

Topology - X400416

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These notes are based on Topology (2nd edition) by James R. Munkres.

1 Topological Spaces

The motivation behind defining a topological space is to generalize the notion of a metric space. Recall that a metric on a set X is a map $d : X \times X \rightarrow [0, \infty)$ that satisfies

1. $d(x, y) = d(y, x)$
2. $d(x, x) = 0$
3. $d(x, y) > 0, x \neq y$
4. $d(x, y) \leq d(x, z) + d(z, y)$

Then we say that a set $U \subset X$ is open if for all $x \in U$ and some $r > 0$

$$B(x, r) := \{y \in X \mid d(x, y) < r\} \subset U.$$

In other words, around every point in U there is a "ball" that is contained in U . In Analysis I one learns about continuity using the classic ε - δ definition which requires a metric. As it turns out, we don't really need a metric to define a continuous function, only open sets:

Definition. *A function between metric spaces is continuous if and only if the preimage of an open set is open.*

Using this definition, different seeming metric can yield the same notions of which functions are continuous! We call the collection of open subsets of X defined by some metric $d : X \times X \rightarrow [0, \infty)$ a *topology*. This open sets satisfy some important properties. Namely: (1) X and \emptyset are open, (2) arbitrary unions of open sets are open and (3) finite intersections of open sets is open. It turns out that a metric is not required to define a topology, only these three properties:

Definition. *Let X be a set. Then a topology on X is a set $\mathcal{T} \subset \mathcal{P}(X)$ such that*

1. $\emptyset \in \mathcal{T}, X \in \mathcal{T}$
2. *If $\{U_\alpha\} \subset \mathcal{T}$ then $\bigcup_\alpha U_\alpha \in \mathcal{T}$*
3. *If $\{U_i\}_{i=0}^n \subset \mathcal{T}$ then $\bigcap_{i=0}^n U_i \in \mathcal{T}$*

A topological space is the pair (X, \mathcal{T})

Then we say that $U \subset X$ is open if $U \in \mathcal{T}$. Note that which sets are open depends on the topology, which might conflict with your notion of open set as defined above. For example, in the topology $\mathcal{P}(\mathbb{R})$ every subset of the real line is open, while in the topology $\{\emptyset, \mathbb{R}\}$ only the empty set and \mathbb{R} are open. We often say that U is open in X without giving a specific topology, which simply

means that the statement that follows will hold for any topology we define on X and any element in that topology.

If \mathcal{T} and \mathcal{T}' are two topologies on X such that $\mathcal{T} \subseteq \mathcal{T}'$ then we say that \mathcal{T}' is *finer* (or *strictly finer* if the containment is proper) than \mathcal{T} . We similarly say that \mathcal{T} is *coarser* (or *strictly coarser*) than \mathcal{T}' . It might also be the case that two topologies are not *comparable*.

2 Basis for a Topology

Specifying topologies directly is often not possible, due to the enormous size of many topologies. So we often define a topology using a smaller subset called a *basis*.

Definition. If X is a set, then a *basis* of a topology is a collection \mathcal{B} of subsets of X such that

1. $\forall x \in X, \exists B \in \mathcal{B}$ such that $x \in B$
2. If $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$ then there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

If \mathcal{T} is a topology generated by a basis \mathcal{B} then U is open if for all $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. Also $B \in \mathcal{T}$ for all $B \in \mathcal{B}$. The proof that \mathcal{T} is indeed a topology is not included. An alternative construction of a topology from a basis is given by the following lemma

Lemma 2.1. Let X be a set and \mathcal{B} a basis for a topology \mathcal{T} . Then \mathcal{T} equals the collection of all unions of elements in \mathcal{B} .

Using this lemma is sometime easier in practice; given a basis \mathcal{B} and $U \subset X$, if one can write U as a union of elements in \mathcal{B} then U is open.

We can also go in the reverse direction: from topology to a basis.

Lemma 2.2. Let X be a topological space. If \mathcal{C} is a collection of open sets such that for each open set U and each $x \in U$ there is $C \in \mathcal{C}$ such that $x \in C \subset U$, then \mathcal{C} is a basis.

When topologies are given in terms of basis, we can already determine which one is finer using the following criterion

Lemma 2.3. Let \mathcal{B} and \mathcal{B}' be bases for the topologies \mathcal{T} and \mathcal{T}' respectively. Then $\mathcal{T} \subset \mathcal{T}'$ if and only if for each $x \in X$ and $B \in \mathcal{B}$ containing x there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

It might be tricky to remember the direction of the inclusion. One way to think about it is since \mathcal{T}' has more subsets of X it needs to have smaller basis elements.

Lastly, we define the notion of a *subbasis*.

