Topology - X400416

Yoav Eshel

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Contents

These notes are based on Topology (2^{nd} edition) by James R. Munkres.

1 Topological Spaces

The motivation behind defining a topological space is to generalize the notion of a metric space. Recall that a metric on a set X is a map $d : X \times X \to [0, \infty)$ that satisfies

1. $d(x, y) = d(y, x)$
2. $d(x, x) = 0$
3. $d(x, y) > 0, x \neq y$
4. $d(x, y) \leq d(x, z) + d(z, y)$

Then we say that a set $U \subset X$ is open if for all $x \in U$ and some $r > 0$

$$B(x, r) := \{y \in X \mid d(x, y) < r\} \subset U.$$

In other words, around every point in U there is a "ball" that is contained in U. In Analysis I one learns about continuity using the classic ε-δ definition which requires a metric. As it turns out, we don't really need a metric to define a continuous function, only open sets:

Definition. *A function between metric spaces is continuous if and only if the preimage of an open set is open.*

Using this definition, different seeming metric can yield the same notions of which functions are continuous! We call the collection of open subsets of X defined by some metric $d : X \times X \to [0, \infty)$ a *topology*. This open sets satisfy some important properties. Namely: (1) X and $\varnothing$ are open, (2) arbitrary unions of open sets are open and (3) finite intersections of open sets is open. It turns out that a metric is not required to define a topology, only these three properties:

Definition. *Let X be a set. Then a topology on X is a set $\mathcal{T} \subset \mathcal{P}(x)$ such that*

1. $\varnothing \in \mathcal{T}, X \in \mathcal{T}$
2. *If* $\{U_\alpha\} \subset \mathcal{T}$ *then* $\bigcup_\alpha U_\alpha \in \mathcal{T}$
3. *If* $\{U_i\}_{i=0}^n \subset \mathcal{T}$ *then* $\bigcap_{i=0}^n U_i \in \mathcal{T}$

A topological space is the pair $(X, \mathcal{T})$

Then we say that $U \subset X$ is open if $U \in \mathcal{T}$. Note that which sets are open depends on the topology, which might conflict with your notion of open set as defined above. For example, in the topology $\mathcal{P}(\mathbb{R})$ every subset of the real line is open, while in the topology $\{\varnothing, \mathbb{R}\}$ only the empty set and $\mathbb{R}$ are open. We often say that U is open in X without giving a specific topology, which simply means that the statement that follows will hold for any topology we define on X and any element in that topology.

If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X such that $\mathcal{T} \subseteq \mathcal{T}'$ than we say that $\mathcal{T}'$ is *finer* (or *strictly finer* if the containment is proper) than $\mathcal{T}$. We similarly say that $\mathcal{T}$ is *coarser* (or *strictly coarser*) than $\mathcal{T}'$. It might also be that case that two topologies are not *comparable*.

2 Basis for a Topology

Specifying topologies directly is often not possible, due the enormous size of many topologies. So we often define a topology using a smaller subset called a *basis*.

Definition. *If X is a set, then a basis of a topology is a collection $\mathcal{B}$ of subsets of X such that*

1. $\forall x \in X, \exists B \in \mathcal{B}$ *such that* $x \in B$
2. *If* $x \in B_1 \cap B_2$ *with* $B_1, B_2 \in \mathcal{B}$ *then there exists* $B_3 \in \mathcal{B}$ *with* $x \in B_3 \subset B_1 \cap B_2$.

If $\mathcal{T}$ is a topology generated by a basis $\mathcal{B}$ then U is open if for all $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. Also $B \in \mathcal{T}$ for all $B \in \mathcal{B}$. The proof that $\mathcal{T}$ is indeed a topology is not included. An alternative construction of a topology from a basis is given by the following lemma

Lemma 2.1. *Let X be a set and $\mathcal{B}$ a basis for a topology $\mathcal{T}$. Then $\mathcal{T}$ equals the collection of all unions of elements in $\mathcal{B}$.*

Using this lemma is sometime easier in practice; given a basis $\mathcal{B}$ and $U \subset X$, if one can write U as a union of elements in $\mathcal{B}$ then U is open.

We can also go in the reverse direction: from topology to a basis.

