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PH240F
Assignment 2
2/23/17

Question 1. Map Functions

1. Express the recombination fraction θ in terms of the chiasma count distribution $\{\pi_n\}$.

Let R be an indicator variable as defined below:

$$R = \begin{cases} 1 & \text{if a particular chromatid is recombinant} \\ 0 & \text{otherwise} \end{cases}$$

Let X be the number of chiasmata between two loci A and B.

1. No-chromatid-interference (NCI)

$$\theta = \Pr(R=1) = \sum_{n=1}^{\infty} \Pr(R=1 \mid X=n) \Pr(X=n)$$

Since a particular chromatid is recombinant only when it is involved in an odd number of chiasmata, I can write the following

$$\theta = \Pr(R=1) = \sum_{n=1}^{\infty} \Pr\left(\begin{matrix}\text{chromatid is involved in} \\ \text{odd \# of chiasmata}\end{matrix} \,\middle|\, X=n\right) \Pr(X=n)$$

I now define a new random variable Y_i:

$$\text{Let } Y_i = \begin{cases} 1 & \text{if the chromatid was involved in the } i^{th} \\ & \text{chiasmata event} \\ 0 & \text{otherwise} \end{cases}$$

$$\theta = Pr(R=1) = \sum_{n=1}^{\infty} Pr\left(\sum_{i=1}^{n} Y_i \text{ is odd} \mid X=n\right) Pr(X=n)$$

$$= \sum_{n=1}^{\infty}\Bigg[Pr\left(\sum_{i=1}^{n} Y_i \text{ is odd} \,\Big|\, \sum_{i=1}^{n-1} Y_i \text{ is even}\right) Pr\left(\sum_{i=1}^{n-1} Y_i \text{ is even}\right)$$
$$+ Pr\left(\sum_{i=1}^{n} Y_i \text{ is odd} \,\Big|\, \sum_{i=1}^{n-1} Y_i \text{ is odd}\right) Pr\left(\sum_{i=1}^{n-1} Y_i \text{ is odd}\right)\Bigg] Pr(X=n)$$

$$= \sum_{n=1}^{\infty}\Bigg[Pr\left(Y_n=1 \,\Big|\, \sum_{i=1}^{n-1} Y_i \text{ is even}\right) Pr\left(\sum_{i=1}^{n-1} Y_i \text{ is even}\right)$$
$$+ Pr\left(Y_n=0 \,\Big|\, \sum_{i=1}^{n-1} Y_i \text{ is odd}\right) Pr\left(\sum_{i=1}^{n-1} Y_i \text{ is odd}\right)\Bigg] Pr(X=n)$$

Since each of the four pairs of non-sister chromatids is equally likely to be involved in a chiasma, independent of strand involvement in other chiasmata, the expression simplies as follows:

$$\theta = Pr(R=1) = \sum_{n=1}^{\infty}\left[\frac{1}{2} Pr\left(\sum_{i=1}^{n-1} Y_i \text{ is even}\right) + \frac{1}{2} Pr\left(\sum_{i=1}^{n-1} Y_i \text{ is odd}\right)\right] Pr(X=n)$$

Note that $Pr\left(\sum_{i=1}^{n-1} Y_i \text{ is even}\right) + Pr\left(\sum_{i=1}^{n-1} Y_i \text{ is odd}\right) = 1$

Since these two events are complementary. Hence the following simplification:

$$\theta = Pr(R=1) = \sum_{n=1}^{\infty} \frac{1}{2} Pr(X=n)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} \pi_n$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \pi_n$$

$$\boxed{\theta_{NCI} = Pr(R=1) = \frac{1}{2}(1-\pi_0)}$$

This is Mather's formula for map functions under the assumptions of NCI.

1. (continued) Express the recombination fraction θ in terms of the chiasma count distribution $\{\pi_n\}$.

2. Complete-negative-chromatid-interference (Neg CI)

Under the assumptions of Neg CI, we have

$$\theta = \Pr(R=1 | X=n) = \begin{cases} 0, & n=2m \text{ (even)} \\ 1/2, & n=2m+1 \text{ (odd)} \end{cases}$$

So we can write the following:

$$\theta = \Pr(R=1)$$

$$= \sum_{n=1}^{\infty} \Pr(R=1|X=n)\Pr(X=n)$$

$$= \sum_{m=0}^{\infty} \Pr(R=1|X=2m+1)\Pr(X=2m+1)$$

$$= \sum_{m=0}^{\infty} \frac{1}{2}\Pr(X=2m+1)$$

$$= \sum_{m=0}^{\infty} \frac{1}{2}\pi_{2m+1}$$

$$\boxed{\theta_{NegCI} = \frac{1}{2}\sum_{m=0}^{\infty}\pi_{2m+1}}$$

1. (continued) Express the recombination fraction θ in terms of the chiasma count distribution $\{\pi_n\}$.

