Analysis of the Swing of a Pendulum **12/5/2023**

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Introduction:

Lab 1:

Pendulums are a fundamental illustration of damped harmonic oscillation. As pendulums swing, damping forces cause the dissipation of their mechanical energy into the surroundings. These forces are often attributed to air resistance and friction, but the effect of these damping forces differ between every unique model, resulting in every distinct setup having slight differences in motion. This report explores the effects of these unknown damping forces through a comparison of real, experimental data with the provided theoretical equations and models to uncover the intricate interplay between theory and reality.

With our experimental setup, the motion between period [s] and angle with respect to the vertical [rad] was found to be a second order polynomial in the form

$$T = ax^2 + bx + c \tag{1}$$

Where $a = 0.110 \pm 0.004\ [s \cdot m^{-2}]$, $b = -0.001 \pm 0.004\ [s \cdot m^{-1}]$, and $c = 1.590 \pm 0.002\ [s]$.

Through the consideration of frame error, it was found that the ax^2 term could be ignored and the period-length relationship could be called constant for $|x|[m] < 0.002$ [rad]. A second analysis that instead considered residuals found the range to be $|x|[m] < 0.0193\ [rad]$.

This report also explores the relationship between the pendulum's amplitude [rad] and time [s], which is theoretically modeled by the equation

$$\theta(t) = ae^{-t/\tau} \tag{2}$$

Applying this model to the experimental data yielded parameter values of $a = 0.997 \pm 0.001\ [rad]$, and $\tau = 82 \pm 1\ [s]$. However, upon analysis, it was evident that the data exhibited a poor fit to the model. This discrepancy was attributed to the inclusion of large angles, reaching to greater than 70º, whereas the model is specifically designed for small angles.

Thus, to better fit the model for a more accurate determination of the Q factor, significant angles (≈ greater than 0.158 [rad], 0.158 being an arbitrary value) were disregarded and the relationship between the amplitude and time was reformulated. This refined relationship of the form $\theta(t) = ae^{-t/\tau}$ had parameter values $a = 0.1590 \pm 0.0004\ [rad]$, and $\tau = 157 \pm 1\ [s]$. This equation (formulated through the inclusion of only small angle data) was used to characterize the pendulum's damping effect, yielding Q-factor estimates of around 296 ± 1 and 304 ± 2 via two distinct methods.

This leads to the second part of the lab, which explores the factor of length. With our model, the relationship between period and length was found to be fit the theoretically proposed model

$$T = kL^n \tag{3}$$

with parameter values $n = 0.52 \pm 0.01$ and $k = 2.05 \pm 0.03\ [s \cdot m^{-0.52}]$. These parameter values were very close to the predicted values of $n = 0.5$ and $k = 2\ [s \cdot m^{-0.5}]$.

In part 2 of the lab, a strong relationship between Q factor and length was also found, which was a negative parabolic curve represented by

$$Q = -64.8 + 950L - 1240L^2 \tag{4}$$

Throughout the lab, data was collected using iPhone 13 video footage (1920 pixels x 1080 pixels). The videos were recorded in 240 fps (slow motion) to minimize frame error, and the camera lens was positioned directly in line with the pendulum's pivot to reduce perspective issues that could affect results. The videos were then processed in Tracker, a physics data collection software, where the position and angle of the pendulum were manually tracked frame by frame with a ± 5 pixel error. Subsequently, the information was exported to Google Sheets and a Python best-fit modeling program to generate graphs and draw conclusions.

Methods and Procedure:

The pendulum setup underwent various changes. The first model consisted of a light, non-stretchable string (length 56.02 ± 0.05 cm) securely tied to a rubber ball (diameter

5.01 ± 0.05 cm) using two perpendicularly crossed rubber bands.

Figure 1., Close Up of Ball in Setup One

This model was quickly modified to address the elliptical 3D motion induced by the single string's unrestricted freedom. The updated design implemented a second string, identical in length, positioned 8.00 ± 0.05 [*cm*] apart from the first one at the pivot (± 0.05 [*cm*] being the measurement error). The second string was affixed to the ball at the same point as the first around the rubber bands, restricting the ball's swing to a vertical plane in 2D motion, thus decreasing the data's imprecision. Additionally, a test for symmetry was conducted by measuring the two angles created between the two strings and the vertical and adjusting the lengths of rope so the angles were equal to a ± 1^{o} uncertainty.