Lemma 2.2. *Let X be a topological space. If $\mathcal{C}$ is a collection of open sets such that for each open set U and each $x \in U$ there is $C \in \mathcal{C}$ such that $x \in C \subset U$, then $\mathcal{C}$ is a basis.*

When topologies are given in terms of basis, we can already determine which one is finer using the following criterion

Lemma 2.3. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively. Then $\mathcal{T} \subset \mathcal{T}'$ if and only if for each $x \in X$ and $B \in \mathcal{B}$ containing x there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

It might be tricky to remember the direction of the inclusion. One way to think about it is since $\mathcal{T}'$ has more subsets of X it needs to have smaller basis elements.

Lastly, we define the notion of a *subbasis*.

Definition. *A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by $\mathcal{S}$ is the collection of all unions of finite intersection of elements of $\mathcal{S}$.*

To conclude this section we define 3 topologies on the real line using the notion of a basis:

1. The *standard topology* generated by the collection of all open intervals (a, b) with $a < b$ (it is not a recursive definition. Here we use open in the familiar metric sense).
2. The *lower limit topology* is generated by half-open intervals $[a, b)$. When $\mathbb{R}$ is given in the lower limit topology we denote it $\mathbb{R}_l$.
3. The *K-topology* is generate by open intervals and sets of the form $(a, b)-K$ where $K = \{1/n \mid n \in \mathbb{N}\}$. When $\mathbb{R}$ is given in this topology we denote it $\mathbb{R}_k$.

One maybe surprising property is that both $\mathbb{R}_l$ and $\mathbb{R}_k$ are finer than the standard topology, but are not comparable with one another.

3 The Product Topology on $X \times Y$

4 The Subspace Topology

If X is a topological space with topology $\mathcal{T}_x$ and $Y \subset X$ then the subspace topology on Y is defined as

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}_X\}.$$

The fact that the collection $\mathcal{T}_Y$ has all the properties of a topology follows from the $\mathcal{T}_X$ being a topology. Then if $\mathcal{B}_X$ is a basis for a topology on X, the basis of the subspace topology on Y is given by

$$\mathcal{B}_Y = \{Y \cap B \mid B \in \mathcal{B}_X\}.$$

Open sets in the subspace topology on Y are not necessarily open in X. If A is an open set in Y, then A is open in X if Y is open in X.

5 Closed Sets and Limit Points

If X is a topological space and $A \subset X$ then A is closed if $X - A$ is open. Closed sets have similar properties to open sets:

Theorem 5.1. *Let X be a topological space. Then*

1. *X and $\varnothing$ are closed*

2. *Arbitrary intersections of closed sets are close*

3. *Finite unions of closed sets are closed.*

One can just as well define a topology in terms of closed sets, but the definition using open sets is much more common. Of course, mathematics wouldn't be fun if there wasn't any space for confusion. In a topology, a set can be open, closed, neither or both. So don't think of sets as doors.

If Y is a subspace of a topological space X, then a closed set in X is not necessarily closed in Y. A set A is closed in Y if and only if it equals the intersection of a closed set of X with Y. This is easy to verify since if A is closed in Y then $Y - A$ is open in Y and so it equals $U \cap Y$ for some open set U in X. Then $X - U$ is closed and $A = Y \cap (X - U)$. The other direction is similarly proved.

Let $A \subset X$. THen the smallest closed set that contain A is called the closure of A and is denoted $\bar{A} = \bigcap_{C \text{ closed}, A \subset C} C$. The smallest open set containing A is called the interior of A and is denoted $\text{Int}A = \bigcup_{U \text{ open}, U \subset A}$. Then clearly

$$\text{Int}A \subset A \subset \bar{A}.$$

A point $x \in X$ is in the closure of A if and only if for any basis element B containing x the intersection $A \cap B$ is non empty.

To add to the soup of metaphors, we define the *neighborhood* of $x \in X$ is an open set U containing x. Then a *limit point* of $A \subset X$ is defined as

Definition. *A limit point of $A \subset X$ is a point $x \in A$ such that every neighborhood of x intersects A in a point different than x*

In other words, x is a limit point of A if for every open set U containing x the intersection $U \cap (A - x)$ is non empty. Now denote the set of limit points of A by A'. Then $\bar{A} = A' \cup A$. This gives us an alternative condition for a set being closed. Namely, if $A' \subset A$ then $\bar{A} = A$ and so A is closed. Alternatively, if A is closed, then $\bar{A} = A$ and so $A' \subset A$. Hence a set is closed if and only if it contains all of its limit points.

