Galois Theory - 5122GALO6Y

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Contents

1 Introduction

Galois theory is about studying Polynomials with coefficients in a field ($\mathbb{Q}, \mathbb{R}, \mathbb{C}$ etc.). Let

$$f(T) = T^n + \cdots + a_1 T + a_0 \in \mathbb{Q}[T].$$

Then $f(T)$ splits completely in $\mathbb{C}[T]$ as

$$f(T) = (T - \alpha_1) \cdots (T - \alpha_n)$$

with $\alpha_1, \ldots \alpha_n \in \mathbb{C}$ are the roots of f. Galois theory studies permutation of the the roots that preserve algebraic relations between these roots. The allowed permutation of the roots give rise to a group denoted $\mathrm{Gal}(f)$. Not a very useful definition:

Definition. *Let $\sigma : \mathbb{C} \to \mathbb{C}$ be a field automorphism and $\alpha \in \mathbb{C}$ a root of $F(T) \in \mathbb{Q}[T]$. Since $\sigma(1) = 1$ it follows that $sigma(n) = n$ for all integers and so $\sigma(a/b) = \sigma(a)/\sigma(b) = a/b$ is the identity on $\mathbb{Q}$. Then*

$$\begin{aligned} f(\sigma(\alpha)) &= \sigma(\alpha)^n + \cdots + a_1\sigma(\alpha) + a_0 \\ &= \sigma(f(\alpha)) \\ &= 0. \end{aligned}$$

Then each automorphism σ is a permutation of the roots which is precisely the Galois group of the polynomial $Gal(f) \subset S_n$. In other words we have a group action

$$Aut(\mathbb{C}) \times \{\alpha_1, \ldots, \alpha_n\} \to \{\alpha_1, \ldots, \alpha_n\}$$

Then $Gal(f) := Im(\phi)$ where $\phi : Aut(\mathbb{C}) \to S_n$ mapping $\sigma \mapsto (\alpha_i \mapsto \sigma(alpha_i))$

$\mathrm{Gal}(f) \subset S_n$ is transitive subgroup if and only if f is irreducible (what is a transitive subgroup?). Review group actions

2 Symmetric Polynomials

Let $f = (T - \alpha_1) \cdots (T - \alpha_n) \in \mathbb{C}[T]$. Then its discriminant

$$\Delta_f = \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

When f is quadratic this definition coincides with the high-school definition of the discriminant. However in high school the discriminant is usually written in terms of the coefficients rather than the roots. This is the main point of Symmetric polynomials. Fact: $\sqrt{\Delta_f} \in \mathbb{Q} \iff \mathrm{Gal}(f) \subset A_n$. We will prove this later on.

Consider the ring $R = \mathbb{Z}[X_1, \ldots, X_n]$. If $\sigma \in S_n$ is a permutation then

$$\sigma(X_i) := X_{\sigma(i)}$$

which extends to a group action of S_n on R. Then a polynomial is said to be *symmetric* if $\sigma(f) = f$ for all $\sigma \in S_n$. We denote the the subset of symmetric polynomials by $R^{S_n} \subset R$ which is a subring.

Define the *universal polynomial* $f^{\text{univ}} \in R[T]$ with roots $X_1, \ldots, X_n$ as

$$\begin{aligned} f^{\text{univ}}(T) :&= (T - X_1) \cdots (T - X_n) \\ &= T^n - (X_1 + \cdots + X_2)T^{n-1} + (X_1X_2 + X_1X_3 + \cdots)T^{n-1} + \\ &\quad \cdots + (-1)^n(X_1 \cdots X_n). \end{aligned}$$

Then *elementary symmetric polynomials* $s_1, \ldots, s_n$ are the coefficients of f^{univ}. Every element in R^{S_n} is a combination of the elementary symmetric polynomials

Theorem 2.1. *The map*

$$\begin{aligned} \Phi : \mathbb{Z}[Y_1, \ldots, Y_n] &\to R^{S^n} \\ Y_i &\mapsto s_i \end{aligned}$$

is an isomorphism of rings.

Algorithm for writing symmetric polynomials in term of elementary symmetric polynomials:

1.

The following theorem is useful when applying the algorithm above.

Theorem 2.2 (Orbit Stabilizer Theorem)**.** *Let G be a group acting on set S. For any $x \in S$ let $G_x = \{g \in G \mid g \cdot x = x\}$ denote the stabilizer of x, and let $G \cdot x = \{g \cdot x \mid g \in G\}$ denote the orbit of x. Then*

$$|G| = |G \cdot x||G_x|$$

Since S_n is acting on the set $\{T_1, \ldots, T_2\}$ we can find the number of elements in a given sum. Since $|S_n| = n!$ the orbit of an elementary is given by

$$\frac{n!}{\text{size of stabilizer}}$$

3 Exercises

3.1 Symmetric Polynomial

Exercise 14.10

Express the symmetric polynomials $\sum_n T_1^2 T_2$ and $\sum_n T_1^3 T_2$ in the elementary symmetric polynomials.

