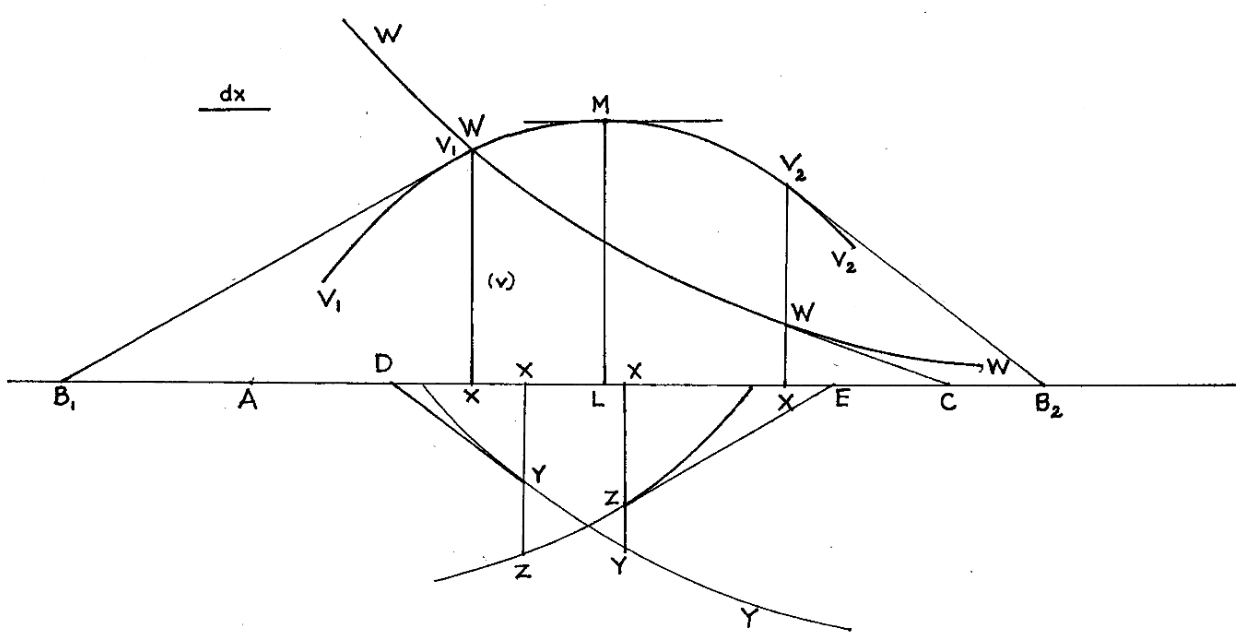
# Readings and Notes for Junior Mathematics




St. John's College, Annapolis

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# Contents

# Selections from Aristotle's *Physics* to read with Galileo's *Two New Sciences*

The following four passages on the infinite and continuity from Aristotle's *Physics* may be useful in reading Galileo's *Two New Sciences*, pages 77–93. The translations are by Joe Sachs (*Aristotle's Physics: A Guided Study*, 1995).

**Book III, 206a7–b28** That, then, there is no actually infinite body, is clear from these things. But that, if there is no infinite simply, many impossible things follow, is clear. For there will be a beginning and an end of time, as well as magnitudes not divisible into magnitudes, and number will not be infinite. But, whenever such a distinction has been made and neither way seems possible, there is a need for discrimination, and it is clear that in one way the infinite is, and in another way it is not. Now being is said of what is potentially or of what is in complete activity, and there is an infinite by addition or by division. And that there is no magnitude actually infinite has been said, but there is magnitude that is infinite by division; for it is not difficult to refute indivisible lines. What is left, then
is that the infinite is as potentiality. But it is necessary | not to take 206a20
the being-potentially in the same way as if something were potentially a statue, since this will also *be* a statue, and thus there would also be an infinite which would be at-work. But since there are many ways of being, just as day is, or the athletic games which always come about one after the other, so also with the infinite. (For also with these things there is both being-potentially and being-at-work; for there are Olympic games both in the sense that the games are capable of happening and that they are happening.) And this is evident in different ways in time, in human beings, and in the division of magnitudes. In general the infinite is in this way: it is in what is taken always one after the other, while what is taken
is | always finite, but always another and another. So being is meant in 206a30
many ways, and the infinite must not be taken as a *this,* such as a man or a house, but in the way that day or the games are meant, to which being belongs not as to a thing, but in a constant coming into being and passing away, finite, but always other and other. But in magnitudes what is taken
| remains, while with time and human beings it is always perishing in such 206b1
a way as not to run out.

But the infinite by addition is in some way the same as that by division, for the latter comes about in the finite by addition turned back the other way. For where a division is seen to be to infinity, there is obviously an addition to what is cut off. For if someone taking a marked-off part of a finite magnitude keeps taking from it in the same ratio (not including the piece of the whole magnitude already taken), the pieces will not exhaust
the finite thing. | But if in the same way one increases the ratio so as 206b10
always to include the same amount, they do exhaust it, through the whole finite thing's being used up by whatever part is marked off. So it is in

no other way, but in this way there is an infinite, in potentiality and by exhaustion (but it is also at-work, in the way we say day and the games to be). And it *is* thus in the way material is, potentially, and not on its own in the way the finite is. And the infinite by addition is surely in potentiality in the same way, which we say is in a certain way the same as that by division. For there will always be something outside to take, and it will not exceed | every magnitude, just as in the division it does go beyond every marked-off piece and there will always be a smaller piece. 206b20

Therefore to exceed everything by addition is not even possible potentially, unless there is accidentally an actual infinite, as the writers on nature say that which is outside the body of the cosmos, being of air or some other such thing, is infinite. But if it is impossible for there to be an actually infinite sensible body in this way, it is clear that not even potentially could there be one by addition, other than in the way described, by a reversed division.

**Book III, 206b.33–207a.10** The infinite turns out to be the opposite of the way people speak of it. For this is the infinite: not that outside of which there is nothing, but that outside of which there is always something. Here is a sign: people speak of rings which do not have stone-settings as endless because there is always something beyond to take, speaking in accordance with a likeness though not strictly. For it is necessary both that this condition be present and that at no time the same part be taken; but in the circle it does not happen that way, but only the succeeding part is always different. Infinite, then, is that of which, to those taking it by quantities, there is always something beyond to take. That of which nothing is outside is complete and whole; that is how we define the whole, as that of which nothing is absent, as a whole human being or box.

**Book V, 227a.10–17** That which, being next in series to something, is touching it, is next to it. The continuous is that which is next to something, but I call them continuous only when the limits at which they are touching become one and the same, and, as the name [συνεχής] implies, hold together [συνέχειν]. And this is not possible if the extremities are two. And it is clear from this definition that the continuous is among those things out of which some one thing naturally comes into being as a result of their uniting. And in whatever way the continuous becomes one, so too will the whole be one, such as by a bolt or glue or a mortise joint, or by growing into one another.

**Book VI, 232b.24–25** I call continuous that which is always divisible into divisible parts.

# An Approach to the Arithmetic of Infinites: where we also show that a greatest number or an infinite number of all numbers is impossible or nothing; and show by examples that some things that are held to be axioms can be demonstrated

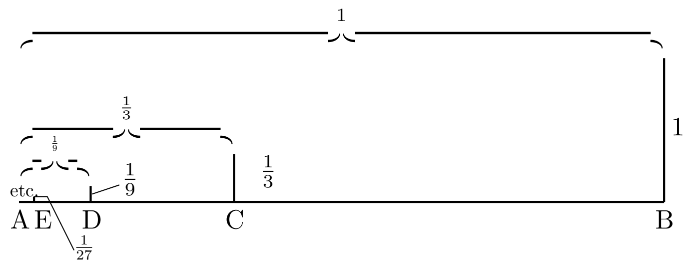
It has been established that the science of the least and the greatest, or of the indivisible and the infinite, is among the greatest pieces of evidence which the human mind uses in laying claim to its own incorporeality. Who indeed, following his senses, could persuade himself that there can be *given* no line so short that it does not contain both infinitely many points and also infinitely many lines (and accordingly actually contains an infinite number of parts, all separated from each other) having a finite ratio to the *given* line, if demonstrations did not compel him? And how truly wonderful it is to calculate the sum of infinitely many decreasing quantities! or to prescribe limits to quantities increasing or decreasing in a finite space! or to generate finite figures and demonstrate their proportions by multiplying infinites by one another! Note 1, page 15

Archimedes long ago used the arithmetic of infinites and the geometry of indivisibles, as well as inscribed and circumscribed figures, in *The Measurement of the Circle*, *On the Sphere and the Cylinder*, and *The Quadrature of the Parabola*. In our time, Cavalieri has revived the geometry of indivisibles (while Galileo served as his midwife and gave his approval), Wallis the arithmetic of infinites, and James Gregory inscribed and circumscribed figures. And certainly, if a new light from indivisibles and infinites does not shine on it and the art of analysis does not progress, there is no hope of great progress in geometry.