3. Complete-positive-chromatid-interference (Pos CI)

Under the assumptions of Pos CI, we have

$$Pr(R=1|X=n) = \begin{cases} 0, & n=4m \\ 1/2, & n=4m+1 \\ 1, & n=4m+2 \\ 1/2, & n=4m+3 \end{cases}$$

Hence, we can write the following:

$$\theta = Pr(R=1) = \sum_{n=1}^{\infty} Pr(R=1|X=n)\, Pr(X=n)$$

$$= \sum_{m=0}^{\infty} Pr(R=1|X=4m+1)\, Pr(X=4m+1)$$

$$+ \sum_{m=0}^{\infty} Pr(R=1|X=4m+2)\, Pr(X=4m+2)$$

$$+ \sum_{m=0}^{\infty} Pr(R=1|X=4m+3)\, Pr(X=4m+3)$$

$$= \sum_{m=0}^{\infty} \frac{1}{2} Pr(X=4m+1) + \sum_{m=0}^{\infty} Pr(X=4m+2) + \sum_{m=0}^{\infty} \frac{1}{2} Pr(X=4m+3)$$

$$= \frac{1}{2}\left[\sum_{m=0}^{\infty} \pi_{4m+1} + \sum_{m=0}^{\infty} \pi_{4m+3}\right] + \sum_{m=0}^{\infty} \pi_{4m+2}$$

Note that $\sum_{m=0}^{\infty} \pi_{4m+1} + \sum_{m=0}^{\infty} \pi_{4m+3} = \sum_{m=0}^{\infty} \pi_{2m+1}$ because we are adding up the probabilities when number of chiasmata events are odd. Hence,

$$\boxed{\theta_{PosCI} = \frac{1}{2} \sum_{m=0}^{\infty} \pi_{2m+1} + \sum_{m=0}^{\infty} \pi_{4m+2}}$$

This is equivalent to

$$\theta_{PosCI} = \frac{1}{2} Pr(X \text{ is odd}) + Pr(X \text{ is even but not divisible by } 4)$$

Question 1. Map Functions

2. For a Poisson chiasma process, derive a simple expression for the recombination fraction θ.

Given interval I with genetic map length d Morgans (M), under the Poisson chiasma process,

$$P(N(I)=n) = \pi_n = \frac{e^{-(\lambda d)}(\lambda d)^n}{n!}$$, where λ is the rate of a chiasma event taking place

1. NCI:

$$\theta_{NCI} = \frac{1}{2}(1-\pi_0)$$

$$\pi_0 = \frac{e^{-(\lambda d)}(\lambda d)^0}{0!} = e^{-(\lambda d)}$$

So $\theta_{NCI} = M(d) = \frac{1}{2}(1-e^{-(\lambda d)})$, which is the Haldane map function.

Question 1. Map Functions

2. (continued)

2. NegCI:

$$\theta_{NegCI} = \frac{1}{2}\sum_{m=0}^{\infty} \pi_{2m+1}$$

$$= \frac{1}{2}\sum_{m=0}^{\infty} \frac{e^{-(\lambda d)}(\lambda d)^{2m+1}}{(2m+1)!}$$

$$= \frac{1}{2} e^{-(\lambda d)} \sum_{m=0}^{\infty} \frac{(\lambda d)^{2m+1}}{(2m+1)!}$$

$$= \frac{1}{2} e^{-(\lambda d)} \left[\frac{(\lambda d)^1}{1!} + \frac{(\lambda d)^3}{3!} + \frac{(\lambda d)^5}{5!} + \cdots \right]$$

$$= \frac{1}{2} e^{-(\lambda d)} \left\{ \frac{2}{2} \left[\frac{(\lambda d)^1}{1!} + \frac{(\lambda d)^3}{3!} + \frac{(\lambda d)^5}{5!} + \cdots \right] \right\}$$

$$= \frac{1}{2} e^{-(\lambda d)} \frac{1}{2} \left[\frac{2(\lambda d)^1}{1!} + \frac{2(\lambda d)^3}{3!} + \frac{2(\lambda d)^5}{5!} + \cdots \right]$$

$$= \frac{1}{4} e^{-(\lambda d)} \left[\frac{(\lambda d)^1}{1!} + \frac{(\lambda d)^1}{1!} + \frac{(\lambda d)^3}{3!} + \frac{(\lambda d)^3}{3!} + \cdots \right]$$

