Binary Quintic Forms, Quintic Rings & Sextic Resolvents

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1 Introduction

By a construction of Birch and Merriman, we can attach to a binary quintic form f a quintic ring R_f. Not all quintic rings are produced this way; the forms describe a certain subclass of quintic rings. What is special about this subclass? Which quintic rings are these? We can also associate to a quintic ring a sextic resolvent ring; perhaps there is something special about the sextic resolvent?

Another problem we can hope to understand is the number of $GL_2(\mathbb{Z})$-classes of binary quintic forms of bounded discriminant. The collection of all quintic rings is described by the space $\mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$, which naturally has an action of $\Gamma = GL_4(\mathbb{Z}) \times GL_5(\mathbb{Z})$. If we can find a discriminant-preserving, orbit-preserving map from binary quintic forms to this space, then we can hope to translate our knowledge of orbits in $\mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$ to study classes of binary quintics.

1.1 Results

Denote a generic element $f \in (Sym^5\mathbb{Z}^2)^*$ by $f = f_0x^5 + f_1x^4y + f_2x^3y^2 + f_3x^2y^3 + f_4xy^4 + f_5y^5$.

Theorem 1. *There is a map $\Phi : (Sym^5\mathbb{Z}^2)^* \to \mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$ which respects the two constructions of quintic rings. Furthermore, it is discriminant- and orbit-preserving. Explictly:*

$$\Phi(f) = \begin{pmatrix} 0 & t_3 & -t_2 & t_1 & 0 \\ -t_3 & 0 & -f_0t_1 - f_1t_2 & -f_2t_2 - f_3t_3 & -t_4 \\ t_2 & f_0t_1 + f_1t_2 & 0 & -f_4t_3 - f_5t_4 & t_3 \\ -t_1 & f_2t_2 + f_3t_3 & f_4t_3 + f_5t_4 & 0 & -t_2 \\ 0 & t_4 & -t_3 & t_2 & 0 \end{pmatrix}$$

For $A \in \mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$, denote the associated based quintic ring by $R(A)$ and the based sextic resolvent ring by $S(A)$. This theorem tells us that $R_f = R(\Phi(f))$, but the more interesting ramification is that it gives us access to the sextic resolvent ring $S(\Phi(f))$, which we denote by S_f. We can explicitly compute its multiplicative structure, and find out that the sextic resolvent rings coming from binary quintic forms have the following unusual structure:

Theorem 2. *Let f be a binary quintic form of non-zero discriminant $\Delta(f)$. Let the basis of S_f be denoted by $\{1, \beta_1, \ldots, \beta_5\}$. Denote the dual basis with respect to the trace pairing by $\{\beta_0^*, \beta_1^*, \ldots, \beta_5^*\}$, i.e. $\beta_i^* \in S_f \otimes \mathbb{Q}$ such that $Tr(\beta_i^* \beta_j) = \delta_{ij}$.*

Let $x = \beta_1^*, y = \beta_5^*$, then $(\beta_2, \beta_3, \beta_4) \equiv 8\Delta(f) \cdot (x^2, 2xy, y^2)$ mod $\mathbb{Q}$.

Nakagawa proved that if f and g are $GL_2(\mathbb{Z})$-equivalent binary quintic forms, their associated quintic rings R_f and R_g are isomorphic, but have different canonical bases; the $GL_2(\mathbb{Z})$ action induces this change of basis of the quintic ring, and can equally be viewed as a $GL_2(\mathbb{Z})$ action on $R_f/\mathbb{Z} \simeq \mathbb{Z}^4$. Similarly, in the sextic resolvent ring, the $GL_2(\mathbb{Z})$ action induces a change of basis and a corresponding change of dual basis, which respect the special property above:

Theorem 3. *Let f, g be binary quintic forms with $\gamma \in GL_2(\mathbb{Z})$ such that $g = \gamma \cdot f$. Then, S_f and S_g are isomorphic, with different canonical bases. The induced change of basis and change of dual basis respect the property from Theorem 2: namely, γ acts linearly on $sp\{\beta_1^*, \beta_5^*\}$ and quadratically on $sp\{\beta_2, \beta_3, \beta_4\}$.*

We would like to exactly pin down the quintic ring / sextic resolvent pairs (R, S) given by binary quintics. Thus, we need to know if this property is sufficient for the pair to have come from a binary quintic. With an extra condition on the alternating matrix A, there is indeed a basis of R such that it comes from a binary quintic form:

Theorem 4. *Let $A \in \mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5$, with sextic resolvent ring $S(A)$ having associated basis $\{1, \beta_1, \ldots, \beta_5\}$.*

Let $H \leqslant SL_5(\mathbb{Z})$ be the group of all matrices of the following form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & * \\ * & 0 & 1 & 0 & * \\ * & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

Then, A is an $SL_4(\mathbb{Z}) \times H$ translate of $\Phi(f)$ for some integral binary quintic form f if and only if the following two conditions hold:

- $A(e_1, e_5) = 0$
- $(\beta_2, \beta_3, \beta_4) \equiv 8\, Disc(R) \cdot ((\beta_1^*)^2, 2\beta_1^*\beta_5^*, (\beta_5^*)^2)$ mod $\mathbb{Q}$

In this case, there is a basis of $R(A)$ arising from the binary quintic f.

The other direction in which we would like to take this work is to count classes of binary quintic forms of bounded discriminant. The number of Γ-orbits in $\mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5$ of bounded discriminant is understood, and the map Φ is discriminant- and orbit-preserving, so if we understand Φ well then we can count classes of binary quintics. Specifically, we need to know how many classes of binary quintics can possibly land in one Γ-orbit in $\mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$. The following result goes some way towards this:

Theorem 5 (In Progress)**.** *Given a binary quintic $f \in (Sym^5\mathbb{Z}^2)^*$, there is an associated Segre cubic threefold $V_f \subseteq \mathbb{P}^4$. The Fano variety of lines on V_f has a distinguished 2-dimensional component $D \subseteq \mathbb{P}^9$. Certain integral points on the cone over D in $\mathbb{A}^{10}$ correspond to classes of forms which land in the same Γ-orbit as $\Phi(f)$. Furthermore, we can provide explicit polynomials cutting out a 2-dimensional subvariety of $\mathbb{A}^{10}$, whose integral points are precisely those in question.*

This theorem is proved, except for the statement that the subvariety in question is of dimension 2. I have only proved this in the case of $f = x^5 + y^5$. In the general case, I currently know that it is of dimension at most 2.

