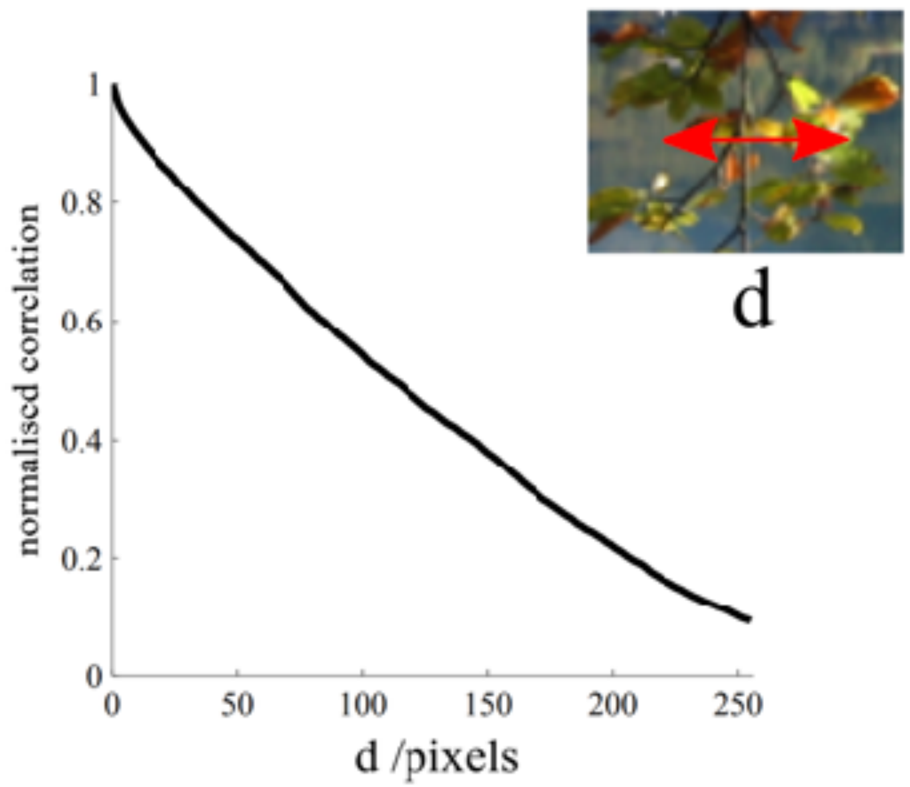
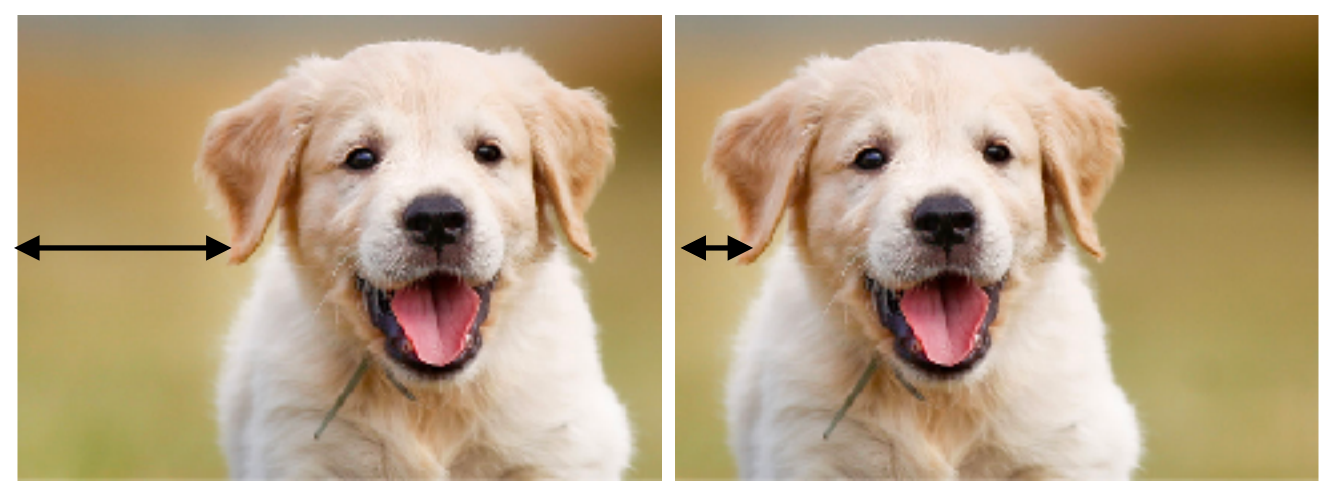




Locality **of *pixel statistics***

Property of the data

http://www.gatsby.ucl.ac.uk/~turner/teaching/4g3/2013/stats-recep-field.pdf

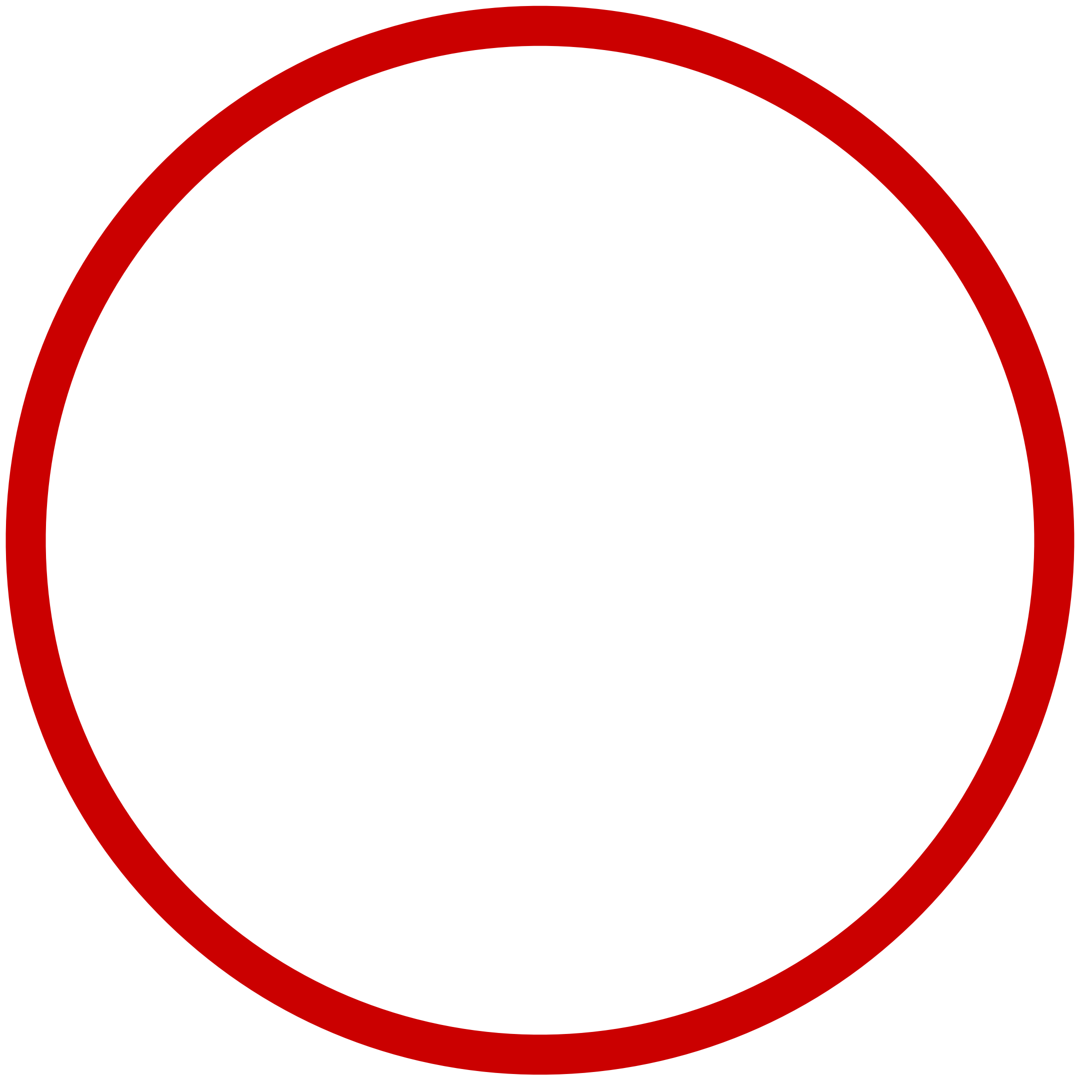


Translational invariance **of classification**

Property of the task

https://www.healthypawspetinsurance.com/Images/V3/DogAndPuppyInsurance/Dog_CTA_Desktop_HeroImage.jpg

OVERVIEW



What is symmetry?

What Is Symmetry?

Set of input transformations leaving f invariant

$$f(\mathbf{I}) = f(\mathcal{T}_\theta[\mathbf{I}])$$

image

function/
feature mapping

transformation

Notational aside:

e.g. Geometric translation

$$\mathcal{T}_\theta[\mathbf{I}](\mathbf{x}) = \mathbf{I}(\mathbf{x} - \boldsymbol{\theta})$$

e.g. Geometric rotation

$$\mathcal{T}_\theta[\mathbf{I}](\mathbf{x}) = \mathbf{I}(\mathbf{R}_{\boldsymbol{\theta}}^{-1}\mathbf{x})$$

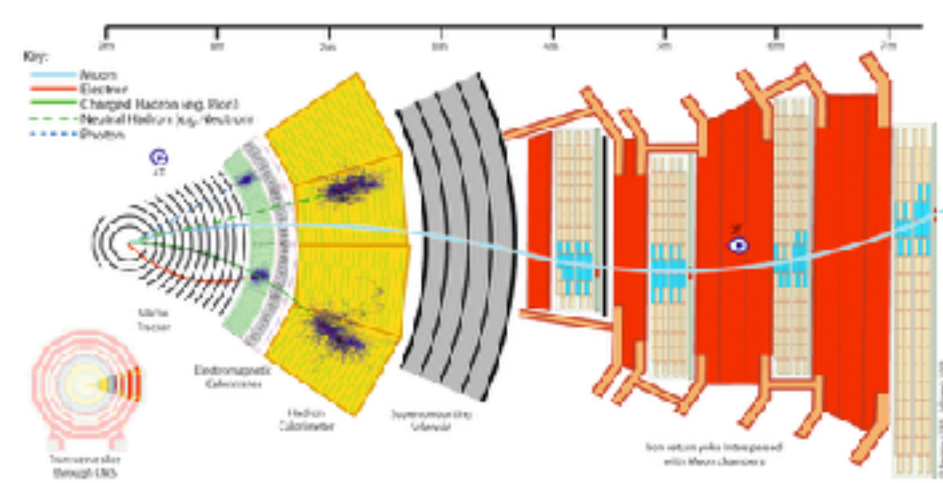
e.g. Pixel normalisation

$$\mathcal{T}[\mathbf{I}] = (\mathbf{I} - \boldsymbol{\mu})/.\boldsymbol{\sigma}^{-1}$$

Symmetry is a property of functions/tasks, e.g.

Classification

Disentangling
(cocktail party)



Signal discovery/detection

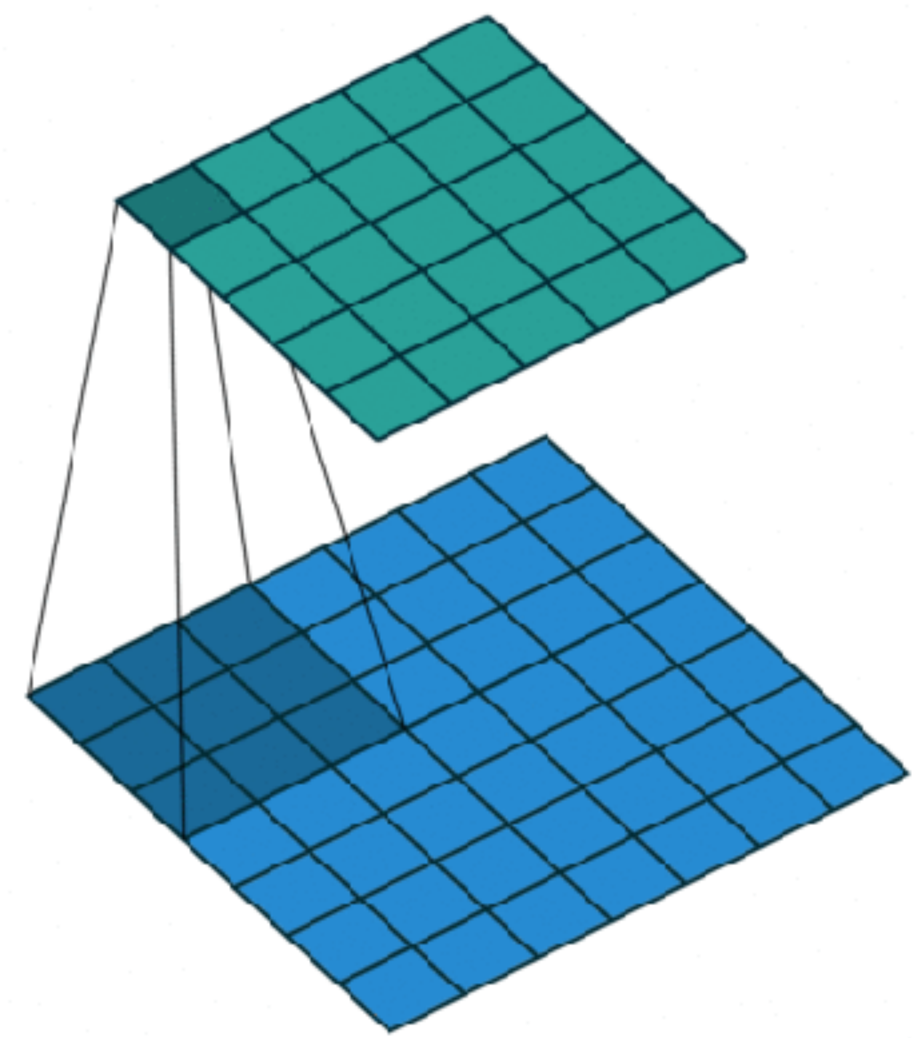
EQUIVARIANCE

Different *representations* of same transformation

Mapping preserves algebraic structure of transformation

$$\mathcal{S}_\theta[f](\mathbf{I}) = f(\mathcal{T}_\theta[\mathbf{I}])$$

transformation in feature space



Invariance

$$\mathcal{S}_{\boldsymbol{\theta}} = \mathrm{Id}$$

Convolution (and correlation)