We conclude this section with a definition of convergence in a topological space.

Definition. *Let $(X, \mathcal{T})$ be a topological space. If x_n is a sequence in X then $x_n \to x$ as $n \to \infty$ if*

$$\forall U \in \mathcal{T}, x \in U \, \exists N \in \mathbb{N} : \{x_n\}_{n \geq N} \subset U$$

Note that this definition does not require any metric. The downside is that limit are not necessarily unique and sequences can converge to any number of points. To avoid this horrific phenomenon, we add an extra condition to rid our definition of topological spaces of such situations. Topological spaces in which limits are unique are called Hausdorff spaces.

6 Exercises

6.1 Basis for a topology

Exercise 13.1
Let X be a topological space; let $A \subset X$. Suppose that for each $x \in A$ there is an open set U such that $x \in U \subset A$. Show that A is open.

Proof. For every $x \in A$, let U_x denote the open set containing x such that $U_x \subset A$. Then $U = \bigcap_{x \in A} U_x \subset A$ since each U_x is contained in A. For the other inclusion, take $x \in A$. Then $x \in U$ since x is in U_x by definition. Hence $A \subset U$ and it follows that $A = U$. Since each U_x and arbitrary unions of open sets are open it follows that A is open. $\square$

Exercise 13.4

1. If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X,how that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

2. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology containing all the all the collection of $\mathcal{T}_\alpha$ and a unique largest topology contained in all $\mathcal{T}_\alpha$

3. If $X = \{a, b, c\}$ let

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}.$$

   Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Solution.

1. (a) Since $\varnothing, X \in \mathcal{T}_\alpha$ for all α it follows that $\varnothing, X \in \bigcap \mathcal{T}_\alpha$. (b) If $U_\beta \in \bigcap \mathcal{T}_\alpha$, then $U_\beta \in \mathcal{T}_\alpha$ for all α and so $\bigcup U_\beta \in \mathcal{T}_\alpha$ for all α since $\mathcal{T}_\alpha$ is a topology. Hence $\bigcup U_\beta \in \bigcap \mathcal{T}_\alpha$. (c) If $U_1, U_2 \in \bigcap \mathcal{T}_\alpha$ then $U_1, U_2 \in \mathcal{T}_\alpha$ for all α and so $U_1 \cap U_2 \in \mathcal{T}_\alpha$ for all α. Therefore $U_1 \cap U_2 \in \bigcap \mathcal{T}_\alpha$. It follows by induction that $\bigcap \mathcal{T}_\alpha$ is closed under countable intersections. Hence an intersections of topologies is a topology.

   Let $X = \{a, b, c\}$. Then $\mathcal{T}_1 = \{\varnothing, X, \{a\}\}$ and $\mathcal{T}_2 = \{\varnothing, X, \{b\}\}$ are topologies on X. But $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b\}\}$ is not a topology since $\{a\}, \{b\} \in \mathcal{T}_1 \cup \mathcal{T}_2$ but $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$. Hence a union of topologies is, in general, not a topology.

2. Let $\mathcal{S} = \bigcup_\alpha \mathcal{T}_\alpha$. Then $X \in \mathcal{S}$ since X is in each individual $\mathcal{T}_\alpha$ as they are all topologies. It follows that $X = \bigcup_{S \in \mathcal{S}} S$ and so $\mathcal{S}$ is a sub-basis. Let $\mathcal{B}$ be the basis generated by $\mathcal{S}$ and $\mathcal{T}_s$ be the topology generated by $\mathcal{B}$. Fix some $\mathcal{T}_\alpha$ and take $U \in \mathcal{T}_\alpha$. Then $U \in \mathcal{S} \subset \mathcal{B} \subset \mathcal{T}_{\mathcal{S}}$ by construction. Hence

$U \in \mathcal{T}_{\mathcal{S}}$ and it follows that $\mathcal{T}_\alpha \subset \mathcal{T}_{\mathcal{S}}$ for all α. Is it the smallest topology with such property? Let $\mathcal{T}'$ be a topology on X such that $\mathcal{T}_\alpha \subset \mathcal{T}'$ for all α and take $U \in \mathcal{T}_{\mathcal{S}}$. Then U is an arbitrary union of finite intersections of elements of $\mathcal{S} = \bigcup \mathcal{T}_\alpha \subset \mathcal{T}'$. Since $\mathcal{T}'$ is a topology it is closed under arbitrary unions and finite intersections and so $U \in \mathcal{T}'$. Hence $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}'$ and it follows that $\mathcal{T}_{\mathcal{S}}$ is the smallest topology containing all $\mathcal{T}_\alpha$.