2 Preliminaries regarding n-ic rings

To add:

- Cubic and quartic rings (+ resolvents)
- More detail on quintic rings / sextic resolvents
- n-ic rings from binary n-ic forms
- Relevant material on n-ic rings in general

3 Maps from $(Sym^5\mathbb{Z}^2)^* \to \mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$

The maps below mimic closely the maps from binary quartics to $\mathbb{Z}^2 \otimes Sym^2\mathbb{Z}^3$ defined by Wood.

Denote a generic element $f \in (Sym^5\mathbb{Z}^2)^*$ by $f = f_0x^5 + f_1x^4y + f_2x^3y^2 + f_3x^2y^3 + f_4xy^4 + f_5y^5$.

Theorem 6. *The following map $\Phi : (Sym^5\mathbb{Z}^2)^* \to \mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$ respects the two constructions of based quintic rings:*

$$\Phi(f) = \begin{pmatrix} 0 & t_3 & -t_2 & t_1 & 0 \\ -t_3 & 0 & -f_0t_1 - f_1t_2 & -f_2t_2 - f_3t_3 & -t_4 \\ t_2 & f_0t_1 + f_1t_2 & 0 & -f_4t_3 - f_5t_4 & t_3 \\ -t_1 & f_2t_2 + f_3t_3 & f_4t_3 + f_5t_4 & 0 & -t_2 \\ 0 & t_4 & -t_3 & t_2 & 0 \end{pmatrix}$$

Proof. Computation of the SL_5-invariants leads to the multiplicative structure of $R(\Phi(f))$, which is seen to match that of the ring R_f defined by Birch and Merriman. $\square$

However, the derivation of this map was not entirely guesswork. It relies on noticing that the sub-pfaffians can be taken to be quite simple quadrics, as explained by the following results, which massively simplifies brute forcing a candidate map Φ.

Proposition 3.1. *Let f be a binary n-ic form of non-zero discriminant. Recall that we may attach a set of n points in $\mathbb{P}^{n-2}$ to the based n-ic ring R_f, denoted by X_{R_f}. Then, $X_{R_f} = \{(x^{n-2} : x^{n-3}y : \ldots : xy^{n-3} : y^{n-2}) : f(x,y) = 0\}$.*

Proof. (Insert this in an earlier section with preliminary results on n-ic rings.) □

Proposition 3.2. *Let f have non-zero discriminant and let R_f be given by $A_f \in \mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$. The sub-pfaffians of A_f span the space of quadrics vanishing on X_{R_f}. One such basis for this space is seen to be:*

$$Q_1 = t_1t_3 - t_2^2 \tag{2}$$
$$Q_2 = t_1t_4 - t_2t_3 \tag{3}$$
$$Q_3 = t_2t_4 - t_3^2 \tag{4}$$
$$Q_4 = f_0t_1t_2 + f_1t_2^2 + f_2t_2t_3 + f_3t_3^2 + f_4t_3t_4 + f_5t_4^2 \tag{5}$$
$$Q_5 = f_0t_1^2 + f_1t_1t_2 + f_2t_2^2 + f_3t_2t_3 + f_4t_3^2 + f_5t_3t_4 \tag{6}$$

Proof. It is known from Higher Composition Laws IV that the sub-pfaffians of non-degenerate $A \in \mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$ span the space of quadrics vanishing on $X_{R(A)}$.

For the latter part of the claim, we use Proposition 3.1. The forms Q_1, Q_2, Q_3 cut out the rational normal curve in $\mathbb{P}^3$, which contains X_{R_f}. The forms Q_4, Q_5, when evaluated on the rational normal curve, are then rephrasings of the equations $x \cdot f(x,y) = 0$ and $y \cdot f(x,y) = 0$. These are easily seen to span the space of quadrics vanishing on X_{R_f} - by explicit computation, or by a dimension count. □

Corollary 3.3. *The map Φ is discriminant-preserving.*

Proof. For $A \in \mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$, the discriminant of $R(A)$ was shown by Bhargava to be $Disc(A)$. Similarly, Birch and Merriman showed that the discriminant of R_f is $\Delta(f)$. The result then follows from Theorem 6. □

Theorem 7. *If $\gamma \in GL_2(\mathbb{Z})$, then $\Phi(\gamma \cdot f)$ and $\Phi(f)$ are $GL_4(\mathbb{Z}) \times GL_5(\mathbb{Z})$-equivalent.*

Explicitly, define $\sigma : GL_2(\mathbb{Z}) \to GL_4(\mathbb{Z}) \times GL_5(\mathbb{Z})$ as follows:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7}$$

$$\sigma(\gamma) = (\psi(\gamma), \rho(\gamma)) \tag{8}$$

$$\psi(\gamma) = (ad - bc) \begin{pmatrix} a^3 & a^2b & ab^2 & b^3 \\ 3a^2c & 2abc + a^2d & 2abd + b^2c & 3b^2d \\ 3ac^2 & 2acd + bc^2 & 2bcd + ad^2 & 3bd^2 \\ c^3 & c^2d & cd^2 & d^3 \end{pmatrix} \tag{9}$$

$$\rho(\gamma) = \begin{pmatrix} d & 0 & 0 & 0 & -c \\ 0 & a^2 & 2ab & b^2 & 0 \\ 0 & ac & bc + ad & bd & 0 \\ 0 & c^2 & 2cd & d^2 & 0 \\ -b & 0 & 0 & 0 & a \end{pmatrix} \tag{10}$$

Recall that $H \leqslant SL_5(\mathbb{Z})$ is the group of all matrices of the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & * \\ * & 0 & 1 & 0 & * \\ * & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

Then, there exists $h \in H$ such that $\Phi(\gamma \cdot f) = (1, h) \cdot \sigma(\gamma) \cdot \Phi(f)$.

Proof. This can be checked by explicit computation, but the derivation of the result in the first place is illuminating.

Let $g = \gamma \cdot f$. Nakagawa proved that $R_f = R_g$, with their bases related (mod $\mathbb{Z}$) by $\psi(\gamma)$. Hence, if $\Phi(g) = (\alpha, \beta) \cdot \Phi(f)$ for some $(\alpha, \beta) \in \Gamma$, then $\alpha = \psi(\gamma)$.