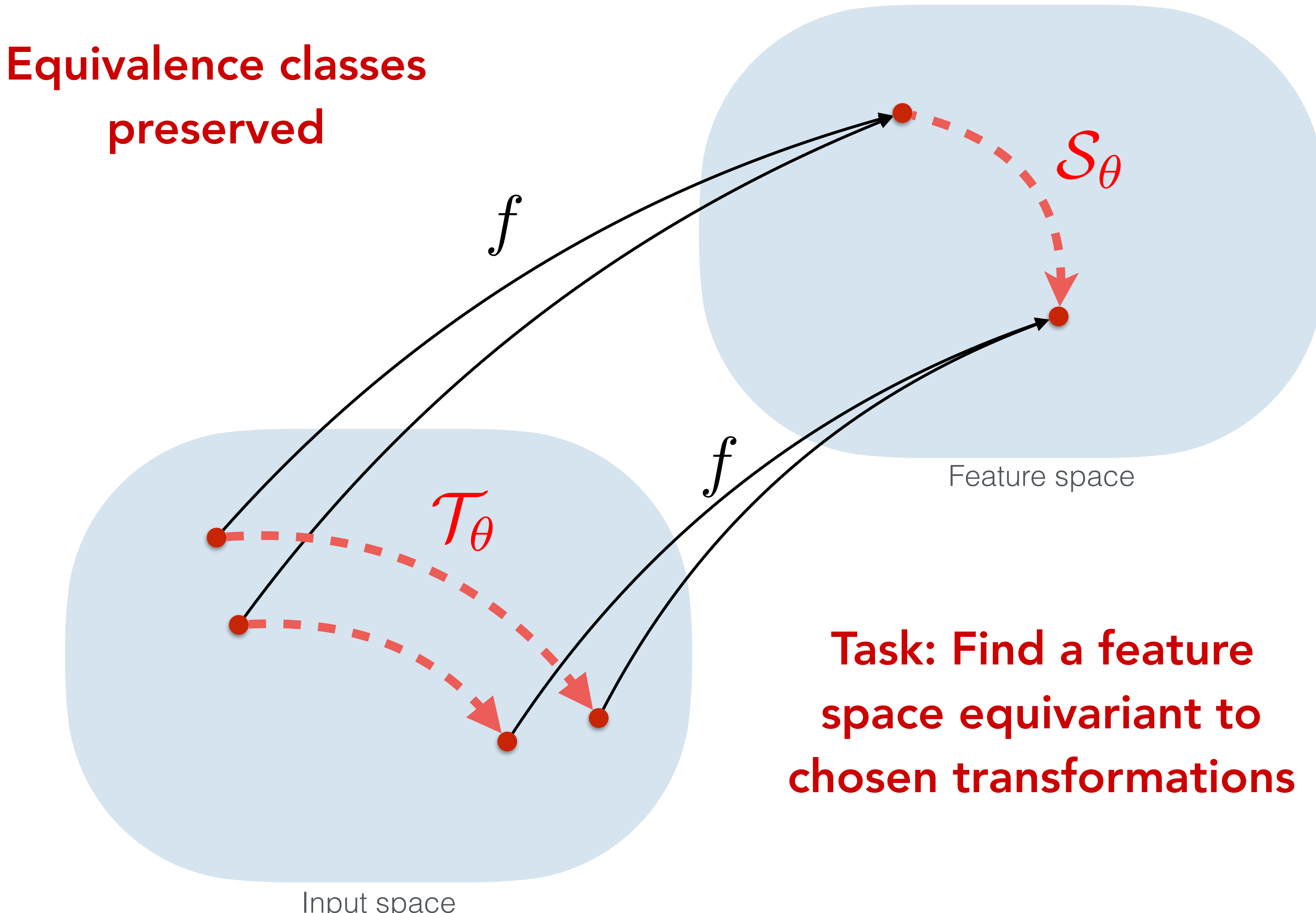
$$[\mathbf{I} * \mathbf{W}](\mathbf{x} - \boldsymbol{\theta}) = \mathcal{T}_{\boldsymbol{\theta}}[\mathbf{I}] * \mathbf{W}(\mathbf{x})$$

Fourier spectrum phase

$$e^{-\iota\boldsymbol{\omega}^\top\boldsymbol{\theta}} \int_{\mathbb{R}^N} \mathbf{I}(\mathbf{x}) e^{-\iota\boldsymbol{\omega}^\top\mathbf{x}}\, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} \mathbf{I}(\mathbf{x} - \boldsymbol{\theta}) e^{-\iota\boldsymbol{\omega}^\top\mathbf{x}}\, \mathrm{d}\mathbf{x}$$

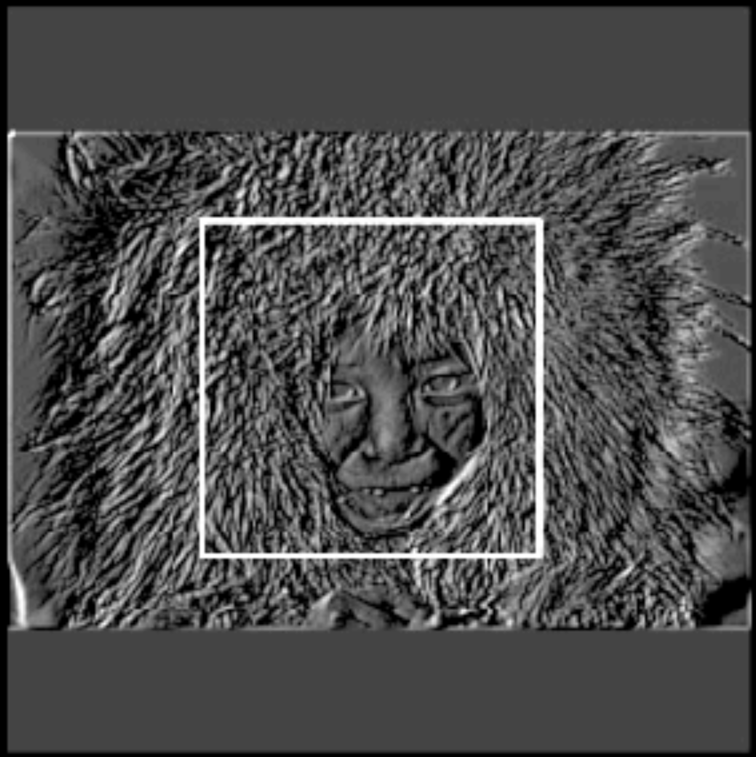
https://github.com/vdumoulin/conv_arithmetic

EQUIVARIANCE INTUITION

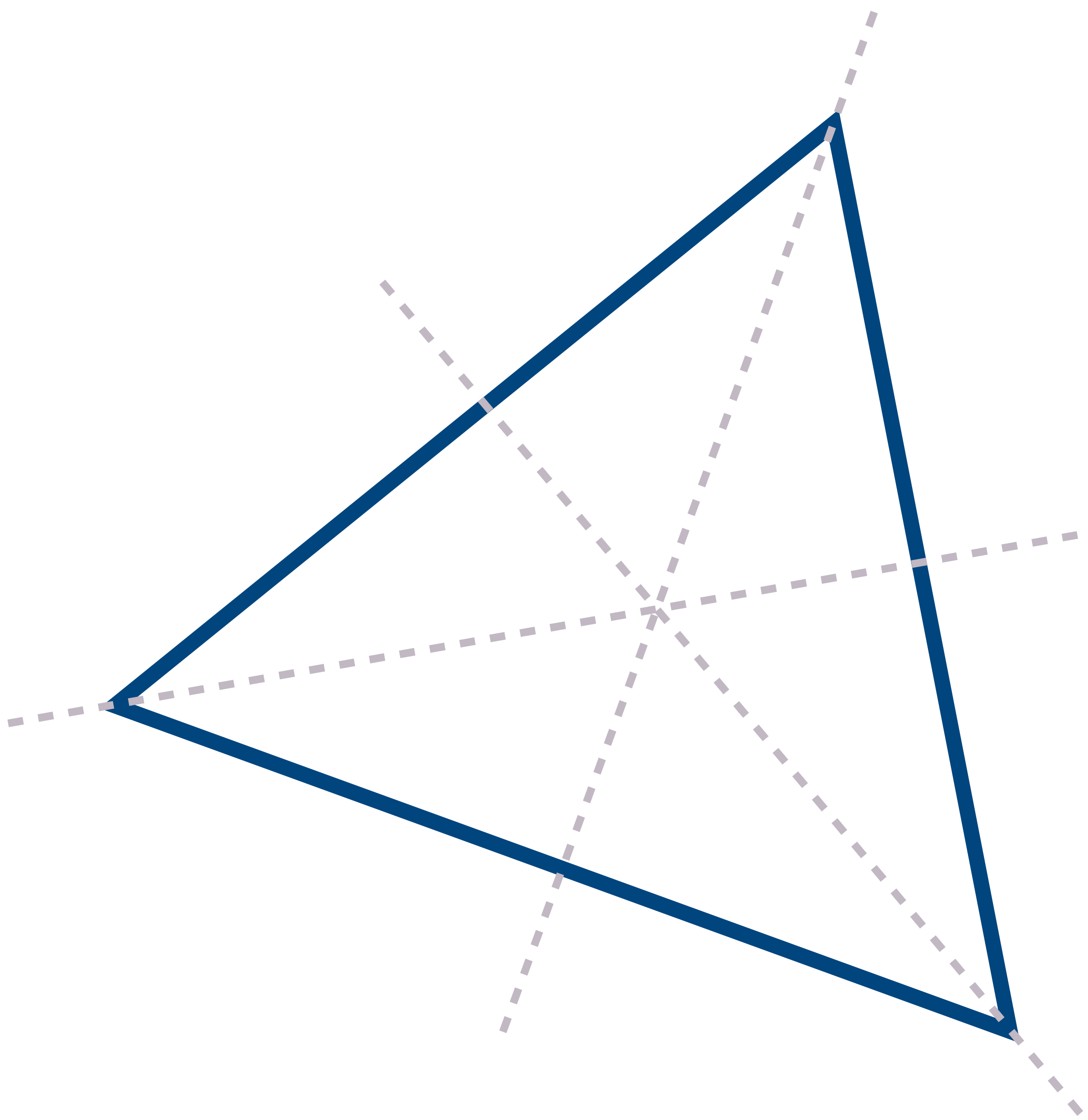


Thanks to Andrea Vedaldi for this insight

LACK OF KNOWN ROTATION **EQUI**VARIANCE:



Overview



Group symmetries

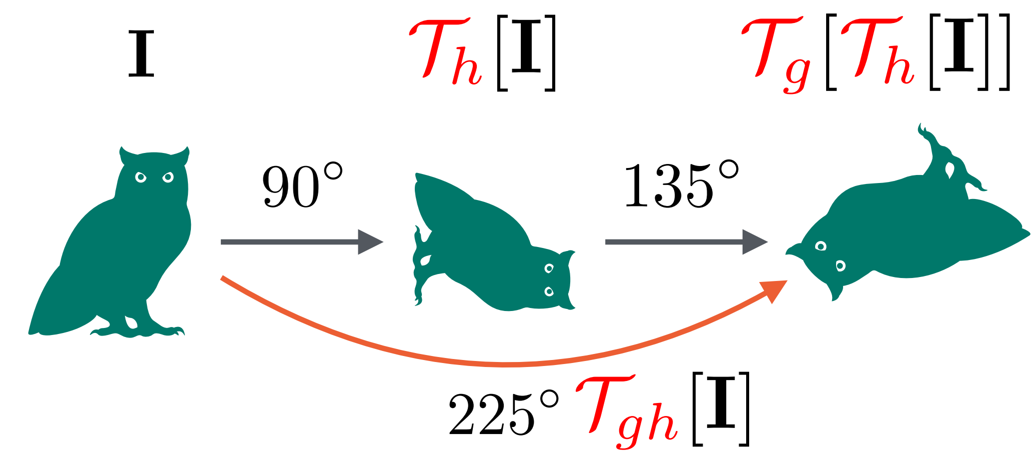
What Is Group Theory?