Definition. A subbasis \mathcal{S} for a topology on X is a collection of subsets of X whose union equals X . The topology generated by \mathcal{S} is the collection of all unions of finite intersection of elements of \mathcal{S} .

To conclude this section we define 3 topologies on the real line using the notion of a basis:

1. The *standard topology* generated by the collection of all open intervals (a, b) with $a < b$ (it is not a recursive definition. Here we use open in the familiar metric sense).
2. The *lower limit topology* is generated by half-open intervals $[a, b)$. When \mathbb{R} is given in the lower limit topology we denote it \mathbb{R}_l .
3. The *K -topology* is generated by open intervals and sets of the form $(a, b) - K$ where $K = \{1/n \mid n \in \mathbb{N}\}$. When \mathbb{R} is given in this topology we denote it \mathbb{R}_k .

One maybe surprising property is that both \mathbb{R}_l and \mathbb{R}_k are finer than the standard topology, but are not comparable with one another.

3 The Product Topology on $X \times Y$

4 The Subspace Topology

If X is a topological space with topology \mathcal{T}_X and $Y \subset X$ then the subspace topology on Y is defined as

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}_X\}.$$

The fact that the collection \mathcal{T}_Y has all the properties of a topology follows from the \mathcal{T}_X being a topology. Then if \mathcal{B}_X is a basis for a topology on X , the basis of the subspace topology on Y is given by

$$\mathcal{B}_Y = \{Y \cap B \mid B \in \mathcal{B}_X\}.$$

Open sets in the subspace topology on Y are not necessarily open in X . If A is an open set in Y , then A is open in X if Y is open in X .

5 Closed Sets and Limit Points

If X is a topological space and $A \subset X$ then A is closed if $X - A$ is open. Closed sets have similar properties to open sets:

Theorem 5.1. *Let X be a topological space. Then*

1. X and \emptyset are closed

2. *Arbitrary intersections of closed sets are close*

3. *Finite unions of closed sets are closed.*

One can just as well define a topology in terms of closed sets, but the definition using open sets is much more common. Of course, mathematics wouldn't be fun if there wasn't any space for confusion. In a topology, a set can be open, closed, neither or both. So don't think of sets as doors.

If Y is a subspace of a topological space X , then a closed set in X is not necessarily closed in Y . A set A is closed in Y if and only if it equals the intersection of a closed set of X with Y . This is easy to verify since if A is closed in Y then $Y - A$ is open in Y and so it equals $U \cap Y$ for some open set U in X . Then $X - U$ is closed and $A = Y \cap (X - U)$. The other direction is similarly proved.

Let $A \subset X$. Then the smallest closed set that contain A is called the closure of A and is denoted $\bar{A} = \bigcap_{C \text{ closed}, A \subset C} C$. The smallest open set containing A is called the interior of A and is denoted $\text{Int}A = \bigcup_{U \text{ open}, U \subset A} U$. Then clearly

$$\text{Int}A \subset A \subset \bar{A}.$$

A point $x \in X$ is in the closure of A if and only if for any basis element B containing x the intersection $A \cap B$ is non empty.

To add to the soup of metaphors, we define the *neighborhood* of $x \in X$ is an open set U containing x . Then a *limit point* of $A \subset X$ is defined as

Definition. A *limit point* of $A \subset X$ is a point $x \in A$ such that every neighborhood of x intersects A in a point different than x

In other words, x is a limit point of A if for every open set U containing x the intersection $U \cap (A - x)$ is non empty. Now denote the set of limit points of A by A' . Then $\bar{A} = A' \cup A$. This gives us an alternative condition for a set being closed. Namely, if $A' \subset A$ then $\bar{A} = A$ and so A is closed. Alternatively, if A is closed, then $\bar{A} = A$ and so $A' \subset A$. Hence a set is closed if and only if it contains all of its limit points.

We conclude this section with a definition of convergence in a topological space.

Definition. Let (X, \mathcal{T}) be a topological space. If x_n is a sequence in X then $x_n \rightarrow x$ as $n \rightarrow \infty$ if

$$\forall U \in \mathcal{T}, x \in U \exists N \in \mathbb{N} : \{x_n\}_{n \geq N} \subset U$$

Note that this definition does not require any metric. The downside is that limit are not necessarily unique and sequences can converge to any number of points. To avoid this horrific phenomenon, we add an extra condition to rid our definition of topological spaces of such situations. Topological spaces in which limits are unique are called Hausdorff spaces.

6 Exercises

6.1 Basis for a topology

Exercise 13.1

Let X be a topological space; let $A \subset X$. Suppose that for each $x \in A$ there is an open set U such that $x \in U \subset A$. Show that A is open.