To find a candidate for β, we consider the action on sub-pfaffians. Denote the five 4×4 signed sub-pfaffians of $\Phi(f)$ by $P_{1,f}, \ldots, P_{5,f}$, and do the same for g. If $\Phi(g) = (\alpha, \beta) \cdot \Phi(f)$ then Higher Composition Laws IV tells us that:

$$\begin{pmatrix} P_{1,g} \\ P_{2,g} \\ P_{3,g} \\ P_{4,g} \\ P_{5,g} \end{pmatrix} = (\det \beta)(\beta^{-1})^t \begin{pmatrix} \alpha P_{1,f} \alpha^t \\ \alpha P_{2,f} \alpha^t \\ \alpha P_{3,f} \alpha^t \\ \alpha P_{4,f} \alpha^t \\ \alpha P_{5,f} \alpha^t \end{pmatrix} \tag{12}$$

Using $\alpha = \psi(\gamma)$, we compute this all and see that β would have to be of the form $h \cdot \rho(\gamma)$ for a unique h.

Then, we check with this β that indeed $\Phi(g) = (\alpha, \beta) \cdot \Phi(f)$. □

There is a relative Φ' of Φ, which is $GL_2(\mathbb{Z})$-equivariant on the nose (i.e. not up to some $h \in H$), but is only defined for integer-matrix quintic forms. They are related in the sense that $\Phi(f)$ and $\Phi'(f)$ are $GL_5(\mathbb{Z})$-translates:

Proposition 3.4. *There exists $\Phi' : Sym^5\mathbb{Z}^2 \to \mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$ and $\sigma : GL_2(\mathbb{Z}) \to GL_4(\mathbb{Z}) \times GL_5(\mathbb{Z})$ as above, such that $\Phi'(\gamma \cdot f) = \sigma(\gamma) \cdot \Phi'(f)$. We also have that $R(\Phi'(f)) = R_f$ as based rings.*

In full detail, $\Phi(f) = t_1A_1 + t_2A_2 + t_3A_3 + t_4A_4$ for:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ & 0 & -f_0 & -\frac{2f_1}{5} & 0 \\ & & 0 & -\frac{f_2}{10} & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} \tag{13}$$

$$A_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ & 0 & -\frac{3f_1}{5} & -\frac{3f_2}{5} & 0 \\ & & 0 & -\frac{3f_3}{10} & 0 \\ & & & 0 & -1 \\ & & & & 0 \end{pmatrix} \tag{14}$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & -\frac{3f_2}{10} & -\frac{3f_3}{5} & 0 \\ & & 0 & -\frac{3f_4}{5} & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} \tag{15}$$

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & 0 & -\frac{f_3}{10} & -\frac{2f_4}{5} & -1 \\ & & 0 & -f_5 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} \tag{16}$$

Proof. We aim to find a $GL_5(\mathbb{Z})$-translate of $\Phi(f)$, which respects the $GL_2(\mathbb{Z})$ action on the nose and does not need slight adjustment by $h \in H$. Thus, we look for a polynomial map $P(f) \in H$ such that $\Phi'(f) = P(f) \cdot \Phi(f)$ is $GL_2(\mathbb{Z})$-equivariant, i.e. we want $\Phi'(\gamma \cdot f) = \sigma(\gamma) \cdot \Phi'(f)$.

Our work from Theorem 7 in fact tells us that $\Phi(\gamma \cdot f) = (1, h(\gamma, f)) \cdot \sigma(\gamma) \cdot \Phi(f)$, for $h(\gamma, f)$ polynomial in γ and f. We calculate:

$$\begin{aligned} \Phi'(\gamma \cdot f) &= P(\gamma \cdot f) \cdot \Phi(\gamma \cdot f) && (17) \\ &= P(\gamma \cdot f) \cdot (1, h(\gamma, f)) \cdot \sigma(\gamma) \cdot \Phi(f) && (18) \\ &= P(\gamma \cdot f) \cdot (1, h(\gamma, f)) \cdot \sigma(\gamma) \cdot P(f)^{-1}\Phi'(f) && (19) \\ &= (\psi(\gamma), P(\gamma \cdot f)h(\gamma, f)\rho(\gamma)P(f)^{-1}) \cdot \Phi'(f) && (20) \end{aligned}$$

Thus, we see that we need P such that $P(\gamma \cdot f)h(\gamma, f)\rho(\gamma)P(f)^{-1} = \rho(\gamma)$. From explicit calculation, the following P arises and indeed does the job, leading to $\Phi'(f)$:

$$P(f) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{2}{5}f_1 & 1 & 0 & 0 & \frac{2}{5}f_2 \\ -\frac{1}{10}f_2 & 0 & 1 & 0 & \frac{1}{10}f_3 \\ -\frac{2}{5}f_3 & 0 & 0 & 1 & \frac{2}{5}f_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

□

This map is only defined for integer-matrix forms, but there is a third, related map which respects the action of $GL_2(\mathbb{Z})$ and is defined for all binary quintic forms. Inspired by the analogous map in the case of binary quartics, we can map to a quotient of $\mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$:

Corollary 3.5. *Define* $\bar{\Phi} : Sym^5\mathbb{Z}^2 \to H\backslash(\mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5)$ *by* $\bar{\Phi}(f) = H\Phi(f)$*. Then* $\bar{\Phi}(\gamma \cdot f) = \sigma(\gamma) \cdot \bar{\Phi}(f)$*.*

Proof. Follows from Theorem 7. [Note that this claim is well-defined: $\sigma(\gamma)\cdot\bar{\Phi}(f)$ is a left H-coset because $\sigma(\gamma)H\sigma(\gamma)^{-1} = H$.] □

Corollary 3.6. *The maps* Φ' *and* $\bar{\Phi}$ *are both discriminant-preserving.*

Proof. The based quintic ring in question is unchanged, so the same proof as that of Corollary 3.3 applies. □

4 The sextic resolvent ring S_f

Binary quartic forms have the special property that they describe quartic rings whose cubic resolvent is monogenic. Given a binary quintic form, with associated quintic ring and sextic resolvent, what special properties does the resolvent ring have?

For binary quartic forms, the basis of the cubic resolvent ring depends only on the $GL_2(\mathbb{Z})$-orbit of the binary quartic in question. This is because the relevant map $GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}) \times GL_3(\mathbb{Z})$ has trivial $GL_2(\mathbb{Z})$ component. This means that the structure of the cubic resolvent can be understood in terms of the GL_2 invariants of the binary quartic form.

For binary quintic forms, since the map σ lands non-trivially in $GL_5(\mathbb{Z})$, this is not the case; just as the structure coefficients of the based ring R_f depend on f and not just its equivalence class, so the same will be true for the sextic resolvent $S_f = S(\Phi(f))$.

Yet, the block matrix form of $\rho(\gamma)$ is intriguing, and suggests there may be special structure in the resolvent ring; indeed, this has to do with the relation between $\{\beta_1^*, \beta_5^*\}$ and $\{\beta_2, \beta_3, \beta_4\}$.