- **Field of abstract mathematics**
  - Language to describe invertible transformations

- **Criteria to be a group G**

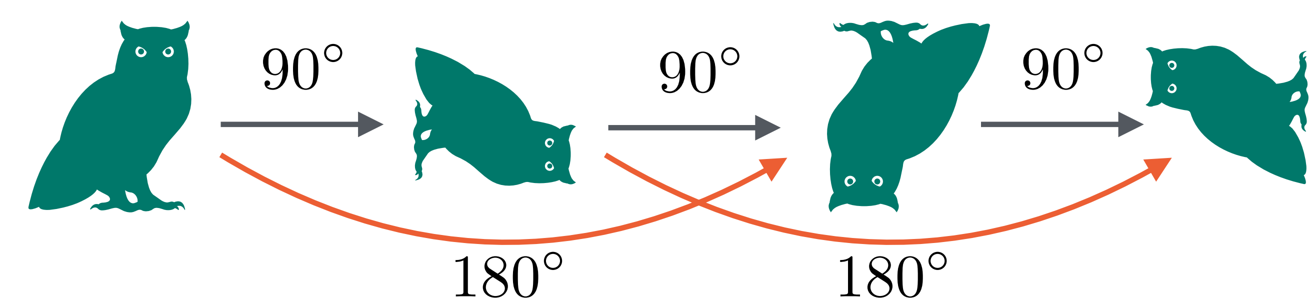
1. **Closure:** two transformations in a row is a transformation

$$gh \in G$$



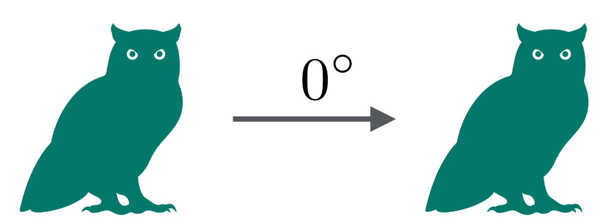
2. **Associativity:** transforming twice same as single compound transformation

$$g(hk) = (gh)k \in G$$



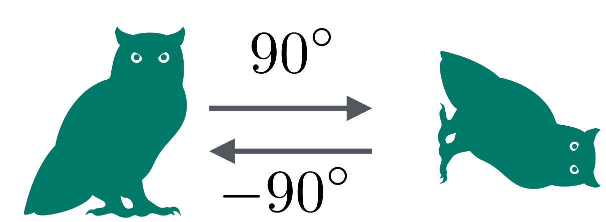
3. **Identity:** there is a “do nothing” transformation

$$eg = ge = g$$

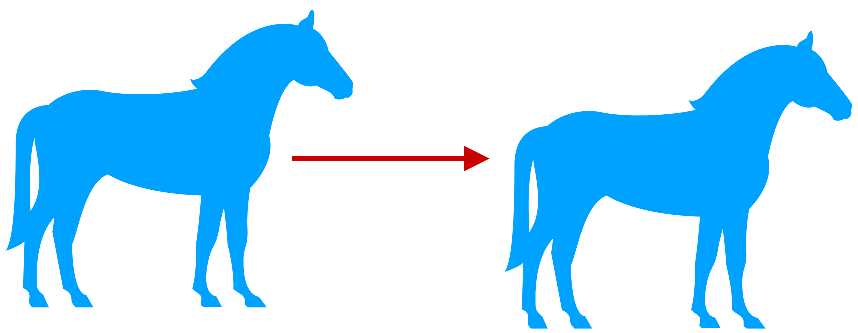


4. **Invertibility:** every transformation can be reversed perfectly

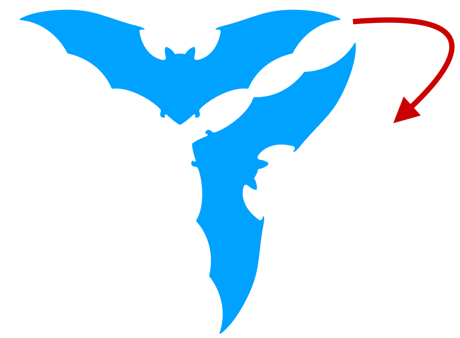
$$g^{-1}g = gg^{-1} = e$$



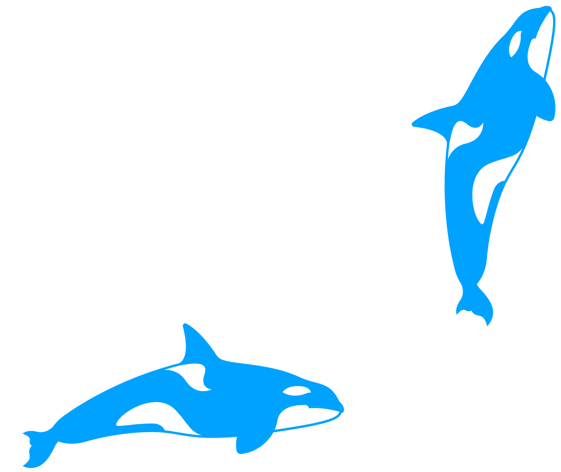
Group Examples



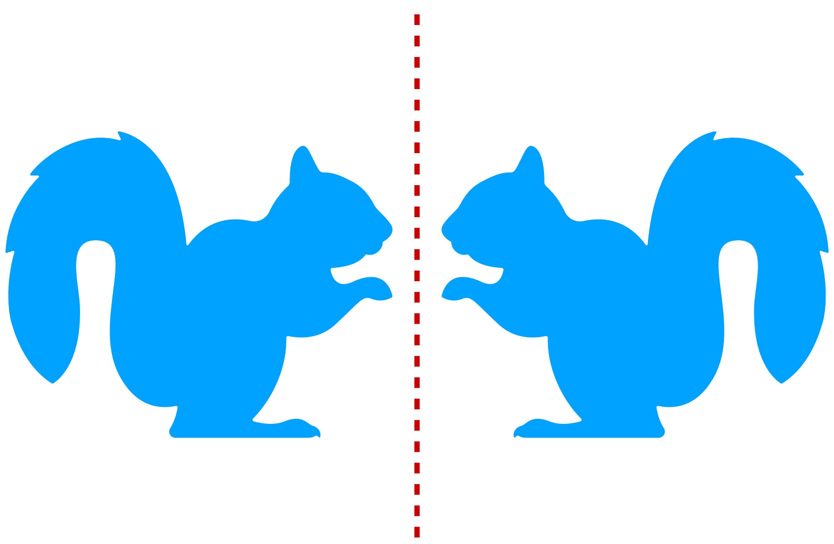
Translation



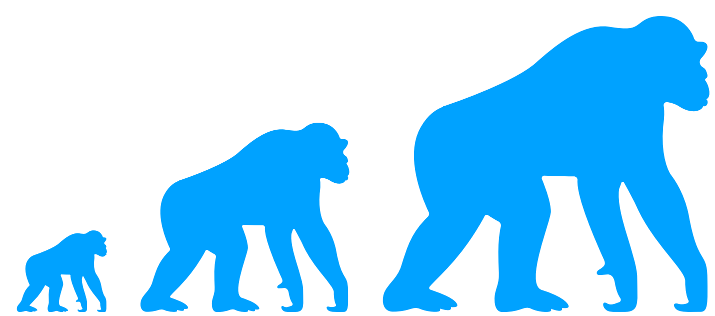
Rotation



Roto-translation

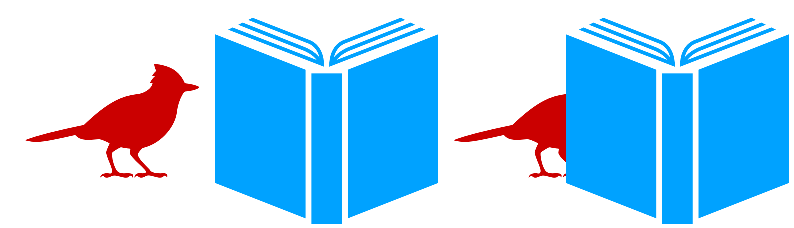


Reflections



Scalings*

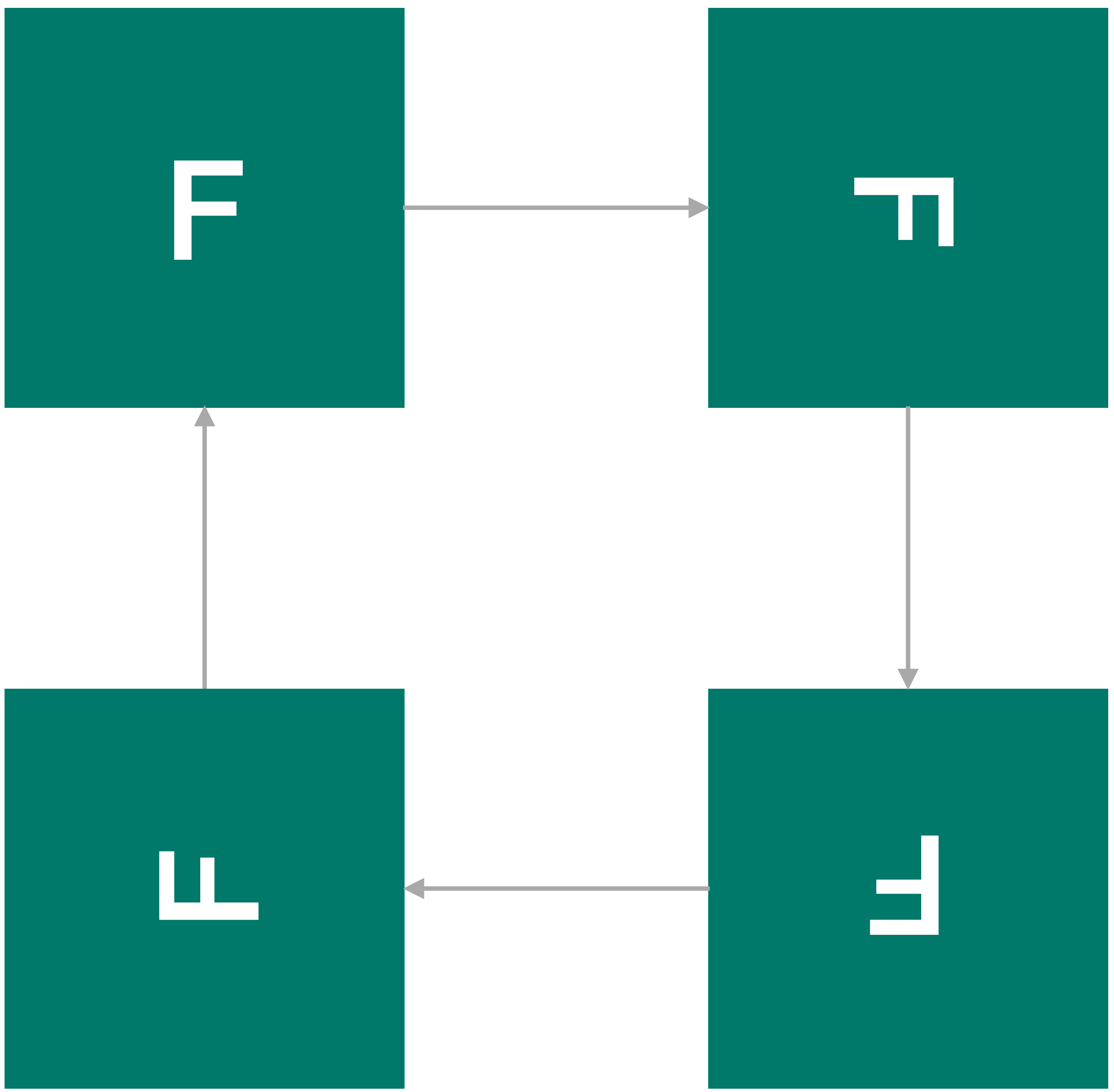
Non-example



Occlusions

*Scalings are probably better modelled as semigroups, i.e. groups without the invertibility condition

OVERVIEW



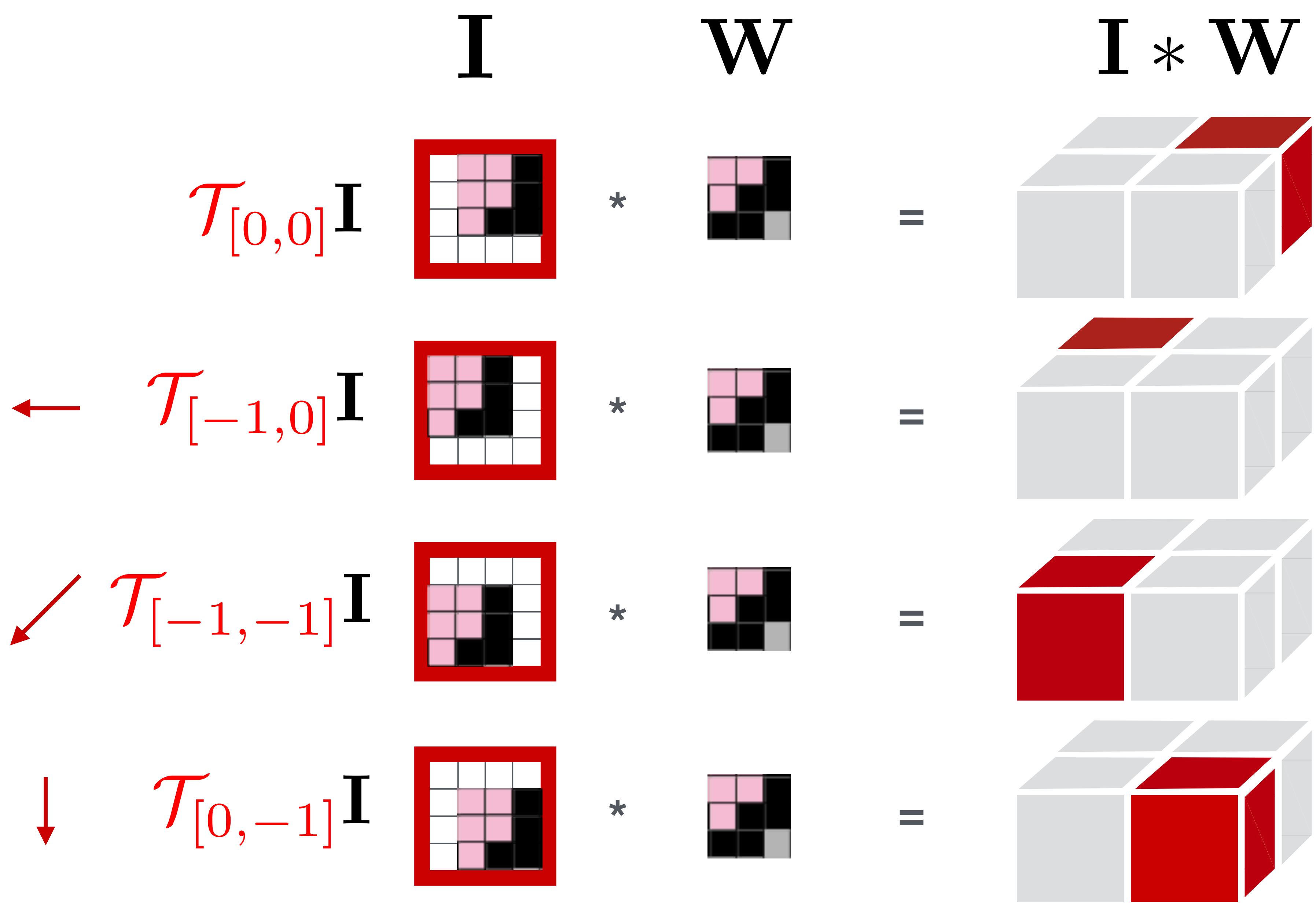
Group convolutions

CONVOLUTIONS ON TRANSLATION GROUP

Standard (yet much beloved) convolution

$$[\mathbf{I} * \mathbf{W}](\mathbf{y}) = \sum_{\mathbf{x}\in\mathbb{Z}^2} \mathbf{I}(\mathbf{x})\mathbf{W}(\mathbf{x}-\mathbf{y})$$