Figure 2. - Model 1 (left) and 2 (right) Comparison

With our second model, the camera setup was also optimized to achieve direct eye-to-eye alignment between the camera lens and the pendulum's pivot, done to a ± 1 [*cm*] uncertainty. This alignment was facilitated by the use of camera grid lines, carefully matched with the ground and table surface, guaranteeing parallel orientation to the pendulum's motion plane. These considerations helped enhance the accuracy of the data measurements, minimizing the impact of error and imprecision that could be caused from camera angle skewing and distortion.

Utilizing the improved setup, 240 fps videos were then captured to gather data for the *Period [s] vs. Angle [rad]* graph. Each of the five videos documented a distinct range of maximum angles, representing the angles between the string and vertical when the pendulum's velocity hit zero. For instance, the first video recorded the pendulum's oscillation between maximum angles of magnitudes 57.28^{o} to 34.27^{o}, while the second video recorded the pendulum's oscillation between max angles of magnitudes of 70.00^{o} to 54.84^{o}. Given the videos were recorded at 240 fps, there was a video capture error of ± $1/480$ seconds.

The videos were then imported into "Tracker", a physics software designed for data tracking, offering diverse information related to the pendulum's oscillation. One of Tracker's functions is allowing the tracking of a specified object's position and angle concerning a user-defined axis. This process was employed to determine the ball's position, time, and angle at the pendulum's outermost points, where velocity reaches zero. This data was used to create the Period vs Angle and Amplitude vs Time graphs.

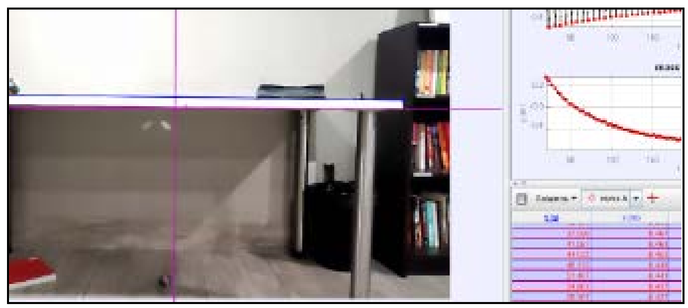

Figure 3. A Snippet of The Tracker Setup – The purple cross serves as the reference axes for gathering data on pendulum angle and position. Data is displayed on the right side of the video display.

For the second part of the lab, only minor adjustments were made to the setup. This is because our Lab 1 setup did not present any significant error, meaning no substantial alterations were necessary. However, a notable addition is the implementation of a protractor, set at the pendulum's pivot. This modification was made specifically for the Q-factor versus length graph to ensure a uniform drop angle could be made for the pendulum across varying lengths to obtain precise Q-factor measurements.

To acquire data for the second part of the lab, the string was adjusted to 12 varying lengths, precisely measured using a ruler with ± 0.5 [*mm*] uncertainty. Then, for each length, a four minute, 240 FPS video was taken of the pendulum's oscillation from a release angle of 25 ± 1^{o}. The recorded videos were then analyzed using Tracker, where the pendulum's amplitude was manually tracked at its outermost swing points. By averaging the time taken for each complete oscillation (from one outermost point to the same outermost point), the average period of the pendulum's drop was determined for each length. Additionally, the Q-factor for each trial was calculated using the formula $Q = \pi \frac{\tau}{T}$, where τ was found using a best-fit model python program. The resulting length, Q-factor, and period values were used to construct graphs depicting the relationships between Q-factor and Length, as well as Period and Length.