4.1 The structure of the sextic resolvent ring

Theorem 8. *Let f be a binary quintic form of non-zero discriminant $\Delta(f)$. Let the basis of S_f be denoted by $\{1, \beta_1, \ldots, \beta_5\}$. Denote the dual basis with respect to the trace pairing by $\{\beta_0^*, \beta_1^*, \ldots, \beta_5^*\}$, i.e. $\beta_i^* \in S_f \otimes \mathbb{Q}$ such that $Tr(\beta_i^* \beta_j) = \delta_{ij}$.*

Let $x = \beta_1^*, y = \beta_5^*$, then $(\beta_2, \beta_3, \beta_4) \equiv 8\Delta(f) \cdot (x^2, 2xy, y^2)$ mod $\mathbb{Q}$.

Proof. Proof is by computation of the SL_4-invariants of $\Phi(f)$, from which the multiplicative structure of S_f can be computed as explained in Higher Composition Laws IV. [See Appendix for multiplication table.] $\square$

Two notes:

- The basis of S_f is determined only mod $\mathbb{Z}$, so it wouldn't make sense for this theorem to be stated on the nose with an equality.
- But, then why is it stated mod $\mathbb{Q}$ and not mod $\mathbb{Z}$? This is because the three expressions $8\Delta(f) \cdot ((\beta_1^*)^2, 2\beta_1^*\beta_5^*, (\beta_5^*)^2)$ may not lie in S_f, but some $\mathbb{Q}$-translate of each of them does. These will in turn be congruent to $\beta_2, \beta_3, \beta_4$ mod $\mathbb{Z}$.

[Note: If we had a different way to prove the Segre cubic = trace cubed property of Theorem 10, that would result in an alternative proof of the above result. Currently, the proof of Theorem 10 also relies on the multiplication table of S_f, though it is true for all $S(A)$.]

Using this result, we can say more about x, y:

Lemma 4.1. $Tr(x^3) = Tr(x^2y) = Tr(xy^2) = Tr(y^3) = 0$.

Proof. Since $Tr(\beta_i \beta_j^*) = 0$ for $i \neq j$ and $Tr(\beta_j^*) = 0$ for $j \neq 0$, choosing $i \in \{2, 3, 4\}$, $j \in \{1, 5\}$ and applying Theorem 8 does the job. $\square$

Lemma 4.2. $(8\Delta(f))^2 \cdot Tr(x^i y^j) \in \mathbb{Z}$ *for* $i, j \geq 0, i + j = 5$.

Proof. For $i, j \in \{2, 3, 4\}$, $k \in \{1, 5\}$, $\mathbb{Z} \ni d_{ij}^k = Tr(\beta_i \beta_j \beta_k^*)$ and this expression is invariant under translating β_i, β_j by elements of $\mathbb{Q}$. Hence:

$$d^1_{22} = (8\Delta(f))^2 \quad \cdot Tr(x^5) \tag{22}$$
$$d^5_{22} = (8\Delta(f))^2 \quad \cdot Tr(x^4y) \tag{23}$$
$$d^1_{23} = 2(8\Delta(f))^2 \cdot Tr(x^4y) \tag{24}$$
$$d^5_{23} = 2(8\Delta(f))^2 \cdot Tr(x^3y^2) \tag{25}$$
$$d^1_{24} = (8\Delta(f))^2 \quad \cdot Tr(x^3y^2) \tag{26}$$
$$d^5_{24} = (8\Delta(f))^2 \quad \cdot Tr(x^2y^3) \tag{27}$$
$$d^1_{33} = 4(8\Delta(f))^2 \cdot Tr(x^3y^2) \tag{28}$$
$$d^5_{33} = 4(8\Delta(f))^2 \cdot Tr(x^2y^3) \tag{29}$$
$$d^1_{34} = 2(8\Delta(f))^2 \cdot Tr(x^2y^3) \tag{30}$$
$$d^5_{34} = 2(8\Delta(f))^2 \cdot Tr(xy^4) \tag{31}$$
$$d^1_{44} = (8\Delta(f))^2 \quad \cdot Tr(xy^4) \tag{32}$$
$$d^5_{44} = (8\Delta(f))^2 \quad \cdot Tr(y^5) \tag{33}$$

□

Lemma 4.3. *$Tr((u\beta_5^* - v\beta_1^*)^5) = -10f(u,v)$*

Proof. Computing the structure coefficients d^k_{ij}, we see that for $i, j \in \{2,3,4\}, k \in \{1,5\}$, they are all integer multiples of some f_i:

$$d^1_{22} = 10f_5 \;||\; d^5_{22} = -2f_4 \tag{34}$$
$$d^1_{23} = -4f_4 \;||\; d^5_{23} = 2f_3 \tag{35}$$
$$d^1_{24} = f_3 \quad ||\; d^5_{24} = -f_2 \tag{36}$$
$$d^1_{33} = 4f_3 \quad ||\; d^5_{33} = -4f_2 \tag{37}$$
$$d^1_{34} = -2f_2 \;||\; d^5_{34} = 4f_1 \tag{38}$$
$$d^1_{44} = 2f_1 \quad ||\; d^5_{44} = -10f_0 \tag{39}$$

Lemma 4.2 then completes the proof. □

Corollary 4.4. *A based sextic ring S can arise from at most one binary quintic form as $S_f = S(\Phi(f))$.*

Proof. If $S = S_f$ for some f, then f is encoded in β_1^*, β_5^* as in Lemma 4.3. □

4.2 An associated Segre cubic threefold

There is a variety called a Segre cubic 3-fold attached to an element $A \in \mathbb{C}^4 \otimes \wedge^2\mathbb{C}^5$. This is work of Seok Hyeong (Sean) Lee [All original work? If not, which parts are due to Seok Hyeong?]. The construction is as follows:

Given a quadruple $A = (A_1, A_2, A_3, A_4)$ of skew-symmetric 5×5 matrices, for each point $x = (x_1 : x_2 : x_3 : x_4)$ in $\mathbb{P}^3$, the 5×5 skew-symmetric matrix $x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4$ has even rank. We denote this matrix by $A(x)$.