Question 1. Map Functions

2. (continued)

$$\theta_{NegCI} = \frac{1}{4} e^{-(\lambda d)} \left\{ \left[\frac{(\lambda d)^1}{1!} + \frac{(\lambda d)^1}{1!} + \frac{(\lambda d)^3}{3!} + \frac{(\lambda d)^3}{3!} + \cdots \right] + \left[\frac{(\lambda d)^0}{0!} - \frac{(-\lambda d)^0}{0!} + \frac{(\lambda d)^2}{2!} - \frac{(-\lambda d)^2}{2!} + \cdots \right] \right\}$$

$$= \frac{1}{4} e^{-(\lambda d)} \left\{ \left[\frac{(\lambda d)^0}{0!} + \frac{(\lambda d)^1}{1!} + \frac{(\lambda d)^2}{2!} + \frac{(\lambda d)^3}{3!} + \cdots \right] - \left[\frac{(-\lambda d)^0}{0!} - \frac{(\lambda d)^1}{1!} + \frac{(-\lambda d)^2}{2!} - \frac{(\lambda d)^3}{3!} + \cdots \right] \right\}$$

$$= \frac{1}{4} e^{-(\lambda d)} \left\{ \left[\frac{(\lambda d)^0}{0!} + \frac{(\lambda d)^1}{1!} + \frac{(\lambda d)^2}{2!} + \frac{(\lambda d)^3}{3!} + \cdots \right] - \left[\frac{(-\lambda d)^0}{0!} + \frac{(-\lambda d)^1}{1!} + \frac{(-\lambda d)^2}{2!} + \frac{(-\lambda d)^3}{3!} + \cdots \right] \right\}$$

$$= \frac{1}{4} e^{-(\lambda d)} \sum_{n=0}^{\infty} \left(\frac{(\lambda d)^n}{n!} - \frac{(-\lambda d)^n}{n!} \right)$$

$$= \frac{1}{4} e^{-(\lambda d)} \left[e^{\lambda d} - e^{-\lambda d} \right]$$

Question 1. Map Functions

2. (Continued)

$$\boxed{\theta_{NegCI} = M(d) = \frac{1}{4}\left[1 - e^{-(2\lambda d)}\right]}$$

Pos CI:

$$\theta_{PosCI} = M(d) = \frac{1}{2}\sum_{m=0}^{\infty} \pi_{2m+1} + \sum_{m=0}^{\infty} \pi_{4m+2}$$

$$= \frac{1}{2}\sum_{m=0}^{\infty} \frac{e^{-(\lambda d)}(\lambda d)^{2m+1}}{(2m+1)!} + \sum_{m=0}^{\infty} \frac{e^{-(\lambda d)}(\lambda d)^{4m+2}}{(4m+2)!}$$

$$= \frac{1}{2} e^{-(\lambda d)} \sum_{m=0}^{\infty}\left(\frac{(\lambda d)^{2m+1}}{(2m+1)!} + 2\frac{(\lambda d)^{4m+2}}{(4m+2)!}\right)$$

Note: Derivation of θ_{NegCI} showed the following:

$$\sum_{m=0}^{\infty} \frac{(\lambda d)^{2m+1}}{(2m+1)!} = \frac{1}{2}\sum_{m=0}^{\infty}\left(\frac{(\lambda d)^m}{m!} - \frac{(-\lambda d)^m}{m!}\right)$$

Therefore, we have:

$$\theta_{PosCI} = M(d) = \frac{1}{2} e^{-(\lambda d)} \sum_{m=0}^{\infty}\left[\frac{1}{2}\left(\frac{(\lambda d)^m}{m!} - \frac{(-\lambda d)^m}{m!}\right) + 2\frac{(\lambda d)^{4m+2}}{(4m+2)!}\right]$$

$$= \frac{1}{2} e^{-(\lambda d)}\left[\frac{1}{2}\left(e^{(\lambda d)} - e^{-(\lambda d)}\right) + 2\frac{(\lambda d)^{4m+2}}{(4m+2)!}\right]$$

Question 1. Map Functions

2. (continued)

To finish the derivation of θ_{posCI}, I will first focus on simplifying $\sum_{m=0}^{\infty} 2 \frac{(\lambda d)^{4m+2}}{(4m+2)!}$.