Solution. To get the polynomial $\sum_n T_1^2 T_2$ we start with

$$s_1 s_2 = \sum_n T_1 \sum_n T_1 T_2 = \sum_n T_1^2 T_2 + 3 \sum_n T_1 T_2 T_3 = \sum_n T_1^2 T_2 + 3 s_3$$

Thus

$$\sum_n T_1^2 T_2 = s_1 s_2 - 3 s_3$$

Similarly, to transform the polynomial $\sum_n T_1^3 T_2$ we start with

$$\begin{aligned} s_1^2 s_2 &= \left(\sum_n T_1 \right)^2 \sum_n T_1 T_2 \\ &= \left(\sum_n T_1^2 + 2 \sum_n T_1 T_2 \right) \sum_n T_1 T_2 \\ &= \sum_n T_1^2 \sum_n T_1 T_2 + 2 s_2^2 \\ &= \sum_n T_1^3 T_2 + \sum_n T_1^2 T_2 T_3 + 2 s_2^2. \end{aligned}$$

And since

$$s_1 s_3 = \sum_n T_1 \sum_n T_1 T_2 T_3 = \sum_n T_1^2 T_2 T_3 + 4 \sum_n T_1 T_2 T_3 T_4$$

it follows that $\sum_n T_1^2 T_2 T_3 = s_1 s_3 - 4 s_4$ and so

$$\sum_n T_1^3 T_2 = s_1^2 s_2 - s_1 s_3 + 4 s_4 - 2 s_2^2$$

Exercise 14.21

Express $p_4 = \sum_n T_1^4$ in elementary symmetric polynomials

Solution. Let $n \geq 4$. Starting with

$$\begin{aligned} s_1^4 &= \left(\sum_n T_1 \right)^4 \\ &= \sum_n T_1^4 + 4 \sum_n T_1^3 T_2 + 12 \sum_n T_1^2 T_2 T_3 + 6 \sum_n T_1^2 T_2^2 + 24 \sum_n T_1 T_2 T_3 T_4. \end{aligned}$$

To understand how to coefficients of the sum are obtained, consider the number of ways the T_i can be arranged. For example, $T_1^4 = T_1T_1T_1T_1$ can only be arranged in 1 way but $T_1^2T_2T_3 = T_1T_1T_2T_3$ can be arrange in $\frac{4!}{2} = 12$ ways (where we divided by 2 since the two T_1 can be swapped in any given arrangement). Then

$$s_1^2 s_2 = \left(\sum_n T_1\right)^2 s_2 = \left(\sum_n T_1^2 + 2\sum_n T_1T_2\right) s_2 = \sum_n T_1^3T_2 + \sum_n T_1^2T_2T_3 + 2s_2^2.$$

So far we have

$$\begin{aligned} p_4 &= s_1^4 - 4\left(s_1^2s_2 - 2s_2^2 - \sum_n T_1^2T_2T_3\right) - 12\sum_n T_1^2T_2T_3 - 6\sum_n T_1^2T_2^2 - 24\sum_n T_1T_2T_3T_4 \\ &= s_1^4 - 4s_1^2s_2 + 8s_2^2 - 24s_4 - 6\sum_n T_1^2T_2^2 - 8\sum_n T_1^2T_2T_3. \end{aligned}$$

So continuing with $\sum_n T_1^2T_2^2$ we get

$$s_2^2 = \left(\sum_n T_1T_2\right)^2 = \sum_n T_1^2T_2^2 + 2\sum_n T_1^2T_2T_3 + 6\sum_n T_1T_2T_3T_4.$$

Finding the coefficients here is slightly trickier since s_2 contains pairs not all arrangements are allowed. For example, $T_1^2T_2^2$ can only come from the pair T_1T_2. On the other hand $T_1T_2T_3T_4$ can come from T_1T_2 and T_3T_4 or T_1T_4 and T_2T_3 and so on. We choose the first pair ($\binom{4}{2} = 6$ ways) which also fixes the second pair and so there are 6 ways to get $T_1T_2T_3T_4$. Hence

$$\begin{aligned} p_4 &= s_1^4 - 4s_1^2s_2 + 8s_2^2 - 24s_4 - 6\left(s_2^2 - 2\sum_n T_1^2T_2T_3 - 6s_4\right) - 8\sum_n T_1^2T_2T_3 \\ &= s_1^4 - 4s_1^2s_2 + 2s_2^2 + 12s_4 + 4\sum_n T_1^2T_2T_3. \end{aligned}$$

Using Exercise 14.10 we get

$$\begin{aligned} p_4 &= s_1^4 - 4s_1^2s_2 + 2s_2^2 + 12s_4 + 4(s_1s_3 - 4s_4) \\ &= s_1^4 - 4s_1^2s_2 + 2s_2^2 - 4s_4 + 4s_1s_3 \end{aligned}$$

Exercise 14.22

A rational function $f \in \mathbb{Q}[T_1, \ldots, T_n]$ is called symmetric if it is invariant under all permutations of the variables T_i. Prove that every symmetric rational function is a rational function in the elementary symmetric functions.