Translating kernel



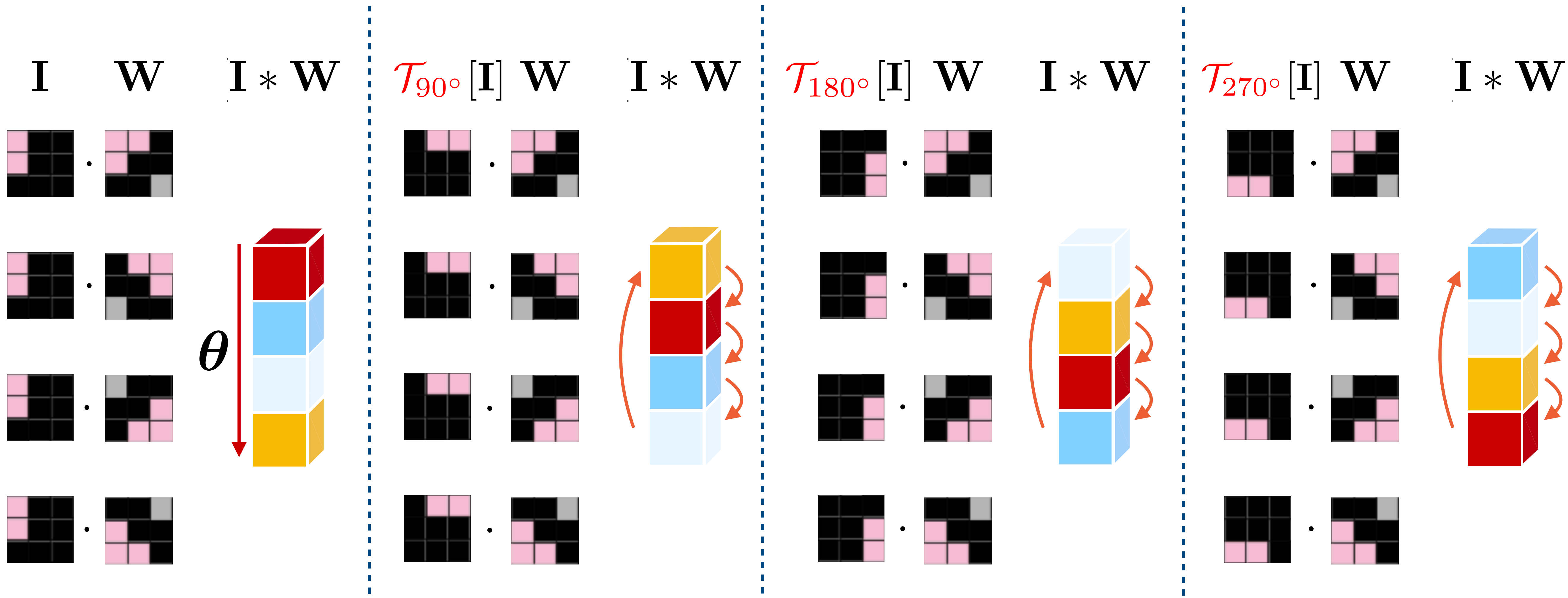
Convolutions on Rotation Group

Group convolution **Notice index of convolution**

Ref.
$$[\mathbf{I} * \mathbf{W}](\mathbf{y}) = \sum_{\mathbf{x}\in\mathbb{Z}^2} \mathbf{I}(\mathbf{x})\mathbf{W}(\mathbf{x}-\mathbf{y})$$

$$[\mathbf{I} * \mathbf{W}](\boldsymbol{\theta}) = \sum_{\mathbf{x}\in\mathbb{Z}^2} \mathbf{I}(\mathbf{x})\mathbf{W}(\mathbf{R}_{\theta}^{-1}\mathbf{x})$$

Rotating kernel



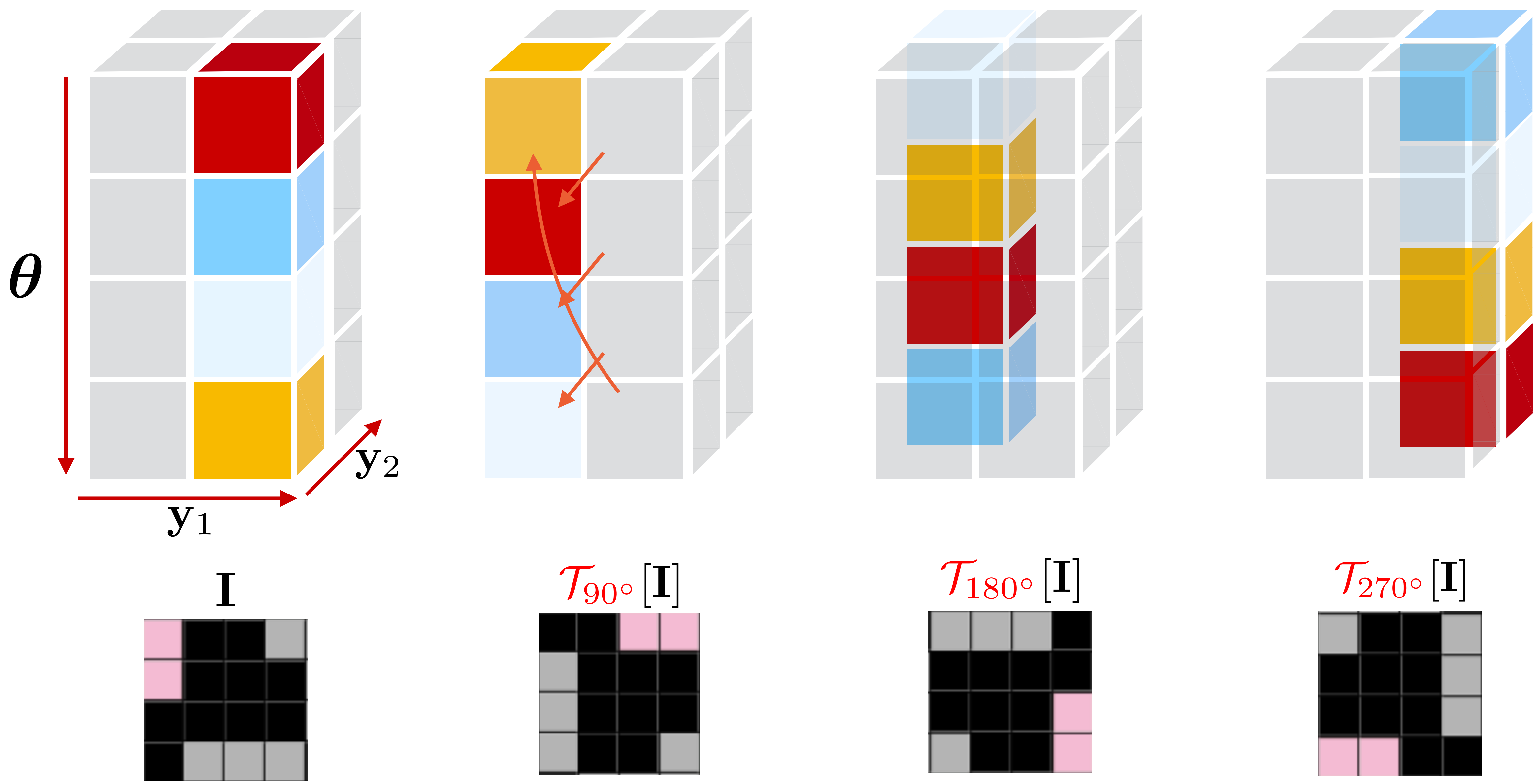
@ hidden layer $$[\mathbf{I} * \mathbf{W}](\psi) = \sum_{\boldsymbol{\theta}\in\Theta} \mathbf{I}(\boldsymbol{\theta})\mathbf{W}(\theta - \psi)$$

Convolutions on Roto-Translation Group

Group convolution

$$[\mathbf{I} * \mathbf{W}](\boldsymbol{\theta}, \mathbf{y}) = \sum_{\mathbf{x} \in \mathbb{Z}^2} \mathbf{I}(\mathbf{x}) \mathbf{W}(\mathbf{R}_{\theta}^{-1}\mathbf{x} - \mathbf{y})$$

Roto-translating kernel



@ hidden layer $$[\mathbf{I} * \mathbf{W}](\boldsymbol{\psi}, \mathbf{z}) = \sum_{(\boldsymbol{\theta}, \mathbf{Z}) \in (\Theta \times \mathbb{Z}^2)} \mathbf{I}(\boldsymbol{\theta}, \mathbf{y}) \mathbf{W}(\mathbf{R}_{\psi}^{-1}\mathbf{y} - \mathbf{z}, \boldsymbol{\theta} - \boldsymbol{\psi})$$

CONVOLUTIONS ON DISCRETE GROUPS

Convolution examples

$$[\mathbf{I} * \mathbf{W}](\mathbf{y}) = \sum_{\mathbf{x} \in \mathbb{Z}^2} \mathbf{I}(\mathbf{x}) \mathbf{W}(\mathbf{x} - \mathbf{y})$$

$$[\mathbf{I} * \mathbf{W}](\boldsymbol{\theta}) = \sum_{\mathbf{x} \in \mathbb{Z}^2} \mathbf{I}(\mathbf{x}) \mathbf{W}(\mathbf{R}_\theta^{-1} \mathbf{x})$$

$$[\mathbf{I} * \mathbf{W}](\boldsymbol{\theta}, \mathbf{y}) = \sum_{\mathbf{x} \in \mathbb{Z}^2} \mathbf{I}(\mathbf{x}) \mathbf{W}(\mathbf{R}_\theta^{-1} \mathbf{x} - \mathbf{y})$$

For general discrete groups

@ input layer

$$[\mathbf{I} * \mathbf{W}](\boldsymbol{\theta}) = \sum_{\mathbf{x} \in \mathbb{Z}^2} \mathbf{I}(\mathbf{x}) \mathcal{T}_\theta[\mathbf{W}](\mathbf{x})$$

@ hidden layer

$$[\mathbf{I} * \mathbf{W}](\psi) = \sum_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \mathbf{I}(\boldsymbol{\theta}) \mathcal{T}_\psi[\mathbf{W}](\boldsymbol{\theta})$$

$$= \sum_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \mathbf{I}(\boldsymbol{\theta}) \mathbf{W}(\psi^{-1} \boldsymbol{\theta})$$

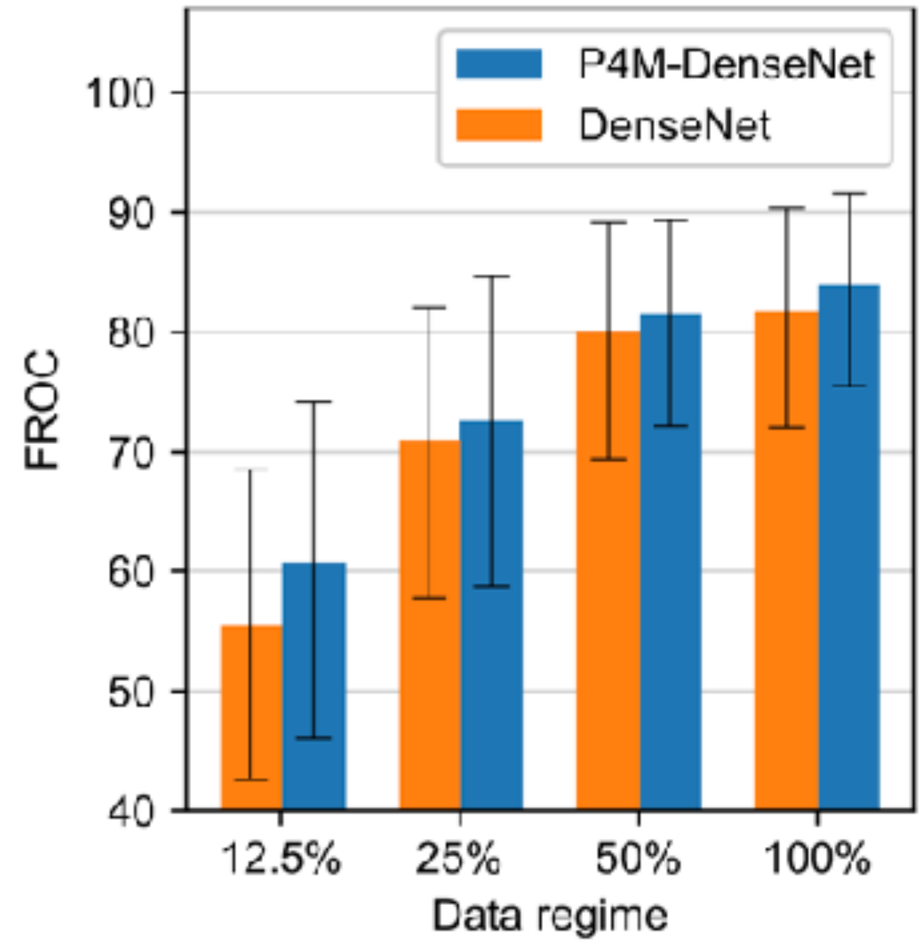
Read: Group Equivariant Convolutional Networks, Cohen & Welling, 2016

EXAMPLES

Rotation Equivariant CNNs for Digital Pathology

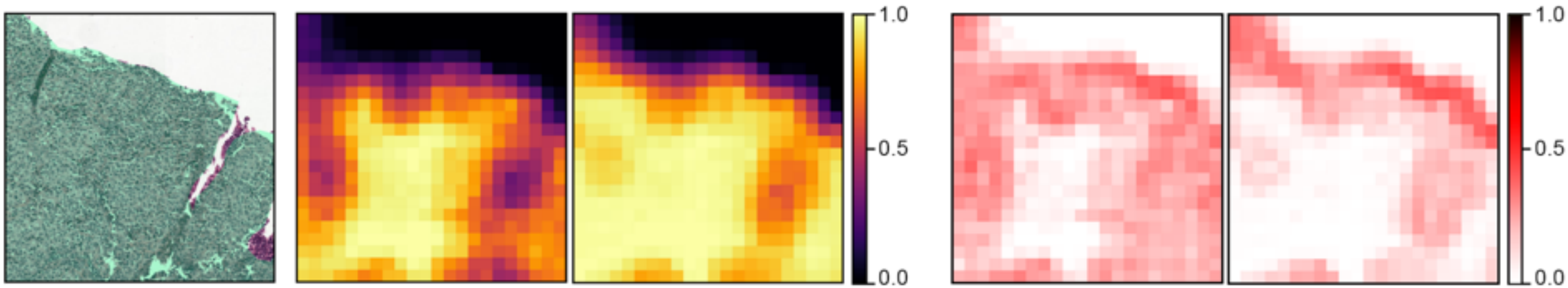
Bastiaan S. Veeling*, Jasper Linmans*, Jim Winkens*, Taco Cohen, and Max Welling

