Complex Analysis - X400386

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These notes are based on Complex Variables and Applications by James Ward Brown (9^{th} edition).

1 Regions in the Complex Plane

For $z_0 \in \mathbb{C}$, its *neighborhood* is the set

$$|z - z_0| \leq \varepsilon, \quad \varepsilon > 0.$$

It consists of all the point that lie inside (but not on!) the circle of radius ε centered at z_0. A *deleted neighborhood* is the same set as the neighborhood minus the origin, i.e

$$0 < |z - z_0| < \varepsilon.$$

A point z_0 is an *interior point* of some set S, if there exists a z_0-neighborhood that contains only points in S. It is an *exterior point* if there exists a neighborhood containing no points in S. If z_0 is neither, then it is a *boundary point.*

A set is *open* if it does not contain any of its boundary points ($\iff$ every point is an interior point) and *closed* if it contains all of them. A set can be neither or both. The disk $0 < |z| \leq 1$ is neither while the entire complex plane is both.

An open set S is *connected* if there for any two points in the set there is a finite number of straight lines (polygonal line), all in S, that connect them. A nonempty open set that is connected is called a *domain* (e.g. every neighborhood is a domain). A connected set that is not open is called a *region.*

A set is *bounded* if it can be contained in some circle, otherwise it is *unbounded.*

A point z_0 is an *accumulation point* of a set S if every deleted neighborhood of z_0 contains at least one point of S (i.e. we can get arbitrarily close to z_0 while staying in S). For example, the point 0 is an accumulation point of the set $S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$.

2 Analytic Functions

We say that $f(z) \to w_0$ as $z \to z_0 = (x_0, y_0)$ if for every $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$0 < |z - z_0| < \delta \implies |f(z) - w_0| < \varepsilon$$

where $|z - z_0| = |x - x_0 + (y - y_0)i| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$. If such w_0 exists then it is unique.

If $f : \mathbb{C} \to \mathbb{C}$ is given by

$$f(z) = f(x + yi) = u(x, y) + iv(x, y)$$

for u, v real valued and

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0, \qquad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$

exist, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x+yi) = u_0 + v_0 i.$$

Therefore the limit rules for real valued functions are easily applied to complex functions.

If we center a unit sphere on the origin of the complex plane, then every point z on the plane is mapped to a unique point P on the surface of the sphere, where P is the intersection of the line from z to the north pole of the sphere. Note that points outside the unit circle are mapped to the northern hemisphere while points inside the unit circle are mapped to the southern hemisphere. Then the unit circle is mapped to the equator and the origin is mapped to the south pole. The *point of infinity* is defined as the point that is mapped to the north pole. Then for small ε the set $|z| > \frac{1}{\varepsilon}$ is called a *neighborhood of* ∞. Thus we can simply replace the neighborhoods in the definition of a limit by neighborhoods of ∞.

The derivative in complex analysis is defined in the same manner as in usual calculus. In other words, if

$$f'(z) := \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists then f is differentiable at z.

2.1 Cauchy-Riemann Equations

Let $f : \mathbb{C} \to \mathbb{C}$ given by $f(x+yi) = u(x,y) + v(x,y)i$. Suppose f is differentiable at some z_0 and $'f(z_0) = a + bi$. Then we can approximate f using

$$f(z_0 + h) = f(z_0) + f'(z_0) \cdot h + \mathcal{O}(h)$$

Let $h = x + yi$. Then

$$\begin{aligned}(a+bi)(x+yi) &= \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} \\ &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\end{aligned}$$

and it follows that the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is the Jacobian of f viewed as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$! Hence $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$. This is a necessary condition for a complex function to be differentiable. If the partial derivatives exist and are continuous around z_0 and satisfy the relation above then $f'(z_0)$ exists. In other words, it means that a complex differentiable function is locally a rotation and a scaling. It follows that if f is differentiable then

$$f'(x+yi) = \partial_x u(x,y) + i\partial_x v(x,y)$$

3 Exercises

3.1 Continuity

Exercise 18.1c
Prove that

$$\lim_{z\to 0} \frac{\overline{z}^2}{z} = 0$$

Proof. Let $\varepsilon, \delta > 0$ and suppose that

$$|z| = |\overline{z}| < \delta.$$

Let $z = re^{i\theta}$ and so

$$\left|\frac{\overline{z}^2}{z}\right| = \left|\frac{r^2e^{-2i\theta}}{re^{i\theta}}\right| = \left|re^{-i\theta}e^{-2i\theta}\right| = \left|\overline{z}e^{-2i\theta}\right| < \delta\left|e^{-2i\theta}\right| = \delta$$

so $\delta = \varepsilon$ would do.

$\square$

Exercise 18.2b
Show that $\lim_{z\to z_0} z^2 + c = z_0^2 + c$.

Proof. Let $\varepsilon > 0$, $0 < \delta < 1$ and suppose that $|z - z_0| < \delta$. Then

$$\left|z^2 + c - z_0^2 - c\right| = \left|z^2 - z_0^2\right| = |z - z_0||z + z_0| \leq |z - z_0|(2|z_0|+|z - z_0|) \leq \delta(2|z_0|+1)$$

Hence $\delta < \min\left(\frac{\varepsilon}{2|z_0|+1}, 1\right)$ would do. $\square$