MARKOV CHAIN TRANSITION MATRIX REVERSAL

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Abstract

Sometimes we want to "reverse" a Markov Chain process. Taking the inverse of the transition matrix allows this to work, but the inverse result is not a transition matrix. If I wanted to model a population going to work, and then going back home, negative and greater than 1 probabilities in the inverse matrix will cause issues. I propose a method to compute the "inverse" of a transition matrix, and the result is still a transition matrix.

1. Introduction

A Markov Chain is a probabilistic system that connects agents in a state based system.

Pretend I go to work 0.8 of the time, and I am 0.2 sick. Then, from work, I return home always 1.0 of the time. So, we can model this as a Markov Chain, as it shows a connected agent based system.

To model this scenario, we place these probabilities in a transition matrix as follows:

$$T = \begin{bmatrix} 0.2 & 1.0 \\ 0.8 & 0.0 \end{bmatrix} \tag{1}$$

Note, you can see that the columns of the matrix add to 1. This is a probability law, as all probabilities must add to 1, and must be between 0 and 1.

Now, pretend I am a one person population and I want to model how I move between these states. To do this, we take the powers of the Transition

Matrix T. Power of 1 means one interaction of the model (or in this case - 1/2 of a day). And, a power of 2 means 1 whole day.

$$T^1 = \begin{bmatrix} 0.2 & 1.0 \\ 0.8 & 0.0 \end{bmatrix} \tag{2}$$

$$T^2 = \begin{bmatrix} 0.84 & 0.2 \\ 0.16 & 0.8 \end{bmatrix} \tag{3}$$

Now, I want to see after 1/2 a day, how "much" of me is at work and how "much" is at home. First, let's start from our home. Then we can conclude:

$$P_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4}$$

$$T^1 = \begin{bmatrix} 0.2 & 1.0 \\ 0.8 & 0.0 \end{bmatrix} \tag{5}$$

$$T^1 P_1 = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} \tag{6}$$

Now, at the end of the day, I want to see how I move around. I then apply the 2nd power of T, and we get:

$$P_2 = T^2 P_1 = \begin{bmatrix} 0.84 \\ 0.16 \end{bmatrix} \tag{7}$$

Now, my question is: Is there a transition matrix such that we can "revert" the population or reverse the Markov Chain to the first state when I started moving from home?

In other words, find a matrix R(reversal) such that given some T:

$$RP_2 = P_1 \tag{8}$$

2. Applications

I wanted a reverse Markov process, as I was modelling how a disease spread through a population. I had computed probabilistic movements for transiting people, but the problem was that I didn't want to compute night and day cycles separately (it would be too much work). So instead, I wanted to find a method to reverse my transition matrix to account for day and night cycles. Inverse couldn't work, as I needed real probabilities. So, I decided to make a new method.

This process can be used to do less work when modelling Markov systems. If you want to model mass gatherings, and then they had to go back home after some event, you can reverse the processes easily using this technique.

Likewise, if you wanted to somehow find what a reversed system or "inversed" Markov process would look like, you can use this method to go "backwards" in time.

3. Markov Reversal

We can apply the inverse, as we want some matrix R (reversal) such that:

$$RP_2 = P_1 \tag{9}$$

Now, using our old T transition matrix:

$$TP_1 = P_2 \tag{10}$$

$$T^{-1}TP_1 = T^{-1}P_2 \tag{11}$$

$$P_1 = T^{-1}P_2 \tag{12}$$

$$R = T^{-1} \tag{13}$$

However, when we compute this for our first example, we get:

$$T^{-1} = \begin{bmatrix} 0 & 1.25 \\ 1 & -0.25 \end{bmatrix} \tag{14}$$

Which clearly is NOT a transition matrix. There are negative values, and probabilities cannot be negative. Also some are larger than one.

To solve this issue, I propose this method:

$$R_t = \left(\frac{T_{t-1}(P_{t-1} \circ I_n)}{P_t \circ J_n} \right)^T \tag{15}$$

Where J_n is the unit matrix full of ones, and I_n is the identity matrix. The circle is the Hadamard product or element wise multiplication. The division is element wise division. T is transpose.

Applying this to our first problem, we find that R_2 is:

$$R_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \tag{16}$$

$$R_2 P_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{17}$$

Now, I made a random Markov Transition Matrix to see if my method works:

$$T = \begin{bmatrix} 0.8 & 0.1 & 0.3 \\ 0.1 & 0.7 & 0.1 \\ 0.1 & 0.2 & 0.6 \end{bmatrix} \tag{18}$$

$$P_1 = \begin{bmatrix} 10 \\ 100 \\ 1000 \end{bmatrix} \tag{19}$$

$$P_2 = T^1 P_1 = \begin{bmatrix} 318 \\ 171 \\ 621 \end{bmatrix} \tag{20}$$

Using (14), we find the reverse Transition Matrix using inverses as:

$$R_2 = \begin{bmatrix} 1.3333 & 0 & -0.6666 \\ -0.1666 & 1.5 & -0.1666 \\ -0.1666 & -0.5 & 1.8333 \end{bmatrix} \tag{21}$$

Now, applying (15), we find that my method produces:

$$R_2 = \begin{bmatrix} 0.0252 & 0.0058 & 0.016 \\ 0.0314 & 0.4094 & 0.0322 \\ 0.9434 & 0.5848 & 0.9662 \end{bmatrix} \tag{22}$$

And using Julia Programming, I can guarantee that $R_2 P_2 = P_1$ We also see that for each column, it adds to 1, and all $0 <= P <= 1$

4. Derivation

First, let's make a new transition matrix and population matrix:

$$T = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \tag{23}$$

$$P_1 = \begin{bmatrix} 10(A) \\ 100(B) \end{bmatrix} \tag{24}$$

$$P_2 = \begin{bmatrix} 58(A) \\ 52(B) \end{bmatrix} \tag{25}$$

So, from A, 2 goes to B and 8 stays. From B, 50 goes to A and 50 stays. Now, at time = 2, A has population 58, and B has 52.

Now, we want from B, 2 to go back to A. From A, we want 50 to go back to B. Also, we want the rest to stay in A and B respectively. So we can compute the ratios like so:

$$R_2 = \begin{bmatrix} \frac{8}{58} & \frac{2}{52} \\ \frac{50}{58} & \frac{50}{52} \end{bmatrix} \tag{26}$$

So, to get that matrix, we did:

$$\frac{\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} || \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \end{bmatrix}}{\begin{bmatrix} 58 & 58 \\ 52 & 52 \end{bmatrix}} \tag{27}$$

$$= \frac{\begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \left(\begin{bmatrix} 10 \\ 0 \end{bmatrix} || \begin{bmatrix} 0 \\ 100 \end{bmatrix} \right)}{P_2 \circ J_2} \tag{28}$$

$$= \frac{T_1(P_1 \circ I_2)}{P_2 \circ J_2} = R_2^T \tag{29}$$

Therefore,

$$R_2 = \left(\frac{T_1(P_1 \circ I_2)}{P_2 \circ J_2} \right)^T \tag{30}$$