Proof. For every $x \in A$, let U_x denote the open set containing x such that $U_x \subset A$. Then $U = \bigcap_{x \in A} U_x \subset A$ since each U_x is contained in A . For the other inclusion, take $x \in A$. Then $x \in U$ since x is in U_x by definition. Hence $A \subset U$ and it follows that $A = U$. Since each U_x and arbitrary unions of open sets are open it follows that A is open. \square

Exercise 13.4

1. If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X , how that $\bigcap \mathcal{T}_\alpha$ is a topology on X . Is $\bigcup \mathcal{T}_\alpha$ a topology on X ?
2. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X . Show that there is a unique smallest topology containing all the all the collection of \mathcal{T}_α and a unique largest topology contained in all \mathcal{T}_α
3. If $X = \{a, b, c\}$ let

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing \mathcal{T}_1 and \mathcal{T}_2 and the largest topology contained in \mathcal{T}_1 and \mathcal{T}_2 .

Solution.

1. (a) Since $\emptyset, X \in \mathcal{T}_\alpha$ for all α it follows that $\emptyset, X \in \bigcap \mathcal{T}_\alpha$. (b) If $U_\beta \in \bigcap \mathcal{T}_\alpha$, then $U_\beta \in \mathcal{T}_\alpha$ for all α and so $\bigcup U_\beta \in \mathcal{T}_\alpha$ for all α since \mathcal{T}_α is a topology. Hence $\bigcup U_\beta \in \bigcap \mathcal{T}_\alpha$. (c) If $U_1, U_2 \in \bigcap \mathcal{T}_\alpha$ then $U_1, U_2 \in \mathcal{T}_\alpha$ for all α and so $U_1 \cap U_2 \in \mathcal{T}_\alpha$ for all α . Therefore $U_1 \cap U_2 \in \bigcap \mathcal{T}_\alpha$. It follows by induction that $\bigcap \mathcal{T}_\alpha$ is closed under countable intersections. Hence an intersections of topologies is a topology.
Let $X = \{a, b, c\}$. Then $\mathcal{T}_1 = \{\emptyset, X, \{a\}\}$ and $\mathcal{T}_2 = \{\emptyset, X, \{b\}\}$ are topologies on X . But $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b\}\}$ is not a topology since $\{a\}, \{b\} \in \mathcal{T}_1 \cup \mathcal{T}_2$ but $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$. Hence a union of topologies is, in general, not a topology.
2. Let $\mathcal{S} = \bigcup_\alpha \mathcal{T}_\alpha$. Then $X \in \mathcal{S}$ since X is in each individual \mathcal{T}_α as they are all topologies. It follows that $X = \bigcup_{S \in \mathcal{S}} S$ and so \mathcal{S} is a subbasis. Let \mathcal{B} be the basis generated by \mathcal{S} and \mathcal{T}_s be the topology generated by \mathcal{B} . Fix some \mathcal{T}_α and take $U \in \mathcal{T}_\alpha$. Then $U \in \mathcal{S} \subset \mathcal{B} \subset \mathcal{T}_s$ by construction. Hence

$U \in \mathcal{T}_S$ and it follows that $\mathcal{T}_\alpha \subset \mathcal{T}_S$ for all α . Is it the smallest topology with such property? Let \mathcal{T}' be a topology on X such that $\mathcal{T}_\alpha \subset \mathcal{T}'$ for all α and take $U \in \mathcal{T}_S$. Then U is an arbitrary union of finite intersections of elements of $\mathcal{S} = \bigcup \mathcal{T}_\alpha \subset \mathcal{T}'$. Since \mathcal{T}' is a topology it is closed under arbitrary unions and finite intersections and so $U \in \mathcal{T}'$. Hence $\mathcal{T}_S \subset \mathcal{T}'$ and it follows that \mathcal{T}_S is the smallest topology containing all \mathcal{T}_α .

From part one we know that $\bigcap \mathcal{T}_\alpha$ is a topology, and by definition it is contained in \mathcal{T}_α for all α . If $\mathcal{T}' \subset \mathcal{T}_\alpha, \forall \alpha$ is a topology, then for every $U \in \mathcal{T}'$, $U \in \bigcap \mathcal{T}_\alpha$ and so $\mathcal{T}' \subset \bigcap \mathcal{T}_\alpha$. Therefore $\bigcap \mathcal{T}_\alpha$ is the largest topology that is contained in all \mathcal{T}_α .

3. Apply part (2).

Exercise 13.5

Show that that topology \mathcal{T} on X generated by a basis \mathcal{B} is equal to the intersections of all the topologies on X that contain \mathcal{B} .