The ancients have given us a rule for calculating a sum of fractions or ratios decreasing indefinitely in a geometric progression. For if a quantity, exhibited by the line $AB$ [see the diagram below], is given, and this line is continually cut and recut so that the ratio of a subsection, such as $AD$, to a section, such as $AC$, is as the ratio of the section $AC$ to the whole, $AB$, or so that the ratios $\frac{AB}{AC} = \frac{AC}{AD} = \frac{AD}{AE}$, etc., are equal; then, the ratio of $CB$ (the remainder when the section $AC$ is taken away from the whole $AB$) to the whole $AB$ will be the same as the ratio of the whole $AB$ to a whole composed of the whole and in addition first the section, then the section of the section, etc., all taken simultaneously; that is,

$$\frac{CB}{AB} = \frac{AB}{AB + AC + AD + AE + \text{etc.}}.$$

I have seen a demonstration of this rule attempted by certain learned men, but I have not seen an absolute demonstration; I not only demonstrate it from a universal principle but also draw from it an elegant consequence, namely: if Note 2, page 16 we take continually decreasing fractions whose numerators are unity, but whose denominators are the terms in a certain geometric progression, then the sum of all the fractions of the given progression will be the first fraction of the preceding




geometric progression, so that

$$\begin{aligned} \frac{1}{2}+\frac{1}{4}+\frac{1}{8}\text{ etc.} &= \frac{1}{1},\text{ and}\\ \frac{1}{3}+\frac{1}{9}+\frac{1}{27}\text{ etc.} &= \frac{1}{2},\text{ and}\\ \frac{1}{4}+\frac{1}{16}+\frac{1}{64}\text{ etc.} &= \frac{1}{3}, \end{aligned}$$

Note 3, page 17 and so on.

But this is not enough; let us go on to some things for which there are as yet no rules. I thought that I should write down for you, most distinguished man,[1] some ideas that came to my mind about helping the arithmetic of infinites grow.

When I once told the Illustrious Huygens that I had certain ways of summing a few series that decreased indefinitely, and whose computation had not yet been published, he proposed the following to me: he told me to look for the sum of the fractions whose numerators are unity, but whose denominators are the triangular numbers $0^1 1^2 3^3 6^4 10^5 15^6 21^7 28$ etc., namely, the numbers whose Note 4, page 18 differences are natural numbers. He said that once, when he was thinking about calculations for dice and other games of chance, he needed this sum and had found it, but it was not yet published. I looked, and found that the sum is binary, that is, $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}$ etc. $= 2$. After I had shown this to Huygens, he admitted that it is true and also agrees with his own calculation.

I, however, had found at the same time a universal method of summing series of fractions or ratios not only of this progression of *triangulars*, where the differences of terms are natural numbers, but also of *pyramidals*, where the differences of terms are triangular numbers, and of *triangulo-triangulars*, where the differences are pyramidal, and of *triangulo-pyramidals*, where the differences are triangulo-triangular, and of *pyramido-pyramidals*, where the differences are Note 5, page 18 triangulo-pyramidal, and so on indefinitely. Examine Table 1.

And these are the numbers whose series some call numerical orders, others combinatoric orders, and others the numbers of a symmetric progression. Pascal set forth their many uses in *The Arithmetic Triangle*, the treatise he wrote that

[1] Leibniz intended to send this manuscript to Jean Gallois, the editor of the Journal des Sçavans.

Table 1

| zero | units | naturals | triangulars | pyramidals | triangulo-triangulars | triangulo-pyramidals | pyramido-pyramidals |
|---|---|---|---|---|---|---|---|
| 0 | | | | | | | |
| | 1 | | | | | | |
| 0 | | 1 | | | | | |
| | 1 | | 1 | | | | |
| 0 | | 2 | | 1 | | | |
| | 1 | | 3 | | 1 | | |
| 0 | | 3 | | 4 | | 1 | |
| | 1 | | 6 | | 5 | | 1 |
| 0 | | 4 | | 10 | | 6 | |
| | 1 | | 10 | | 15 | | 7 |
| 0 | | 5 | | 20 | | 21 | |
| | 1 | | 15 | | 35 | | 28 |
| 0 | | 6 | | 35 | | 56 | |
| | 1 | | 21 | | 70 | | 84 |
| | | 7 | | 56 | | 126 | |
| | | | 28 | | 126 | | 210 |
| | | | | 84 | | 252 | |
| | | | | | 210 | | 462 |
| | | | | | | 462 | |
| | | | | | | | 924 |

is dedicated to them.[2] I usually call them the numbers of a *replicated arithmetic progression*; for any numbers whatever (for example, binary or ternary numbers) can be substituted for the units, while arbitrary numbers of an arithmetic progression beginning from its own difference can be substituted for the natural numbers (for example, 2, 4, 6, 8 etc., for 1, 2, 3, 4, etc.), and the table will be proportionally the same; indeed, if the generator is binary, we simply double all the terms, and if it is ternary, we triple them, etc. Moreover, we can make a universal rule for the sums of fractions, whatever the generator may be, if only we understand the numerator of the fractions to be the generator; for example, if the generator is 2, we should substitute $\frac{2}{2}+\frac{2}{6}+\frac{2}{12}$ etc. for $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}$ etc. And after we divide all the numbers in the former series by the generator, it is the same as the latter.

But as I was saying, there will be a rule for finding sums [see Table 2, below]: the sum of a series of fractions whose numerators are the generator, and whose denominators are the terms of a certain replicated arithmetic progression, or, what amounts to the same thing, the sum of ratios whose antecedents are unity, and whose consequents are the terms of a certain replicated arithmetic progression having unity as its generator—this sum, I say, is the fraction or ratio whose numerator or antecedent is the exponent of the immediately preceding series, that is, the penultimate series (taking the given series as the ultimate series), but whose denominator or consequent is the exponent of the series immediately preceding the preceding series, that is, of the antepenultimate series. By *exponent* I mean here the number of the series or the ordinal number of its replication, namely, the number that expresses the place of its replication in the series of replications. Thus the exponent in the first series, $\frac{1}{1}$, $\frac{1}{1}$, $\frac{1}{1}$, etc., is 1, and the exponent in the second series, 1, 2, 3, 4, etc., is 2. For while in the first series only the unit generator is repeated, in the second the replications themselves or the repetitions are replicated, and in the third, 1, 3, 6, 10, etc., the replications of the replications are repeated; but if the generator is the unit, the number of a series or the exponent of its degree coincides with the first number in it after the unit. I call the number of the series its *exponent* because I am following the example of the geometric progression; in a geometric progression the exponent of the roots is 1, the exponent of the squares is 2, the exponent of the cubes is 3, etc., just as in this case the exponent of the generators is 1, the exponent of the naturals is 2, the exponent of the triangulars is 3, etc.

Therefore it follows that the sum of the series of triangular fractions

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28} \text{ etc.}$$

is

$$=\frac{2}{1};$$

[2]An excerpt of this treatise is translated into English by Anna Savitsky in D. E. Smith's *A Source Book in Mathematics*, 1929, pages 67–79. The whole work, in Latin and French, is included Pascal's *Oeuvres*, Mesnard ed., vol. II, pp. 1166–1332, Desclée de Brouwer, Paris, 1970.

