Options Pricing and BS Convergence

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Binomial Pricing Covergence

In this analysis we implement 3 pricing formulas for a European call option:

- Cox-Ross-Rubinstein
- Leisen-Reimer.
- Black-Scholes.

The first two are based on the discrete time approach and we proceed to evaluate the convergence of them to the Black-Scholes model, which is set on continuous time.
We find that the Leisen-Reimer adaptation converges with less time steps in between than the Cox-Ross-Rubinstein.

Inputs

All models share some of the same inputs, which in our case are presented in **table 1.** And the models CRR and L-R are Cox-Ross-Rubinstein and Leisen-Reimer respectively.
The only input that changes along models is that the Black-Scholes formula does not require the **n** timestep, because of the mathematical construction. And all models can be assumed to have the same dividend adjusted risk free rate **b** = risk free rate **r**, since we are assuming 0 dividends.

Binomial Pricing: Cox-Ross-Rubinstein

The Cox-Ross-Rubinstein (CRR) approach assumes that the probabilities of an asset of moving up and down are given by:

$$u = e^{\sigma\sqrt{\Delta t}}$$

Where:

- ***u*** is the probability of the asset of moving up.
- ***σ*** is the annualized volatility of the asset
- Δt is the time for which the u is calculated, so for each time step ***n*** that is implemented it would be

$$\Delta t = \frac{1}{n}$$

From this ***u*** we can get the inverse ***d:***

$$ud = 1; \;\; u = \frac{1}{d}$$

Table 1

Input	100	Meaning	Models
S	100	Spot Price	All
r	1.00%	Risk-free rate	All
vol	20%	Asset volatilty	All
T	1	Time in years	All
K	100	Strike Price	All
n	Range from 10 to 1,000	Timesteps	CRR, L-R
b	1.00%	Dividend adjusted Risk-free rate	L-R

Binomial Pricing: Cox-Ross-Rubinstein

The ***u*** and ***d*** are then used to also calculate the probability for each timestep of moving up ***p*** or moving down ***(1-p).***
The estimation of the probability of the asset moving up within each timestep is then:

$$\boldsymbol{p = \frac{e^{r\Delta t} - d}{u - d}}$$

Finally, the implementation of the CRR approach is simply estimate the payoff at each possible binomial combination of moving up or down. Then multiply this payoffs by the associated probabilities and discount them to the present period 0.

The formula used to estimate the call price for every n timesteps is presented below:

$$C = \frac{1}{R_f^n}\left[\sum_{j=a}^{n}\left(\frac{n!}{j!\,(n-j)!}\right)p^j(1-p)^{n-j}\left[u^j d^{n-j} S - K\right]\right]$$

Where:

- ***C*** is the price of the call at any time
- $\frac{1}{R_f^n}$ is the discount factor which uses the risk-free rate ***r***
- ***j*** represents the number of up movements or upticks in the stock price over the total number of periods ***n***
- ***S*** is the spot price
- ***K*** is the strike price

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Black-Scholes Pricing

After implementing the CRR pricing we compare it with the Black-Scholes approach, which is based on continuous time, and under a large ***n,*** the CRR model should converge to the Black-Scholes one.
The Black-Scholes formula is given by:

$$C(S_t, t) = N(d1)S_t - N(d2)Ke^{-r(T-t)}$$

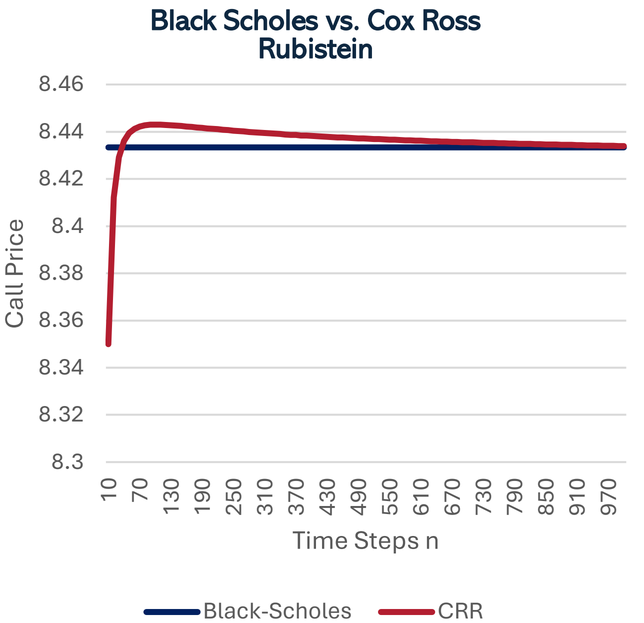
$$d1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)\right]$$

$$d2 = d1 - \sigma\sqrt{T-t}$$

Where:

- S_t is the spot price at time ***t***
- ***K*** is the strike price.
- ***r*** is the risk-free rate.
- ***T*** it the maturity.
- ***t*** is the time left in the option, or the current time if you will.
- ***σ*** is the volatility
- ***N(d1), N(d2)*** are the normal cumulative distribution functions of ***d1*** and ***d2*** respectively.

The Black Scholes formula was calculated using the inputs of table 1, as was the CRR method. The results are presented below:



We see that the CRR method seems to converge to Black Scholes after some 500-time steps n, which is not ideal. Nevertheless, the difference in prices is under 1% from the starting point and after 200 time steps the difference is just under 0.1%.

