Topology - X400416

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These notes are based on Topology (2^{nd} edition) by James R. Munkres.

1 Topological Spaces

The motivation behind defining a topological space is to generalize the notion of a metric space. Recall that a metric on a set X is a map $d : X \times X \to [0, \infty)$ that satisfies

1. $d(x, y) = d(y, x)$
2. $d(x, x) = 0$
3. $d(x, y) > 0, x \neq y$
4. $d(x, y) \leq d(x, z) + d(z, y)$

Then we say that a set $U \subset X$ is open if for all $x \in U$ and some $r > 0$

$$B(x, r) := \{y \in X \mid d(x, y) < r\} \subset U.$$

In other words, around every point in U there is a "ball" that is contained in U. In Analysis I one learns about continuity using the classic ε-δ definition which requires a metric. As it turns out, we don't really need a metric to define a continuous function, only open sets:

Definition. *A function between metric spaces is continuous if and only if the preimage of an open set is open.*

Using this definition, different seeming metric can yield the same notions of which functions are continuous! We call the collection of open subsets of X defined by some metric $d : X \times X \to [0, \infty)$ a *topology*. This open sets satisfy some important properties. Namely: (1) X and $\varnothing$ are open, (2) arbitrary unions of open sets are open and (3) finite intersections of open sets is open. It turns out that a metric is not required to define a topology, only these three properties:

Definition. *Let X be a set. Then a topology on X is a set $\mathcal{T} \subset \mathcal{P}(x)$ such that*

1. *$\varnothing \in \mathcal{T}, X \in \mathcal{T}$*
2. *If $\{U_\alpha\} \subset \mathcal{T}$ then $\bigcup_\alpha U_\alpha \in \mathcal{T}$*
3. *If $\{U_i\}_{i=0}^n \subset \mathcal{T}$ then $\bigcap_{i=0}^n U_i \in \mathcal{T}$*

A topological space is the pair $(X, \mathcal{T})$

Then we say that $U \subset X$ is open if $U \in \mathcal{T}$. Note that which sets are open depends on the topology, which might conflict with your notion of open set as defined above. For example, in the topology $\mathcal{P}(\mathbb{R})$ every subset of the real line is open, while in the topology $\{\varnothing, \mathbb{R}\}$ only the empty set and $\mathbb{R}$ are open. We often say that U is open in X without giving a specific topology, which simply means that the statement that follows will hold for any topology we define on X and any element in that topology.

If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X such that $\mathcal{T} \subseteq \mathcal{T}'$ than we say that $\mathcal{T}'$ is *finer* (or *strictly finer* if the containment is proper) than $\mathcal{T}$. We similarly say that $\mathcal{T}$ is *coarser* (or *strictly coarser*) than $\mathcal{T}'$. It might also be that case that two topologies are not *comparable*.

2 Basis for a Topology

Specifying topologies directly is often not possible, due the enormous size of many topologies. So we often define a topology using a smaller subset called a *basis*.

Definition. *If X is a set, then a basis of a topology is a collection $\mathcal{B}$ of subsets of X such that*

1. $\forall x \in X, \exists B \in \mathcal{B}$ *such that* $x \in B$
2. *If* $x \in B_1 \cap B_2$ *with* $B_1, B_2 \in \mathcal{B}$ *then there exists* $B_3 \in \mathcal{B}$ *with* $x \in B_3 \subset B_1 \cap B_2$.

If $\mathcal{T}$ is a topology generated by a basis $\mathcal{B}$ then U is open if for all $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. Also $B \in \mathcal{T}$ for all $B \in \mathcal{B}$. The proof that $\mathcal{T}$ is indeed a topology is not included. An alternative construction of a topology from a basis is given by the following lemma

Lemma 2.1. *Let X be a set and $\mathcal{B}$ a basis for a topology $\mathcal{T}$. Then $\mathcal{T}$ equals the collection of all unions of elements in $\mathcal{B}$.*

Using this lemma is sometime easier in practice; given a basis $\mathcal{B}$ and $U \subset X$, if one can write U as a union of elements in $\mathcal{B}$ then U is open.

We can also go in the reverse direction: from topology to a basis.