Table 2: Series of fractions of a replicated arithmetic progression

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | etc. | exponents |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | | Series of fractions of a replicated arithmetic progression with unit generator |
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{6}$ | $\frac{1}{7}$ | | |
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{10}$ | $\frac{1}{15}$ | $\frac{1}{21}$ | $\frac{1}{28}$ | | |
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{4}$ | $\frac{1}{10}$ | $\frac{1}{20}$ | $\frac{1}{35}$ | $\frac{1}{56}$ | $\frac{1}{84}$ | | |
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{5}$ | $\frac{1}{15}$ | $\frac{1}{35}$ | $\frac{1}{70}$ | $\frac{1}{126}$ | $\frac{1}{210}$ | | |
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{6}$ | $\frac{1}{21}$ | $\frac{1}{56}$ | $\frac{1}{126}$ | $\frac{1}{252}$ | $\frac{1}{462}$ | | |
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{7}$ | $\frac{1}{28}$ | $\frac{1}{84}$ | $\frac{1}{210}$ | $\frac{1}{462}$ | $\frac{1}{924}$ | | |
| $\frac{0}{0}$ | $\frac{0}{0}$ | $\frac{1}{0}$ | $\frac{2}{1}$ | $\frac{3}{2}$ | $\frac{4}{3}$ | $\frac{5}{4}$ | $\frac{6}{5}$ | etc. | sums |

for the series preceding the series 1, 3, 6 etc., namely 1, 2, 3 etc., has exponent 2, and the series preceding *this* series, namely 1, 1, 1 etc., has exponent 1; hence we get $\frac{2}{1}$ or 2. And the sum of the series of pyramidal fractions

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \frac{1}{35} \text{ etc.}$$

is

$$= \frac{3}{2},$$

or the ratio of the exponent of the triangulars to the exponent of the naturals.

This is easier to see by looking at Table 2.

But since such a method of finding and demonstrating is quite lengthy and requires many lemmas, I will wait until I have more time to put it in order before I publish it, along with many other things of the same kind. Note 6, page 20

Yet I cannot pass up here a chance I have to make a certain point about the nature of the *infinite number of all numbers*. Galileo[3] compares it to the *unit*, and he reasoned as follows: every number indefinitely has its own square, its own cube, etc; for if it is multiplied by itself, its square, cube, etc. is produced; therefore there are as many cubes and as many squares as there are roots or simple numbers, which is impossible; for there are always many other non-squares placed between the square numbers, and still more non-cubes between

[3] In the *Two New Sciences*, First Day, on page 83 in Volume VIII of Galileo's *Opere*, edited by A. Favaro; on Page 45 in Stillman Drake's translation, Toronto, 1989.

the cubes. What then? Do the attributes "equal" and "greater" or "less" have no place in the infinite? He also suggests that, if any number is infinite, the unit is; for it has that property that the infinite number of all numbers needs to have: that there are as many roots in it as squares and cubes; for the square and cube etc. of the unit is the unit. I, however, conclude that if there is any such infinite number, it is zero or nothing, or, what amounts to the same thing, that such a number is nothing or $= 0$.

Such an infinite number has not only the property that Galileo observed in it —that there are as many powers of every kind in it as there are roots—but also the property that there are in it as many numbers taken simply, that is, both evens and odds together, as there are even numbers. For the even numbers are the doubles of the numbers taken simply, yet there are as many simple numbers as there are doubles of them. In the same way we conclude that there are not only as many numbers taken simply as there are even numbers (binaries), but also as many as there are ternaries (triples of numbers taken simply), and as there are quaternaries, etc., and triangulars, pyramidals, etc. In the same way we prove that there are as many numbers taken simply as there are numbers of any given progression, arithmetic, geometric, or mixed, or of any replicated progression going on indefinitely; although it is more than manifest that between the binaries or evens there are other, odd, numbers, and that there are still more non-ternary numbers between the ternaries. Therefore, since in such an infinite number there are as many even numbers as there are odd and even numbers together, that is, as many as there are numbers taken simply, it follows that in such an infinite number the axiom that the whole is greater than the part fails (just as Gregory of St. Vincent contends that it fails for the angle of contact[4]). But it is impossible for this axiom to fail, or, what amounts to the same thing, such an axiom never fails, that is, it fails for *nothing*. Therefore such an infinite number is impossible—not one, not whole, but nothing. Therefore such an infinite number $= 0$. And in 0 or zero we certainly find not only the property noticed by Galileo in the unit, but also all the rest; for the square and cube of 0 is 0, and the double and triple of 0 is 0, and $0 + 0$ is $= 0$, the whole to the part. And, so as not to seem to digress too far from the matter at hand, I confirm this by gathering into a sum a series that progresses indefinitely; for when we sum the fractions of a geometric progression it is certain that the sum of any series is the first fraction of the preceding series, and $\frac{1}{3} + \frac{1}{9} + \frac{1}{27}$ etc. $= \frac{1}{2}$, and likewise $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ etc. $= 1$, and therefore $\frac{1}{1} + \frac{1}{1} + \frac{1}{1}$ etc. $= 0$. Now $1 + 1 + 1$ etc. constitutes the infinite number of all numbers. The same thing happens in the above table of the fractions of a replicated arithmetic progression, where it is clear that $\frac{1}{1} + \frac{1}{2} + \frac{1}{3}$ etc. $= \frac{1}{0}$ and $\frac{1}{1} + \frac{1}{1} + \frac{1}{1}$ etc. $= \frac{0}{0} = 0$.

The same topic should remind us that, if we are going to be rigorous, if philosophy is to be perfected, we should accept no proposition unless it either agrees with an immediate sense-observation or is demonstrated from clear and

[4]Gregory apparently argued that the angle formed by a diameter of a circle and a tangent at its endpoint is not greater than the curvilineal angle formed by the same diameter and the circumference. In the diagram to Euclid's *Elements*, Proposition III 16, Gregory would say that the whole right angle $BAE$ is not greater than its part, the curvilineal angle $BAHC$.

distinct imagination, that is, from an *idea*, or from a definition (which is what idea signifies). Obviously the definitions themselves need not be demonstrated, since, as the restorer of philosophy Galileo has emphasized so many times in his writings, they are arbitrary, and cannot be charged with falsity, but only with unsuitability and obscurity.

Since such a proposition—that the whole is greater than the part—has been doubted by the greatest geometers, including Galileo and Gregory of St. Vincent, shall we continue to proclaim that there are other propositions that are known through themselves?

Galileo certainly believed that an infinite number is something or one whole, for he compares it to a unit; but nevertheless he denies that being greater and being less have any place in it; for he denies that there are more numbers taken simply, that is, squares and the non-squares, than there are square numbers, or that the whole is greater than the part.

Hobbes erred in concluding that the truth of all propositions comes from human decision.[5] For, first of all, we should leave out those propositions which are established by sense-perception, such as the proposition that I am *sensed* by myself as sensing; but we should also leave out those propositions that we demonstrate by applying definitions to what we know—for example, when we demonstrate from the preceding proposition that I *sense* or think, and likewise that I *am*. For it is certain by sense-perception that I am sensed by myself as sensing, and therefore that *I, as sensing* am sensed immediately (without any medium); for between myself and me, in the mind, there is no medium. Whatever is immediately sensed is immediately sensible. Whatever is immediately sensible is sensible without error (for all error is from a medium of sensation—I am supposing this as something that is to be demonstrated elsewhere). Whatever is sensible without error, is; hence it follows that *I am sensing*, that is, that the proposition, "I am *sensing*," is true. Consequently I reflect: "*I am sensing*." We should also leave out identical propositions or the affirmation of the same thing about the same thing with the same words. But when we say the same thing about the same thing with equivalent words—for example, when we say the definition of something, or when we take different definitions of the same thing and apply them to each other in turn, or when we take a part of one definition of something and apply it to that thing or to some other definition of it—it is manifest that the truth of our proposition is by human decision; for definition is by human decision. And all axioms that do not depend on sense-perception—indeed all theorems in the sciences that are independent of sense perception and experience—are propositions of this sort, as Aristotle also noticed, who set down *definition* as the unique principle of demonstration.[6] And in fact all the axioms that Euclid puts at the beginning of his elements are demonstrable from definitions. "Then," you will say, "what are we learning when we investigate the theorems of such sciences?" Nothing, I would say, except how to think more quickly and distinctly for practical purposes, or how to

[5]Hobbes concludes something like this in *On Body* (the first part of his *Elements of Philosophy*), Chapter 3, Paragraph 8.