$$\sum_{m=0}^{\infty} 2 \frac{(\lambda d)^{4m+2}}{(4m+2)!}$$

$$= \frac{2(\lambda d)^2}{2!} + \frac{2(\lambda d)^6}{6!} + \frac{2(\lambda d)^{10}}{10!} + \frac{2(\lambda d)^{14}}{14!} + \cdots$$

$$= 1 - 1 + \frac{(\lambda d)^2}{2!} + \frac{(\lambda d)^2}{2!} + \frac{(\lambda d)^6}{6!} + \frac{(\lambda d)^6}{6!} + \frac{(\lambda d)^{10}}{10!} + \frac{(\lambda d)^{10}}{10!} + \cdots$$

$$= 1 - 1 + \frac{(\lambda d)^2}{2!} + \frac{(\lambda d)^2}{2!} + \frac{(\lambda d)^6}{6!} + \frac{(\lambda d)^6}{6!} + \frac{(\lambda d)^{10}}{10!} + \frac{(\lambda d)^{10}}{10!} + \cdots$$
$$+ \frac{(\lambda d)^4}{4!} - \frac{(\lambda d)^4}{4!} + \frac{(\lambda d)^8}{8!} - \frac{(\lambda d)^8}{8!} + \frac{(\lambda d)^{12}}{12!} - \frac{(\lambda d)^{12}}{12!} + \cdots$$

$$= \left[1 + \frac{(\lambda d)^2}{2!} + \frac{(\lambda d)^4}{4!} + \frac{(\lambda d)^6}{6!} + \frac{(\lambda d)^8}{8!} + \cdots\right]$$
$$- \left[1 - \frac{(\lambda d)^2}{2!} + \frac{(\lambda d)^4}{4!} - \frac{(\lambda d)^6}{6!} + \frac{(\lambda d)^8}{8!} - \cdots\right]$$

Question 1. Map Functions

2. (continued)

$$\sum_{m=0}^{\infty} 2\frac{(\lambda d)^{4m+2}}{(4m+2)!} = \sum_{m=0}^{\infty}\frac{(\lambda d)^{2m}}{(2m)!} - \sum_{m=0}^{\infty}(-1)^{m+1}\frac{(\lambda d)^{2m}}{(2m)!}$$

$$= \cosh(\lambda d) - \cos(\lambda d)$$

Plug this result into θ_{PosCI}:

$$\theta_{PosCI} = M(d) = \frac{1}{2}e^{-(\lambda d)}\left[\frac{1}{2}\left(e^{(\lambda d)} - e^{-(\lambda d)}\right) + \cosh(\lambda d) - \cos(\lambda d)\right]$$

$$= \frac{1}{2}e^{-(\lambda d)}\left[\frac{1}{2}e^{(\lambda d)} - \frac{1}{2}e^{-(\lambda d)} + \cosh(\lambda d) - \cos(\lambda d)\right]$$

$$= \frac{1}{2}e^{-(\lambda d)}\left[\frac{1}{2}e^{-(\lambda d)}\left(e^{2\lambda d} - 1 + 2e^{\lambda d}\cosh(\lambda d) - 2e^{\lambda d}\cos(\lambda d)\right)\right]$$

$$= \frac{1}{2}e^{-(\lambda d)}\left[\frac{1}{2}e^{-(\lambda d)}\left(e^{2\lambda d} - 2e^{\lambda d}\cos(\lambda d) + 2e^{\lambda d}\left[\frac{e^{\lambda d}+e^{-\lambda d}}{2}\right] - 1\right)\right]$$

$$= \frac{1}{2}e^{-(\lambda d)}\left[\frac{1}{2}e^{-(\lambda d)}\left[e^{2\lambda d} - 2e^{\lambda d}\cos(\lambda d) + e^{2\lambda d} + 1 - 1\right]\right]$$

$$= \frac{1}{2}e^{-(\lambda d)}\left[\frac{1}{2}e^{-(\lambda d)}\left[2e^{2\lambda d} - 2e^{\lambda d}\cos(\lambda d)\right]\right]$$

Question 1. Map Functions

2. (continued)

$$\theta_{Pos\,CI} = M(d) = \frac{1}{4} e^{-2\lambda d}\left[2e^{2\lambda d} - 2e^{\lambda d}\cos(\lambda d)\right]$$

$$\boxed{\theta_{Pos\,CI} = M(d) = \frac{1}{2}\left(1 - e^{-(\lambda d)}\cos(\lambda d)\right)}$$

3. For a Poisson chiasma process, determine the range of the map function $M(d)$ relating the recombination fraction θ to the genetic map distance d. Comment on the impact of the strand involvement model on the range of the recombination fraction.

NCI:

$$M(d) = \frac{1}{2}\left(1 - e^{-\lambda d}\right), \quad d \geq 0$$

$\frac{dM(d)}{dd} = e^{-\lambda d}$, so $M(d)$ is a monotonic function that decreases as d increases. So the range is below:

$$M(d) \in \left[0, \frac{1}{2}\right].$$

Maximum value of $M(d)$ is $M(0) = \frac{1}{2}$.

Question 1. Map Functions.

3. (continued)

NCI:

The range of the recombination fraction makes sense because the maximum probability of a chromatid being recombinant is $\frac{1}{2}$ under the NCI assumption.

