STA257 Tutorial Problems

Daniel Flam-Shepherd

University of Toronto

Last update: September 21, 2017

Contents

1 Tutorial 1 September 20

1. True or False :

   i) $A \subset B \Longrightarrow p(A) \leq p(B)$ **True**
   ii) if $p(B) > 0 \Longrightarrow p(A|B) \geq p(A)$ **False** what if $A \cap B = \emptyset$
   iii) $p(A \cap B) \geq p(A) + p(B) - 1$ **True**
   iv) $p(A \cap B^c) = p(A \cup B) - p(B)$ **True**

2. We throw n identical balls into m urns at random, where each urn is equally likely and each throw is independent of any other throw. What is the probability that the i-th urn is empty?

   Solution: Let B_j^i = the event the j^{th} ball is not thrown into the i^{th} urn.

   $p(B_j^i) = 1 - 1/m$. Since all throws are independent

   $$p(\cap_{j=1}^n B_j^i) = p(B_1^i) \cdots p(B_n^i) = (1 - 1/m) \cdots (1 - 1/m) = (1 - 1/m)^n$$

3. A well-shuffled deck of 52 cards is dealt evenly to two players (26 cards each). What is the probability that player 1 gets all the aces?

   Solution : Let A be the event that player 1 gets all the aces.

   $p(A) = |A|/|\Omega| = \binom{48}{22}/\binom{52}{26}$ =# of ways to get all aces/the number of hands (26 cards from 52).

4. We toss two fair coins simultaneously and independently. Let A be the event that the first coin comes up heads, B be the event that the second coin comes up heads, and C be the event that the outcomes of the two coin tosses are the same. Are Events A, B and C are independent?

   Solution: $\Omega = \{HH, HT, TH, TT\},\ A = \{HH, HT\}, B = \{HH, TH\}, C = \{HH, TT\}$. Then $P(A) = P(B) = P(C) = 1/2$. We also have $P(A \cap C) = P(A \cap B) = P(B \cap C) = 1/4$, hence $P(A \cap C) = P(A)P(C); P(A \cap B) = P(A)P(B); P(B \cap C) = P(B)P(C)$ and all 3 events are independent. Knowing C adds no information on whether the coin falls as heads : $P(A|C) = P(A)$.

5. Let A, B, C be independent events. Show A^c, B^c ; $A, B \cup C$ are independent.

   proof :

   $$\begin{aligned}
   p(A^c \cap B^c) &= 1 - p(A \cup B) = 1 - p(A) - p(B) + p(A \cap B) \\
   &= 1 - p(A) - p(B)[1 - p(A)] = [1 - p(A)][1 - p(B)] = p(A^c)p(B^c) \\
   p(A \cap (B \cup C)) &= p((A \cap B) \cup (A \cap C)) = p(A \cap B) + p(A \cap C) - p(A \cap B \cap C) \\
   &= p(A)p(B) + p(A)p(C) - p(A)p(B \cap C) = p(A)[p(B) + p(C) - p(B \cap C)] \\
   &= p(A)p(B \cup C)
   \end{aligned}$$

6. Let A, B, C are events such that $p(A) > 0, p(B) > 0, p(C) > 0$ and $p(A \cap B \cap C) \neq 0$. Prove that $p(C|B \cap A) = p(C|B) \iff p(A|C \cap B) = p(A|B)$.

   proof :

   $$\begin{aligned}
   p(C|B \cap A) = \frac{p(C \cap B)}{p(B)} = p(C|B) &\iff \frac{p(C \cap B \cap A)}{p(B \cap A)} = \frac{p(C \cap B)}{p(B)} \\
   &\iff \frac{p(C \cap B \cap A)}{p(C \cap B)} = \frac{p(B \cap A)}{p(B)} \\
   &\iff p(C|B \cap A) = p(C|B)
   \end{aligned}$$