[6]See *Posterior Analytics* II 3, 90b25.

use certain fitting symbols to order thoughts we have had and ideas we have received through our senses. (These symbols may be either names or characters.) Consider, for example, numbers; who does not see that we learn nothing new in all of arithmetic except the names of numerals and their various recursions, which become harmonic if they are inverted; from here we draw out equations as theorems and it then becomes very clear how useful characters are, when by using the symbols we have made we can notice much that we would not otherwise have seen—for example, when we easily calculate the sum of an entire progression. And this is most apparent in algebra, where no one does not see that we do everything by means of symbols variously transposed, and reap a prodigious harvest, not because we learn new things, but because things are shown naked to the mind. In the same way, if we were to have a philosophical language or at least a philosophical writing, which I spoke about in the *Combinatoric Art*,[7] and which would use the elements of thinking instead of an alphabet, we could write things down by means of their definitions. And just as in algebra there are equations everywhere, here there would be theorems everywhere, and we could propose and solve infinitely many problems and demonstrate theorems with no trouble, and it would be impossible for anyone who does not understand things to use this writing, and everyone would be able to reason without error, as in arithmetic. And algebra, both numerical and specious, is only a part or an example of this universal writing or philosophic characterism; I wonder why this has not been sufficiently noticed by the greatest men. I, however, am preparing an example in morality or justice, in order that it may appear—

Since I have ascertained that every whole is greater than its part, I boldly conclude that such an infinite number or greatest number or sum of all possible units, which you may also call most infinite or the number of all numbers, is 0 or nothing. And we can give a new demonstration, for instance by using the fact that a greatest number is the sum of all units or the number of all numbers. And the sum of all numbers is necessarily greater than the number of numbers (as $1 + 2 + 3$ etc. is greater than $1 + 1 + 1$ etc.). Therefore, the greatest number is not the greatest number, that is, the greatest number is 0, although I would not therefore straightaway deny that there are infinitely many parts in the continuum or that there is an infinite magnitude in time or space.

Hence it appears that such propositions as: that things equal to the same thing are also equal to each other; that equals added to or taken away from equals make equals; that the whole is greater than the part; that equimultiples are as simples; that if proportionals are added or taken away from proportionals, proportionals are produced, etc.–since these all can be doubted, they must be demonstrated, and accordingly, if they are true, they are demonstrable (from terms or definitions, of course). And the scholastics wished that first truths might become known through inspection of terms, that is that they might be easy to demonstrate and almost definitions; on the other hand there are those who think that these first truths are known through themselves, by means of I

[7] A work Leibniz published in 1666. It is reprinted in Volume 1 of Series VI the Berlin Academy edition of Leibniz's collected works. The relevant passage is in sections 89 and 90.

know not what natural light. For it is well known that some things are placed by some men among the things known through themselves, while the same things are rejected and distinguished from them by others, and that men have no criterion to decide what is known through itself, except perhaps common opinion, which, besides being exposed to doubt, would set down probable things as the foundations of demonstrations, which is to give way to the Pyrrhonists.

"But indeed," someone will say, "if all axioms are demonstrable from definitions of names, all truths will depend on human decision, since the definitions of names are arbitrary; but this opinion in Hobbes is condemned by the learned." I reply that propositions depend on definitions to the extent that they are expressed by words and other symbols, but asymbolic thoughts, that is, the connexions of the ideas themselves, are either from sense perception or from distinct imagination. (We have a distinct imagination when we distinguish a subject matter into parts by examining it and considering it through its circumstances, as long as nothing new happens that is relevant to the matter at hand.) Hence the theorems change as relations change, just as the same city changes its shape depending on which side we see it from. It therefore seems to me that one must distinguish between propositions; for the truth of some, such as those that are based on experiments and observations of nature, depends upon sense-perception, while the truth of others, such as the theorems of arithmetic and geometry, depends upon a clear and distinct imagination or ideas or, if you prefer, definitions (for definition is nothing other than the signification of "idea"). Therefore signs and symbols, whether they are words or characters, are arbitrary, but all nations have the same ideas. Yet in reasoning about very complex things we are accustomed to use symbols, without considering the ideas themselves. These *thoughts* are *blind*, since in them we are content with an analogy with small and simple thoughts distinctly comprehended. For example, when we say "100,000," no one forms all its units in his mind; for he knows that by doing this he can leave them behind the symbols. And the art of devising symbols consists in this: that they be briefer than the ideas themselves and yet free from confusion and suited to revealing (to the extent that this is possible) proportions of every kind in those very ideas no less easily than if they had been resolved into ultimate elements or had been clearly and distinctly understood. And the decimal progression does this quite well for numbers; for without a progression like this it would have been too tedious for mortals to count up huge numbers. Algebra does the same thing for geometry, so that even when we assume impossibilities, such as dimensions beyond the third and surd[8] numbers and numbers less than nothing, it nevertheless succeeds.

Therefore, since when we have found suitable symbols they support our mind like spiritual machines, but those symbols we have now, except in the pure mathematical sciences (although even there I could wish for more), are neither simple, nor complete, nor ordered, it appears that no one would be more deserving in the whole realm of human reasoning than someone who could devise either a

[8]Surd numbers are irrational numbers, that is, numbers that are not equal to fractions where both the numerator and denominator are whole numbers. $\sqrt{2}$ is an example of a surd number. The Latin word "surdus," which Leibniz uses here, literally means mute.

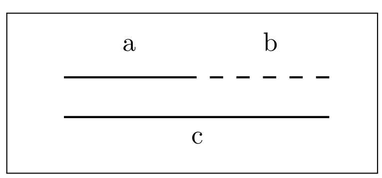
philosophical language or at least a writing to serve for rigorous investigations. I stated this six years ago in the *Dissertation on the Combinatoric Art*, which, although it is a childish work written in an academic manner, I will nevertheless not entirely scorn now. There I pointed out that all propositions of the pure sciences, that is, those independent of sense-perception (although we can also, as it were, examine and confirm their truth by sense-perception), which include also the sciences of action in general, of reasoning, of motion, of utility, and of justice, do nothing but pronounce either a definition or a part of a definition (or a definition of a part or a part of a part, wholly or in part) about something that is defined or about some other definition of the same thing. And I pointed out that the same idea can be expressed by various definitions and this gives birth to a fertile art for constructing theorems. I remember Pascal saying the same thing somewhere or another,[9] where he recommends that we give varied enunciations of the same theorems and says that the whole study of geometers should consist in this; for thus, he says, we open a way to new and untouched things. Cujas, in his Paratitla, also observed that we can usefully propose many definitions of the same name; indeed, definitions in that universal characteristic Note 7, page 24 are the same as equations in algebra.

But let us show, by the deed itself, rather than by words, the demonstrability of the axioms we set out as examples.

**First:** *that things equal to the same third thing are equal to each other.* We immediately understand this axiom from the definition of equality. For let $a = b$ and $b = c$; I say $a = c$. For things are *equal* which have the same quantity or which can be substituted for each other while preserving the quantity; therefore let us substitute either $c$ in place of $b$ in the equation $a = b$ or $a$ in place of $b$ in the equation $b = c$, and either way we will have $a = c$. Q. E. D.

**Second:** *that equals added to or subtracted from equals make equals.* $a = b$ and $c = d$. I say that $a + c = b + d$. For $a + c = b + c$ (because $a = b$) and $b + c = b + d$ (because $c = d$). Therefore $a + c = b + d$.

**Third:** *that the whole is greater than the part.* For if (def. 1) the parts are $a$, $b$, the whole (def. 2) will be $a + b$.




Note 8, page 24 Again, if $a$ is less (def. 3), then $c = a + b$ will be *greater* (def. 4). If we put the definitions together the demonstration will complete: the *whole*

[9]In the *Traité des Ordres Numériques* (*Treatise on Numerical Orders*), *Oeuvres*, Mesnard edition, volume II, page 1329.

$= a + b$ (def. 2), $a + b = c$ (def. 4), $c =$ greater (def. 4), the part $= a$ (def. 1), and $a =$ less (def. 3).

**Fourth:** *that equimultiples are as simples*; e.g., as three are to four, so are twice three to twice four.

$\frac{ca}{cb} = \frac{a}{b}$. For $\frac{ca}{cb} = \frac{c}{c} \cap \frac{a}{b}$. Now $\frac{c}{c} = 1$ and $1 = \frac{1}{1}$, and therefore

$$\frac{ca}{cb} = \frac{1}{1} \cap \frac{a}{b} = \frac{1a}{1b} = \frac{a}{b}.$$

So that there cannot be any doubt left, I prove that $\frac{ca}{cb} = \frac{c}{c} \cap \frac{a}{b}$ as follows:

$$\frac{c}{c} \cap \frac{a}{b} = \frac{c \cap \frac{a}{b}}{c} = \frac{\frac{ca}{b}}{c} = \frac{ca}{cb}.$$

Here $\cap$ is the multiplication sign.