From part one we know that $\bigcap \mathcal{T}_\alpha$ is a topology, and by definition it is contained in $\mathcal{T}_\alpha$ for all α. If $\mathcal{T}' \subset \mathcal{T}_\alpha, \forall \alpha$ is a topology, then for every $U \in \mathcal{T}'$, $U \in \bigcap \mathcal{T}_\alpha$ and so $\mathcal{T}' \subset \bigcap \mathcal{T}_\alpha$. Therefore $\bigcap \mathcal{T}_\alpha$ is the largest topology that is contained in all $\mathcal{T}_\alpha$.

3. Apply part (2).

Exercise 13.5
Show that that topology $\mathcal{T}$ on X generated by a basis $\mathcal{B}$ is equal to the intersections of all the topologies on X that contain $\mathcal{B}$.

Proof. Let $T = \{\mathcal{T}_\beta \mid \mathcal{B} \subset \mathcal{T}_\beta\}$ be the collection of all topologies on X that contain $\mathcal{B}$. Let $u \in \mathcal{T}$. Then U can be written as a union of element in $\mathcal{B}$, i.e.

$$U = \bigcup_\alpha B_\alpha, \quad B_\alpha \in \mathcal{B}$$

Since $\mathcal{T}_\beta$ is a topology and $\mathcal{B} \subset \mathcal{T}_\beta$ for all $\mathcal{T}_\beta \in T$ it follows that $U = \bigcup_\alpha B_\alpha \in \mathcal{T}_\beta$ for all $\mathcal{T}_\beta \in T$ and so

$$\mathcal{T} \subset \bigcap_{\mathcal{T}_\beta \in T} \mathcal{T}_\beta.$$

Since $\mathcal{B} \subset \mathcal{T}$ by definition of a basis it follows that $\mathcal{T} \in T$ and so

$$\bigcap_{\mathcal{T}_\beta \in T} \mathcal{T}_\beta \subset \mathcal{T}.$$

Hence $\bigcap_{\mathcal{T}_\beta \in T} \mathcal{T}_\beta = \mathcal{T}$. □

Exercise 13.7

Solution.

Exercise 13.8

1. Show that the collection

$$\mathcal{B} = \{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

generates the standard topology on $\mathbb{R}$.

2. Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a, b \in \mathbb{Q}\}$$

generates a topology different from the lower limit topology.

Solution.

1. Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$ and $\mathcal{T}_{\mathcal{B}}$ the topology generated by $\mathcal{B}$. Let $x \in (a, b) \in \mathcal{T}$. Since the rationals are dense in $\mathbb{R}$, there exist $a', b' \in \mathbb{Q}$ such that $x \in (a', b') \subset (a, b)$. Hence $\mathcal{T} \subset \mathcal{T}_{\mathcal{B}}$. The other inclusion is trivial since every basis element $(a, b) \in \mathcal{B}$ is a basis element of $\mathcal{T}$. We conclude that $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$.

2. Let $\mathcal{T}_c$ be the topology generated by $\mathcal{C}$ and let $\mathcal{B}$ be the basis of $\mathcal{T}_l$, the lower limit topology on $\mathbb{R}$. Then for any $[a, b) \in \mathcal{C}$, $[a, b) \in \mathcal{B}$, and so $\mathcal{T}_{\mathcal{C}} \subset \mathcal{T}_l$. To show that this inclusion is strict we need to prove the statement

$$\begin{aligned} &\neg\left(\forall B \in \mathcal{B}\, \forall x \in B\, \exists C \in \mathcal{C} : x \in C \subset B\right) \\ \iff &\exists B \in \mathcal{B}\, \exists x \in B\, \forall C \in \mathcal{C} : x \notin C \vee C \not\subset B \\ \iff &\exists B \in \mathcal{B}\, \exists x \in B\, \forall C \in \mathcal{C} : x \in C \implies C \not\subset B \end{aligned}$$

Let $[x, b) \in \mathcal{B}$ with $x \notin \mathbb{Q}$. Then $[a, c) \in \mathcal{C}$ can contain x only if $a < x$ since x is irrational. Therefore there is no element in $\mathcal{C}$ that contains x and is a subset of $[x, b)$. This proves that $\mathcal{T}_{\mathcal{C}} \subsetneq \mathcal{T}_l$.