Results and Data Analysis:

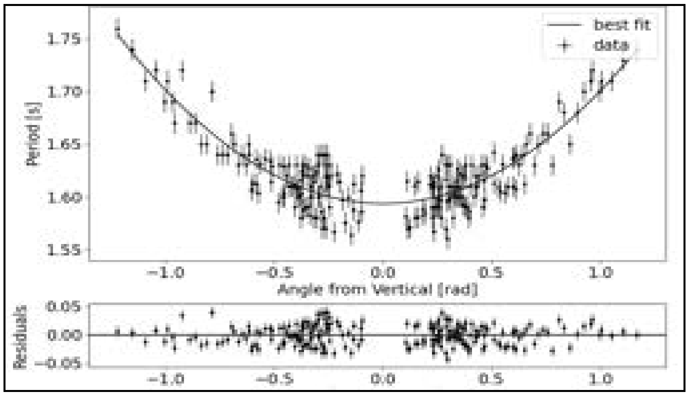

Figure 4. Period [s] vs. Angle [rad]

Trend line is of the form $T = ax^2 + bx + c$, *where* $a = 0.110 \pm 0.004\ [s{\cdot}m^{-2}]$, $b = -0.001 \pm 0.004\ [s{\cdot}m^{-1}]$, *and* $c = 1.590 \pm 0.002$ *(values from python best-fit modeling program).*

Figure 4. illustrates the correlation between the period [s] and angle [rad] of the pendulum, displaying a compelling non-linear trend resembling a parabola. This deviation from the predicted model suggests a dependency between angle and period, where greater magnitudes of angles correspond to longer periods, and smaller magnitudes of angles smaller periods. Possible explanations for this could be that the effects of resistive forces (drag and friction) cause a non-linear shape, or perhaps the shape is simply a result of the inherent physics of a pendulum oscillating under gravity. The graph also disagrees with the prediction that period, T, will equal $2\sqrt{L}$, as T is clearly not a constant value.

This analysis also addresses the range of amplitudes that is 'small enough' that the value of $a = 0.110 \pm 0.004$ can be ignored and the relationship is constant. For this model, the ax^2 term can be ignored when the range of amplitudes is smaller than the period uncertainty. The period uncertainty in this case is equivalent to the time uncertainty of $\pm 1/480$ or ± 0.002 meaning that for this specific model, the value of ax^2 can only be ignored when $|x| < 0.002$.

An alternative approach to determine when ax^2 can be disregarded involves examining the residuals of the graph. It is evident in Figure 4. that for amplitudes close to the amplitude of zero, there is a wider range of period values. An increased spread and range of residuals indicates a less pronounced parabolic relationship at the corresponding angles. Therefore, ax^2 can also be considered negligible for the range of x's that is less than the average value of the magnitude of the residuals, or equivalently when $|x| < 0.0193$.

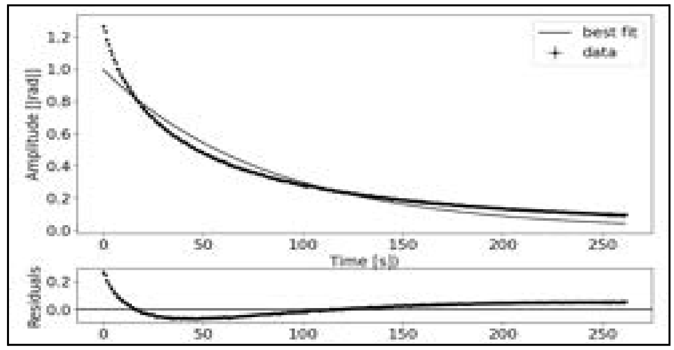

Figure 5. Amplitude [rad] vs. Time [rad]

Trend line in the theoretical form $ae^{-t/\tau}$, *where* $\tau = 82 \pm 1\ [s]$ *and* $a = 1.00 \pm 0.01\ [rad]$

Figure 5. shows the relationship between the pendulum's amplitude *[rad]* versus time *[s]*. From the graph, it is evident that the trendline provided by the python best-fit program does not fit the experimental data's distinct shape well. At greater angles (≈ θ > 0.43 [rad]), the experimental data's amplitude decays at a much faster rate than the theoretical best-fit model, and at smaller angles (≈ θ < 0.43 [rad]), the amplitude decays at a slower rate.