**Fifth:** *that if proportionals are added or taken away from proportionals, proportionals are produced.* For example, since 4 is to 8 as 3 is to 6, so 4+3, or 7, will have the same ratio to to 8+6, or 14; that is, if $\frac{a}{b} = \frac{c}{d}$, then they both $= \frac{a+c}{b+d}$. But first of all let me demonstrate this *lemma*: $bc = ad$; for because $\frac{a}{b} = \frac{c}{d}$, by multiplying each side by $d$, $\frac{ad}{b}$ will be $= \frac{c}{1}$, and therefore by multiplying each side by $b$, we will have $ad = bc$. Now, continuing, if

$$\frac{a+c}{b+d} \times \frac{a}{b} = 1,$$

then will Note 9, page 24

$$\frac{a+c}{b+d} = \frac{a}{b}.$$

That the latter equation follows from the former is obvious; for

$$\frac{a+c}{b+d} \times \frac{a}{b} = \frac{a+c}{b+d} \cup \frac{a}{b} = \frac{a + c \cup \frac{a}{b}}{b+d}$$

$$= \frac{a + c \cup a \cup \frac{1}{b}}{b+d} = \frac{a + c \cap b \cup \frac{b}{b}}{b + d \cap a} = \frac{a + c \cap b \cup 1}{b + d \cap a}$$

$$= \frac{a + c \cap b}{b + d \cap a} = \frac{a+c}{b+d} \times \frac{a}{b}.$$

I prove the former equation as follows:

$$\frac{a+c}{b+d} \times \frac{a}{b} = \frac{ba+bc}{ab+ad},$$

and because, by the preceding lemma, $bc = ad$, it follows that

$$\frac{ba+bc}{ab+ad} = \frac{ba+bc}{ab+bc} = 1.$$

From this last example we see that this proposition, which we made our fifth axiom, is no easier to demonstrate than some others that are counted as theorems. For example, it is a theorem that *if two ratios are equal, their inverses are also equal.* We easily demonstrate this as follows: $\frac{a}{b} = \frac{c}{d}$, I say $\frac{b}{a} = \frac{d}{c}$. For if $\frac{b}{a} \times \frac{d}{c} = 1$, then will $\frac{b}{a} = \frac{d}{c}$. I prove the former equation as follows: $\frac{b}{a} \times \frac{d}{c} = \frac{bc}{da} = \frac{bc}{bc} = 1$; for by the stated lemma $da = bc$.

These examples should be enough to support our observation—an observation that, although it is scarcely believed, is nonetheless necessary to establish the rigor of the sciences against the Pyrrhonists.

# Notes on Leibniz's "An Approach to the Arithmetic of Infinites"

Leibniz wrote this paper for the *Journal des Sçavans* (Journal of the Learned) in late 1672. The paper is written in Latin, and we have translated it from a text published in 1976 in an edition of Leibniz's collected works put out by the Berlin Academy of Sciences, the *Sämtliche Schriften und Briefe* (Collected Writings and Letters), in Series III, Volume 1, on pages 1–20.

## Note 1

In a short text titled "On the Use and Necessity of Demonstrations of the Immortality of the Soul," Leibniz says more about how the incorporeality of the soul is connected to the science of the indivisible and the infinite:

> But I shall say nothing about Mind except what can be both clearly perceived and distinctly demonstrated. What I shall say about Mind will be no more difficult than what geometers say about a point and angles. Indeed, the theory of points and angles, of the instant, of endeavor (by *endeavor* I mean a last or least motion, that is, a motion which happens in an instant, within a point), will be for me the key to explaining the nature of thought. For I shall demonstrate that Mind exists in a point, that thought is endeavor or least motion, and that a body may have many endeavors at one time, although it has only one motion. Whence it will follow that a mind may no more be destroyed than a point. For a point is indivisible, and therefore cannot be destroyed. Therefore though a body may be burned up and scattered to all the corners of the earth, the Mind will remain safe and untouched in its point. For who will burn up a point?[10]

In this first paragraph of "An Approach to the Arithmetic of Infinites," Leibniz briefly alludes to four examples. It is not necessary to understand these examples to go on, but it may be helpful to say a few words about them here.

1. Any line can be cut into infinitely many separate lines.

2. We can "calculate the sum of infinitely many decreasing quantities." Leibniz will give a few examples of this later in the paper.

3. We can "prescribe limits to quantities increasing or decreasing in a finite space." Here Leibniz may be thinking of constructions like those in Proposition 2 of Book XII of Euclid's *Elements*. There Euclid shows how to construct an infinite series of polygons inscribed in a circle, such that each polygon contains the the previous polygon in the series. These inscribed polygons are "quantities increasing ... in a finite space," and the

[10] See Series II, volume 1, page 113, of the Berlin Academy's edition of Leibniz's collected works. This text was written in 1671.

limit which is prescribed for them is the circle. Similarly, Euclid shows in the same proposition how to construct an infinite series of polygons circumscribed around a circle, such that each polygon is contained by the previous polygon in the series. These circumscribed polygons are "quantities decreasing in a finite space," and the limit which is prescribed for them is again the circle.

4. We can"generate finite figures and demonstrate their proportions by multiplying infinites by one another." Infinites may be used to find proportions between curvilinear figures. Here is an argument using infinity to find the area of circle:[11]

   > We wish to find the relation between the area of a circle and its circumference. For simplicity we suppose that the radius of the circle is 1. Now, the circle can be thought of as composed of infinitely many straight-line segments, all equal to each other and infinitely short. The circle is then the sum of infinitesimal triangles, all of which have altitude 1. For a triangle the area is half the base times the altitude. Therefore the sum of the areas of the triangles is half the sum of the bases. But the sum of the areas of the triangles is the area of the circle, and the sum of the bases of the triangles is its circumference. Therefore the area of the circle of radius 1 is equal to one half its circumference.

   In taking a circle to be a sum of infinitely many infinitely small triangles, we may be said to generate it by multiplying an infinite (one triangle, which is infinitely small) by another infinite (the number of these triangles).

## Note 2

Leibniz does not give a demonstration of this rule in this paper, but here is a roughly Euclidean demonstration. We first note that

$$CB + DC + ED + \text{ etc.} = AB.$$

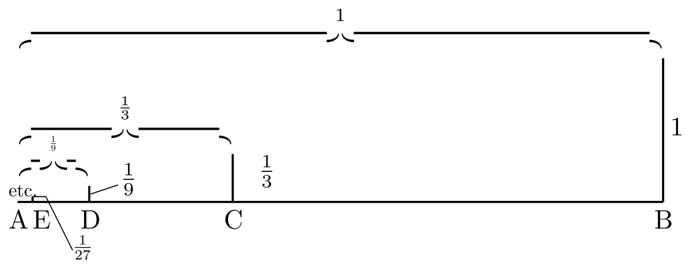
Next, according to Proposition 19 in Book V of Euclid's *Elements*, because

$$AB : AC :: AC : AD,$$

it follows that

$$(AB - AC) : (AC - AD) :: AB : AC, \text{ that is, } CB : DC :: AB : AC.$$

[11] This passage is from the the Santa Fe Junior Mathematics Manual (2007 edition, page 12). It appears to be quoted from *The Mathematical Experience*, by Philip J. Davis and Reuben Hersh (on pages 262 and 263 of the 1995 study edition, published by Birkäuser). Davis and Hersh attribute the quote to Nicholas of Cusa, but do not give a reference.




Alternating (Euclid, V 16), we get

$$CB : AB :: DC : AC.$$

Similar arguments show that

$$DC : AC :: ED : AD.$$

We could go on indefinitely in the same way, getting an infinite series of proportions:

$$CB : AB :: DC : AC :: ED : AD \ldots$$

Now according to Euclid V 12, "if any number of magnitudes be proportional, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents." If we suppose Euclid's proposition is true not only for any *finite* number of magnitudes, but also for an *infinite* number of magnitudes, then as one of the antecedents ($CB$) is to one of the consequents ($AB$), so will all of the antecedents ($CB + DC + ED +$ etc.) be to all of the consequents ($AB + AC + AD + AE +$ etc.). Expressing this proportion in a equation, we get:

$$\frac{CB}{AB} = \frac{CB + DC + ED + \text{ etc.}}{AB + AC + AD + \text{ etc.}} = \frac{AB}{AB + AC + AD + \text{ etc.}}$$

This is Leibniz's equation.