Proof. Let $T = \{\mathcal{T}_\beta \mid \mathcal{B} \subset \mathcal{T}_\beta\}$ be the collection of all topologies on X that contain \mathcal{B} . Let $u \in \mathcal{T}$. Then U can be written as a union of element in \mathcal{B} , i.e.

$$U = \bigcup_{\alpha} B_{\alpha}, \quad B_{\alpha} \in \mathcal{B}$$

Since \mathcal{T}_β is a topology and $\mathcal{B} \subset \mathcal{T}_\beta$ for all $\mathcal{T}_\beta \in T$ it follows that $U = \bigcup_{\alpha} B_{\alpha} \in \mathcal{T}_\beta$ for all $\mathcal{T}_\beta \in T$ and so

$$\mathcal{T} \subset \bigcap_{\mathcal{T}_\beta \in T} \mathcal{T}_\beta.$$

Since $\mathcal{B} \subset \mathcal{T}$ by definition of a basis it follows that $\mathcal{T} \in T$ and so

$$\bigcap_{\mathcal{T}_\beta \in T} \mathcal{T}_\beta \subset \mathcal{T}.$$

Hence $\bigcap_{\mathcal{T}_\beta \in T} \mathcal{T}_\beta = \mathcal{T}$. □

Exercise 13.7

Solution.

Exercise 13.8

1. Show that the collection

$$\mathcal{B} = \{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

generates the standard topology on \mathbb{R} .

2. Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

generates a topology different from the lower limit topology.

Solution.

1. Let \mathcal{T} be the standard topology on \mathbb{R} and $\mathcal{T}_{\mathcal{B}}$ the topology generated by \mathcal{B} . Let $x \in (a, b) \in \mathcal{T}$. Since the rationals are dense in \mathbb{R} , there exist $a', b' \in \mathbb{Q}$ such that $x \in (a', b') \subset (a, b)$. Hence $\mathcal{T} \subset \mathcal{T}_{\mathcal{B}}$. The other inclusion is trivial since every basis element $(a, b) \in \mathcal{B}$ is a basis element of \mathcal{T} . We conclude that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$.
2. Let \mathcal{T}_c be the topology generated by \mathcal{C} and let \mathcal{B} be the basis of \mathcal{T}_l , the lower limit topology on \mathbb{R} . Then for any $[a, b) \in \mathcal{C}$, $[a, b) \in \mathcal{B}$, and so $\mathcal{T}_c \subset \mathcal{T}_l$. To show that this inclusion is strict we need to prove the statement

$$\begin{aligned} & \neg(\forall B \in \mathcal{B} \forall x \in B \exists C \in \mathcal{C} : x \in C \subset B) \\ \iff & \exists B \in \mathcal{B} \exists x \in B \forall C \in \mathcal{C} : x \notin C \vee C \not\subset B \\ \iff & \exists B \in \mathcal{B} \exists x \in B \forall C \in \mathcal{C} : x \in C \implies C \not\subset B \end{aligned}$$

Let $[x, b) \in \mathcal{B}$ with $x \notin \mathbb{Q}$. Then $[a, c) \in \mathcal{C}$ can contain x only if $a < x$ since x is irrational. Therefore there is no element in \mathcal{C} that contains x and is a subset of $[x, b)$. This proves that $\mathcal{T}_c \subsetneq \mathcal{T}_l$.

6.2 The Subspace Topology

Exercise 16.1

Show that if Y is a subspace of X , and A is a subset of Y then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X .

Proof. Let \mathcal{T} be a topology on X , \mathcal{T}_Y subspace topology on Y . Let \mathcal{T}'_A be the topology A inherits as a subset of Y . Then

$$\begin{aligned} \mathcal{T}'_A &= \{A \cap U \mid U \in \mathcal{T}_Y\} \\ &= \{A \cap U \mid U \in \{Y \cap V \mid V \in \mathcal{T}\}\} \\ &= \{A \cap U \mid U = Y \cap V, V \in \mathcal{T}\} \\ &= \{A \cap (Y \cap V) \mid V \in \mathcal{T}\} \\ &= \{(A \cap Y) \cap V \mid V \in \mathcal{T}\} \\ &= \{A \cap V \mid V \in \mathcal{T}\} \end{aligned}$$

which is by definition the topology A inherits as a subset of X . \square

Exercise 16.3**Exercise 16.4**

Show that $\pi_1 : X \times Y \rightarrow X$ is an open map.

Proof. Let U be open in $X \times Y$ and take $(x, y) \in U$. Then there exists a basis element $B_x \times B_y$ such that $(x, y) \in B_x \times B_y \subset U$. For any $b \in B_x$, $(b, y) \in B_x \times B_y \subset U$ and so $b = \pi_1(b, y) \in \pi_1(U)$. It follows that $B_x \subset \pi_1(U)$. Since the basis of a product topology is the the product of open sets, B_x is open in X which means that for every $x \in \pi_1(U)$ there is an open set $B_x \subset X$ such that $x \in B_x \subset \pi_1(U)$. From Exercise 13.1 it follows that $\pi_1(U)$ is open in X . \square

6.3 Closed Sets and Limit Points