University of Amsterdam, The Netherlands



Mean prediction

Standard deviation



Input

DenseNet

Rotation equivariant DenseNet

DenseNet

Rotation equivariant DenseNet

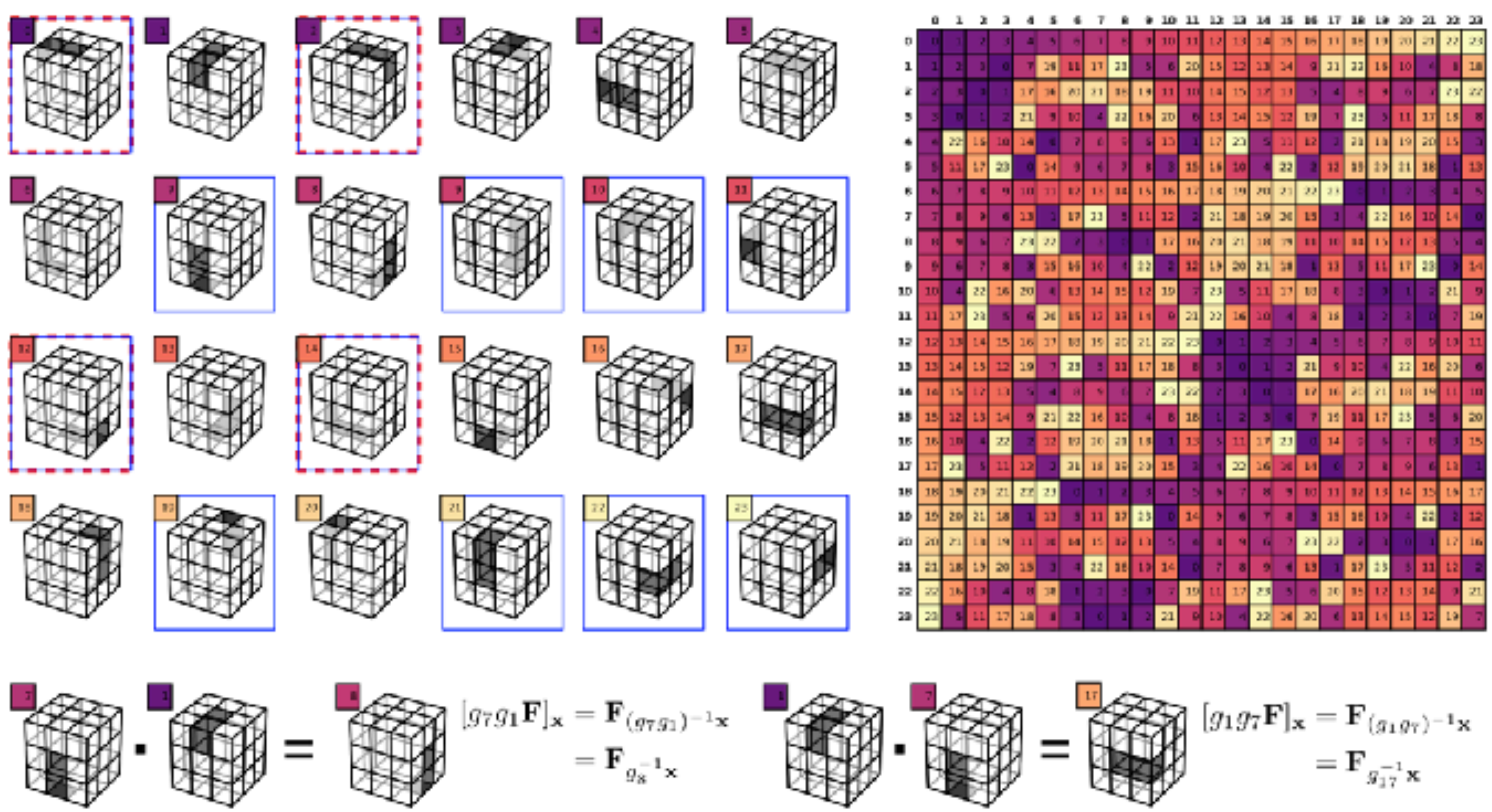
EXAMPLES

CubeNet: Equivariance to 3D Rotation and Translation

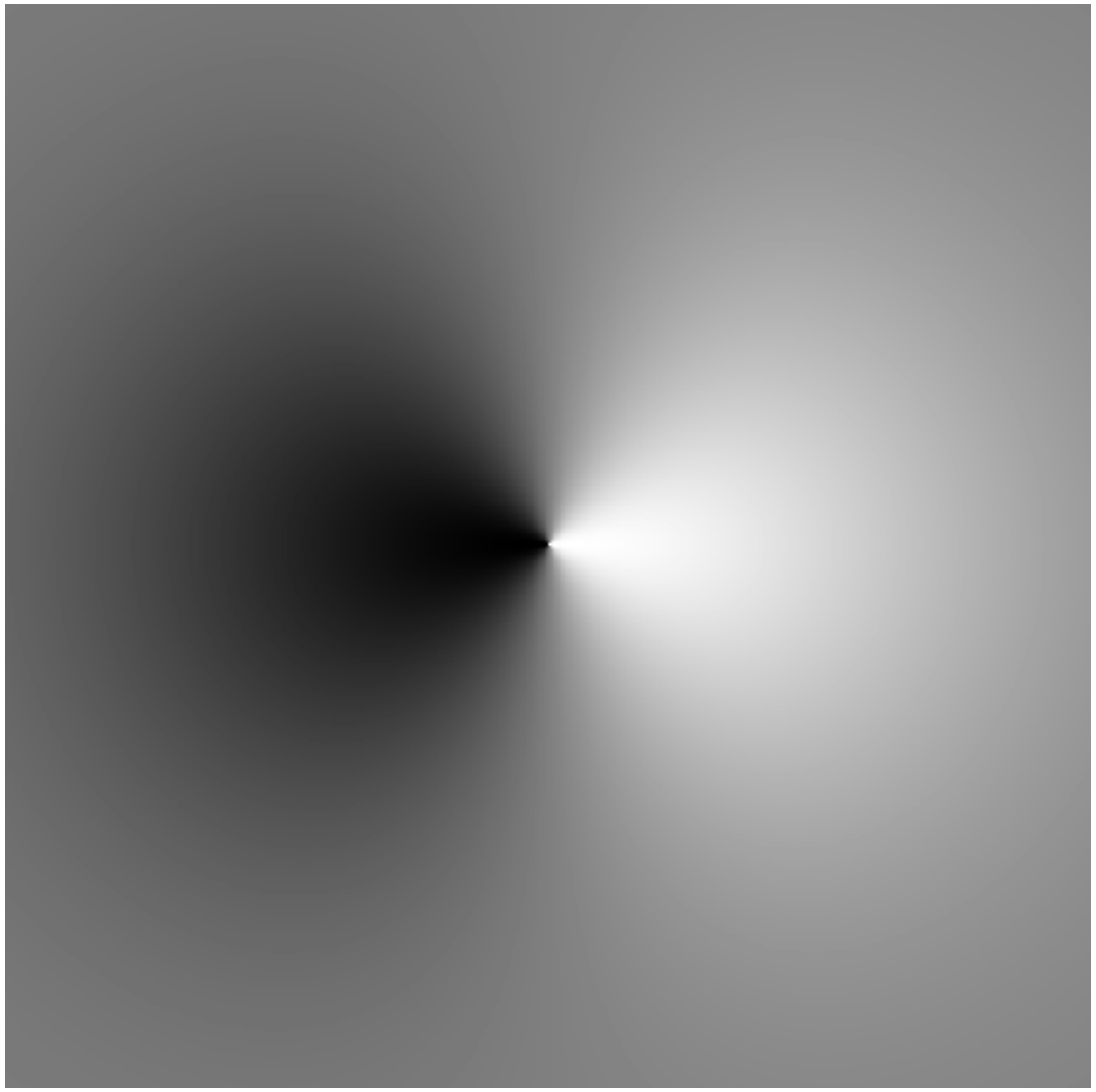
Daniel Worrall and Gabriel Brostow

Computer Science Department, University College London, UK
{d.worrall,g.brostow}@cs.ucl.ac.uk

Method	ModelNet10	# params ($\times 10^6$, 2 s.f.)
3D ShapeNets [41]	0.8354	12
Xu & Todovoric [48]	0.8800	0.080
3D-GAN [49]	0.9100	11
VRN [42]	0.9133	18
VoxNet [40]	0.9200	0.92
Fusion-Net [50]	0.9311	120
ORION [43]	0.9380	0.91
Ours T_4	0.9127	4.5
Our V (average)	0.9372	4.5
Ours V (best model single-view)	0.9120	4.5
Ours V (best model rotation averaged)	**0.9460**	4.5
VRN Ensemble [42]	0.9714	108



Overview



Harmonic/steerable convolutions

ROTATION EQUIVARIANCE METHODS

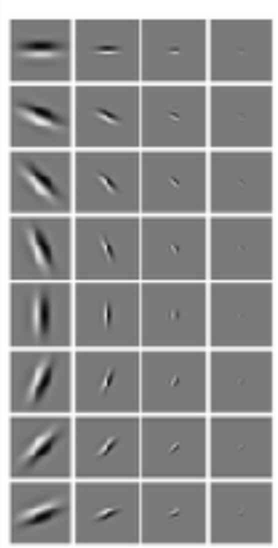
Rotate in network

Rotate filters

Rotate activations

Exploiting Cyclic Symmetry
(Dieleman et al., 2016)

Store full response



Roto-Translation Scattering
(Oyallon et al., 2014)
Group Equivariant CNNs
(Cohen et al., 2016)
Oriented Response Networks
(Zhou et al., 2017)
Steerable CNNs
(Cohen et al., 2017)
Steerable Filters G-CNNs
(Weiler et al. 2018)

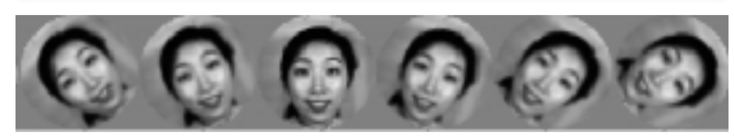
Store max/mean response

Roto-Translation Invariant Scattering
(Sifre et al., 2013)
Rotation Equivariant Vector Fields
(Marcos et al., 2017)
And more…

Rotate input

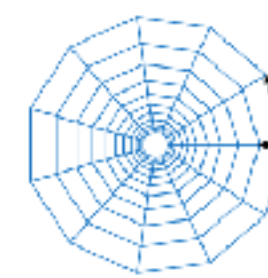
Brute-force

Rotation-Invariant Neoperceptron
(Fasel et al., 2006)
TI-Pooling
(Laptev et al., 2016)



Log-polar trans.

Warped Convolutions
(*Henriques et al., 2017)*
Polar Transformer Nets
(Esteves et al., 2017)



The Fourier Shift Theorem

Interestingly this generalises beyond 1D shifts to any transformation if considering groups

$$\int_{-\pi}^{\pi} I(\phi - \theta) e^{-\iota m \phi} \, \mathrm{d}\phi = \int_{-\pi-\theta}^{\pi-\theta} I(\phi') e^{-\iota m (\phi' + \theta)} \, \mathrm{d}\phi'$$

$$m \in \mathbb{Z} \qquad = \underbrace{e^{-\iota m \theta}}_{\text{Equivariant}} \underbrace{\int_{-\pi}^{\pi} I(\phi') \boxed{e^{-\iota m \phi'}} \, \mathrm{d}\phi'}_{\text{Invariant}}$$