## Note 3

We can obtain these sums by substituting in numerical values for the lines $AB$, $AC$, $AD$, etc. To get the first sum, let $AB = 1$ and $AC = \frac{1}{2}$. Then $AD$ is likewise half of $AC$ (because $AD : AC :: AC : AB$), and $AE$ is half of $AD$, and so on; therefore

$$AD = \frac{1}{4},\ AE = \frac{1}{8}, \text{ etc.}$$

Then $CB = AB - AC = 1 - \frac{1}{2} = \frac{1}{2}$. We substitute all these values into Leibniz's equation,

$$\frac{CB}{AB} = \frac{AB}{AB + AC + AD + AE + \text{etc.}},$$

to get

$$\frac{\frac{1}{2}}{1} = \frac{1}{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \text{ etc.}}$$

Inverting both sides, we get

$$\frac{2}{1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \text{ etc.}$$

Dividing both sides by 2, we get

$$\frac{1}{1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \text{ etc.,}$$

which is Leibniz's first sum. To get the second and third sums, we proceed in the same way, but set $AC = \frac{1}{3}$ and $AC = \frac{1}{4}$, respectively.

**Problem 1**

Show that if $AB = 1$ and $AC = x$, then

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \text{ etc.}$$

## Note 4

Leibniz's notation here may be confusing. The triangular numbers are simply the numbers 0, 1, 3, 6, 10, 15, 21, 28, etc. The numbers 1, 2, 3, 4, 5, 6, and 7, written between and above the triangular numbers, are the differences of the pairs of consecutive triangular numbers: the difference of 0 and 1 is 1, the difference of 1 and 3 is 2, the difference of 3 and 6 is 3, etc.

Triangular numbers may also be defined as numbers whose units may be evenly arranged into equilateral triangles. This is the way Nicomachus defines them, in Book II, Chapter 8 of his *Introduction to Arithmetic*.

## Note 5

Each number in this table is equal to the difference of the two nearest numbers to its right. For example, the triangular number 6 is equal to the difference of the pyramidal numbers 10 and 4, the triangulo-pyramidal 252 is equal to the difference of the pyramido-pyramidals 462 and 210, etc. Or, what amounts to the same thing, each number is the sum of the number above it and the number above it and to its left: $10 = 6 + 4$, $462 = 252 + 210$, etc. This gives an easy way to generate all the entries in the table by adding. For example, to fill in one more row in the table, we note that the next natural number after 7 is 8, and therefore the next triangular number after 28 must be $28 + 8 = 36$, the next pyramidal number after 84 must be $84 + 36 = 120$, the next triangulo-triangular number after 210 must be $210 + 120 = 330$, etc. We could go on indefinitely in this way, filling out the entries of the table both down and to the right.

Table 3

| zero | units | naturals | triangulars | pyramidals | triangulo-triangulars | triangulo-pyramidals | pyramido-pyramidals |
|---|---|---|---|---|---|---|---|
| 0 | | | | | | | |
| | 1 | | | | | | |
| 0 | | 1 | | | | | |
| | 1 | | 1 | | | | |
| 0 | | 2 | | 1 | | | |
| | 1 | | 3 | | 1 | | |
| 0 | | 3 | | 4 | | 1 | |
| | 1 | | 6 | | 5 | | 1 |
| 0 | | 4 | | 10 | | 6 | |
| | 1 | | 10 | | 15 | | 7 |
| 0 | | 5 | | 20 | | 21 | |
| | 1 | | 15 | | 35 | | 28 |
| 0 | | 6 | | 35 | | 56 | |
| | 1 | | 21 | | 70 | | 84 |
| | | 7 | | 56 | | 126 | |
| | | | 28 | | 126 | | 210 |
| | | | | 84 | | 252 | |
| | | | | | 210 | | 462 |
| | | | | | | 462 | |
| | | | | | | | 924 |

The pyramidal numbers are those numbers whose units may be evenly arranged into a pyramid, that is, into a regular four-sided solid. Pyramidal numbers are thus a kind of solid numbers, just as triangular numbers are a kind of plane numbers. The other numbers could be said to have dimension greater than three, and Leibniz names them in analogy with Viète's ladder magnitudes.[12] Just as Viète calls fourth degree numbers square-squares, fifth degree numbers square-cubes, etc., so Leibniz calls the numbers after the pyramidals triangulo-triangulars, and the the numbers in the next column triangulo-pyramidals, etc.

## Note 6

These sums follow from an important fundamental principle that Leibniz gives elsewhere:

> *The sum of the successive differences of any series of terms is equal to the difference of its extreme terms.*[13]

We will call it the *principle of sums of differences.*

The easiest way to understand the principle is through examples. Suppose our series of terms is the series of the first five square numbers, beginning from 1:

$$1, 4, 9, 16, 25.$$

Then the successive differences are:

$$4 - 1 = 3,\ 9 - 4 = 5,\ 16 - 9 = 7,\ \text{and } 25 - 16 = 9.$$

According to the principle of sums of differences, the sum of these differences is equal to the difference of the extreme terms of the series, namely to $25 - 1$. To see why this is so, examine the following table:

| series | sum of differences |
|---|---|
| 1 | |
| 4 | $4 - 1$ |
| 9 | $+9 - 4$ |
| 16 | $+16 - 9$ |
| 25 | $+25 - 16$ |
| | $= 25 - 1$ |

If we take the sum of all the terms in the right column, all the terms cancel except for $-1$ in the first difference and the 25 in the last. The same thing will happen when we sum the differences of any series whatever: all the intermediate terms will cancel, and we will be left with the difference of the extreme terms.

[12] See his *Introduction to the Analytic Art.*

[13] Leibniz gives this principle in "The History and Origin of the Differential Calculus," which is translated by J. M. Child in *The Early Mathematical Manuscripts of Leibniz.* The quote is on pages 30 and 31 of Child's translation, and in Latin on page 396 in Volume V of C. I. Gerhardt's edition of Leibniz's mathematical writings.

Table 4: Series of fractions of a replicated arithmetic progression

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | etc. | exponents |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | | Series of fractions of a replicated arithmetic progression with unit generator |
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{6}$ | $\frac{1}{7}$ | | |
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{10}$ | $\frac{1}{15}$ | $\frac{1}{21}$ | $\frac{1}{28}$ | | |
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{4}$ | $\frac{1}{10}$ | $\frac{1}{20}$ | $\frac{1}{35}$ | $\frac{1}{56}$ | $\frac{1}{84}$ | | |
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{5}$ | $\frac{1}{15}$ | $\frac{1}{35}$ | $\frac{1}{70}$ | $\frac{1}{126}$ | $\frac{1}{210}$ | | |
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{6}$ | $\frac{1}{21}$ | $\frac{1}{56}$ | $\frac{1}{126}$ | $\frac{1}{252}$ | $\frac{1}{462}$ | | |
| $\frac{0}{0}$ | $\frac{1}{1}$ | $\frac{1}{7}$ | $\frac{1}{28}$ | $\frac{1}{84}$ | $\frac{1}{210}$ | $\frac{1}{462}$ | $\frac{1}{924}$ | | |
| $\frac{0}{0}$ | $\frac{0}{0}$ | $\frac{1}{0}$ | $\frac{2}{1}$ | $\frac{3}{2}$ | $\frac{4}{3}$ | $\frac{5}{4}$ | $\frac{6}{5}$ | etc. | sums |

Now each of the columns in Leibniz's Table 2, beginning with the column with exponent 3, is in fact proportional to the series of differences of the preceding column. For example, each term in the column with exponent 3 is equal to twice the difference of two successive terms in the column with exponent 2:

| term in column 3 | = | 2 × diff. of terms in col. 2: |
|---|---|---|
| $\frac{1}{1}$ | = | $2\left(\frac{1}{1}-\frac{1}{2}\right)$ |
| $\frac{1}{3}$ | = | $2\left(\frac{1}{2}-\frac{1}{3}\right)$ |
| $\frac{1}{6}$ | = | $2\left(\frac{1}{3}-\frac{1}{4}\right)$ |
| etc. | | etc. |

Therefore

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}\text{ etc.} = 2\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\text{ etc.}\right]$$

Now the quantity in brackets on the right side of this equation is a sum of differences; but it is an infinite sum, and not a finite sum. If we had taken only three differences, then according to the principle of sums of differences, the sum

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)$$

would have been

$$\frac{1}{1}-\frac{1}{4}.$$

If we had taken four differences, then the sum

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)$$

would have been

$$\frac{1}{1}-\frac{1}{5}.$$

If we had taken five differences, then the sum would have been

$$\frac{1}{1}-\frac{1}{6},$$

and so on. Thus as we take more differences these sums become closer and closer to $\frac{1}{1}$, and in fact the differences between the sums and $\frac{1}{1}$, namely,

$$\frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \text{etc.},$$

become less than any given quantity. Therefore, when we take infinitely many differences, we may say that the sum is equal to $\frac{1}{1}$:

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\text{ etc.}=\frac{1}{1}.$$

Therefore

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}\text{ etc.}=2\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\text{ etc.}\right]=\frac{2}{1}.$$

In a manuscript written a few years later, Leibniz writes:

> Whenever it is said that a certain infinite series of numbers has a sum, I am of the opinion that all that is being said is that any finite series with the same rule has a sum, and that the error always diminishes as the series increases, so that it becomes as small as we would like.[14]

### Problem 2

Show that each term in the column with exponent 4 is equal to $\frac{3}{2}$ times the difference of two successive terms in the column with exponent 3. Use this, and the principle of sums of differences, to show that (as Leibniz claims)

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\text{ etc.}=\frac{3}{2}.$$

[14] "Infinite Numbers," from April of 1676. The translation is from Richard T. W. Arthur's *The Labyrinth of the Continuum*, Yale: New Haven and London, 2001, page 99. The Latin text is in Arthur's book, as well as Series VI, Volume 3, pages 496–504 of the Berlin Academy edition.

