MATH 222 - Assignment 4

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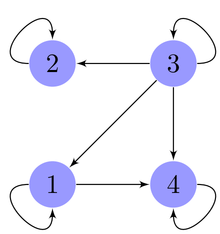
1. (a) In order for a relation to be both symmetric and antisymmetric, it must only contain reflexive elements. For each reflexive pair (x, x), there are 2 options: either the pair is in the relation, or it is not.
$\therefore$ there are 2^n relations on A that fulfill the requirements

   (b) If R is a relation on A that is antisymmetric, then it may not contain any 2 pairs such that (x, y) and (y, x) are both in R unless $x = y$.
$\therefore$ we can count the number of elements in the upper diagonal half of the corresponding relation matrix.
This is equivalent to $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

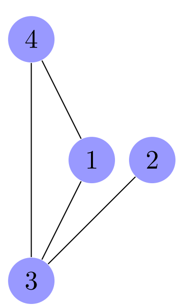
   (c) For each pair, either (x, y) or (y, x) may be in R, unless $x = y$.
$\therefore$ there are $\frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$ pairs that may be swapped out for their inverse.
As in part (a), we have 2 options for each of our pairs, so we have $2^{\frac{n(n-1)}{2}}$ relations

   (d) If R has domain A and is a function, then each element may map to only 1 other element in A, by the definition of a function.
Since R is reflexive, for every $x \in A$ the pair (x, x) must be in R.
Since every value in the domain is already mapped to itself, we know that no value can map to another due to our function restriction.
$\therefore$ R is made up of all of the reflexive pairs (x, x) where $x \in A$

2. (a)



   (b)



(c) A total order must be antisymmetric, but must also contain at least one pair for every $x, y \in A$. There are currently 2 sets of pairs that are not included in R, these being $(2,1)/(1,2)$ and $(2,4)/(4,2)$.
For each of these 2 pairs there are 2 choices, with that choice being which will be in our total order.
$\therefore$ there are $2^2 = 4$ total orders containing our partial order

3. (a) In order for R to be an equivalence relation, it must be symmetric, reflexive, and transitive.

   i. Multiplication is commutative, so if $xy > 0$, we know that $yx > 0$
   This means that if xRy then yRx
   $\therefore R$ is symmetric

   ii. Given any non-zero numbers x, y we know that $xy \neq 0$
   $\therefore$ the product must be less than 0 or greater than 0
   For any x we know xRx since x^2 will also be positive
   $\therefore R$ is reflexive

   iii. If xRy then x and y are both positive or both negative
   If yRz also, then z has the same sign as both x and y
   If this is the case, then xRz since they have the same sign
   $\therefore R$ is transitive

   $\therefore R$ is an equivalence relation on S
   [1] and [-1] are examples of equivalence classes on R than give a partition of S, as the 2 partitions in this case are the set of positive numbers and the set of negative numbers.

   (b) R_2 is not an equivalence relation because x^2 is always positive, and therefore not in R_2. This means that the relation is not reflexive, and is therefore not an equivalence relation.

4. (a) In order for R_Y to be a partial order on Y it must be anti-symmetric, reflexive, and transitive.

   i. Since for any $a, b \in Y$, (a, b) is in R_Y if and only if it is in R, then (b, a) cannot also be in R_Y since R is already a partial order, and therefore cannot contain (b, a)
   $\therefore R_Y$ is anti-symmetric

   ii. Since for any $a \in Y$, (a, a) is in R, we know that it is also in R_Y since both of the inclusion conditions are met.
   $\therefore R_Y$ is reflexive

   iii. Since for any $a, b, c \in Y$, if (a, b) and (b, c) are both in R then we know that (a, c) must also be in R since it is a partial order
   $\therefore R_Y$ is transitive