For generic $x \in \mathbb{P}^3$, it has rank 4, but for bad x - which turn out to be the five points of $X_{R(A)}$ - its rank will be 2. Consider the following subvariety of $\mathbb{P}^3 \times \mathbb{P}^4$: $V = \{(x, y) : y \in ker(A(x))\}$. When we consider the map $V \to \mathbb{P}^3$, away from the five bad points identified, there is a unique point above each point $x \in \mathbb{P}^3$, since $dim(ker(A(x))) = 1$ generically. At the five bad points, by virtue of $dim(ker(A(x))) = 3$, there is a plane above each such point. So, V is $\mathbb{P}^3$ blown up at five points. If we now consider the image of V in $\mathbb{P}^4$, it turns out that this is a Segre cubic 3-fold, which is a dimension 3 cubic variety with ten singularities, which are all nodal.

Theorem 9 (S. H. Lee?)**.** *Let $A \in \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^5$, $y, z \in \mathbb{C}^5$ with $y = (y_1, y_2, y_3, y_4, y_5)^t$ and $z = (z_1, z_2, z_3, z_4, z_5)^t$. The determinant $A_1 y \wedge A_2 y \wedge A_3 y \wedge A_4 y \wedge z$ factors as $\langle y, z\rangle \cdot F_A(y)$, where $\langle y, z\rangle = \sum y_i z_i$ is the usual bilinear dot product and $F_A(y)$ is a cubic form in y. Then, F_A cuts out the Segre cubic threefold associated to A.*

Lemma 4.5. *Let $(g, h) \in GL_4(\mathbb{C}) \times GL_5(\mathbb{C})$, and let $A' = (g, h) \cdot A$. Then:*

$$F_{A'}(y) = \det(g) \det(h)\, F_A(h^t y) \tag{40}$$

Proof.

$$\begin{aligned}
\langle y, z\rangle \cdot F_{A'}(y) &= A'_1 y \wedge A'_2 y \wedge A'_3 y \wedge A'_4 y \wedge z && (41)\\
&= \det(g) \cdot \\
&\qquad (hAh^t)_1 y \wedge (hAh^t)_2 y \wedge (hAh^t)_3 y \wedge (hAh^t)_4 y \wedge z && (42)\\
&= \det(g) \det(h) \cdot \\
&\qquad (Ah^t)_1 y \wedge (Ah^t)_2 y \wedge (Ah^t)_3 y \wedge (Ah^t)_4 y \wedge h^{-1} z && (43)\\
&= \det(g) \det(h) \langle h^t y, h^{-1} z\rangle \cdot F_A(h^t y) && (44)\\
&= \det(g) \det(h) \langle y, z\rangle \cdot F_A(h^t y) && (45)
\end{aligned}$$

□

Before we state how the form F_A is related to the arithmetic of the sextic resolvent ring, we take a moment to note that the formulae of Higher Composition Laws IV define a commutative, associative sextic $\mathbb{C}$-algebra $S(A)$ when we extend them to $A \in \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^5$, because these properties are algebraic and $\mathbb{Z}^4 \otimes \wedge^2 \mathbb{Z}^5$ is Zariski dense in $\mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^5$. Furthermore, $SL_4(\mathbb{C}) \times SL_5(\mathbb{C})$ acts on the basis of $S(A)/\mathbb{C}$ in the same way that $SL_4(\mathbb{Z}) \times SL_5(\mathbb{Z})$ acts on the basis of $S(A)/\mathbb{Z}$ for integral A, because $SL_n(\mathbb{Z})$ is Zariski dense in $SL_n(\mathbb{C})$ [proof / source?].

Now we can state the meaning of F_A in the sextic algebra $S(A)$:

Theorem 10. *Let A, F_A, y be as in Theorem 9, but with $Disc(A) \neq 0$. The sextic resolvent algebra $S = S(A)$ comes equipped with a basis of $S/\mathbb{C}$, and a corresponding dual basis of $\tilde{S}$ which we denote by $\{\beta_1^*, \ldots, \beta_5^*\}$. Then:*

$$8\, Disc(A) \cdot Tr((y_1 \beta_1^* + \ldots + y_5 \beta_5^*)^3) = -3\, F_A(y) \tag{46}$$

Proof. The $G = GL_4(\mathbb{C}) \times GL_5(\mathbb{C})$ representation $\mathbb{C}^4 \otimes \wedge^2\mathbb{C}^5$ is a prehomogeneous vector space, meaning that is has a dense open orbit. This orbit is comprised of the elements of non-zero discriminant. So, if we can prove that the equation is G-invariant and that it holds for some A of non-zero discriminant, then it will hold for all A of non-zero discriminant, as desired.

Taking $A = \Phi(f)$ for any binary quintic of non-zero discriminant, our knowledge of the structure coefficients of $S(A)$ enables us to prove explicitly that this formula holds.

Now to prove G-invariance. Let $(g, h) \in G$ and $A' = (g, h) \cdot A$. We know from Lemma 4.5 that $F_{A'}(y) = \det(g)\det(h)\,F_A(h^t y)$. We will prove that the left hand side of the equation transforms in the same way, by looking at the actions of $SL_4(\mathbb{C}) \times SL_5(\mathbb{C})$ and the scalar matrices $(\lambda I_4, \mu I_5)$ separately.

Let $(g, h) \in SL_4(\mathbb{C}) \times SL_5(\mathbb{C})$. From our note above on the action of $SL_4(\mathbb{C}) \times SL_5(\mathbb{C})$ on the basis of $S(A)/\mathbb{C}$, we know that the action of (g, h) on the basis $\{\beta_1^*, \ldots, \beta_5^*\}$ of $\tilde{S(A)}$ is just by h. Transferring this change of basis to a transformation on y amounts to replacing y by $h^t y$. The polynomial $Disc(A)$ is invariant under $SL_4(\mathbb{C}) \times SL_5(\mathbb{C})$. Thus, both sides of the equation transform in the same way: by applying h^t to y.

