MATH 222 Assignment #2

Due: Friday, February 10, 2017

Assignments are due in class **before** the start of lecture. Late assignments will be accepted up to 24 hours late, but will be deducted a 20% late penalty.

All answers must include full explanations. For full marks, your work must be stapled, neatly written, and contain enough detail that it is clear how you arrived at your solutions.

1. Consider the n-dimensional cube Q_n. In class, we see that Q_n is bipartite by considering the partition $\{V_1, V_2\}$ where V_1 denotes the set of vertices that have an odd number of 1s and V_2 denotes the set of vertices that have an even number of 1s. Suppose n is even. Explain why there is no Hamiltonian path in Q_n that starts at the vertex $00\ldots0$ and ends at the vertex $11\ldots1$.

2. A vertex v in a graph G is called a *cut-vertex* if $\kappa(G - v) > \kappa(G)$ (where $\kappa(G)$ denotes the number of components in G). So a cut-vertex in a connected graph G is any vertex v for which $G - v$ is disconnected. Prove that a vertex v in a connected graph G is a cut-vertex if and only if there exist vertices u and w such that v is on every $u - w$ path of G.

3. (a) Show that when any edge is removed from K_5, the resulting subgraph is planar.

   (b) Let $G = (V, E)$ be a connected graph with $|V| \geq 11$. Prove that either G or $\overline{G}$ must be nonplanar.

4. Let G be a graph with exactly one cycle. Prove that $\chi(G) \leq 3$.

5. State the converse of the 4-colour Theorem: If G is planar, then $\chi \leq 4$. Is the converse true?

6. Let T be a tree on n vertices. Prove that if the maximum degree of any vertex in T is k, then T has at least k leaves (vertices of degree 1).

7. Use a combinatorial proof (i.e. a counting argument) to show that $\binom{2n}{2} = 2\binom{n}{2} + n^2$ for all $n \geq 2$, $n \in \mathbb{Z}^+$.

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8. For any positive integer n, consider the expression $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^k\binom{n}{k} + \cdots + 2^n\binom{n}{n}$.

   (a) Use the Binomial Theorem with appropriate values for x and y to simplify the expression.

   (b) Now consider the identity you get by stating that $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^k\binom{n}{k} + \cdots + 2^n\binom{n}{n}$ is equal to your answer to (a). You know that the identity is correct because the Binomial Theorem has been proven. However, you can also prove it directly using a combinatorial argument.

   Define a set of elements (which could be mathematical objects or something else) such that when you count the elements in two different ways you get each side of the identity.