   $\therefore R_Y$ is a partial order on Y

(b) For R to be a total order, then for any $x, y \in X$ either the pair (x, y) or (y, x) is in R
If this is the case, then for any $x, y \in Y$ either (x, y) or (y, x) will be in R_Y
$\therefore R_Y$ is a total order on Y

5. (a) $g(x) = \frac{x^3}{1-x^2}$
$= x^3 \frac{1}{1-x^2}$
$= x^3 \sum\limits_{n=0}^{\infty} (x^2)^n$
$= x^3 \sum\limits_{n=0}^{\infty} x^{2n}$
$= x^3(1 + x^2 + x^4 + ...)$
$= x^3 + x^5 + x^7 + ...$
$0, 0, 0, 1, 0, 1, 0, ...$

(b) $g(x) = (4x - 1)^3$
$= -(1 - 4x)^3$
$= -\sum\limits_{n=0}^{3} \binom{3}{n} (-4x)^n$
$= -[1 + 3(-4x) + 3(-4x)^2 + (-4x)^3]$
$= -(1 - 12x + 48x^2 - 64x^3)$
$= -1 + 12x - 48x^2 + 64x^3$
$-1, 12, -48, 64, 0, 0, ...$

(c) $g(x) = \frac{1-x}{1+x}$
$= \frac{1}{1+x} - x\frac{1}{1+x}$
$= \sum\limits_{n=0}^{\infty} (-1)^n x^n - x \sum\limits_{n=0}^{\infty} (-1)^n x^n$
$= (1 - x + x^2 - x^3 + ...) - x(1 - x + x^2 - x^3 + ...)$
$= (1 - x + x^2 - x^3 + ...) - (x - x^2 + x^3 - x^4 + ...)$
$= 1 - 2x + 2x^2 - 2x^3 + ...$
$1, -2, 2, -2, 2, ...$

6. (a) This sequence looks similar to $g(x) = \frac{1}{1+x}$ in that the signs alternate
This gives us $1, -1, 1, -1, ...$
Unfortunately the corresponding sequence does not start with 0, but we can fix this by amending our guess to $g(x) = \frac{x}{1+x}$
This gives us $0, 1, -1, 1, -1, ...$
Now we need to deal with the increasing powers of 2. Since these numbers are clearly tied to n, we can multiply x in the denominator by 2, so that when the sum $\sum\limits_{n=0}^{\infty} (-1)^n x^n$ is evaluated we will get $(2x)^n$ and subsequently 2^n in front of our x
$\therefore g(x) = \frac{x}{1+2x}$

(b) It is immediately apparent that some values of n are skipped in this sequence, but we will start with $g(x) = \frac{1}{1+x}$ once again due to the alternating signs

This gives us $1, -1, 1, -1, ...$

We will perform the same procedure as above by making $g(x) = \frac{x}{1+x}$

This give us $0, 1, -1, 1, -1, ...$

Now we must deal with the fact that numbers are skipped. Since every other number is skipped, we are dealing with x^{2n} in our sum. To accomplish this in our generating function, we can square x in the denominator

$\therefore g(x) = \frac{x}{1+x^2}$

(c) We can begin to solve this problem by noticing that for a coefficient a_y to exist, there must be an x^y in our final expression. The only time that x^y will exist, is when y can be made up of our $5, 10, 25, 100$ dollar denominations.

We can set up a set of expressions to represent our final expression as follows:

$g(x) = (1+x^5+x^{10}+...)(1+x^{10}+x^{20}+...)(1+x^{25}+...)(1+x^{100}+...)$

$= (\sum\limits_{n=0}^{\infty} x^{5n})(\sum\limits_{n=0}^{\infty} x^{10n})(\sum\limits_{n=0}^{\infty} x^{25n})(\sum\limits_{n=0}^{\infty} x^{100n})$

$= \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} \cdot \frac{1}{1-x^{25}} \cdot \frac{1}{1-x^{100}}$

$= \frac{1}{(1-x^5)(1-x^{10})(1-x^{25})(1-x^{100})}$