Lemma 2.2. *Let X be a topological space. If $\mathcal{C}$ is a collection of open sets such that for each open set U and each $x \in U$ there is $C \in \mathcal{C}$ such that $x \in C \subset U$, then $\mathcal{C}$ is a basis.*

When topologies are given in terms of basis, we can already determine which one is finer using the following criterion

Lemma 2.3. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively. Then $\mathcal{T} \subset \mathcal{T}'$ if and only if for each $x \in X$ and $B \in \mathcal{B}$ containing x there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

It might be tricky to remember the direction of the inclusion. One way to think about it is since $\mathcal{T}'$ has more subsets of X it needs to have smaller basis elements.

Lastly, we define the notion of a *subbasis*.

Definition. *A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by $\mathcal{S}$ is the collection of all unions of finite intersection of elements of $\mathcal{S}$.*

To conclude this section we define 3 topologies on the real line using the notion of a basis:

1. The *standard topology* generated by the collection of all open intervals (a, b) with $a < b$ (it is not a recursive definition. Here we use open in the familiar metric sense).

2. The *lower limit topology* is generated by half-open intervals $[a, b)$. When $\mathbb{R}$ is given in the lower limit topology we denote it $\mathbb{R}_l$.

3. The *K-topology* is generate by open intervals and sets of the form $(a, b)-K$ where $K = \{1/n \mid n \in \mathbb{N}\}$. When $\mathbb{R}$ is given in this topology we denote it $\mathbb{R}_k$.

One maybe surprising property is that both $\mathbb{R}_l$ and $\mathbb{R}_k$ are finer than the standard topology, but are not comparable with one another.

3 The Product Topology on $X \times Y$

4 The Subspace Topology

If X is a topological space with topology $\mathcal{T}_x$ and $Y \subset X$ then the subspace topology on Y is defined as

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}_X\}.$$

The fact that the collection $\mathcal{T}_Y$ has all the properties of a topology follows from the $\mathcal{T}_X$ being a topology. Then if $\mathcal{B}_X$ is a basis for a topology on X, the basis of the subspace topology on Y is given by

$$\mathcal{B}_Y = \{Y \cap B \mid B \in \mathcal{B}_X\}.$$

Open sets in the subspace topology on Y are not necessarily open in X. If A is an open set in Y, then A is open in X if Y is open in X.

5 Closed Sets and Limit Points

If X is a topological space and $A \subset X$ then A is closed if $X - A$ is open. Closed sets have similar properties to open sets:

Theorem 5.1. *Let X be a topological space. Then*

1. *X and $\varnothing$ are closed*

2. *Arbitrary intersections of closed sets are closed*

3. *Finite unions of closed sets are closed.*

One can just as well define a topology in terms of closed sets, but the definition using open sets is much more common. Of course, mathematics wouldn't be fun if there wasn't any space for confusion. In a topology, a set can be open, closed, neither or both. So don't think of sets as doors.

If Y is a subspace of a topological space X, then a closed set in X is not necessarily closed in Y. A set A is closed in Y if and only if it equals the intersection of a closed set of X with Y. This is easy to verify since if A is closed in Y then $Y - A$ is open in Y and so it equals $U \cap Y$ for some open set U in X. Then $X - U$ is closed and $A = Y \cap (X - U)$. The other direction is similarly proved.

6 Exercises

6.1 Basis for a topology

Exercise 13.1
Let X be a topological space; let $A \subset X$. Suppose that for each $x \in A$ there is an open set U such that $x \in U \subset A$. Show that A is open.

Proof. For every $x \in A$, let U_x denote the open set containing x such that $U_x \subset A$. Then $U = \bigcap_{x \in A} U_x \subset A$ since each U_x is contained in A. For the other inclusion, take $x \in A$. Then $x \in U$ since x is in U_x by definition. Hence $A \subset U$ and it follows that $A = U$. Since each U_x and arbitrary unions of open sets are open it follows that A is open. $\square$

Exercise 13.4

1. If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X,how that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

2. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology containing all the all the collection of $\mathcal{T}_\alpha$ and a unique largest topology contained in all $\mathcal{T}_\alpha$

3. If $X = \{a, b, c\}$ let

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}.$$

   Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Solution.

1. (a) Since $\varnothing, X \in \mathcal{T}_\alpha$ for all α it follows that $\varnothing, X \in \bigcap \mathcal{T}_\alpha$. (b) If $U_\beta \in \bigcap \mathcal{T}_\alpha$, then $U_\beta \in \mathcal{T}_\alpha$ for all α and so $\bigcup U_\beta \in \mathcal{T}_\alpha$ for all α since $\mathcal{T}_\alpha$ is a topology. Hence $\bigcup U_\beta \in \bigcap \mathcal{T}_\alpha$. (c) If $U_1, U_2 \in \bigcap \mathcal{T}_\alpha$ then $U_1, U_2 \in \mathcal{T}_\alpha$ for all α and so $U_1 \cap U_2 \in \mathcal{T}_\alpha$ for all α. Therefore $U_1 \cap U_2 \in \bigcap \mathcal{T}_\alpha$. It follows by induction that $\bigcap \mathcal{T}_\alpha$ is closed under countable intersections. Hence an intersections of topologies is a topology.