A possible explanation for this may be that for greater amplitudes, the ball travels a greater distance per cycle than the theoretical model accounts for (if it accounts for it at all), meaning that there is more work, Fd, being done by drag and friction forces. This causes more energy to be dissipated into the surroundings than expected, resulting in relatively more decay.

Relating to the decay is the Q-factor, which was found by manually counting the number of oscillations until the amplitude was 20% of the original value and then multiplying that value by 2. This method produced a Q factor of $68 \pm 1 \times 2 = 136 \pm 1$ (uncertainty of ± 1 from possible errors in counting). However, this value is highly inaccurate as the Q-factor is a value whose accuracy depends on the experimental data following the trendline form precisely, as the Q-factor value is based off of the theoretical equation $ae^{-\frac{t}{\tau}}$.

Thus, a second, better method utilizing the small angle approximation was used to find a more accurate Q-factor. With this method, large amplitudes that were greater than 0.158 [*rad*] were simply disregarded and removed from the graph. The small amplitude points that remained were then shifted left across the time axis so that the leftmost point had a time value of zero. This new graph of Small Angle Amplitudes vs Time is depicted below in *Figure 6.*

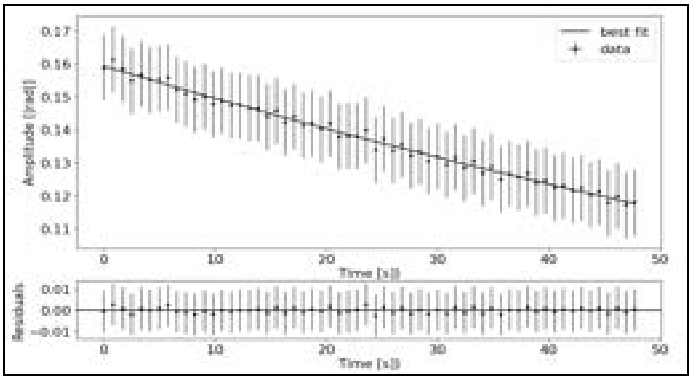

Figure 6. Small Amplitudes [rad] vs. Time [rad]

Trend line is in the theoretical form $ae^{-\frac{t}{\tau}}$, *with* $a = 0.159 \pm 0.004$ [*rad*], *and* $\tau = 157 \pm 2$ [*s*]. Note that x-error bars, equivalent to the frame uncertainty, are shown but are so small that they are not displayed. Amplitude error bars are equal to 1° (0.02 radians).

Using the equation $Q = \pi\frac{\tau}{T}$, where $\tau = 157 \pm 2$ *[s]* and *T* is the average period value of the data points in *Figure 6.*, or *1.620 ± 0.002 [s]*, the new Q-factor was found to be $Q = \pi\frac{157 \pm 2}{1.62 \pm 0.002}$, or $Q = 304.463 \pm 3$.

The second method used to calculate the *Q* factor was counting the number of oscillations until the amplitude became *69%* of the original amplitude of *0.162 ± 0.02 [rad] (*uncertainty of 1° from Tracker) and then multiplying that value by 8.47 (this was derived from the equation $e^{-\pi/8.47} \sim 69\%$). Using this method, the *Q* factor came out to be $\frac{70 \pm 1}{2} \times 8.47 = 296.45 \pm 1$, ± 1 accounting for potential error in counting.

The two Q-factor values do not agree with each other well as they are not within the uncertainty of each other. In future experiments, this lack of agreement could be minimized by taking a larger range of data to analyze, or finding another way to measure Q-factor derived from a better-fitting model than $ae^{-\frac{t}{\tau}}$.

In the second part of the lab, the relationships of period [*s*] vs length [*m*] and Q factor vs. length [*m*] were found.