Constraint on angular component of weights

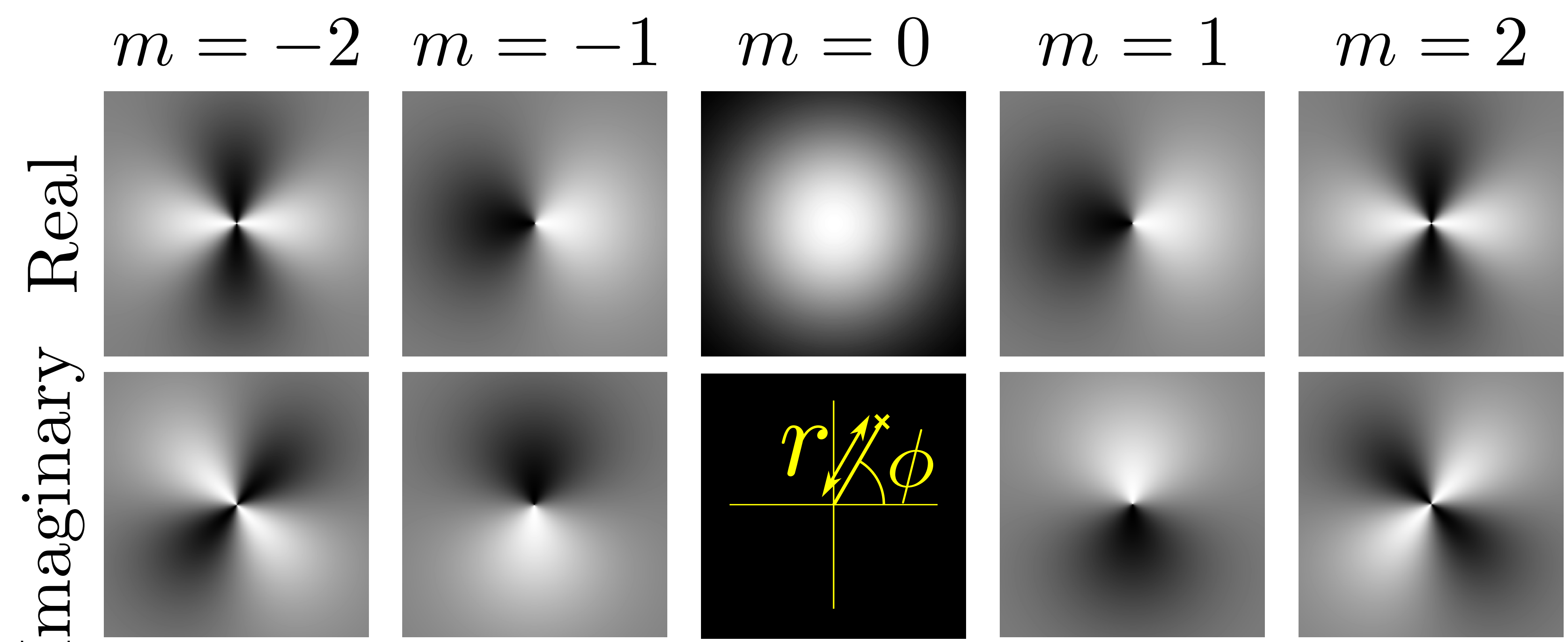
Circular Harmonics

Note: the sign flip in the angular component

Live in orthogonal spaces, so don't interact

$$W_m(r;\phi) = R(r)e^{\iota(m\phi+\beta)}$$

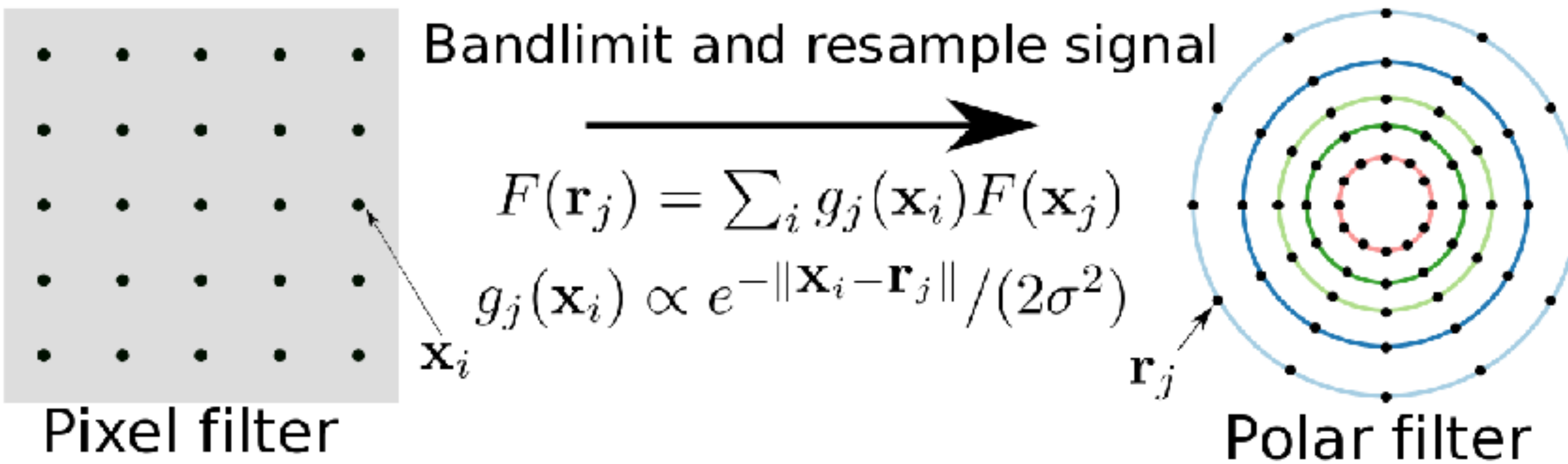
Free Constrained



Example with Gaussian radial component

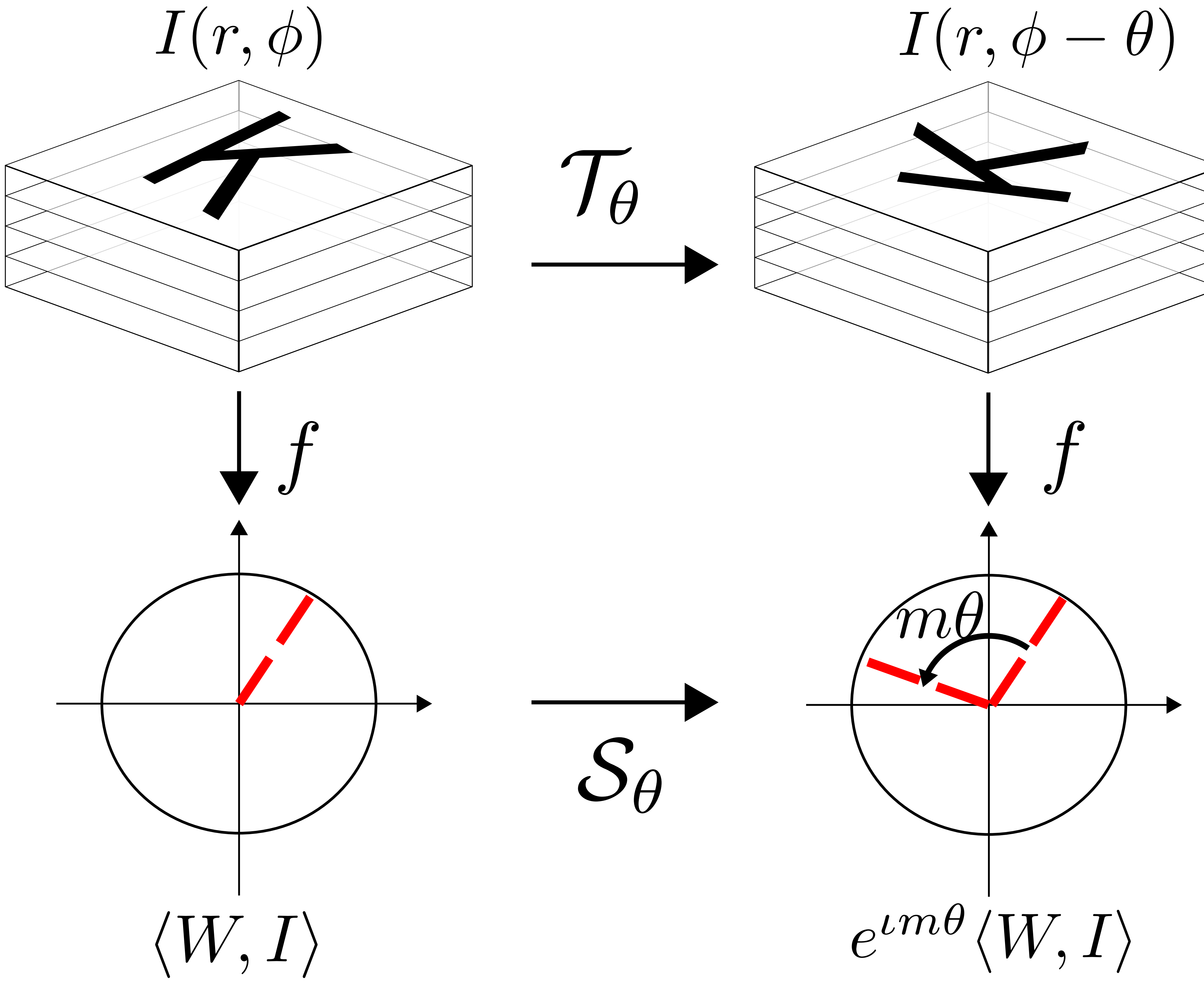
CIRCULAR HARMONIC WEIGHTS: IN PRACTICAL TERMS

Circular harmonics are defined in polar coordinates, so **resample** with Gaussian **anti-aliasing filter**.



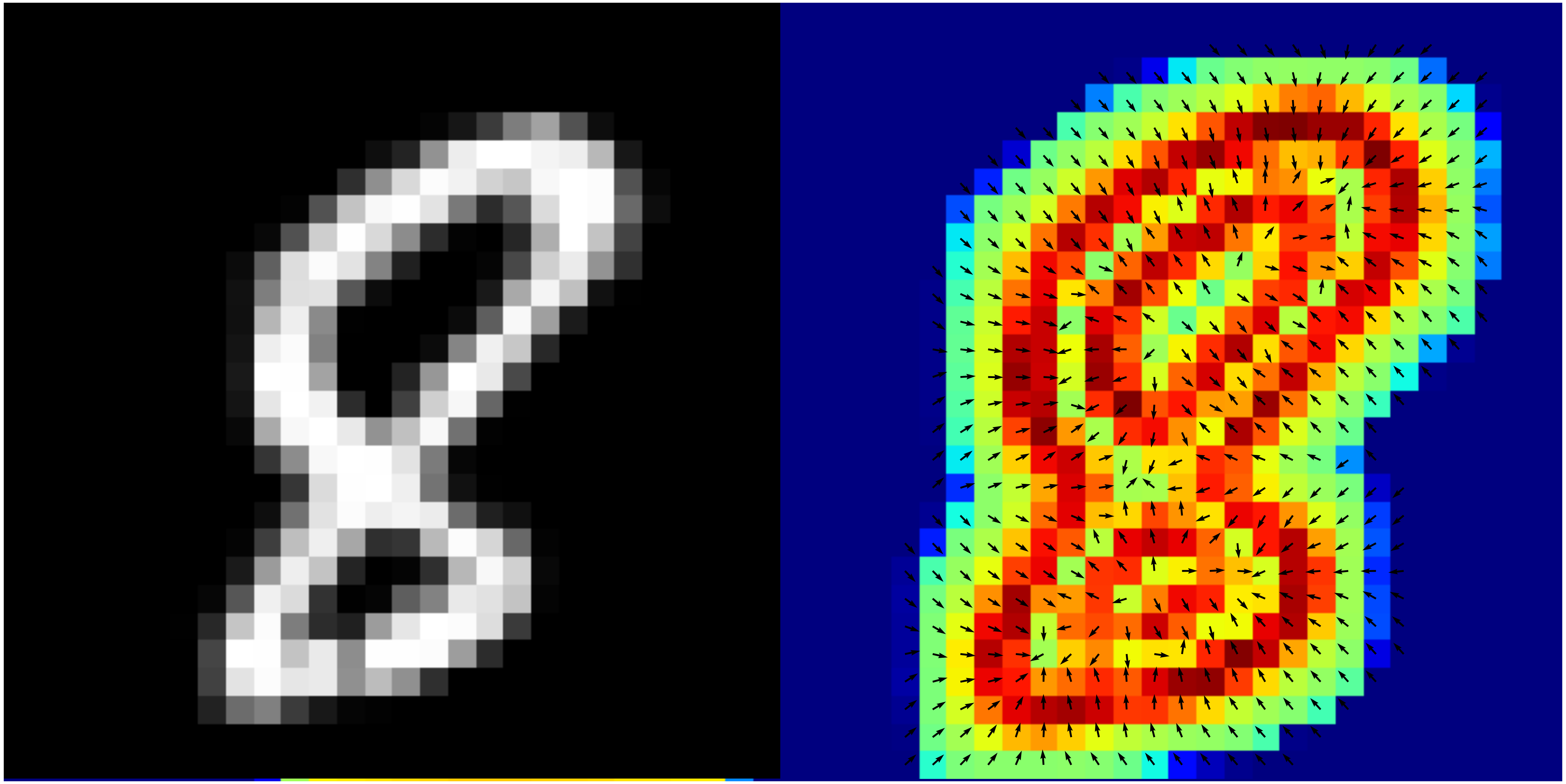
$$\mathbf{I} * \mathbf{W}_m = \underbrace{(\mathbf{I}^{\mathrm{Re}} * \mathbf{W}_m^{\mathrm{Re}} - \mathbf{I}^{\mathrm{Im}} * \mathbf{W}_m^{\mathrm{Im}})}_{\text{Real}} + \iota\underbrace{(\mathbf{I}^{\mathrm{Im}} * \mathbf{W}_m^{\mathrm{Re}} - \mathbf{I}^{\mathrm{Re}} * \mathbf{W}_m^{\mathrm{Im}})}_{\text{Imaginary}}$$

