Dynamical Systems - X400637

Yoav Eshel

February 2, 2021

Contents

These notes are based on Differential Equations and Dynamical Systems (third edition) by Lawrence Perko.

1 Linear Systems

A linear system of ordinary differential equations is written as

$$\dot{\mathbf{x}} = A\mathbf{x}$$

with $x \in \mathbb{R}^n$, A an $n \times n$ matrix and

$$\dot{\mathbf{x}} = \begin{pmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{d_t} \end{pmatrix}$$

. If A is a diagonal matrix with n real and distinct eigenvalues and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is any set of corresponding eigenvectors then

$$PAP^{-1} = \operatorname{diag}[\lambda_1, \dots, \lambda_n]$$

with $P = (\mathbf{v}_1, \dots, \mathbf{v}_n)$.

Hence any linear systems with diagonal matrix can be transformed to an uncoupled system by substituting $\mathbf{y} = P\mathbf{x}$. Then the solution is

$$\mathbf{x}(t) = P \operatorname{diag}\left[e^{\lambda_1 t}, \dots, e^{\lambda_n t}\right] P^{-1}\mathbf{x}(0).$$

Let $L(\mathbb{R}^n)$ denote the space of linear operators on $\mathbb{R}^n$. For a linear transformation $L(\mathbb{R}^n) \ni T : \mathbb{R}^n \to \mathbb{R}^n$ the *operator norm* is defined as

$$\|T\| = \max_{|\mathbf{x}| \leq 1} |T(\mathbf{x})|$$

with $|\mathbf{x}|$ denoting the Euclidean norm. The operator norm has all the usual properties one might expect of a norm.

A sequence of linear operators $T_k \in L(\mathbb{R}^n)$ converges to an operator $T \in L(\mathbb{R}^n)$ if

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N} : k \geq N \implies \|T_k - T\| < \varepsilon.$$

Then we write

$$\lim_{k \to \infty} T_k = T.$$

Lemma 1.1. *For $S, T \in L(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ it holds that*

1. $|T(\mathbf{x})| \leq \|T\| |\mathbf{x}|$
2. $\|TS\| \leq \|T\| \|S\|$
3. $\left\|T^k\right\| \leq \|T\|^k, k \in \mathbb{N}_0$

Theorem 1.2. *Givne $T \in L(\mathbb{R}^n)$ and $t_0 > 0$ the series*

$$\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}$$

converges uniformly and absolutely for all $|t| \le t_0$.

The theorem is easily proved using the observation that

$$\left\| \frac{T^k t^k}{k!} \right\| \le \frac{a^k t_0^k}{k!}$$

where $a = \|T\|$. Thus

$$\left\| \sum_{k=0}^{\infty} \frac{T^k t^k}{k!} \right\| \le \sum_{k=0}^{\infty} \left\| \frac{T^k t^k}{k!} \right\| \le \sum_{k=0}^{\infty} \frac{a^k t_0^k}{k!} = e^{a t_o}$$

And so we can define

$$e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$$

without any problems. It then follows that $e^T \in L(\mathbb{R}^n)$ and $\left\| e^T \right\| \le e^{\|T\|}$ (why? explain).

$$e^{PTP^{-1}} = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{(PTP^{-1})^k}{k!} = P \lim_{n \to \infty} \sum_{k=0} \frac{T^k}{k!} P^{-1} = P e^T P^{-1}$$

If $PAP^{-1} = \operatorname{diag}[\lambda_1, \dots, \lambda_n]$ then

$$e^{At} = P \operatorname{diag}[e^{\lambda_1 t}, \dots, e^{\lambda_n t}] P^{-1}$$

If $S, T \in L(\mathbb{R}^n)$ with $ST = TS$ then $e^{S+T} = e^S e^T$. The the inverse of transformation e^T is $\left(e^T\right)^{-1} = e^{-T}$.

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \begin{pmatrix} \Re(\lambda^k) & -\Im(\lambda^k) \\ \Im(\lambda^k) & \Re(\lambda^k) \end{pmatrix} \\ &= \begin{pmatrix} \Re(e^\lambda) & -\Im(e^\lambda) \\ \Im(e^\lambda) & \Re(e^\lambda) \end{pmatrix} \\ &= e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix} \end{aligned}$$

and If $a = 0$ then e^A is a rotation of b radians.

If the eigenvalues of a matrix A are real and distinct then we can solve the equation $\dot{\mathbf{x}} = A\mathbf{x}$ using the following algorithm:

1. Compute eigenvalues $\lambda_1, \ldots, \lambda_n$
2. Compute eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$
3. Define $E = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ which is invertible since the eigenvalues are distinct. Let $D = \text{diag}\,[\lambda_1, \ldots, \lambda_n]$
4. Then $AE = ED$ or $A = EDE^{-1}$
5. Let $\mathbf{y} = E^{[} - 1]\mathbf{x}$. Then

   $$\dot{\mathbf{y}} = E^{=1}\dot{\mathbf{x}} = E^{-1}A\mathbf{x} = E^{-1}AE\mathbf{y} = D\mathbf{y}$$

   is decoupled.
6. Then a solution is

   $$\mathbf{y}(t) = \text{diag}\left[e^{-\lambda_1 t}, \ldots, e^{-\lambda_n t}\right]\mathbf{y}_0.$$

7. Going back to $\mathbf{x}$ we get

   $$\mathbf{x}(t) = E\,diag\left[e^{-\lambda_1 t}, \ldots, e^{-\lambda_n t}\right]E^{-1}\mathbf{x}_0$$

An algorithm for calculating e^{tA} for any square matrix A:

1. Compute eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$
2. Compute $a_1(t), \ldots, a_n(t)$ given by the recursive definition

   $$\begin{aligned} a_1(t) &= e^{\lambda_1 t} \\ a_k(t) &= \int_0^t e^{\lambda_k(t-s)} a_{k-1}(s)\,ds. \end{aligned}$$

3. Compute $A_1, \ldots, A_n$ given by

   $$\begin{aligned} A_1 &= I \\ A_k &= (A - \lambda_{k-1} I)A_{k-1} \end{aligned}$$

4. Then

   $$e^{tA} = a_1(t)A_1 + \cdots + a_n(t)A_n$$