Proof. Let $f \in \mathbb{Q}[T_1, \ldots, T_n]$ be a symmetric rational function. Then $f = g/h$ for g, h polynomials. If h is a symmetric polynomial then $g = fh$ is symmetric as well. By the fundamental theorem of symmetric polynomial both g and h

can be written in terms of elementary symmetric polynomials and we're done. If h is not symmetric, then let

$$\tilde{h} = \prod_{\sigma \in S_n \setminus \{e\}} \sigma(h)$$

and then $h\tilde{h}$ is symmetric so $f = \frac{g\tilde{h}}{h\tilde{h}}$ which is again the case above. □

Exercise 14.23

Write $\sum_n T_1^{-1}$ and $\sum_n T_1^{-2}$ as rational functions in $\mathbb{Q}[s_1, \ldots, s_n]$

Solution. Starting with

$$\sum_n T_1^{-1} = \frac{1}{T_1} + \cdots + \frac{1}{T_n}.$$

We multiply by $1 = \frac{s_n}{s_n}$ and simplify

$$\begin{aligned} \frac{s_n}{s_n} \sum_n T_1^{-1} &= \frac{T_1 T_2 \cdots T_n}{T_1 T_2 \cdots T_n} \left(\frac{1}{T_1} + \cdots + \frac{1}{T_n} \right) \\ &= \frac{s_{n-1}}{s_n} \end{aligned}$$

For the second expression we present to approaches.

1. Observing that

$$\left(\sum_n T_1^{-1} \right)^2 = \sum_n T_1^{-2} + 2 \sum_n T_1^{-1} T_2^{-1}$$

we can write using the previous part

$$\sum_n T_1^{-2} = \frac{s_{n-1}^2}{s_n^2} - 2 \sum_n T_1^{-1} T_2^{-1}$$

and multiplying by the second term by $\frac{s_n}{s_n}$ we get

$$\sum_n T_1^{-2} = \frac{s_{n-1}^2}{s_n^2} - 2 \left(\frac{1}{T_1 T_2} + \cdots + \frac{1}{T_{n-1} T_n} \right) \frac{T_1 \cdots T_n}{T_1 \cdots T_n} = \frac{s_{n-1}^2}{s_n^2} - 2 \frac{s_{n-2}}{s_n}.$$

Hence $\sum_n T_1^{-2} = \frac{s_{n-1}^2 - 2 s_{n-2} s_n}{s_n^2}$.

2. The second approach is slightly more involved. We start by multiplying by 1 in a clever (but different) way

$$\left(\sum_n T_1^{-2} \right) \frac{s_n^2}{s_n^2} = \left(\frac{1}{T_1^2} + \cdots + \frac{1}{T_n^2} \right) \frac{T_1^2 \cdots T_n^2}{T_1^2 \cdots T_n^2} = \frac{\sum_n T_1^2 \cdots T_{n-1}^2}{s_n^2}.$$

Then $\sum_n T_1^2 \cdots T_{n-1}^2$ is obviously (condescending much?) a symmetric polynomial and so we can use our trusty algorithm. Starting with

$$\begin{aligned} s_1^{2-2} s_2^{2-2} \cdots s_{n-1}^{2-0} &= s_{n-1}^2 \\ &= \left(\sum_n T_1 \cdots T_{n-1} \right)^2 \\ &= \sum_n T_1^2 \cdots T_{n-1}^2 + 2 \sum_n T_1^2 \cdots T_{n-2}^2 T_{n-1} T_n. \end{aligned}$$

Moving to the second term

$$\begin{aligned} s_1^{2-2} \cdots s_{n-2}^{2-1} s_{n-1}^{1-1} s_n^1 &= s_{n-2} s_n \\ &= \left(\sum_n T_1 \cdots T_{n-2} \right) T_1 \cdots T_n \\ &= \sum_n T_1^2 \cdots T_{n-2}^2 T_{n-1} T_n \end{aligned}$$

and it follows that

$$\sum_n T_1^2 \cdots T_{n-1}^2 = s_{n-1}^2 - 2 s_{n-2} s_n.$$

So we conclude that

$$\sum_n T_1^{-2} = \frac{s_{n-1}^2 - 2 s_{n-2} s_n}{s_n^2}$$

which is reassuring.

Note that in the first approach we stumbled upon something rather interesting:

$$\sum_n T_1^{-1} \cdots T_k^{-1} = \frac{s_{n-k}}{s_n}$$

the proof of which is left as an exercise to the reader.

3.2 Field Extensions

3.3 Finite Fields

3.4 Separable and Normal Extensions