Neg CI:

$$M(d) = \frac{1}{4}\left(1-e^{-2\lambda d}\right), \quad d \geq 0$$

$$\frac{d\,M(d)}{dd} = -\frac{1}{4}(-2\lambda)e^{-2\lambda d}$$

$$= \frac{\lambda}{2}e^{-2\lambda d}, \quad d \geq 0$$

M(d) is a monotonic function that decreases as d increases. So the range is below:

$$M(d) \in \left[0, \frac{1}{4}\right]$$

Maximum value of M(d) is $M(0) = \frac{1}{4}$.

Question 1. Map Functions

3. (continued)

NegCI:

The range of $M(d) \in [0, \frac{1}{4}]$ is reasonable under the NegCI model because only two out of the chromatids are involved in the chiasma. And the two chromatids that are involved in chiasma events have a ½ chance of being recombinant since they are recombinant only when odd number of chiasma events took place. The other two uninvolved chromatids have a probability of zero of being a recombinant chromatid. Hence, the maximum probability of any given chromatid being a recombinant chromatid is 1/4.

Question 1. Map Functions

3. (continued)

PosCI:

$$M(d) = \frac{1}{2}\left(1 - e^{-(\lambda d)} \cos(\lambda d)\right), \quad d \geq 0$$

$$\frac{dM(d)}{dd} = -\frac{1}{2} e^{-\lambda d}(-\sin(\lambda d))\lambda - \frac{1}{2} e^{-\lambda d}(-\lambda)\cos(\lambda d)$$

$$= \frac{1}{2}\lambda e^{-\lambda d} \sin(\lambda d) + \frac{1}{2}\lambda e^{-\lambda d} \cos(\lambda d)$$

$$= \frac{1}{2}\lambda e^{-\lambda d}\left(\sin(\lambda d) + \cos(\lambda d)\right), \quad d \geq 0$$

$M(d)$ is not bound by ½ and is not monotone.

Under the Pos CI model, probability of a given chromatid being recombinant can be greater than ½ because every even chiasmata event not divisible by four will have all the chromatids as recombinant.

Minimum value takes place at $d=0$:

$$M(d) = \frac{1}{2}\left(1 - e^{-\lambda(0)} \cos(\lambda(0))\right)$$

$$= \frac{1}{2}(1-1) = 0$$

Question 1. Map Functions

Maximum value takes place at $d \approx 1.1781$:

$M(d) \approx 0.53351$.

Hence, range for $M(d)$:

$M(d) \in [0, 0.53351]$

Question 2. Coincidence Coefficients

a. Prove that the coincidence coefficient for consecutive intervals of genetic map length d and δ can be expressed as

$$C = C(d, \delta) = \frac{M(d) + M(\delta) - M(d+\delta)}{2M(d)M(\delta)}$$

Let $\pi_{1.}$ be the probability of a recombinant event in interval I_1

Let $\pi_{0.}$ be the probability of no recombinant event in interval I_1.

Let $\pi_{.1}$ be the probability of a recombinant event in interval I_2.

Let $\pi_{.0}$ be the probability of no recombinant event in interval I_2.

$\pi_{1.} = M(d) = \pi_{10} + \pi_{11}$ $\qquad$ $M(d+\delta) = \pi_{10} + \pi_{01}$

$\pi_{.1} = M(\delta) = \pi_{01} + \pi_{11}$

$\pi_{11} = \pi_{1.} - \pi_{10}$

$\pi_{11} = M(d) - \pi_{10}$

$\pi_{11} = \pi_{.1} - \pi_{01}$

$\pi_{11} = M(\delta) - \pi_{01}$

Question 2. Coincidence Coefficients

a. (continued)

$$2\pi_{11} = M(d) - \pi_{10} + M(\delta) - \pi_{01}$$

$$2\pi_{11} = M(d) + M(\delta) - (\pi_{10} + \pi_{01})$$

$$2\pi_{11} = M(d) + M(\delta) - M(d+\delta)$$

$$\pi_{11} = \frac{1}{2}\left(M(d) + M(\delta) - M(d+\delta)\right)$$

$$C(I_1, I_2) = C(d, \delta)$$

$$= \frac{\pi_{11}}{(\pi_{10} + \pi_{11})(\pi_{01} + \pi_{11})}$$

$$= \frac{\frac{1}{2}\left(M(d) + M(\delta) - M(d+\delta)\right)}{M(d)\, M(\delta)}$$

$$\boxed{C(d, \delta) = \frac{M(d) + M(\delta) - M(d+\delta)}{2\, M(d)\, M(\delta)}}$$

