Topology - X400416

Yoav Eshel

February 6, 2021

Contents

These notes are based on Topology (2^{nd} edition) by James R. Munkres.

1 Topological Spaces and Continuous Functions

A metric on a set X is a map $d : X \times X \to [0, \infty)$ that satisfies

1. $d(x, y) = d(y, x)$
2. $d(x, x) = 0$
3. $d(x, y) > 0, x \neq y$
4. $d(x, y) \leq d(x, z) + d(z, y)$

A set $U \subset X$ is open if for all $x \in U$ and some $r > 0$

$$B(x, r) = \{y \in X \mid d(x, y) < r\}$$

s.t. $B(x, r) \subset U$. Union (finite, countable or uncountable) of open sets is open and finite intersections of open sets is open (infinite intersections need not be open)

A function between metric spaces is continuous if and only if a preimage of an open set is open.

Definition. *Let X be a set. Then a topology on X is a set $\mathcal{T} \subset \mathcal{P}(x)$ s.t.*

1. $\varnothing \in \mathcal{J}, X \in \mathcal{J}$
2. *For* $U_\alpha \in \mathcal{J}, \bigcup_\alpha U_\alpha \in \mathcal{J}$
3. *For* $(U_i)_{0 \leq i \leq n} \subset \mathcal{J}, \bigcap_{i=0}^n U_i \in \mathcal{J}$

A topological space is the pair $(X, \mathcal{J})$

If X is a set, the a *basis* of a topology is a collection $\mathcal{B}$ of subsets of X s.t.

1. $\forall x \in X, \exists B \in \mathcal{B}$ s.t. $x \in B$
2. If $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$ then there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

2 Exercises

2.1 Basis for a topology

Exercise 13.1

Let X be a topological space; let $A \subset X$. Suppose that for each $x \in A$ there is an open set U such that $x \in U \subset A$. Show that A is open.

Proof. For every $x \in A$, let U_x denote the open set containing x such that $U_x \subset A$. Then $U = \bigcap_{x \in A} U_x \subset A$ since each U_x is contained in A. For the other inclusion, take $x \in A$. Then $x \in U$ since x is in U_x by definition. Hence $A \subset U$ and it follows that $A = U$. Since each U_x and arbitrary unions of open sets are open it follows that A is open. $\square$

Exercise 13.4

1. If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X,how that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

2. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology containing all the all the collection of $\mathcal{T}_\alpha$ and a unique largest topology contained in all $\mathcal{T}_\alpha$

3. If $X = \{a, b, c\}$ let

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}.$$

   Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Solution.

1. (a) Since $\varnothing, X \in \mathcal{T}_\alpha$ for all α it follows that $\varnothing, X \in \bigcap \mathcal{T}_\alpha$. (b) If $U_\beta \in \bigcap \mathcal{T}_\alpha$, then $U_\beta \in \mathcal{T}_\alpha$ for all α and so $\bigcup U_\beta \in \mathcal{T}_\alpha$ for all α since $\mathcal{T}_\alpha$ is a topology. Hence $\bigcup U_\beta \in \bigcap \mathcal{T}_\alpha$. (c) If $U_1, U_2 \in \bigcap \mathcal{T}_\alpha$ then $U_1, U_2 \in \mathcal{T}_\alpha$ for all α and so $U_1 \cap U_2 \in \mathcal{T}_\alpha$ for all α. Therefore $U_1 \cap U_2 \in \bigcap \mathcal{T}_\alpha$. It follows by induction that $\bigcap \mathcal{T}_\alpha$ is closed under countable intersections. Hence an intersections of topologies is a topology.

   Let $X = \{a, b, c\}$. Then $\mathcal{T}_1 = \{\varnothing, X, \{a\}\}$ and $\mathcal{T}_2 = \{\varnothing, X, \{b\}\}$ are topologies on X. But $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b\}\}$ is not a topology since $\{a\}, \{b\} \in \mathcal{T}_1 \cup \mathcal{T}_2$ but $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$. Hence a union of topologies is, in general, not a topology.

