MATH 222 - Assignment 3

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1. (a) This problem can be modeled as the number of ways to put a plus sign between 16 1s, which is equivalent to $2^{n-1} = 2^{16-1} = 2^{15} = 32768$ compositions.

   (b) This problem can be modeled in the same manner, however this time we will treat each set of 2 1s (1+1) as an object, since our summands must be even.
   $\therefore 2^{(\frac{n}{2})-1} = 2^{8-1} = 2^7 = 128$ compositions

2. (a) Since we are limited to exactly 7 runs, we cannot have 2 runs of A next to each other, as this would result in less than exactly 7. Therefore there are 2 cases:

      i. There are 3 runs of A and 4 runs of B
      In this case, we must distribute 7 wins for A among 3 runs, which is equivalent to counting the number of integer solutions to $x_1 + x_2 + x_3 = 7$, where x_i represents the run of A in the list from left to right.
      We must ensure however that each bin has at least one element, so we can modify our question slightly to be number of integer solutions to $y_1 + y_2 + y_3 = 4$, assuming that each run already contains 1 element.
      $\therefore \binom{w+r-1}{r-1} = \binom{4+3-1}{3-1} = \binom{6}{2}$
      We must also distribute 8 wins for B among 4 runs, which using the same process as above, yields $\binom{5+4-1}{4-1} = \binom{8}{3}$
      We can then multiply these together, since for arrangement of runs of A, there are $\binom{8}{3}$ runs of B.
      $\therefore$ the number of possible winner lists when there are 3 runs of A and 4 runs of B is $\binom{6}{2}\binom{8}{3} = (15)(56) = 840$

      ii. There are 4 runs of A and 3 runs of B
      Using a similar process to that above, we get $\binom{7}{3}\binom{7}{2} = 735$

   We can then add these 2 possibilities together to compute the total number of list outcomes: $840 + 735 = 1575$ lists

   (b) Here we must distribute 15 wins among 7 runs, which is equivalent to counting the number of integer solutions to $x_1 + x_2 + \cdots + x_7 = 15$, where x_i represents the run number from left to right.
   We know that there must be at least 1 win each each run however, as otherwise there would be less than the required 7 runs.
   $\therefore$ we can rewrite our equation to be $x_1 + x_2 + \cdots + x_7 = 8$
   The number of integers that satisfy this is $\binom{w+r-1}{r-1} = \binom{8+7-1}{7-1} = \binom{14}{6}$
   This means that we have $\binom{14}{6}$ arrangements of wins for our runs, however we know that there cannot be an A run and a B run next to each other, as this would diminish the total number of runs.
   Therefore our runs must alternate A and B, so there are 2 cases:

      i. Our list starts with an A run
      ii. Our list starts with a B run

For each of these cases there are $\binom{14}{6}$ ways to arrange our wins, so there are $2\binom{14}{6}$ possible lists in total.
$\therefore 2\binom{14}{6} = 2(3003) = 6006$ possible lists

3. Since we have lower bounds on x_3 and x_4, we can rewrite the equation by assuming that x_3 already contains 3, and that x_4 already contains 4.
The equation then becomes $x_1 + x_2 + x_3 + x_4 = 15$
Due to the upper bounds on our integers, this problem can now be written as an inclusion/exclusion question.

Let N be the total number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 15$ with no restrictions on any of the variables.
$\therefore \binom{r+n-1}{n-1} = \binom{15+4-1}{4-1} = \binom{18}{3} = 816$
Let C_1 be the condition that $x_1 \geq 6$
Let C_2 be the condition that $x_2 \geq 7$
Let C_3 be the condition that $x_3 \geq 7$
Let C_4 be the condition that $x_4 \geq 6$
Then the answer to this question can be computed by $N(\bar{C}_1\bar{C}_2\bar{C}_3\bar{C}_4)$

$N(C_1)$ is the number of solutions to $x_1 + x_2 + x_3 + x_4 = 15$ where $x_1 \geq 6$
This can be re-written as $x_1+x_2+x_3+x_4 = 9$ where x_1 has no restrictions
$\therefore \binom{r+n-1}{n-1} = \binom{9+4-1}{4-1} = \binom{12}{3} = 220$
$N(C_2)$ is the solutions to $x_1+x_2+x_3+x_4 = 8$ where x_2 has no restrictions
$\therefore \binom{r+n-1}{n-1} = \binom{8+4-1}{4-1} = \binom{11}{3} = 165$
$N(C_3)$ and $N(C_4)$ have the same number of solutions as $N(C_2)$ and $N(C_1)$