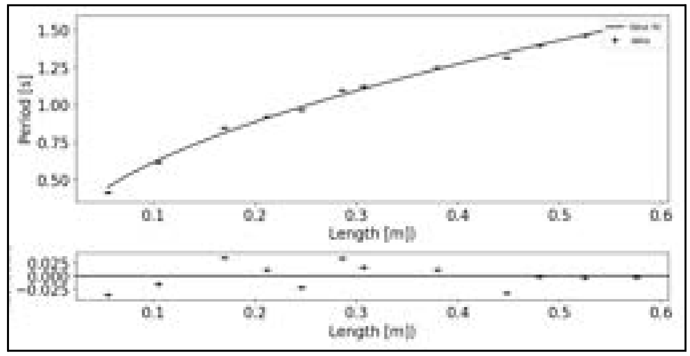

Figure 7. Period [s] Length [m] Graph

Best fit line in the predicted form $T = kL^n$, *with* $k = 2.05 \pm 0.03$ [$s \cdot m^{-0.52}$], *and* $n = 0.52 \pm 0.01$, *with uncertainties from python best fit model. Note that error bars are included; they are just very small. x-error is 0.0005 [m] (ruler measuring error), y-error is 0.002 [s] (frame rate error)*

Figure 7. depicts the strong relationship found between period [s] and length [m]. This relationship follows the predicted $T = 2L^{0.5}$ closely, as the calculated parameter values are within $\frac{(2.05-2) \pm 0.03}{2} = 2.5 \pm 1.5\%$ of the predicted *T* value of 2, and $\frac{(.52-.50) \pm 0.01}{0.50} = 4 \pm 2\%$ of the predicted *n* value of 0.5. This period vs length relationship is also plotted on a log-log scale, shown in Figure 8. below.

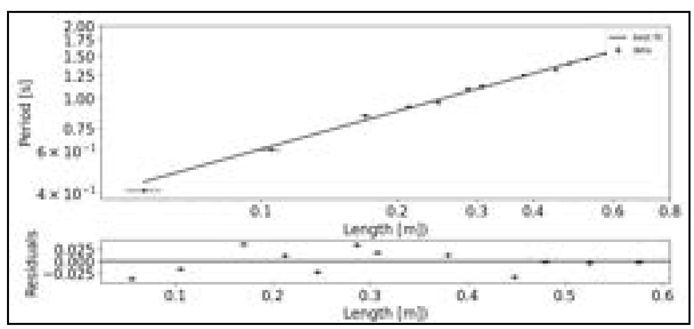

Figure 8. Period [s] Length [m] Log-Log Graph

A depiction of the same data as Figure 7. except in log-log form, helping normalize the curve into a linear form where line of best fit is approximately $y = (0.52 \pm 0.01)x + \log_{10}(2.05 \pm 0.03)$

Figure 7. and 8. both show the strong relationship between period and length. As length increases, period increases as well. This suggests a dependency. This relationship itself may be a result of gravity's impact on the pendulum. It is possible that because gravity acts as a restoring force, there is a greater lever arm for gravity to act on for longer

lengths, meaning greater torque, which results in greater angular acceleration and consequently, a longer period.

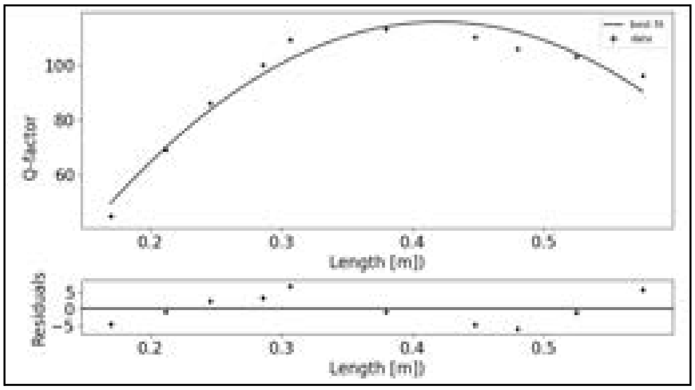

Figure 9. Q-factor vs Length [m] Graph
Best fit line in the form $ax^2 + bx + c$, where $a = -650 \pm 50$ [m^{-2}], $b = 530 \pm 40$ [m^{-1}], and $c = 2 \pm 6$ Note that Error Bars are included; they are just very small.

Figure 9. displays the dependent relationship between Q-factor *[rad]* and Length *[m]*. Note that "length" in this context does not represent the length of the actual strings. Since the setup involves two strings positioned 8 ± 0.05 cm apart at the pivot, "length" refers to the distance between the center of mass of the ball and the midpoint between the two strings at the pivot.