2. Let $\mathcal{S} = \bigcup_\alpha \mathcal{T}_\alpha$. Then $X \in \mathcal{S}$ since X is in each individual $\mathcal{T}_\alpha$ as they are all topologies. It follows that $X = \bigcup_{S \in \mathcal{S}} S$ and so $\mathcal{S}$ is a subbasis. Let $\mathcal{B}$ be the basis generated by $\mathcal{S}$ and $\mathcal{T}_s$ be the topology generated by $\mathcal{B}$. Fix some $\mathcal{T}_\alpha$ and take $U \in \mathcal{T}_\alpha$. Then $U \in \mathcal{S} \subset \mathcal{B} \subset \mathcal{T}_\mathcal{S}$ by construction. Hence

$U \in \mathcal{T}_{\mathcal{S}}$ and it follows that $\mathcal{T}_\alpha \subset \mathcal{T}_{\mathcal{S}}$ for all α. Is it the smallest topology with such property? Let $\mathcal{T}'$ be a topology on X such that $\mathcal{T}_\alpha \subset \mathcal{T}'$ for all α and take $U \in \mathcal{T}_{\mathcal{S}}$. Then U is an arbitrary union of finite intersections of elements of $\mathcal{S} = \bigcup \mathcal{T}_\alpha \subset \mathcal{T}'$. Since $\mathcal{T}'$ is a topology it is closed under arbitrary unions and finite intersections and so $U \in \mathcal{T}'$. Hence $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}'$ and it follows that $\mathcal{T}_{\mathcal{S}}$ is the smallest topology containing all $\mathcal{T}_\alpha$.

From part one we know that $\bigcap \mathcal{T}_\alpha$ is a topology, and by definition it is contained in $\mathcal{T}_\alpha$ for all α. If $\mathcal{T}' \subset \mathcal{T}_\alpha, \forall \alpha$ is a topology, then for every $U \in \mathcal{T}'$, $U \in \bigcap \mathcal{T}_\alpha$ and so $\mathcal{T}' \subset \bigcap \mathcal{T}_\alpha$. Therefore $\bigcap \mathcal{T}_\alpha$ is the largest topology that is contained in all $\mathcal{T}_\alpha$.

3. Apply part (2).

Exercise 13.5

Show that that topology $\mathcal{T}$ on X generated by a basis $\mathcal{B}$ is equal to the intersections of all the topologies on X that contain $\mathcal{B}$.

Proof. Let $T = \{\mathcal{T}_\beta \mid \mathcal{B} \subset \mathcal{T}_\beta\}$ be the collection of all topologies on X that contain $\mathcal{B}$. Let $u \in \mathcal{T}$. Then U can be written as a union of element in $\mathcal{B}$, i.e.

$$U = \bigcup_\alpha B_\alpha, \quad B_\alpha \in \mathcal{B}$$

Since $\mathcal{T}_\beta$ is a topology and $\mathcal{B} \subset \mathcal{T}_\beta$ for all $\mathcal{T}_\beta \in T$ it follows that $U = \bigcup_\alpha B_\alpha \in \mathcal{T}_\beta$ for all $\mathcal{T}_\beta \in T$ and so

$$\mathcal{T} \subset \bigcap_{\mathcal{T}_\beta \in T} \mathcal{T}_\beta.$$

Since $\mathcal{B} \subset \mathcal{T}$ by definition of a basis it follows that $\mathcal{T} \in T$ and so

$$\bigcap_{\mathcal{T}_\beta \in T} \mathcal{T}_\beta \subset \mathcal{T}.$$

Hence $\bigcap_{\mathcal{T}_\beta \in T} \mathcal{T}_\beta = \mathcal{T}$. □

Exercise 13.7

Solution.