   Let $X = \{a, b, c\}$. Then $\mathcal{T}_1 = \{\varnothing, X, \{a\}\}$ and $\mathcal{T}_2 = \{\varnothing, X, \{b\}\}$ are topologies on X. But $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b\}\}$ is not a topology since $\{a\}, \{b\} \in \mathcal{T}_1 \cup \mathcal{T}_2$ but $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$. Hence a union of topologies is, in general, not a topology.

2. Let $\mathcal{S} = \bigcup_\alpha \mathcal{T}_\alpha$. Then $X \in \mathcal{S}$ since X is in each individual $\mathcal{T}_\alpha$ as they are all topologies. It follows that $X = \bigcup_{S \in \mathcal{S}} S$ and so $\mathcal{S}$ is a subbasis. Let $\mathcal{B}$ be the basis generated by $\mathcal{S}$ and $\mathcal{T}_s$ be the topology generated by $\mathcal{B}$. Fix some $\mathcal{T}_\alpha$ and take $U \in \mathcal{T}_\alpha$. Then $U \in \mathcal{S} \subset \mathcal{B} \subset \mathcal{T}_\mathcal{S}$ by construction. Hence

$U \in \mathcal{T}_\mathcal{S}$ and it follows that $\mathcal{T}_\alpha \subset \mathcal{T}_\mathcal{S}$ for all α. Is it the smallest topology with such property? Let $\mathcal{T}'$ be a topology on X such that $\mathcal{T}_\alpha \subset \mathcal{T}'$ for all α and take $U \in \mathcal{T}_\mathcal{S}$. Then U is an arbitrary union of finite intersections of elements of $\mathcal{S} = \bigcup \mathcal{T}_\alpha \subset \mathcal{T}'$. Since $\mathcal{T}'$ is a topology it is closed under arbitrary unions and finite intersections and so $U \in \mathcal{T}'$. Hence $\mathcal{T}_\mathcal{S} \subset \mathcal{T}'$ and it follows that $\mathcal{T}_\mathcal{S}$ is the smallest topology containing all $\mathcal{T}_\alpha$.

   From part one we know that $\bigcap \mathcal{T}_\alpha$ is a topology, and by definition it is contained in $\mathcal{T}_\alpha$ for all α. If $\mathcal{T}' \subset \mathcal{T}_\alpha, \forall \alpha$ is a topology, then for every $U \in \mathcal{T}'$, $U \in \bigcap \mathcal{T}_\alpha$ and so $\mathcal{T}' \subset \bigcap \mathcal{T}_\alpha$. Therefore $\bigcap \mathcal{T}_\alpha$ is the largest topology that is contained in all $\mathcal{T}_\alpha$.

3. Apply part (2).

Exercise 13.5
Show that that topology $\mathcal{T}$ on X generated by a basis $\mathcal{B}$ is equal to the intersections of all the topologies on X that contain $\mathcal{B}$.

Proof. Let $T = \{\mathcal{T}_\beta \mid \mathcal{B} \subset \mathcal{T}_\beta\}$ be the collection of all topologies on X that contain $\mathcal{B}$. Let $u \in \mathcal{T}$. Then U can be written as a union of element in $\mathcal{B}$, i.e.

$$U = \bigcup_\alpha B_\alpha, \quad B_\alpha \in \mathcal{B}$$

Since $\mathcal{T}_\beta$ is a topology and $\mathcal{B} \subset \mathcal{T}_\beta$ for all $\mathcal{T}_\beta \in T$ it follows that $U = \bigcup_\alpha B_\alpha \in \mathcal{T}_\beta$ for all $\mathcal{T}_\beta \in T$ and so

$$\mathcal{T} \subset \bigcap_{\mathcal{T}_\beta \in T} \mathcal{T}_\beta.$$

Since $\mathcal{B} \subset \mathcal{T}$ by definition of a basis it follows that $\mathcal{T} \in T$ and so

$$\bigcap_{\mathcal{T}_\beta \in T} \mathcal{T}_\beta \subset \mathcal{T}.$$

Hence $\bigcap_{\mathcal{T}_\beta \in T} \mathcal{T}_\beta = \mathcal{T}$. ☐

Exercise 13.7

Solution.