Regarding the actual data displayed in Figure 9., there is a significant relationship between Q-factor and period. As the length increases, the Q-factor increases at a slowing rate until it reaches a max at approximately 0.4 m — the apparent optimal length for minimal decay. Then, the Q-factor starts decreasing, although seemingly at a slower magnitude of rate than its ascent.

However, this application of a parabolic model, while useful for a basic understanding of the relationship, oversimplifies the intricacies within the data. To better discern the many patterns within the dataset, a different best fit line should be used. This is done in Figure 10., which depicts the original black best fit line that the python best fit model proposes, and a green best fit line that has been drawn. While there is not a specific equation for this green line, it is statistically a better best fit, as it has smaller residual values than the black best fit line (this is visually evident, although the green curves residuals are not plotted). This green curve mimics the shape of a parabola until reaching its maximum value, where it then transitions into a near linear line with a negative slope.

The green line reveals more insight into the data than the black best fit line, as it shows that after the optimal length for minimal Q-factor is reached, the Q-factor starts decreasing at a somewhat constant rate; not at a faster decreasing rate, which is what the parabola suggests. After all, it does not make sense for the Q-factor to be zero at length 0.8 m (equivalent to the parabola's rightmost x-intercept that is off the graph). Despite this, more data points should be found to confirm this hypothesis, as it is impossible to extrapolate accurately with just 4 points to the right of the peak Q-factor value.

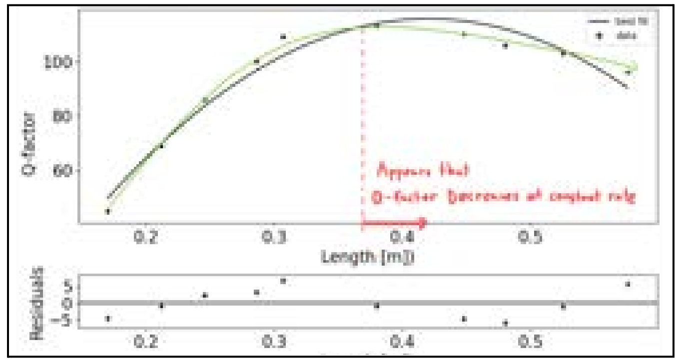

Figure 10. Annotated Q-factor vs Length Graph
Depicts the same data as Figure 9., except with the implementation of an additional green best fit line, which is not represented by any equation.

Another interesting aspect about Figure 9. is the c term of the parabolic best fit equation, $c = 2 \pm 6$, representing the y-intercept, which is very close to (0, 0). The graph itself, Figure 9., visually appears like it will cross the origin. It is important to analyze whether this is a coincidence or not to better understand the relationship – in other words, as the length decreases to almost a negligible amount, does the Q-factor actually decrease to zero?

To answer this question, three more tests were conducted. Using the setup, the length (which includes the radius of the ball) was first adjusted to 10.50 ± 0.05 [cm]. The time it took for the the amplitude to reach 4% from an initial value of 70º ± 1º was then recorded. Two more drops were then conducted in which the lengths were reduced to 5.50 ± 0.05 [cm] and then 3.5 ± 0.05 [cm]. Within each of these decreases in length, there was a dramatic drop in the time taken to reach 4%. For the first drop it took 52.351 ± 0.002 [s], the second 22.141 ± 0.002 [s], and the last 9.231 ± 0.002 [s] (± 0.002 [s] accounting for frame rate uncertainty). The Q-factors were then found by dividing the times by the average period of the pendulum over the recorded interval, since $Q = \frac{time\ to\ reach\ 4\%\ of\ amplitude\ [s]}{avg\ period\ over\ time\ interval\ [s]}$. This yielded Q-factors of 68.8 ± 0.1, 33.0 ± 0.2, and 21.5 ± 0.2 respectively (uncertainty propagated using an online calculator [2]). From these three values, it is evident that the Q-factor is approaching zero as the length becomes very small. A visual representation of this can be viewed here [3]. However, note that in the future, many more tests with smaller lengths should be conducted to confirm this pattern, as three tests are not enough.