**Example 1**

We can use the fact that each column in Table 1 is the series of successive differences of the following column to find the sums. For example, the triangular numbers are the differences of the pyramidals:

$$\begin{aligned} 3 &= 4 - 1 \\ 6 &= 10 - 4 \\ 10 &= 20 - 10 \\ 15 &= 35 - 20, \text{ etc.} \end{aligned}$$

Therefore, by the principle of sums of differences,

$$3 + 6 + 10 + 15 = 35 - 1 = 34.$$

**Problem 3**

Find the following sums using the principle of sums of differences:

a. $4 + 10 + 20 + 35 + 56$

b. $15 + 35 + 70 + 126 + 210$.

**Example 2**

We can use the fact that in Table 2 each column is proportional to the series of successive differences of the preceding column to find sums of *finite* numbers of terms. For example, as we saw above, each triangular fraction is twice the difference of successive natural fractions:

$$\begin{aligned} \tfrac{1}{1} &= 2\left(\tfrac{1}{1} - \tfrac{1}{2}\right) \\ \tfrac{1}{3} &= 2\left(\tfrac{1}{2} - \tfrac{1}{3}\right) \\ \tfrac{1}{6} &= 2\left(\tfrac{1}{3} - \tfrac{1}{4}\right) \text{ etc.} \end{aligned}$$

Therefore, by the principle of sums of differences,

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} = 2\left(\frac{1}{1} - \frac{1}{4}\right) = \frac{3}{2}.$$

**Problem 4**

Find the following sums using the principle of sums of differences:

a. $\frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{1}{28}$

b. $\frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \frac{1}{35} + \frac{1}{56}$.

## Note 7

The "universal characteristic" is the science by means of which we find suitable symbols (that its, characters) for thoughts. In a letter to a friend, Ehrenfried Walther von Tschirnhaus, in 1678, Leibniz writes that

> by means of [the general characteristic] all our thoughts can be, as it were, painted and fixed, and contracted and put in order: *painted*, so that they may be taught to others; *fixed* for us so that we may not forget them; *contracted* so that they may be expressed in few words, *put in order* so that they may all be held in view by those who contemplate them. (*Collected Writings*, Berlin Academy edition, Series III, Volume 2, page 450.)

## Note 8

Leibniz is not citing definitions here, but giving them. The first definition lets us substitute the symbols $a$ and $b$ for the word *part*, or, what amounts to the same thing for Leibniz, gives us two equations: $\textit{part} = a$, and $\textit{part} = b$. The second definition lets us substitute $a + b$ for the word *whole*, that is, it gives us the equation $\textit{whole} = a + b$. The other two definitions may be understood similarly.

## Note 9

Leibniz seems to be using the symbol $\times$ so that, in general,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \cap d}{b \cap c}.$$

Leibniz thus understands $\times$ to mean "compound with the inverse ratio."

Leibniz uses $\cup$ as a kind of division sign,

$$\frac{a}{b} \cup \frac{c}{d} = \frac{a \cup \frac{c}{d}}{b} = \frac{a \cap \frac{d}{c}}{b}.$$

When two quantities are joined by $\cup$, they are thus treated more as numbers, while when they are joined by $\times$, they are treated more as ratios. It turns out that

$$\frac{a}{b} \cup \frac{c}{d} = \frac{a}{b} \times \frac{c}{d},$$

but this requires a demonstration, which Leibniz gives in the following few lines.

# A New Method for Greatests and Leasts, as well as for Tangents, which is not Hindered by Fractional or Irrational Quantities, and a Singular Calculus for these Note 1, p. 41

by Gottfried Wilhelm Leibniz

Let there be (Figure 1) an axis $AX$, and several curves, such as $VV$, $WW$, $YY$, and $ZZ$; let their ordinates normal [that is, perpendicular] to the axis be $VX$, $WX$, $YX$, and $ZX$, let them be called $v$, $w$, $y$, and $z$, respectively, and let $AX$, the abscissa cut off on the axis, be called $x$. Let the tangents be $VB$, $WC$, Note 2, p. 42 $YD$, and $ZE$, meeting the axis at the points $B$, $C$, $D$, and $E$, respectively. Now let some arbitrary straight line be called $dx$, and let the straight line which is to $dx$ as $v$ (or $w$, or $y$, or $z$) is to $XB$ (or $XC$, or $XD$, or $XE$) be called $dv$ (or $dw$, or $dy$, or $dz$) or the difference of the $v$'s themselves (or of the $w$'s, $y$'s, or $z$'s themselves). If we assume these things, the rules of the calculus will be as Note 3, p. 42 follows.

Let $a$ be a given constant quantity; then $da$ will be equal to 0, and $d(ax)$ will be equal to $a\,dx$. If $y = v$ (that is, if each ordinate of the curve $YY$ is equal Note 4, p. 46 to the corresponding ordinate of the curve $VV$), then $dy = dv$.
Now for *Addition* and *Subtraction*: if

$$z - y + w + x = v,$$

then we shall have

$$d(z - y + w + x) = dz - dy + dw + dx.$$

*Multiplication*:

$$d(xv) = x\,dv + v\,dx,$$

that is, setting $y = xv$,

$$dy = x\,dv + v\,dx;$$

for we are free to use either a formula such as $xv$, or a letter as an abbreviation for it, such as $y$. Note that in this calculus we treat $x$ and $dx$ in the same way as $y$ and $dy$ or any other indeterminate letter together with its differential. Also note that we cannot always go backwards from a differential equation, unless we are cautious; but let us not go into that here.

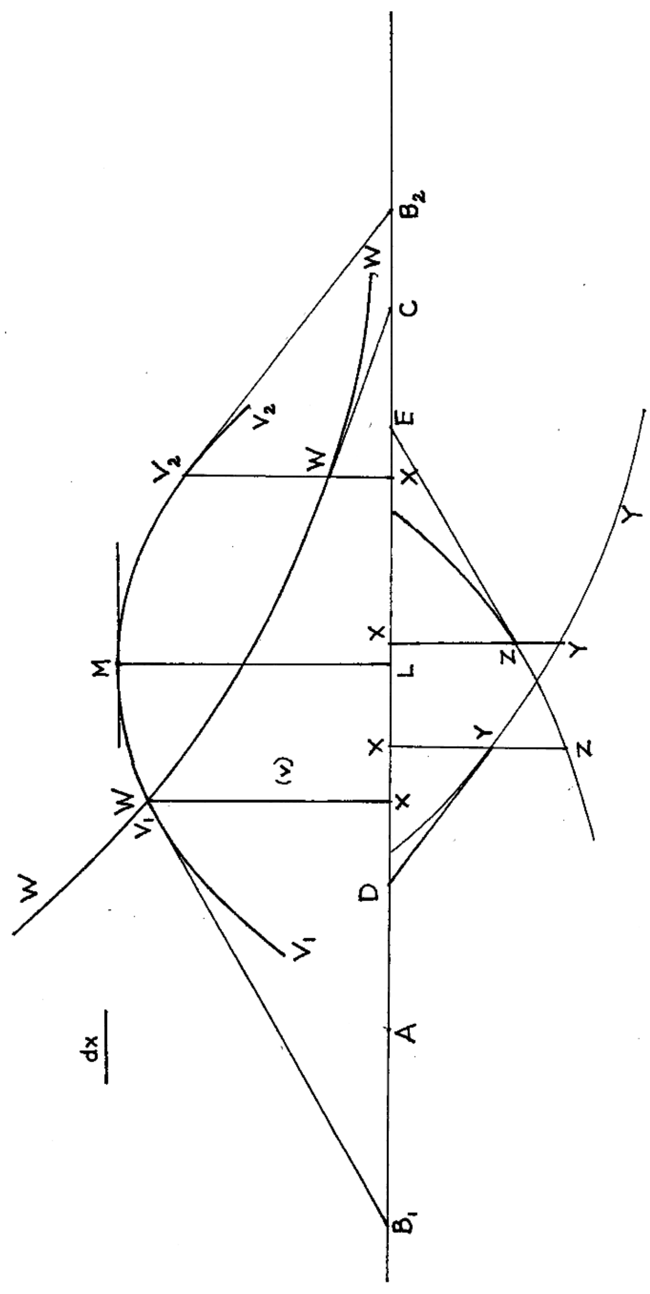



Figure 1: Leibniz's figure, slightly simplified

Next, *Division*:[1]

$$d\left(\frac{v}{y}\right) \text{ or (setting } z = \tfrac{v}{y}) \; dz = \frac{y\,dv - v\,dy}{y^2}.$$