Circular Harmonic Weights



Circular Harmonics

X-correlation w. circular harmonics returns orientation field



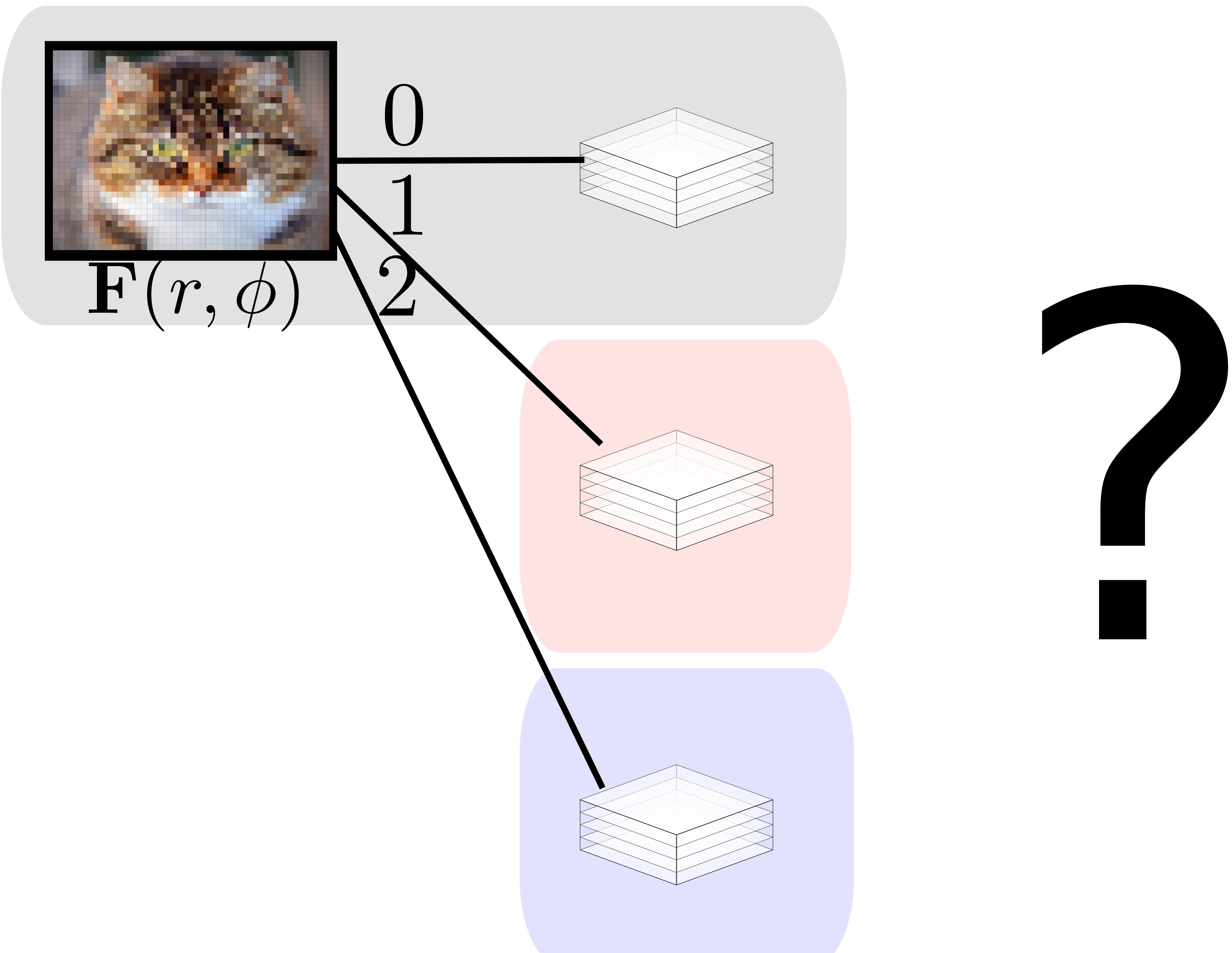
CIRCULAR HARMONIC ALGEBRA

So far we have a mapping $W\star : \mathbf{R}^{M\times N\times K} \to \mathbf{C}^{M\times N\times 1}$

But what about subsequent layers?

$$g(\mathcal{T}_\theta[\mathbf{I}] * \mathbf{W}_{m1}) * \mathbf{W}_{m_2}$$

What About Subsequent Layers?

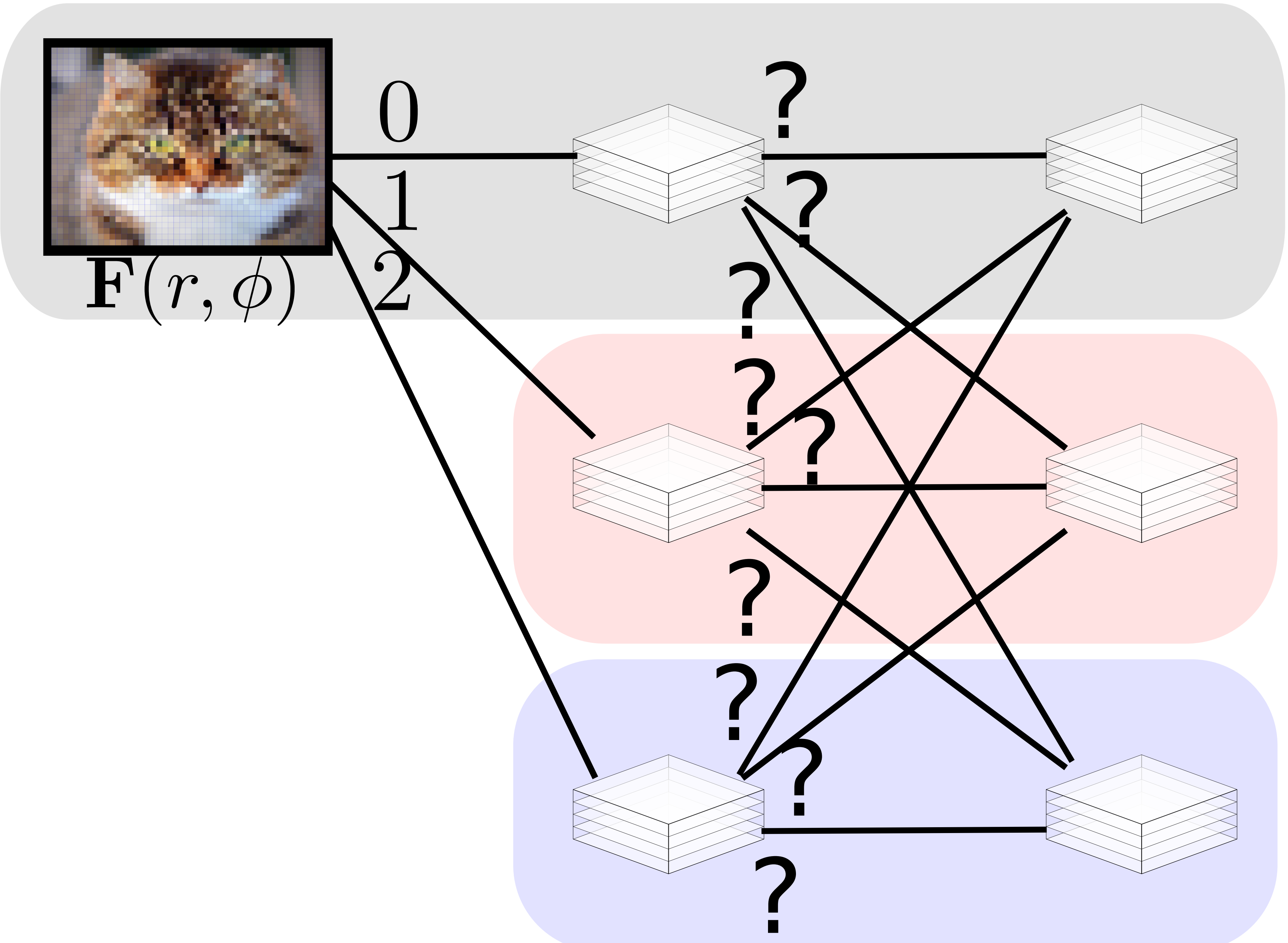


Circular Harmonic Algebra

Sum orders along chained cross-correlations

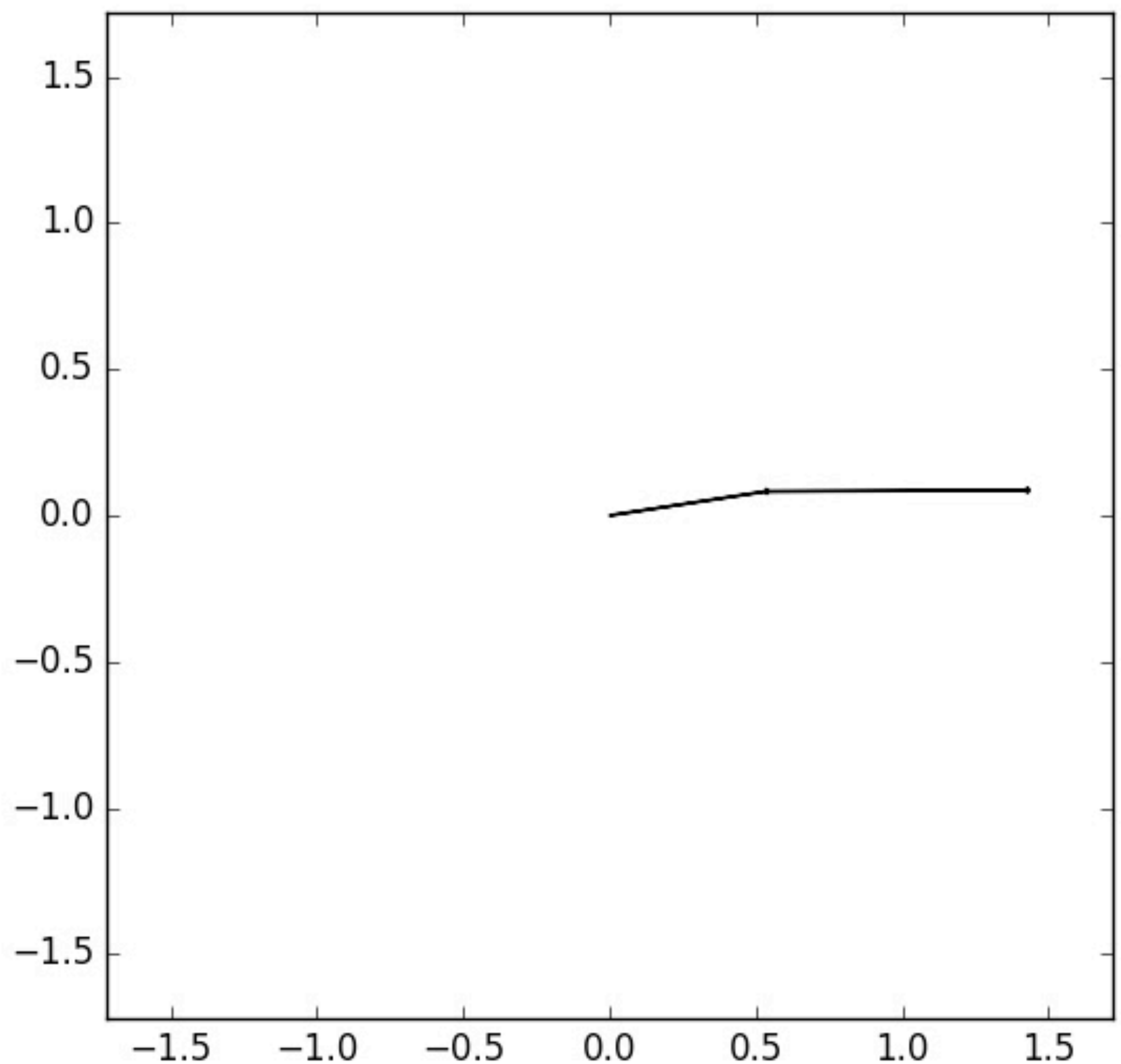
$$g(\mathcal{T}_\theta[\mathbf{I}] * \mathbf{W}_{m1}) * \mathbf{W}_{m_2} = Ye^{\iota(m_1+m_2)\theta}$$

Circular Harmonic Algebra



Circular Harmonic Algebra

At a layer, we sum over incoming feature maps, but need to pay special attention to rotation order



Circular Harmonic Algebra

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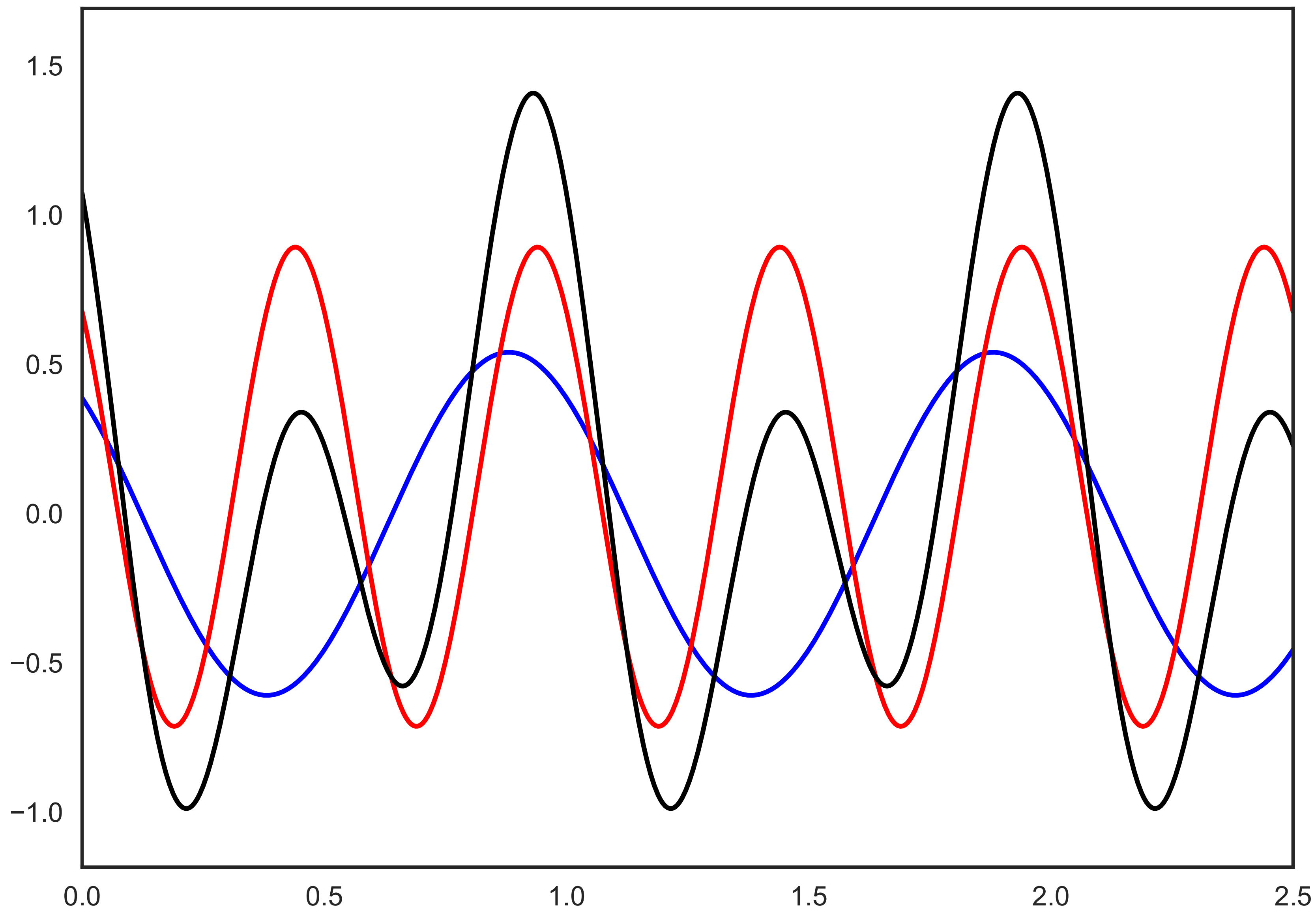