6.2 The Subspace Topology

Exercise 16.1
Show that if Y is a subspace of X, and A is a subset of Y then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Proof. Let $\mathcal{T}$ be a topology on X, $\mathcal{T}_Y$ subspace topology on Y. Let $\mathcal{T}'_A$ be the topology A inherits as a subset of Y. Then

$$\begin{aligned} \mathcal{T}'_A &= \{A \cap U \mid U \in \mathcal{T}_Y\} \\ &= \left\{A \cap U \mid U \in \{Y \cap V \mid V \in \mathcal{T}\}\right\} \\ &= \{A \cap U \mid U = Y \cap V, V \in \mathcal{T}\} \\ &= \{A \cap (Y \cap V) \mid V \in \mathcal{T}\} \\ &= \{(A \cap Y) \cap V \mid V \in \mathcal{T}\} \\ &= \{A \cap V \mid V \in \mathcal{T}\} \end{aligned}$$

which is by definition the topology A inherits as a subset of X. $\square$

Exercise 16.3

Exercise 16.4
Show that $\pi_1 : X \times Y \to X$ is an open map.

Proof. Let U be open in $X \times Y$ and take $(x, y) \in U$. Then there exists a basis element $B_x \times B_y$ such that $(x, y) \in B_x \times B_y \subset U$. For any $b \in B_x$, $(b, y) \in B_x \times B_y \subset U$ and so $b = \pi_1(b, y) \in \pi_1(U)$. It follows that $B_x \subset \pi_1(U)$. Since the basis of a product topology is the the product of open sets, B_x is open in X which means that for every $x \in \pi_1(U)$ there is an open set $B_x \in X$ such that $x \in B_x \subset \pi_1(U)$. From Exercise 13.1 it follows that $\pi_1(U)$ is open in X. □

6.3 Closed Sets and Limit Points

Exercise 17.1
Let $\mathcal{C}$ be a collection of subsets of X. Suppose that $X, \varnothing \in \mathcal{C}$ and that $\mathcal{C}$ is closed under finite unions and arbitrary intersections. Prove that

$$\mathcal{T} = \{X - C \mid C \in \mathcal{C}\}$$

is a topology.

Proof. Since $X - X = \varnothing$ and $X - \varnothing = X$ it follows that $X, \varnothing \in \mathcal{T}$. Let U_α be some collection of elements of $\mathcal{T}$. Then $U_\alpha = X - C_\alpha$ for some $C_\alpha \in \mathcal{C}$. Then

$$\bigcup_\alpha U_\alpha = \bigcup_\alpha (X - C_\alpha) = X - \bigcap_\alpha C_\alpha \in \mathcal{T}$$

Since $\mathcal{C}$ is closed under arbitrary intersections. Lastly let $U_i = X - C_i$, $1 \leq i \leq n$ be a finite collection in $\mathcal{T}$. Then

$$\bigcap_{i=1}^{n} U_i = X - \bigcup_{i=1}^{n} C_i \in \mathcal{T}.$$

It follows that $\mathcal{T}$ is a topology on X. □

Exercise 17.2
Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Proof. Suppose that A is closed in Y and Y is closed in X. Then $Y - A$ is open in Y and so $Y - A = U \cap Y$ for some $U \subset X$ open. Hence $A = (X - U) \cap Y$. Since $X - U$ and Y are closed in A and arbitrary intersections of closed sets are closed, it follows that A is closed in X. □

Exercise 17.6
Let A, B and A_α denote subsets of a space X. Prove the following

1. If $A \subset B$, then $\bar{A} \subset \bar{B}$
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$
3. $\overline{\bigcup_\alpha A_\alpha} \supset \bigcup_\alpha \overline{A_\alpha}$

Proof.

1. Suppose that $A \subset B$. Let $x \in \overline{A}$. Then for every neighborhood U containing x intersects A. Hence $U \cap A$ is non empty, so take $y \in U \cap A$. Then $y \in U \cap B$ since $A \subset B$ and it follows that every neighborhood of x intersects B. Therefore $x \in \overline{B}$ and the result follows.