Consider the action of $(\lambda I_4, \mu I_5)$ on the equation. The structure coefficients of the sextic resolvent algebra are degree 12 polynomials in the entries of A. This means that λI_4 acts on them by λ^{12} and μI_5 acts by μ^{24}. Hence, $(\lambda I_4, \mu I_5)$ act on the basis of the sextic resolvent algebra by $\lambda^{12}\mu^{24}$, and so the action on the dual basis is by $\lambda^{-12}\mu^{-24}$. The polynomial $Disc(A)$ is degree 40 and so (λ, μ) acts by $\lambda^{40}\mu^{80}$. Thus (λ, μ) acts on the left hand side of the equation by $\lambda^4\mu^8$. The action on the right hand side results in $\det(\lambda I_4)\det(\mu I_5)F_A(\mu y) = \lambda^4\mu^8 F_A(y)$ because F_A is cubic, so we have equality. $\square$

Let's compute the equation of the Segre cubic associated to $\Phi(f)$, which we denote by F_f:

$$\begin{aligned} F_f(y) = &-f_0f_2y_2^3 - f_0f_3y_2^2y_3 - f_0f_4y_2y_3^2 + f_0f_5y_2y_3y_4 \\ &-f_0f_5y_3^3 + f_0y_2^2y_5 - f_1f_3y_2^2y_4 - f_1f_4y_2y_3y_4 \\ &-f_1f_5y_3^2y_4 - f_1y_1y_2^2 - f_2f_4y_2y_4^2 - f_2f_5y_3y_4^2 \\ &+f_2y_2y_4y_5 - f_3f_5y_4^3 - f_3y_1y_2y_4 + f_4y_4^2y_5 \\ &-f_5y_1y_4^2 - y_1^2y_2 - y_1y_3y_5 - y_4y_5^2 \end{aligned} \tag{47}$$

Corollary 4.6. *Let $A \in \mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$. With the usual notation β_i, β_j^* for the associated basis of $S(A)$ and dual basis of $\tilde{S(A)}$, we have $(\beta_2, \beta_3, \beta_4) \equiv 8\,Disc(R) \cdot (x^2, 2xy, y^2)$ mod $\mathbb{Q}$ for $x = \beta_1^*, y = \beta_5^*$, if and only if F_A takes the following form:*

$$F_A(y) = -y_1^2y_2 - y_1y_3y_5 - y_4y_5^2 + \text{(lower order terms in } y_1, y_5) \tag{48}$$

Proof. The product $\beta_i^*\beta_j^* \in S \otimes \mathbb{Q}$ has an expansion in terms of the basis $\{1, \beta_1, \ldots, \beta_5\}$ of $S \otimes \mathbb{Q}$. The β_k coefficient is given by $Tr(\beta_i^*\beta_j^*\beta_k^*)$. From Theorem 10, we know that these expressions are encoded in F_A, and the expansions of $(\beta_1^*)^2, 2\beta_1^*\beta_5^*, (\beta_5^*)^2$ will be given by the terms of F_A which are quadratic or cubic in y_1, y_5. $\square$

Lemma 4.7. *Suppose $A \in \mathbb{C}^4 \otimes \wedge^2\mathbb{C}^5$ vanishes on a 2-dimensional subspace $V \subseteq \mathbb{C}^5$. Then $F_A(y) = 0$ for all $y \in V$.*

Proof. Fix non-zero $y \in V$. Consider the following 5×5 matrix:

$$\begin{pmatrix} | & | & | & | & | \\ A_1\, y & A_2\, y & A_3\, y & A_4\, y & \bar{y} \\ | & | & | & | & | \end{pmatrix}^t \tag{49}$$

where $\bar{y}$ denotes complex conjugation.

Its kernel contains $V \cap \{\bar{y}\}^\perp$, hence is non-trivial. So the determinant $F_A(y)\langle y, \bar{y}\rangle$ is zero, so $F_A(y) = 0$. $\square$

We can now say exactly which quartic ring / sextic resolvent pairs are described by binary quintic forms.

First, though, we recall the subgroup $H \leqslant SL_5(\mathbb{Z})$ of all matrices of the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & * \\ * & 0 & 1 & 0 & * \\ * & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{50}$$

Theorem 11. *Let $A \in \mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$, with sextic resolvent ring $S(A)$ having associated basis $\{1, \beta_1, \ldots, \beta_5\}$.*

Then, A is an $SL_4(\mathbb{Z}) \times H$ translate of $\Phi(f)$ for some integral binary quintic form f if and only if the following two conditions hold:

- $A(e_1, e_5) = 0$
- $(\beta_2, \beta_3, \beta_4) \equiv 8\, Disc(R) \cdot ((\beta_1^*)^2, 2\beta_1^*\beta_5^*, (\beta_5^*)^2)\ mod\ \mathbb{Q}$

In this case, there is a basis of $R(A)$ arising from the binary quintic f.

Proof. The forward implication is covered by Theorem 8, because $SL_4(\mathbb{Z})$ does not change basis of $S(A)$ and H does not disturb either of the two properties.

The reverse implication, however, needs proof:

From $A(e_1, e_5) = 0$, we know that A is of the form:

$$\begin{pmatrix} 0 & a & b & c & 0 \\ -a & * & * & * & -d \\ -b & * & * & * & -e \\ -c & * & * & * & -f \\ 0 & d & e & f & 0 \end{pmatrix} \tag{51}$$

for some $a, b, c, d, e, f \in \mathbb{Z}[t_1, t_2, t_3, t_4]$. The entries marked $*$ will be unimportant.

By Corollary 4.6, we know that the terms of F_A cubic in y_1, y_5 all vanish and that the partial derivatives of F_A evaluated at $y_1e_1 + y_5e_5$ take the form:

$$\nabla F_A(y_1e_1 + y_5e_5) = (0, -y_1^2, -y_1y_5, -y_5^2, 0) \tag{52}$$

These properties will impose conditions on a, b, c, d, e, f which will lead us to see that we are $SL_4(\mathbb{Z})$ equivalent to some $\Phi(f)$.

First, note that by Lemma 4.7, the vanishing of terms of F_A which are cubic in y_1, y_5 is already apparent. So, the only interesting conditions are on ∇F_A.

Recall the formula $F_A(y)\langle y, z\rangle = A_1y \wedge A_2y \wedge A_3y \wedge A_4y \wedge z$. Differentiating with respect to y_i, we get:

$$\begin{aligned}\frac{\partial F_A}{\partial y_i}(y)\langle y, z\rangle + F_A(y)z_i = A_1e_i \wedge A_2y \wedge A_3y \wedge A_4y \wedge z + \dots \\ +A_1y \wedge A_2y \wedge A_3y \wedge A_4e_i \wedge z\end{aligned} \tag{53}$$

Taking $y = y_1e_1 + y_5e_5$, this simplifies to:

$$\begin{aligned}\frac{\partial F_A}{\partial y_i}(y)\langle y, z\rangle = A_1e_i \wedge A_2y \wedge A_3y \wedge A_4y \wedge z + \dots \\ +A_1y \wedge A_2y \wedge A_3y \wedge A_4e_i \wedge z\end{aligned} \tag{54}$$