Exercise 13.8

1. Show that the collection

$$\mathcal{B} = \{(a,b) \mid a < b, a, b \in \mathbb{Q}\}$$

   generates the standard topology on $\mathbb{R}$.

2. Show that the collection

$$\mathcal{C} = \{[a,b) \mid a < b, a, b \in \mathbb{Q}\}$$

generates a topology different from the lower limit topology.

Solution.

1. Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$ and $\mathcal{T}_{\mathcal{B}}$ the topology generated by $\mathcal{B}$. Let $x \in (a,b) \in \mathcal{T}$. Since the rationals are dense in $\mathbb{R}$, there exist $a', b' \in \mathbb{Q}$ such that $x \in (a', b') \subset (a,b)$. Hence $\mathcal{T} \subset \mathcal{T}_{\mathcal{B}}$. The other inclusion is trivial since every basis element $(a,b) \in \mathcal{B}$ is a basis element of $\mathcal{T}$. We conclude that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$.

2. Let $\mathcal{T}_c$ be the topology generated by $\mathcal{C}$ and let $\mathcal{B}$ be the basis of $\mathcal{T}_l$, the lower limit topology on $\mathbb{R}$. Then for any $[a,b) \in \mathcal{C}$, $[a,b) \in \mathcal{B}$, and so $\mathcal{T}_{\mathcal{C}} \subset \mathcal{T}_l$. To show that this inclusion is strict we need to prove the statement

$$\begin{aligned}
&\neg\left(\forall B \in \mathcal{B}\, \forall x \in B\, \exists C \in \mathcal{C} : x \in C \subset B\right) \\
\iff &\exists B \in \mathcal{B}\, \exists x \in B\, \forall C \in \mathcal{C} : x \notin C \lor C \not\subset B \\
\iff &\exists B \in \mathcal{B}\, \exists x \in B\, \forall C \in \mathcal{C} : x \in C \implies C \not\subset B
\end{aligned}$$

Let $[x,b) \in \mathcal{B}$ with $x \notin \mathbb{Q}$. Then $[a,c) \in \mathcal{C}$ can contain x only if $a < x$ since x is irrational. Therefore there is no element in $\mathcal{C}$ that contains x and is a subset of $[x,b)$. This proves that $\mathcal{T}_{\mathcal{C}} \subsetneq \mathcal{T}_l$.

6.2 The Subspace Topology

Exercise 16.1
Show that if Y is a subspace of X, and A is a subset of Y then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Proof. Let $\mathcal{T}$ be a topology on X, $\mathcal{T}_Y$ subspace topology on Y. Let $\mathcal{T}'_A$ be the topology A inherits as a subset of Y. Then

$$\begin{aligned}
\mathcal{T}'_A &= \{A \cap U \mid U \in \mathcal{T}_Y\} \\
&= \left\{A \cap U \mid U \in \{Y \cap V \mid V \in \mathcal{T}\}\right\} \\
&= \{A \cap U \mid U = Y \cap V, V \in \mathcal{T}\} \\
&= \{A \cap (Y \cap V) \mid V \in \mathcal{T}\} \\
&= \{(A \cap Y) \cap V \mid V \in \mathcal{T}\} \\
&= \{A \cap V \mid V \in \mathcal{T}\}
\end{aligned}$$

which is by definition the topology A inherits as a subset of X. $\square$

Exercise 16.3

Exercise 16.4
Show that $\pi_1 : X \times Y \to X$ is an open map.

Proof. Let U be open in $X \times Y$ and take $(x, y) \in U$. Then there exists a basis element $B_x \times B_y$ such that $(x, y) \in B_x \times B_y \subset U$. For any $b \in B_x$, $(b, y) \in B_x \times B_y \subset U$ and so $b = \pi_1(b, y) \in \pi_1(U)$. It follows that $B_x \subset \pi_1(U)$. Since the basis of a product topology is the the product of open sets, B_x is open in X which means that for every $x \in \pi_1(U)$ there is an open set $B_x \in X$ such that $x \in B_x \subset \pi_1(U)$. From Exercise 13.1 it follows that $\pi_1(U)$ is open in X. $\square$

6.3 Closed Sets and Limit Points