2. Let $x \in \overline{A \cup B}$. Then every neighborhood U of x intersects $A \cup B$. Since $A \subset \bar{A}$ and $B \subset \bar{B}$ it follows that $A \cup B \subset \bar{A} \cup \bar{B}$ and so U intersects $\bar{A} \cup \bar{B}$. Hence $x \in \bar{A} \cup \bar{B}$ and so $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

   Let $x \in \bar{A} \cup \bar{B}$. Then x is in $\bar{A}$ or $\bar{B}$ and every neighborhood U of x intersects either A or B. Therefore U intersects $A \cup B$ and it follows that $x \in \overline{A \cup B}$. Therefore $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$ and equality follows.

3. Let $x \in \bigcup_\alpha \overline{A_\alpha}$. Then $x \in \overline{A_\alpha}$ for at least one α. Hence every neighborhood U of x intersects A_α and so U intersects $\bigcup_\alpha A_\alpha$. Hence $x \in \overline{\bigcup_\alpha A_\alpha}$ and so $\bigcup_\alpha \overline{A_\alpha} \subset \overline{\bigcup_\alpha A_\alpha}$.

   To show that the other inclusion doesn't hold in general let $A_\alpha = \left(\frac{1}{\alpha}, 1\right)$. Then
   $$\bigcup_{\alpha=1}^{\infty} \overline{A_\alpha} = \bigcup_{\alpha=1}^{\infty} \overline{\left(\frac{1}{\alpha}, 1\right)} = \bigcup_{\alpha=1}^{\infty} \left[\frac{1}{\alpha}, 1\right] = (0, 1]$$
   and
   $$\overline{\bigcup_{\alpha=1}^{\infty} A_\alpha} = \overline{(0,1)} = [0, 1].$$

□

Exercise 17.8

Exercise 17.11
Show that the product of two Hausdorff spaces is Hausdorff

Proof. Let X, Y be Hausdorff spaces and take $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then there are open subsets U_1, U_2 in X such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \varnothing$. Similarly there exists $V_1, V_2 \subset Y$ that are open in Y with $y_1 \in V_1, y_2 \in V_2$ and $V_1 \cap V_2 = \varnothing$. Then $(x_1, y_1) \in U_1 \times V_1, (x_2, y_2) \in U_2 \times V_2$ and

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) = \varnothing \times \varnothing = \varnothing.$$

Since $U_1 \times V_1$ and $U_2 \times V_2$ are open it follows that $X \times Y$ is a Hausdorff space. □

Exercise 17.12
Show that a subspace of Hausdorff space is Hausdorff.

Proof. Let X be a Hausdorff space and $Y \subset X$. Then for every $x_1, x_2 \in Y$ there exists neighborhoods U_1 and U_2 of x_1 and x_2 (respectively) that are disjoint. Then $U_1 \cap Y$ and $U_2 \cap Y$ are open in Y, contain x_1 and x_2 (respectively) and are clearly disjoint. Hence Y is Hausdorff. $\square$

Exercise 17.13
Show that X is Hausdorff if and only if $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

Proof. Suppose X is Hausdorff and consider $(x, y) \notin \Delta$. Since X is Hausdorff, there exists disjoint neighborhoods U and V that contain x and y (respectively). Then take $(x', y') \in U \times V$. Since $U \cap V = \varnothing$ it follows that $x' \neq y'$ and so $(x', y') \notin \Delta$. Since $U \times V$ is open in $X \times X$ it follows that (x, y) is not a limit point of Δ. Hence Δ contains all of its limit points and so it is closed.

Now suppose that Δ is closed in $X \times X$ and consider $(x, y) \notin \Delta$. Since Δ is closed, it contain all of its limit points and so (x, y) is not a limit point of Δ. Hence there exists a neighborhood T of (x, y) that does not intersect Δ. Then there is a basis element $U \times V$ such that $(x, y) \in U \times V \subset T$. Since $U \times V$ does not intersect Δ it follows that for every $(x', y') \in U \times V$ $x' \neq y'$. Hence $U \cap V = \varnothing$. It follows that for every x and y in X there exists disjoint neighborhoods U and V that contain x and y respectively. Therefore X is Hausdorff. $\square$

Exercise 17.14
In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = \frac{1}{n}$ converge?