Question 2. Coincidence Coefficients

b. Derive $M'(d) = 1 - 2C_3(d)M(d)$

$$C(d,\delta) = \frac{M(d) + M(\delta) - M(d+\delta)}{2M(d)M(\delta)}$$

$$2C\,M(d)M(\delta) = M(d) + M(\delta) - M(d+\delta)$$

$$\frac{M(d+\delta) - M(d)}{\delta} = \frac{M(\delta) - 2cM(d)M(\delta)}{\delta}$$

$$\lim_{\delta\to 0} \frac{M(d+\delta) - M(d)}{\delta} = \lim_{\delta\to 0} \frac{M(\delta) - 2cM(d)M(\delta)}{\delta}$$

$$M'(d) = \lim_{\delta\to 0} \frac{M(\delta)}{\delta} - \lim_{\delta\to 0} \frac{2cM(d)M(\delta)}{\delta}$$

$$M'(d) = \lim_{\delta\to 0} \frac{M(\delta)}{\delta} - 2\left(\lim_{\delta\to 0} c\right) M(d) \lim_{\delta\to 0}\left(\frac{M(\delta)}{\delta}\right)$$

Note: Using L'Hopital's Rule,

$$\lim_{\delta\to 0} \frac{M(\delta)}{\delta} \overset{*}{=} \lim_{\delta\to 0} \frac{M'(\delta)}{1} = M'(0) = 1$$

Question 2. Coincidence Coefficients

b. (continued)

$$\boxed{\therefore\ M'(d) = 1 - 2C_3(d)\,M(d)}$$

c. Derive the map function corresponding to the semi-infinitesimal 3-point coincidence function $C_3(d)=1$.

From part b:

$$\frac{d\,M(d)}{dd} = 1 - 2C_3(d)\,M(d)$$

Plug in $C_3(d)=1$. Then,

$$\frac{dd}{d\,M(d)} = \frac{1}{1-2M(d)}$$

$$\int_0^d dx = \int_0^{M(d)} \frac{dy}{1-2y} \qquad \begin{aligned} u &= 1-2y \\ du &= -2dy \end{aligned}$$

$$d = \int_0^{M(d)} \frac{du}{-2}\left(\frac{1}{u}\right)$$

Question 2. Coincidence Coefficients.

d. (continued)

Show this by finding the derivative of

$$M_k(d) = \frac{1}{2}\tanh(2d)$$

$$M_k(d) = \frac{1}{2}\left(\frac{e^{2d}-e^{-2d}}{e^{2d}+e^{-2d}}\right)\left(\frac{e^{2d}}{e^{2d}}\right)$$

$$M_k(d) = \frac{1}{2}\,\frac{e^{4d}-1}{e^{4d}+1}$$

$$2(e^{4d}+1)M_k(d) = e^{4d}-1$$

$$2e^{4d}M_k(d) + 2M_k(d) - e^{4d} + 1 = 0$$

$$e^{4d}(2M_k(d)-1) + 2M_k(d) + 1 = 0$$

$$e^{4d} = \frac{-2M_k(d)-1}{2M_k(d)-1}$$

$$e^{4d} = -\frac{2M_k(d)+1}{2M_k(d)-1}$$

Question 2. Coincidence Coefficients.

d. $\ln e^{4d} = \ln\left(-\frac{2M_k(d)+1}{2M_k(d)-1}\right)$

$$4d = \ln\left(-\frac{2M_k(d)+1}{2M_k(d)-1}\right)$$

$$d = \frac{1}{4}\ln\left(-\frac{2M_k(d)+1}{2M_k(d)-1}\right)$$

$$d = \frac{1}{4}\left(\ln(2M_k(d)+1) - \ln(-2M_k(d)+1)\right)$$

$$\therefore\ d = \frac{1}{2}\tanh^{-1}(2M_k(d)) = \frac{1}{4}\left(\ln(2M_k(d)+1) - \ln(1-2M_k(d))\right)$$

$$\frac{d}{dM_k(d)}(d) = \frac{1}{4}\frac{2}{2M_k(d)+1} - \frac{1}{4}\frac{-2}{1-2M_k(d)}$$

$$= \frac{1}{2}\frac{1}{2M_k(d)+1} + \frac{1}{2}\frac{1}{1-2M_k(d)}$$

$$= \frac{1}{2}\left(\frac{1-2M_k(d) + 2M_k(d)+1}{(2M_k(d)+1)(1-2M_k(d))}\right)$$

$$\frac{d}{dM_k(d)}(d) = \frac{1}{2}\left(\frac{2}{1-4[M_k(d)]^2}\right) = \frac{1}{1-4[M_k(d)]^2}$$

Hence $M_k(d) = \frac{1}{2}\tanh(2d)$ is the Kosambi map function.

Question 2. Coincidence Coefficients.

d. Carter and Falconer map function

Let $C_3(d) = [2M(d)]^3$.