CIRCULAR HARMONIC ALGEBRA

The Equivariance/Coherence Condition

Rotation order of filter

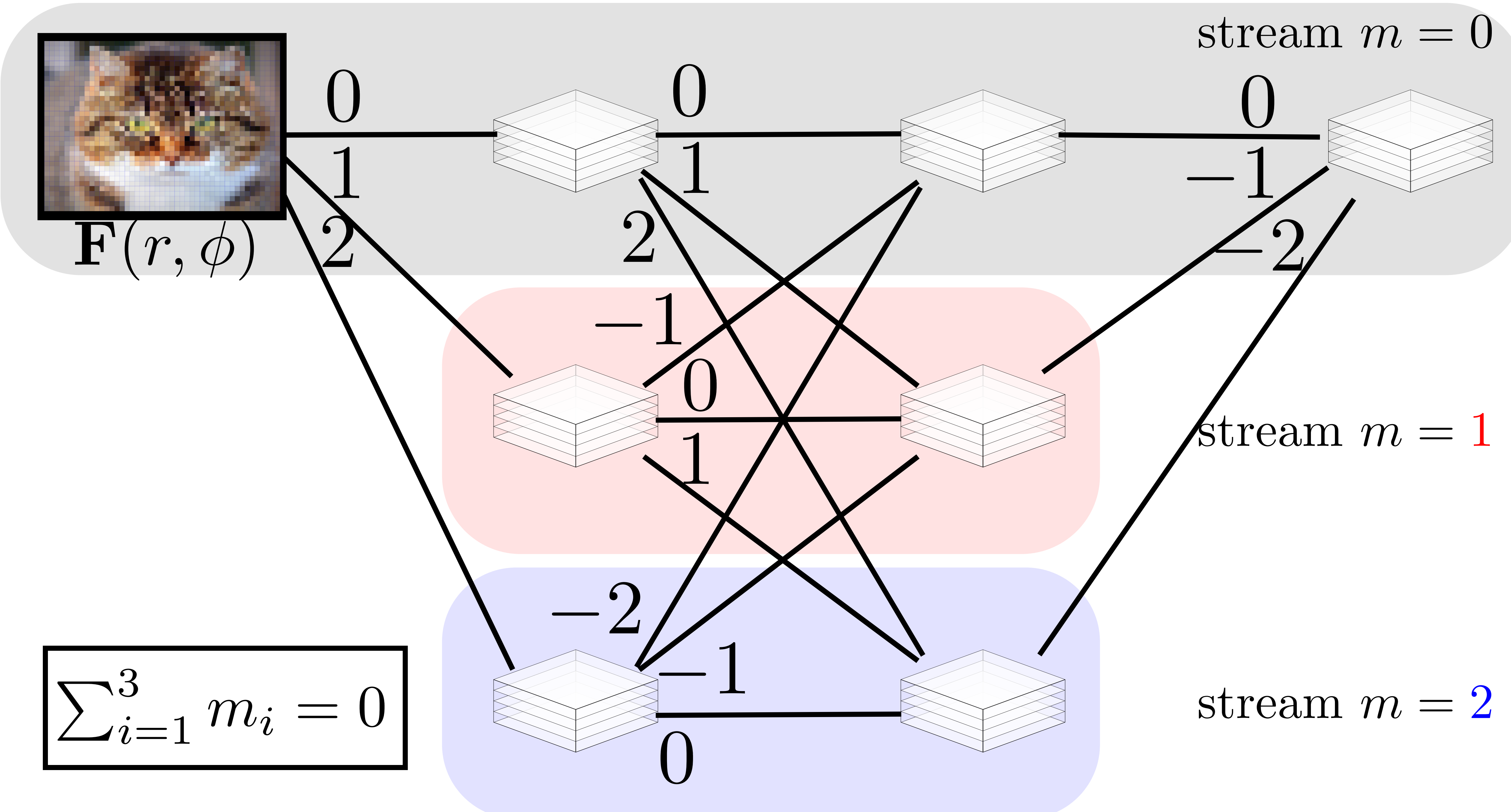
Constant at feature map i

$$\sum_{p \in \mathcal{P}_i} m_p = \kappa_i$$

Path through i

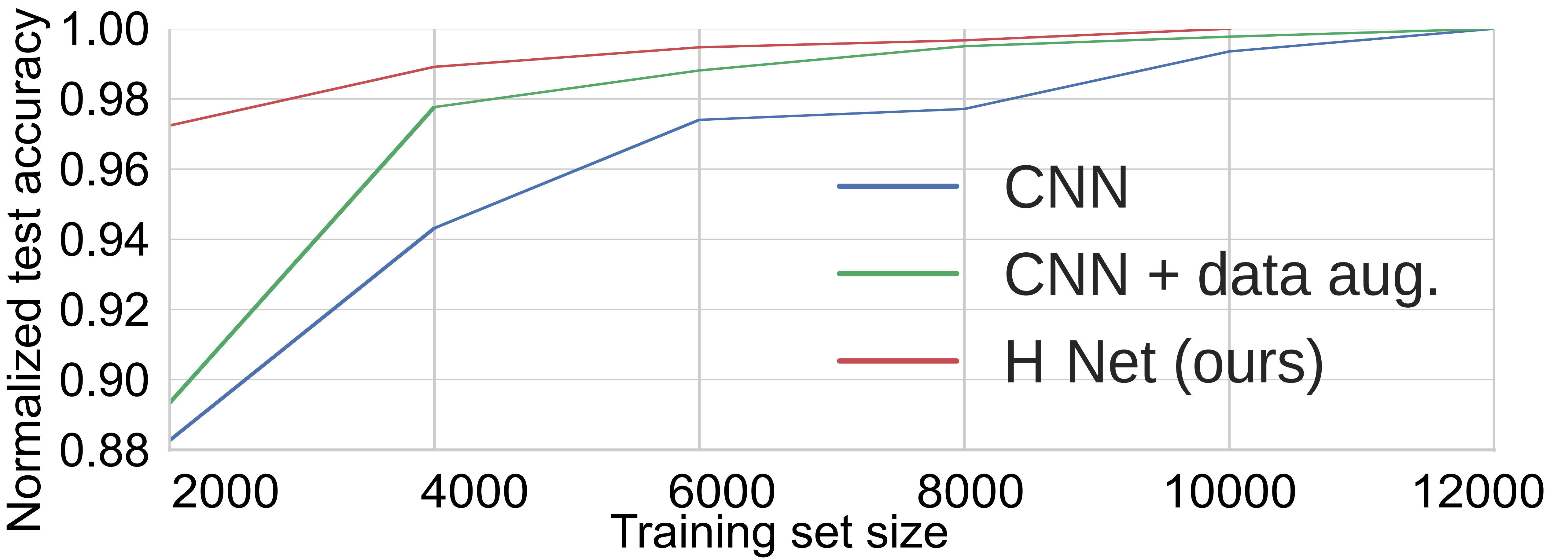
My conjecture: For Lie Groups, streams identified with basis vectors of Lie Algebra, and harmonic filters act as raising/lowering operators, jumping between basis vectors

CIRCULAR HARMONIC ALGEBRA



Harmonic Networks: Deep Rotation And Translation Equivariance, Worrall et al. (2017)

DATA ABLATION

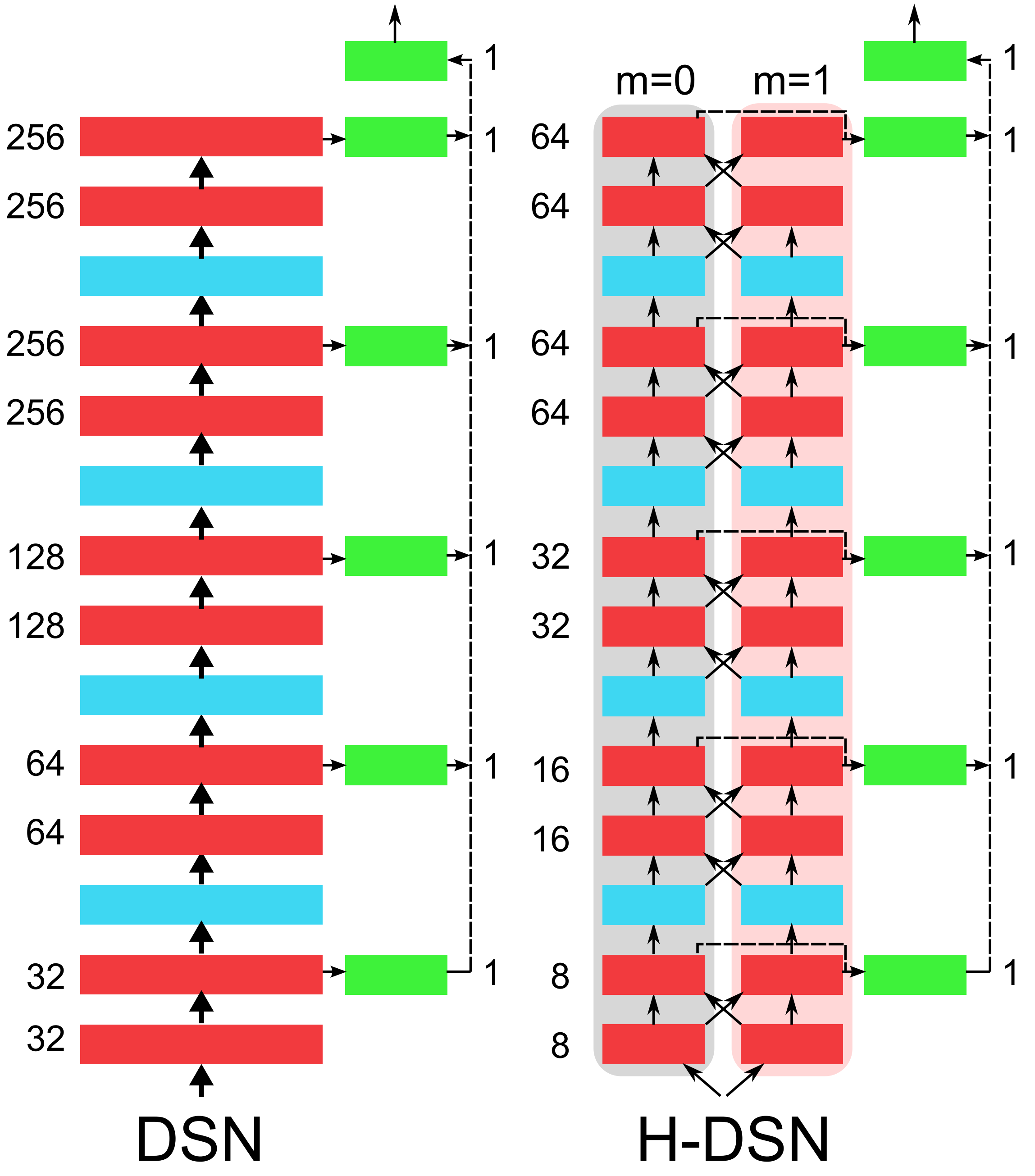


BOUNDARY SEGMENTATION

Boundary segmentation: BSD500

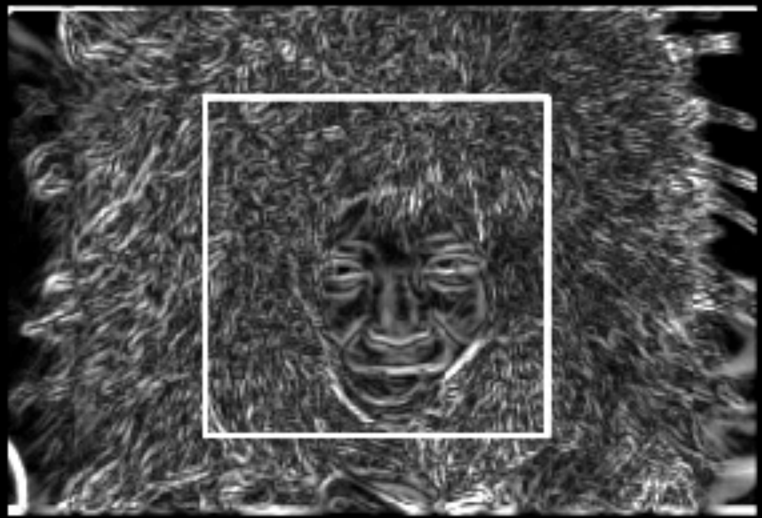
Recent results

Dynamic Steerable ResNet **0.732 0.751**



ORIENTED BOUNDARY SEGMENTATION

ROTATING FACE



ROTATING FACE

Follow Up Work

- G-convs and H-convs both in unified theory *The Quite General Theory of Equivariant Convolutional Networks Cohen et al., (under review), 2018*

- *3D Generalisation: Tensor Field Networks: Rotation- and Translation-Equivariant Neural Networks For 3D Point Clouds, Thomas et al., 2017*

OVERVIEW

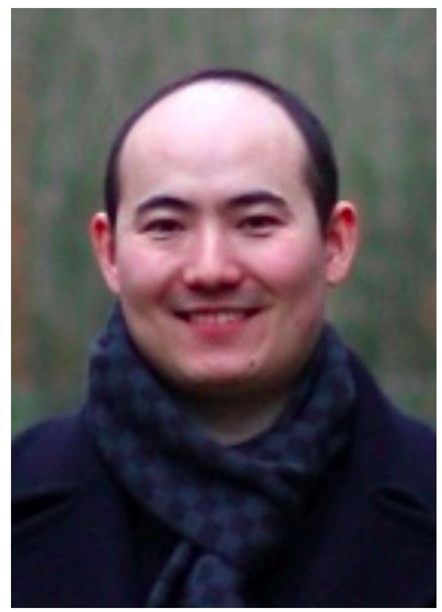
Future challenges

SUMMARY, LIMITATIONS, & FUTURE

- G-convs (good discriminativity, okay equivariance)
- H-convs (good equivariance, okay discriminativity)
- No generalisation guarantees as of yet
- Is group theory too restrictive? I think it is…
- Despite theory have only explored rotations and flips: *what about scale, affine transformation, occlusions…?*

COLLABORATORS

Stephan Garbin



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