Solution. This sequence converges to every point in $\mathbb{R}$! To see why, suppose there exists $a \in \mathbb{R}$ that the sequence does not converge to. Then there exists an open neighborhood U of a such that for all $N \in \mathbb{N}$ there exists an $n \geq N$ for which $x_n \notin U$. There has to be an infinite number of such x_n, for otherwise we could just take N' bigger than the largest n. But then $\mathbb{R} - U$ is not finite nor empty, so it must be all of $\mathbb{R}$. Therefore $U = \varnothing$ which is a contradiction since we assumed that $a \in U$.

Exercise 17.19
For $A \subset X$, the *boundary* of A is

$$\text{Bd } A = \overline{A} \cap \overline{(X - A)}.$$

Show that

1. $\text{Int } A \cap \text{Bd } A = \varnothing$ and $\overline{A} = \text{Int } A \cup \text{Bd } A$.
2. $\text{Bd } A = \varnothing \iff A$ is both open and closed.
3. U is open $\iff \text{Bd } U = \overline{U} - U$

4. If U is open, is it true that $U = \text{Int } \overline{U}$? Justify your answer.

Proof.

1. Let $x \in \text{Int } A$. Then there is a neighborhood U of x that is contained in A, and so U does not intersect $X - A$. Hence x is not a limit of point of $X - A$ and so $x \notin \overline{X - A}$. Therefore x is not in $\overline{A} \cap \overline{(X - A)}$, and so $\text{Int } A \cap \text{Bd } A = \varnothing$.

   Let $x \in X$. If $x \in \text{Int } A$ then clearly $x \in \text{Int } A \cup \overline{(X - A)}$. So suppose $x \notin \text{Int } A$. Then $x \in X - \text{Int } A \subset \overline{X - \text{Int } A}$. Consider any neighborhood U containing x. Since x is not in the interior of A, U is not a subset of A, hence U intersects $X - A$. It follows that x is a limit point of $X - A$ and so $x \in \overline{X - A}$. Therefore $X = \text{Int } A \cup \overline{(X - A)}$ and it follows that

$$\begin{aligned}
\text{Int } A \cup \text{Bd } A &= \text{Int } A \cup \left(\overline{A} \cap \overline{(X - A)}\right) \\
&= \left(\text{Int } A \cup \overline{A}\right) \cap \left(\text{Int } A \cup \overline{(X - A)}\right) \\
&= \overline{A} \cap \left(\text{Int } A \cup \overline{(X - A)}\right) \\
&= \overline{A} \cap X \\
&= \overline{A}.
\end{aligned}$$

2. Suppose $\text{Bd } A = \varnothing$. Then

$$\overline{A} = \text{Int } A \cup \text{Bd } A = \text{Int } A.$$

   Since $\text{Int } A \subset A \subset \overline{A}$ it follows that $A = \text{Int } A = \overline{A}$ and so A is both close and open.

   Now suppose that A is both closed and open. Then $A = \text{Int } A$ and $A = \overline{A}$ so $\text{Int } A = \overline{A}$. From $\overline{A} = \text{Int } A \cup \text{Bd } A$ it follows that $\text{Bd } A \subset \overline{A}$ but

$$\varnothing = \text{Int } A \cap \text{Bd } A = \overline{A} \cap \text{Bd } A.$$

   Hence $\text{Bd } A = \varnothing$.

3. Suppose U is open. Then $U = \text{Int } U$ and $\overline{U} = U \cup \text{Bd } U$. Since $U \cap \text{Bd } U = \varnothing$ it follows that $\text{Bd } U = \overline{U} - U$.

   Now suppose that $\text{Bd } U = \overline{U} - U$ and take $x \in U$. Then $x \in \overline{U}$ and

$$x \notin \overline{U} - U = \text{Bd } U = \overline{U} \cap \overline{(X - U)}.$$

   So $x \notin \overline{(X - U)}$. Then there exists a neighborhood V of x that does not intersect $X - U$. Since for every $y \in V \implies y \notin X - U \implies y \in U$ it follows that $V \subset U$. We showed that for every $x \in U$ there exists an open neighborhood $V \subset U$ that contains x. Therefore U is open.

4. No. Consider $U = (0, 1) \cup (1, 2)$ which is open in the standard topology on $\mathbb{R}$. Then $\overline{U} = [0, 2]$ and $\text{Int } \overline{U} = (0, 2) \neq U$.

□