$N(C_iC_j)$ is shown by $x_1 + x_2 + x_3 + x_4 = 15$ where $x_i \geq 6$ and $x_j \geq 7$
This is equivalent to $x_1 + x_2 + x_3 + x_4 = 2$ with no restrictions
$\therefore \binom{r+n-1}{n-1} = \binom{2+4-1}{4-1} = \binom{5}{3} = 10$
There are 4 such combinations of this case set.
$N(C_iC_j)$ is shown by $x_1 + x_2 + x_3 + x_4 = 15$ where $x_i, x_j \geq 6$
This is equivalent to $x_1 + x_2 + x_3 + x_4 = 3$ with no restrictions
$\therefore \binom{r+n-1}{n-1} = \binom{3+4-1}{4-1} = \binom{6}{3} = 20$
There are 2 such combinations of this case set.
$N(C_iC_j)$ is shown by $x_1 + x_2 + x_3 + x_4 = 15$ where $x_i, x_j \geq 7$
This is equivalent to $x_1 + x_2 + x_3 + x_4 = 1$ with no restrictions
$\therefore \binom{r+n-1}{n-1} = \binom{1+4-1}{4-1} = \binom{4}{3} = 4$
There are 2 such combinations of this case set.

The intersections containing both 3 and 4 conditions all have 0 solutions, as there is no way to sum 3 values that follow any of the conditions and achieve a result that is equal to 15.

$\bar{N} = N - \sum N(C_i) + \sum N(C_iC_j)$
$= 816 - (2*220 + 2*165) + (4*10 + 2*20 + 2*4) = 816 - 770$
$= 88$ integer solutions within the given bounds

4. Let N be the number of permutations of the 9 digits, or 9!
Let C_i be the condition that digit i is immediately followed by $i+1$, where $i = 1, 2, \ldots, 8$
Then the answer to this question can be computed by $N(\bar{C_1}\bar{C_2}\ldots\bar{C_8})$

$N(C_i)$ can be modeled by grouping together i and $i+1$, treating them as a single digit. By doing this, the number of permutations becomes 8!
$N(C_iC_j)$ can be modeled by grouping together the 2 sets of digits, resulting in 7! Note that it does not matter if we have 3 digits consecutively, as the resulting number of movable objects remains the same.

$$\vdots$$

$N(C_1C_2\ldots C_8)$ results in 1! as by this point all digits are grouped.

$\bar{N} = N - \sum N(C_i) + \sum N(C_iC_j) - \cdots + N(C_iC_j\ldots C_8)$
$= 9! - (8)8! + (7)7! - \cdots + 1$
$= 362880 - 322560 + 35280 - 4320 + 600 - 96 + 18 - 4 + 1$
$= 71799$ permutations

5. $\sum\limits_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^m$ can be used to model the number of onto functions 2 given sets, and in our case $n = m = 9$
$= \sum\limits_{k=0}^{8}(-1)^k\binom{8}{k}(8-k)^9$
$= 8^9 - (8)7^9 + (28)6^9 - (56)5^9 + (70)4^9 - (56)3^9 + (28)2^9 - 8$
$= 1451520$ functions
We then need to remove the functions in which $f(x) = x$ for all even x.
There are 4 even numbers in the set, and each represents the number of onto mappings from a set of size 8 to another set of size 8.
Therefore the number of onto mappings in which $f(x) = x$ for any x is:
$= 4\sum\limits_{k=0}^{7}(-1)^k\binom{7}{k}(7-k)^8$
$= 4(7^8 - (7)6^8 + (21)5^8 - (35)4^8 + (35)3^8 - (21)2^8 + 7)$
$= 4(141120)$
$= 564480$ functions
$\therefore 1451520 - 564480$
$= 887040$ functions that fulfill the condition

6. When dividing by 12, there are 12 possible remainders, therefore in 13 chosen integers, at least 2 must have the same remainder.
Our integers a, b can be written as $a = 12k_1 + r$ and $b = 12k_2 + r$
$\therefore a - b = 12k_1 + r - 12k_2 - r$
$\therefore a - b = 12(k_1 - k_2)$
$\therefore 12(a-b) = k_1 - k_2$ which implies that $12|a-b$ since $k_1 - k_2$ is an integer