Show that the solution for

$$\frac{dd}{dM(d)} = \frac{1}{1 - 2[2M(d)]^3 M(d)}$$

$$\frac{dd}{dM(d)} = \frac{1}{1 - 16[M(d)]^4}$$

is $M_{CF}^{-1}(\theta) = \frac{1}{4}(\tanh^{-1}(2\theta) + \tan^{-1}(2\theta))$.

Show this by finding the derivative of $M_{CF}^{-1}(\theta)$:

$$\frac{d M_{CF}^{-1}(\theta)}{d\theta} = \frac{1}{4}\frac{2}{1-(2\theta)^2} + \frac{1}{4}\frac{2}{1+(2\theta)^2}$$

Note: Derivative of $\tanh^{-1}(x)$ was obtained previously.

$$\frac{d}{d\theta}\tan^{-1}(2\theta) = \frac{2}{\frac{d\tan y}{dy}} = \frac{2}{\sec^2 y} = \frac{2}{1+\tan^2 y}$$

$$= \frac{2}{1+(2\theta)^2}$$

Question 2. Coincidence Coefficients.

d. (continued)

$$\Rightarrow \frac{d M_{CF}^{-1}(\theta)}{d\theta} = \frac{1}{2}\left[\frac{(1+4\theta^2)+(1-4\theta^2)}{[1-(2\theta)^2][1+(2\theta)^2]}\right]$$

$$= \frac{1}{2} \frac{2}{1-16\theta^4}$$

$$= \frac{1}{1-16\theta^4}$$

$$= \frac{1}{1-16[M(d)]^4}$$

Hence $M_{CF}^{-1}(\theta) = \frac{1}{4}(\tanh^{-1}(2\theta)+\tan^{-1}(2\theta))$ is the map function when $C_3(d) = [2M(d)]^3$.