Binomial Pricing: Leisen-Reimer

Leisen and Reimer implemented a faster convergence to Black-Scholes in 1996. The novelty in their approach lies in the probability of the underlying moving up or down. In the CRR model, the probabilities are derived from the volatility of the asset, the risk-free rate and the time to maturity. While in the Leisen-Reimer method the probabilities are adjusted by using the first two moments of the binomial distribution to match those of a lognormal, which is assumed in the Black-Scholes model. Therefore, ***N(d1)*** and ***N(d2)*** are used again to converge at a faster rate to the Black-Scholes price.
The modification lies as follows:

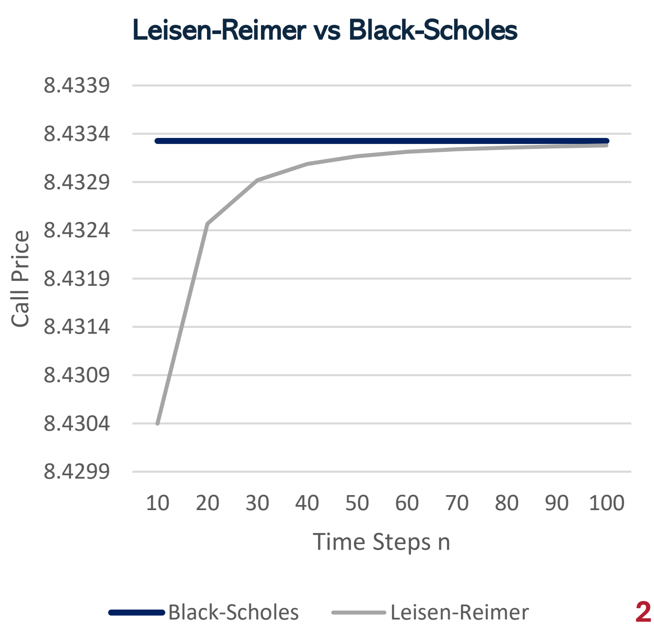
$$p = h^{-1}(d2)$$
$$p' = h^{-1}(d1)$$

Where h^{-1} is the Peizer-Pratt inversion, providing binomial estimates for the normal cumulative distribution function.
Then the probabilities of moving up and down are recalculated as follows:

$$u = e^{(r-q)\Delta t}\frac{p'}{p}$$

$$d = e^{(r-q)\Delta t}\frac{1-p'}{1-p}$$

And the calculation of the call price is calculated with the approach used in the CRR method with the updated ***u*** and ***d***



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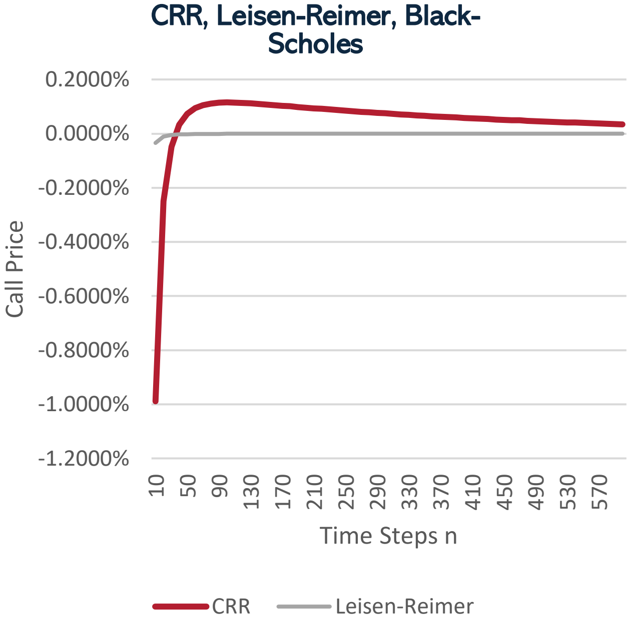
Leisen-Reimer Convergence

As seen in the previous plot, the convergence to the Black-Scholes price is evident within the first 100-time steps.

Leisen-Reimer vs CRR

While the CRR method converges to Black-Scholes, the Leisen-Reimer adjustment converges at a faster rate. In the following plot we can see the price difference vs the Black-Scholes pricing for both approaches.



The plot shows the percentual price difference of each pricing method at each time step count ***n***. We can see that the Leisen-Reimer approach keeps a smaller distance to Black-Scholes even from the beginning of the implementation, while the CRR approach goes from almost 1% difference at the first time steps

Conclusions

The Cox-Ross-Rubistein (CRR) application of pricing binomial options shows a small difference in the price when compared to the Black-Scholes model, being a discrete approach vs a continuous one. Nevertheless, we see that the Leisen-Reimer approach shows a greater rate of convergence to the Black-Scholes formula. This is because of the implementation of the Peizer-Pratt inversion.

All the functions were implemented in a MS Excel workbook, via VBA. The simplest implementation in terms of code is the Black-Scholes, because it does not need a loop to price for all time steps, while the other to need loop cycles.

The Leisen-Reimer used in this report truncates the zeros of the pricing for a faster rate of convergence.

Besides European Call Options, one could see American Call Options or even Put Options both European and American. The pricing for all is to be found in the workbook.

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