Exercise 13.8

1. Show that the collection

$$\mathcal{B} = \{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

generates the standard topology on $\mathbb{R}$.

2. Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

generates a topology different from the lower limit topology.

Solution.

1. Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$ and $\mathcal{T}_{\mathcal{B}}$ the topology generated by $\mathcal{B}$. Let $x \in (a, b) \in \mathcal{T}$. Since the rationals are dense in $\mathbb{R}$, there exist $a', b' \in \mathbb{Q}$ such that $x \in (a', b') \subset (a, b)$. Hence $\mathcal{T} \subset \mathcal{T}_{\mathcal{B}}$. The other inclusion is trivial since every basis element $(a, b) \in \mathcal{B}$ is a basis element of $\mathcal{T}$. We conclude that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$.

2. Let $\mathcal{T}_c$ be the topology generated by $\mathcal{C}$ and let $\mathcal{B}$ be the basis of $\mathcal{T}_l$, the lower limit topology on $\mathbb{R}$. Then for any $[a, b) \in \mathcal{C}$, $[a, b) \in \mathcal{B}$, and so $\mathcal{T}_{\mathcal{C}} \subset \mathcal{T}_l$. To show that this inclusion is strict we need to prove the statement

$$\begin{aligned} &\neg\left(\forall B \in \mathcal{B}\, \forall x \in B\, \exists C \in \mathcal{C} : x \in C \subset B\right) \\ \iff &\exists B \in \mathcal{B}\, \exists x \in B\, \forall C \in \mathcal{C} : x \notin C \vee C \not\subset B \\ \iff &\exists B \in \mathcal{B}\, \exists x \in B\, \forall C \in \mathcal{C} : x \in C \implies C \not\subset B \end{aligned}$$

Let $[x, b) \in \mathcal{B}$ with $x \notin \mathbb{Q}$. Then $[a, c) \in \mathcal{C}$ can contain x only if $a < x$ since x is irrational. Therefore there is no element in $\mathcal{C}$ that contains x and is a subset of $[x, b)$. This proves that $\mathcal{T}_{\mathcal{C}} \subsetneq \mathcal{T}_l$.

2.2 The Subspace Topology

Exercise 16.1

Show that if Y is a subspace of X, and A is a subset of Y then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Proof. Let $\mathcal{T}$ be a topology on X, $\mathcal{T}_Y$ subspace topology on Y. Let $\mathcal{T}'_A$ be the topology A inherits as a subset of Y. Then

$$\begin{aligned} \mathcal{T}'_A &= \{A \cap U \mid U \in \mathcal{T}_Y\} \\ &= \left\{A \cap U \mid U \in \{Y \cap V \mid V \in \mathcal{T}\}\right\} \\ &= \{A \cap U \mid U = Y \cap V, V \in \mathcal{T}\} \\ &= \{A \cap (Y \cap V) \mid V \in \mathcal{T}\} \\ &= \{(A \cap Y) \cap V \mid V \in \mathcal{T}\} \\ &= \{A \cap V \mid V \in \mathcal{T}\} \end{aligned}$$

which is by definition the topology A inherits as a subset of X. $\square$

Exercise 16.3

Exercise 16.4

Show that $\pi_1 : X \times Y \to X$ is an open map.

Proof. Let U be open in $X \times Y$ and take $(x, y) \in U$. Then there exists a basis element $B_x \times B_y$ such that $(x, y) \in B_x \times B_y \subset U$. For any $b \in B_x$, $(b, y) \in B_x \times B_y \subset U$ and so $b = \pi_1(b, y) \in \pi_1(U)$. It follows that $B_x \subset \pi_1(U)$. Since the basis of a product topology is the the product of open sets, B_x is open in X which means that for every $x \in \pi_1(U)$ there is an open set $B_x \in X$ such that $x \in B_x \subset \pi_1(U)$. From Exercise 13.1 it follows that $\pi_1(U)$ is open in X. $\square$