Let's theorize the cause of this identified trend by considering conservation of energy, which tells us that the initial energy—and therefore maximum energy—in a pendulum system is equivalent to its initial gravitational potential energy, mgh_o. Let's also define h of the pendulum as the distance between the lowest point of the pendulum and the height it is dropped from. Thus, for a very short string system compared to a long string system, h_o (initial height) would be smaller and therefore the initial gravitational energy (and energy of the system) would comparably be less as well. However, many damping forces are independent of length. For example, friction at the pivot between tape and string minimally relies on length, instead depending on mass ($F_f = F_n \mu = mg\mu$), and external resistive wind forces are not dependent on length, but on surface area and velocity (the change in surface area from changes in string length is very small and likely negligible). Thus, with very little energy in short-string systems, it becomes very easy for damping to occur since any external force could take away a very large percent of the system's energy. In fact, if we took the limit as the length approached zero, then h, and thus mgh, would approach zero as well. This means that for an infinitesimally small length, any external damping force that is independent of string length would have an extremely significant impact on the system's energy, likely reducing it to zero immediately, resulting in an infinitesimally small Q-factor. This essentially tells us that in the experimental world (where damping forces exist), Q factor would be zero with zero length. This analysis supports the shape of the graph. (Note that for the actual setup, zero length is impossible as we can not reduce the radius of the ball.

Another compelling aspect of Figure 9. is the vertex point, which suggests the existence of an "optimal" length for maximum Q-factor. This optimal length is likely a result of the interplay between multiple factors, two significant ones possibly being slack, and initial energy. If the length of the string is "too long," then the system may experience more decay from damping forces, as there would be more resistive work done by air resistance due to the longer distance covered by the longer swings. On the other hand, if the length is too short, then there is much less initial energy in the system, meaning that damping forces can more easily take away large portions of the system's energy (as explored in the paragraph before). Both these cases result in reduced Q-factor, giving rise to the existence of an optimal length for max Q-factor that is neither "too short" or "too long," but a perfect balance between the two extremums.

Conclusion:

The exploration of the physics of a pendulum revealed various insights and disagreed with multiple of the theoretical predictions. In particular, there was a non-linear, parabolic relationship found between amplitude and period that disagreed with the prediction that there would be a constant relationship. In terms of amplitude versus time, it was revealed that there were perhaps external damping forces working on the pendulum that made the provided best-fit model of $ae^{-\frac{t}{\tau}}$ a subpar trend-line. The discrepancy between experimental and theoretical results suggests that there are multiple external factors that have not been accounted for pulling strings in the background. Regarding period vs length, a strong relationship was found that followed the theoretical prediction. Q-factor and length also followed a unique relationship that was complex and had multiple components to explore. Overall, the greatest source of uncertainty was likely from the measurement of the angles, which was ± 1°. This uncertainty was much greater than the uncertainty in frame rate or length measurement. What makes uncertainty in angle very significant is that each graph relates to angle measurements in some sort of way, meaning that angle uncertainty affects all the results.

Overall, this lab investigation shed light on dynamics of a damped harmonic oscillator for one unique experimental setup, and calls for further exploration to deepen understanding. In future experiments, some improvements may be made. For part 1 of the lab, shorter lengths of string should be used, which would reduce potential slack and decrease initial energy loss. For part 2 of the lab, there are small error bars (which is good) but large gaps between data, meaning that more data points should be made to make the relationships more statistically robust and reliable. Furthermore, for the Q-factor and length relationship, a longer length of string should be implemented to explore a broader range of lengths. This would help us determine the "true" trend of Q-factor as length increased beyond the optimal length, which should be investigated in the future.

Appendix:

[1] Douglas Brown. Tracker 5.0 help
https://physlets.org/tracker/tracker_help.pdf

[2] Propagation of Uncertainty Calculator
https://nicoco007.github.io/Propagation-of-Uncertainty-Calculator/

[3] Daniel Hong. Visual Representation of Differences in Pendulum Decay at Small Lengths
https://docs.google.com/document/d/1FtnU6n5YnspXsdtnTTLfZnyYrFXwJv25G26-L-qokqc/edit?usp=sharing