We can explicitly compute the determinants on the right hand side here, and thus the partial derivatives. The partial derivatives with respect to y_1 and y_5 come out as identically 0, as we expect from Lemma 4.7. The other partial derivatives are more interesting and are listed below. We use the notation $\Delta(pqrs)$ to mean the determinant of the 4×4 matrix whose rows are given by $p, q, r, s \in \mathbb{Z}^4 \simeq \mathbb{Z}[t_1, t_2, t_3, t_4]$.

$$\frac{\partial F_A}{\partial y_2}(y) = -y_1^2\,\Delta(abcd) + y_1y_5\,[\Delta(abdf) - \Delta(acde)] - y_5^2\,\Delta(adef) \tag{55}$$

$$\frac{\partial F_A}{\partial y_3}(y) = -y_1^2\,\Delta(abce) + y_1y_5\,[\Delta(abef) - \Delta(bcde)] - y_5^2\,\Delta(bdef) \tag{56}$$

$$\frac{\partial F_A}{\partial y_4}(y) = -y_1^2\,\Delta(abcf) + y_1y_5\,[\Delta(acef) - \Delta(bcdf)] - y_5^2\,\Delta(cdef) \tag{57}$$

Hence, we need the following equations to be satisfied by a, b, c, d:

$$\Delta(abcd) = 1 \tag{58}$$
$$\Delta(abdf) - \Delta(acde) = 0 \tag{59}$$
$$\Delta(adef) = 0 \tag{60}$$
$$\Delta(abce) = 0 \tag{61}$$
$$\Delta(abef) - \Delta(bcde) = -1 \tag{62}$$
$$\Delta(bdef) = 0 \tag{63}$$
$$\Delta(abcf) = 0 \tag{64}$$
$$\Delta(acef) - \Delta(bcdf) = 0 \tag{65}$$
$$\Delta(cdef) = 1 \tag{66}$$

It is not too hard to solve these by hand, but running them through any computer algebra package will effortlessly inform you that they amount to $e = -a, f = -b, \Delta(abcd) = 1$. This is true over any field, not just over $\mathbb{Z}$.

Thus, the matrix with rows a, b, c, d lies in $SL_4(\mathbb{Z})$, and furthermore the transformation τ taking (a, b, c, d) to $(t_3, -t_2, t_1, t_4)$ also lies in $SL_4(\mathbb{Z})$. Thus, we can write:

$$A' = \tau \cdot A = \begin{pmatrix} 0 & t_3 & -t_2 & t_1 & 0 \\ -t_3 & * & * & * & -t_4 \\ t_2 & * & * & * & t_3 \\ -t_1 & * & * & * & -t_2 \\ 0 & t_4 & -t_3 & t_2 & 0 \end{pmatrix} \tag{67}$$

Finally, it is not hard to see that we can find $h \in H$ to transform A' into a matrix of the form $\Phi(f)$. We just have to find h which kills certain t_i coefficients in the central 3×3 block of A'. For example, in order to kill the t_4 term in $A'(e_2, e_3)$, we translate e_3 by a choice multiple n of e_5, as $A'(e_2, e_3 + ne_5) = A'(e_2, e_3) - nt_4$. These conditions amount to six linear equations in the entries of h, easily seen to have a unique solution; if we didn't have $A'(e_1, e_5) = 0$, these equations would be quadratic and we would have a much harder time. $\square$

[Note: Using the next section, it should be easy to upgrade this theorem to a correspondence between classes of binary quintics and isomorphism classes of quintic ring / sextic resolvent pairs (R, S), where we only allow isomorphisms respecting the special structure of S. The above theorem gives surjectivity of this correspondence. We'll get injectivity because the form f is encoded as $-10f(u, v) = Tr((u\beta_5^* - v\beta_1^*)^5)$ and our notion of isomorphism will fix $\mathbb{Z}\{\beta_1^*, \beta_5^*\}$.]

4.3 The action of $GL_2(\mathbb{Z})$ on the sextic resolvent ring

Recall Theorem 7:

Theorem 12. *Let $f \in (Sym^5\mathbb{Z}^2)^*$ and $\gamma \in GL_2(\mathbb{Z})$. Then there exists $h \in H$ such that $\Phi(\gamma \cdot f) = (1, h) \cdot \sigma(\gamma) \cdot \Phi(f)$*

The GL_5 component of $(1, h) \cdot \sigma(\gamma)$ is of the form:

$$\begin{pmatrix} d & 0 & 0 & 0 & -c \\ * & a^2 & 2ab & b^2 & * \\ * & ac & bc + ad & bd & * \\ * & c^2 & 2cd & d^2 & * \\ -b & 0 & 0 & 0 & a \end{pmatrix} \tag{68}$$

Since $\Phi(f)$ represents a map $\wedge^2\tilde{S} \to \tilde{R}$, this GL_5 element represents the change of basis of $\tilde{S}$, not S.

Denote the basis elements of $S_f/\mathbb{Z}$ by $\beta_{i,f}$, with the usual notation $\beta_{i,f}^*$ for the dual basis of $\tilde{S}_f$. Let $g = \gamma \cdot f$ and denote the basis elements of $S_g/\mathbb{Z}$ and $\tilde{S}_g$ analogously.

The matrix above tells us the relation between the bases of $\tilde{S}_f$ and $\tilde{S}_g$, while the transpose inverse tells us how the bases of $S_f/\mathbb{Z}$ and $S_g/\mathbb{Z}$ are related.

So, we have:

$$\begin{pmatrix} \beta^*_{1,g} \\ \beta^*_{5,g} \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \beta^*_{1,f} \\ \beta^*_{5,f} \end{pmatrix} \tag{69}$$

$$\begin{pmatrix} \beta_{2,g} \\ \beta_{3,g} \\ \beta_{4,g} \end{pmatrix} = \begin{pmatrix} d^2 & -cd & c^2 \\ -2bd & bc+ad & -2ac \\ b^2 & -ab & a^2 \end{pmatrix} \begin{pmatrix} \beta_{2,f} \\ \beta_{3,f} \\ \beta_{4,f} \end{pmatrix} \tag{70}$$

Note that this respects the relation between $\{\beta^*_1, \beta^*_5\}$ and $\{\beta_2, \beta_3, \beta_4\}$.

