MMNMBD HW 5

Daniel Gutierrez

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1 Proof of Laplacian of Circle graph

We need to show that u_k and v_k satisfy $Lu_k = \lambda_k u_k$ and $Lv_k = \lambda_k v_k$.

Let's start with $u_k(i) = \sin\left(\frac{2\pi ki}{n}\right)$.

For each vertex i:

$$(Lu_k)(i) = 2u_k(i) - u_k(i+1) - u_k(i-1)$$

Plugging in the definition of u_k:

$$(Lu_k)(i) = 2\sin\left(\frac{2\pi ki}{n}\right) - \sin\left(\frac{2\pi k(i+1)}{n}\right) - \sin\left(\frac{2\pi k(i-1)}{n}\right)$$

Using the angle addition and subtraction formulas for sine:

$$\sin\left(\frac{2\pi k(i+1)}{n}\right) = \sin\left(\frac{2\pi ki}{n} + \frac{2\pi k}{n}\right) = \sin\left(\frac{2\pi ki}{n}\right)\cos\left(\frac{2\pi k}{n}\right) + \cos\left(\frac{2\pi ki}{n}\right)\sin\left(\frac{2\pi k}{n}\right)$$

$$\sin\left(\frac{2\pi k(i-1)}{n}\right) = \sin\left(\frac{2\pi ki}{n} - \frac{2\pi k}{n}\right) = \sin\left(\frac{2\pi ki}{n}\right)\cos\left(\frac{2\pi k}{n}\right) - \cos\left(\frac{2\pi ki}{n}\right)\sin\left(\frac{2\pi k}{n}\right)$$

Summing these, we get:

$$\sin\left(\frac{2\pi k(i+1)}{n}\right) + \sin\left(\frac{2\pi k(i-1)}{n}\right) = 2\sin\left(\frac{2\pi ki}{n}\right)\cos\left(\frac{2\pi k}{n}\right)$$

Thus,

$$(Lu_k)(i) = 2\sin\left(\frac{2\pi ki}{n}\right) - 2\sin\left(\frac{2\pi ki}{n}\right)\cos\left(\frac{2\pi k}{n}\right) = 2\sin\left(\frac{2\pi ki}{n}\right)(1 - \cos\left(\frac{2\pi k}{n}\right))$$

We recognize the eigenvalue:

$$\lambda_k = 2 - 2\cos\left(\frac{2\pi k}{n}\right)$$

Therefore,

$$(Lu_k)(i) = \lambda_k u_k(i)$$

Similarly, for $v_k(i) = \cos\left(\frac{2\pi ki}{n}\right)$, the same steps apply due to the cosine angle addition and subtraction formulas. Thus, v_k is also an eigenvector with the same eigenvalue.

2 Part 2: Show that u_k and v_k have the Same Eigenvalue

We have already derived that:

$$(Lu_k)(i) = \lambda_k u_k(i)$$

$$(Lv_k)(i) = \lambda_k v_k(i)$$

with $\lambda_k = 2 - 2\cos\left(\frac{2\pi k}{n}\right)$.

Therefore, both u_k and v_k have the eigenvalue $\lambda_k = 2 - 2\cos\left(\frac{2\pi k}{n}\right)$, as required.

3 Plots

We now plot the eigenvectors u_k and v_k for $k \in \{0, 1, 2, 3\}$.

In the next page we can see the plot

4 Summary

We have shown that the functions $u_k(i) = \sin\left(\frac{2\pi ki}{n}\right)$ and $v_k(i) = \cos\left(\frac{2\pi ki}{n}\right)$ are eigenvectors of the graph Laplacian L of the circle graph C_n, and that they have the same eigenvalue $\lambda_k = 2 - 2\cos\left(\frac{2\pi k}{n}\right)$.

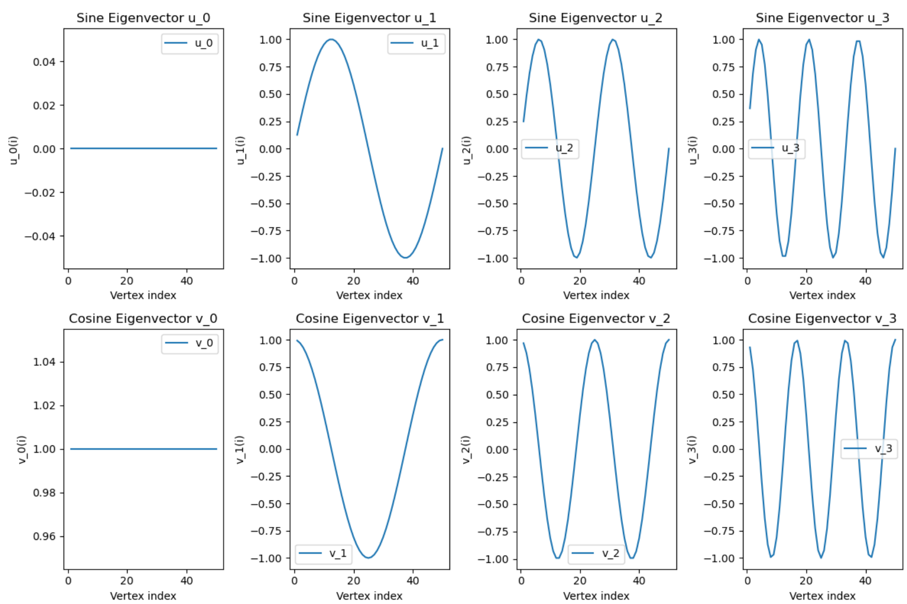

Figure 1: Plots of the eigenvectors u_k and v_k for $k \in \{0, 1, 2, 3\}$.