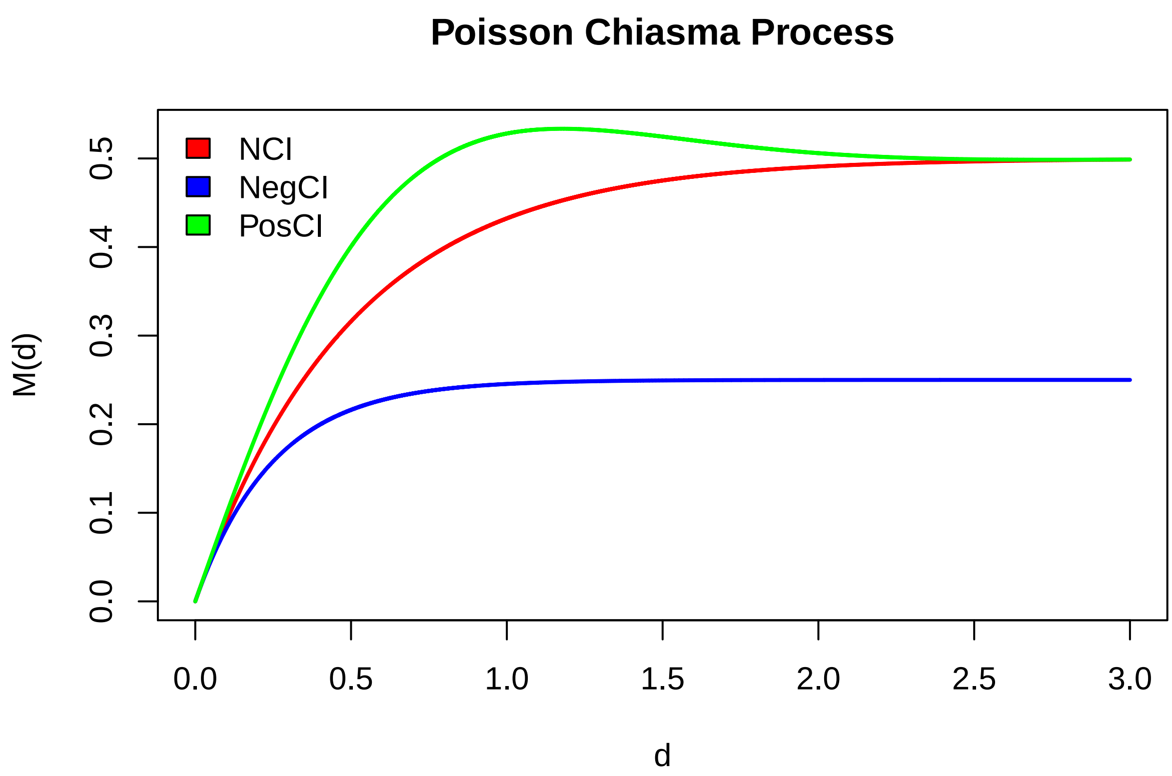
Assignment 2

Daniel Lee

February 23, 2017

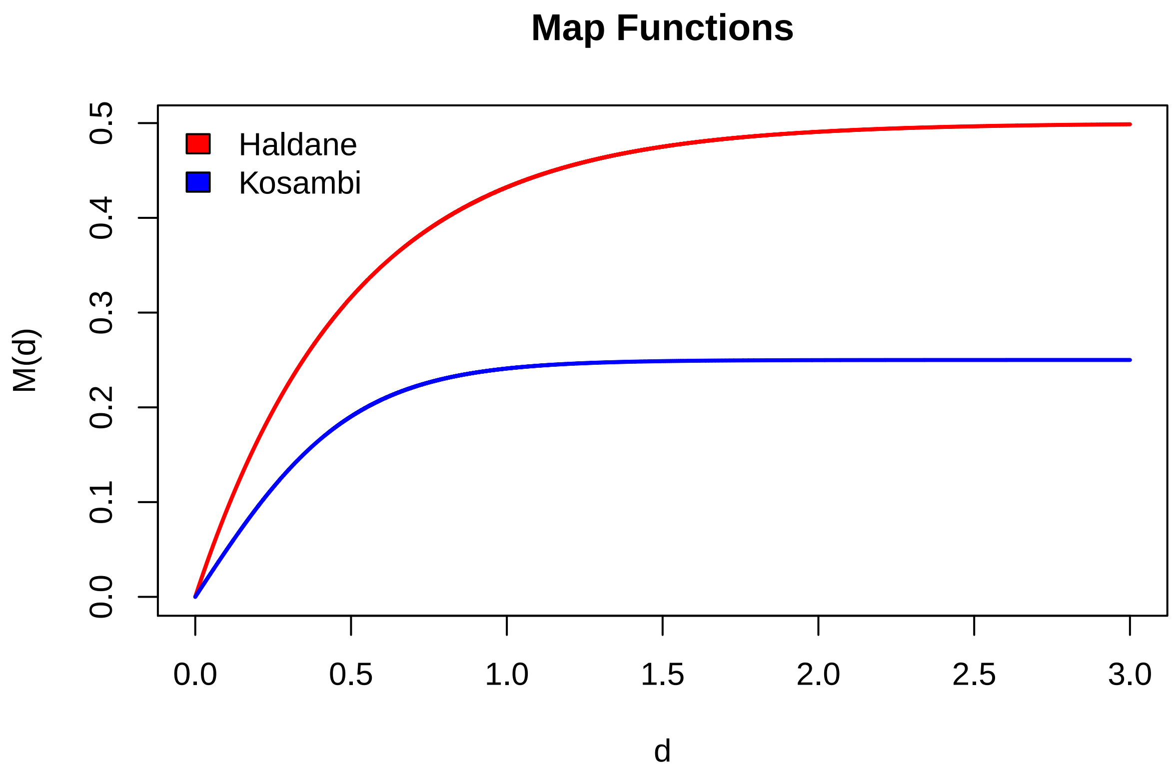
Question 1. Map Functions

```r
d <- seq(from = 0, to = 3, by = 0.0001)
lambda <- 2

#NCI
theta_NCI <- 1/2*(1-exp(-lambda*d))

theta_NegCI <- 1/4*(1-exp(-2*lambda*d))

theta_PosCI <- 1/2*(1-exp(-lambda*d)*cos(lambda*d))



matplot(d, cbind(theta_NCI, theta_NegCI, theta_PosCI),
        type = "l",
        xlab = "d",
        ylab = "M(d)",
        lty = 1, lwd = "2",
        main = "Poisson Chiasma Process",
        col = c("red", "blue", "green"))
legend("topleft", c("NCI", "NegCI", "PosCI"),
       col=c("red", "blue", "green"),
       fill=c("red", "blue", "green"),
       bty = "n")
```



Question 2 Coincidence Coefficients

```
d <- seq(from = 0, to = 3, by = 0.0001)
lambda <- 2

haldane <- 1/2*(1 - exp(-2*d))
kosambi <- 1/4*(tanh(2*d))

matplot(d, cbind(haldane, kosambi),
        type = "l",
        xlab = "d",
        ylab = "M(d)",
        lty = 1, lwd = "2",
        main = "Map Functions",
        col = c("red", "blue"))
legend("topleft", c("Haldane", "Kosambi"),
       col=c("red", "blue"),
       fill=c("red", "blue"),
       bty = "n")
```


Map Functions
Haldane
Kosambi
M(d)
0.0
0.1
0.2
0.3
0.4
0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
d