There is also a subtle point to be made here: We can view $S^*_f \otimes \mathbb{Q}$ as a 6-dimensional representation of $GL_2(\mathbb{Z})$, and likewise for $S_f \otimes \mathbb{Q}$. From the block matrix form of $\rho(\gamma) \in GL_5(\mathbb{Z})$, we see that both these representations break into irreducible components of dimensions 1, 2 and 3. We have an equality $S^*_f \otimes \mathbb{Q} = S_f \otimes \mathbb{Q}$ given by the trace pairing, and we might then think that the subrepresentations occuring in each representation are equal. However, this is not the case; the key point to be noted is that, on this shared space, the two $GL_2(\mathbb{Z})$-actions are in general different, as explained by the following lemma:

Lemma 4.8. *Let K be a field. Suppose that V is a K-vector space with basis $\{v_0, v_1, \ldots, v_{n-1}\}$, and with a non-degenerate pairing $\langle -, - \rangle : V \otimes_K V \to K$. Using $\langle -, - \rangle$, we can identify V^* and V; denote the dual basis by $\{v^*_0, v^*_1, \ldots, v^*_{n-1}\} \subset V$. Let $\sigma \in GL(V)$ such that $\sigma(v_i) = w_i$ and $\sigma(v^*_i) = w^*_i$, where $\{w^*_0, w^*_1, \ldots, w^*_{n-1}\}$ is the dual basis of $\{w_0, w_1, \ldots, w_{n-1}\}$. Then σ stabilises $\langle -, - \rangle$.*

Proof. The dual basis is obtained as follows:

$$\begin{pmatrix} v^*_0 \\ v^*_1 \\ \ldots \\ v^*_{n-1} \end{pmatrix} = \big(\langle v_i, v_j \rangle\big)^{-1}_{i,j} \begin{pmatrix} v_0 \\ v_1 \\ \ldots \\ v_{n-1} \end{pmatrix} \tag{71}$$

Since σ is K-linear, we have:

$$\begin{pmatrix} \sigma(v^*_0) \\ \sigma(v^*_1) \\ \ldots \\ \sigma(v^*_{n-1}) \end{pmatrix} = \big(\langle v_i, v_j \rangle\big)^{-1}_{i,j} \begin{pmatrix} \sigma(v_0) \\ \sigma(v_1) \\ \ldots \\ \sigma(v_{n-1}) \end{pmatrix} \tag{72}$$

But if $\sigma(v_i) = w_i$ and $\sigma(v^*_i) = w^*_i$ then this simplifies:

$$\begin{pmatrix} w^*_0 \\ w^*_1 \\ \ldots \\ w^*_{n-1} \end{pmatrix} = \big(\langle v_i, v_j \rangle\big)^{-1}_{i,j} \begin{pmatrix} w_0 \\ w_1 \\ \ldots \\ w_{n-1} \end{pmatrix} \tag{73}$$

However, we also know to obtain w_i^* in the following way:

$$\begin{pmatrix} w_0^* \\ w_1^* \\ \dots \\ w_{n-1}^* \end{pmatrix} = \big(\langle w_i, w_j \rangle\big)_{i,j}^{-1} \begin{pmatrix} w_0 \\ w_1 \\ \dots \\ w_{n-1} \end{pmatrix} \tag{74}$$

Hence, $\langle w_i, w_j \rangle = \langle v_i, v_j \rangle$ for all i, j, and so σ stabilises the pairing. □

This means that we should be careful when using this representation-theoretic perspective to draw links between S_f and S_f^*.

4.4 The Cayley-Klein resolvent map

Recall from Higher Composition Laws IV the Cayley-Klein resolvent map $\psi : R \to \tilde{S}$, defined as

$$\begin{aligned} \alpha \mapsto \frac{1}{\sqrt{Disc(R)}}(&\alpha^{(1)}\alpha^{(2)} + \alpha^{(2)}\alpha^{(3)} + \alpha^{(3)}\alpha^{(4)} + \alpha^{(4)}\alpha^{(5)} + \alpha^{(5)}\alpha^{(1)} \\ &-\alpha^{(1)}\alpha^{(3)} - \alpha^{(3)}\alpha^{(5)} - \alpha^{(5)}\alpha^{(2)} - \alpha^{(2)}\alpha^{(4)} - \alpha^{(4)}\alpha^{(1)}) \end{aligned} \tag{75}$$

Lemma 4.9. *The Cayley-Klein map $\psi : R_f \to \tilde{S}_f$ has the following property:*

$$\psi(x^3\alpha_1 + x^2y\alpha_2 + xy^2\alpha_3 + y^3\alpha_4) = 4f(x,y)(y\beta_1^* - x\beta_5^*) \tag{76}$$

Proof. Bhargava shows that, for $A \in \mathbb{Z}^4 \otimes \wedge^2\mathbb{Z}^5$ and the associated based quintic ring $R = R(A)$, this map is given by the 4×4 sub-Pfaffians $(P_1, \dots, P_5)$ of A as follows:

$$\psi(t_1\alpha_1 + \dots + t_4\alpha_4) = 4P_1(t)\beta_1^* + \dots + 4P_5(t)\beta_5^* \tag{77}$$

[Q: Are the sub-pfaffians signed here?]

Recall that when $A = \Phi(f)$, the sub-pfaffians are in the span of the following five quadrics:

$$Q_1 = t_1t_3 - t_2^2 \tag{78}$$
$$Q_2 = t_1t_4 - t_2t_3 \tag{79}$$
$$Q_3 = t_2t_4 - t_3^2 \tag{80}$$
$$Q_4 = f_0t_1t_2 + f_1t_2^2 + f_2t_2t_3 + f_3t_3^2 + f_4t_3t_4 + f_5t_4^2 \tag{81}$$
$$Q_5 = f_0t_1^2 + f_1t_1t_2 + f_2t_2^2 + f_3t_2t_3 + f_4t_3^2 + f_5t_3t_4 \tag{82}$$

In fact, the five 4×4 signed sub-pfaffians are easily seen to be:

$$Q(A) = [Q_4, Q_1, Q_2, Q_3, -Q_5] \tag{83}$$

When $A = \Phi(f)$ and $(t_1, \dots, t_4) = (x^3, x^2y, xy^2, y^3)$, because of the special form of the sub-Pfaffians, they simplify to give the stated result. □

This mimics the behaviour of the resolvent maps for binary cubics and quartics:

- For a binary cubic, the resolvent map has the property:

$$x\alpha_1 + y\alpha_2 \mapsto f(x,y)\omega \tag{84}$$

  where ω generates the quadratic resolvent ring.

- For a binary quartic, the resolvent map has the property:

$$x^2\alpha_1 + xy\alpha_2 + y^2\alpha_3 \mapsto f(x,y)\omega \tag{85}$$

  where ω generates the cubic resolvent ring.

So, it seems that the resolvent map picks out some of the key structure of the resolvent ring, when evaluated on a certain family of elements corresponding to the relevant rational normal curve.

5 Counting binary quintics of bounded discriminant

[To be added.]