As far as *Signs* are concerned, we should note that when we substitute the differential of a letter for that letter we keep the same sign, and we write $+dz$ in place of $+z$, and $-dz$ in place of $-z$, as we can see from the addition and subtraction rule we laid down above; but when it comes to the exegesis of values, that is, when we consider the relation of $z$ to $x$, then it becomes clear whether the value of $dz$ is positive or less than zero (negative): when this happens, let us draw the tangent $ZE$ from the point $Z$ not towards $A$, but in the opposite direction (to the right of $X$).[2] This happens when the ordinates $z$ decrease as

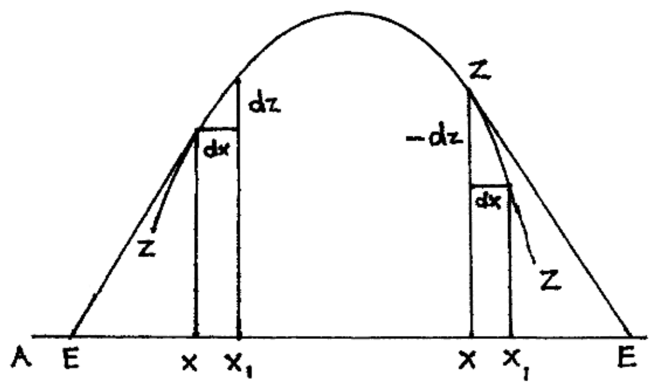



Figure 2: our figure, not Leibniz's

the $x$'s increase. And because the ordinates $v$ sometimes increase and sometimes decrease, $dv$ will sometimes be a positive quantity and sometimes be negative; in the former case we draw the tangent $V_1B_1$ towards $A$, in the latter case in the opposite direction. But we do neither in the middle around $M$, for then the $v$'s themselves neither increase nor decrease, but stand still, and therefore $dv = 0$, and it does not matter whether the quantity is positive or negative,

[1] We have simplified Leibniz's rule. Leibniz does not let the values of his ordinates change signs when a curve crosses the axis, and this makes his formula for the division rule more complicated. Leibniz writes the division rule as follows:

$$d\left(\frac{v}{y}\right) = \frac{\pm v\,dy \mp y\,dv}{y^2}.$$

We will always take our ordinates as negative when the curve is below the axis, following modern conventions and avoiding Leibniz's ambiguous signs $\pm$ and $\mp$.

[2] See Figure 2. At the first point where the ordinate is equal to $z$, that is, at the point on the left, $z_1$ is greater than $z$, and therefore $dz$, which is equal to $z_1 - z$, is positive. At the second point where the ordinate is equal to $z$, that is, the point on the right, $z_1$ is less than $z$, and therefore $dz$ is negative.

for $+0 = -0$; and here $v$, that is, the ordinate $LM$, is the *Greatest* ordinate (or if the convexity turns towards the axis, the *Least*) and we draw the tangent to the curve at $M$ neither towards $A$ (to the left of $X$), approaching the axis there, nor in the opposite direction (to the right of $X$); instead we draw it parallel to the axis. [See Figure 1 or Figures 3 and 4.] If $dv$ is infinite with

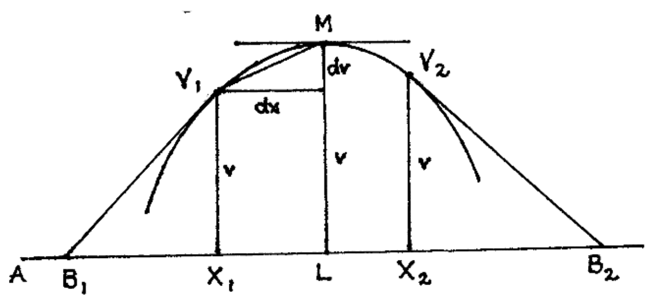



Figure 3: our figure, not Leibniz's

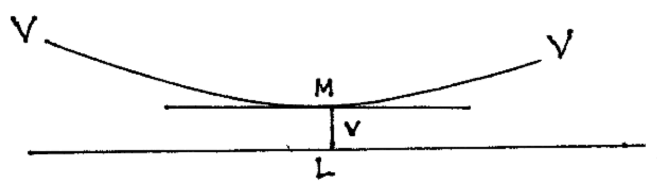



Figure 4: our figure, not Leibniz's

respect to $dx$, then the tangent is perpendicular to the axis, that is, it is itself an ordinate.[3] If $dv$ and $dx$ are equal, the tangent makes half a right angle with the axis [see Figure 6]. If when the ordinates $v$ increase their increments or differences $dv$ also increase (that is, if we suppose that the $dv$'s are positive, then the $ddv$'s,[4] the differences of their differences, are also positive, and if negative, then negative), then the curve turns its *convexity* toward the axis; in the opposite case [that is, where the ordinates increase and their differences decrease], the curve turns its *concavity* toward the axis.[5] But where there is a greatest or least increment, or where the increments go from decreasing to

[3]See Figure 5. Here, as $V_1$ becomes infinitely close to $V_p$, the ratio of $v_1 - v$ (that is, $dv$) to $x_1 - x$ (that is, $dx$), becomes infinite.

[4]The modern notation for $ddv$ is $d^2v$.

[5]See Figure 7, page 30. To say that "the curve turns its convexity toward the axis," is to say that it bends away from the axis. If it turns its concavity toward the axis, then it bends toward the axis. Because we use signs in a different way from Leibniz's (see note 7, page 48), the rule is even simpler for us: when $ddv$ is positive, the curve turns its convexity downward, that is, it bends upward; while if $ddv$ is negative, it turns its concavity downward, that is, it bends downward.

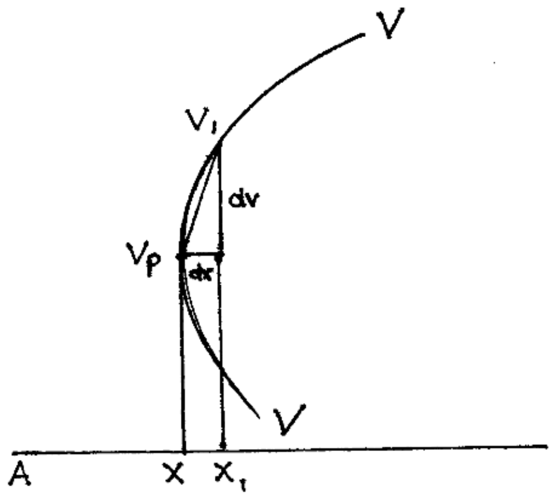



Figure 5: our figure, not Leibniz's

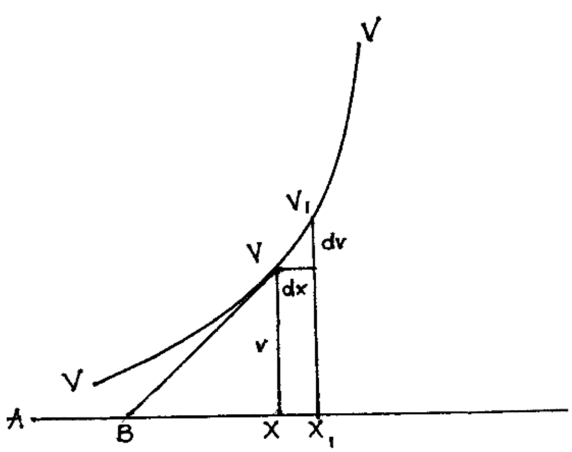



Figure 6: our figure, not Leibniz's

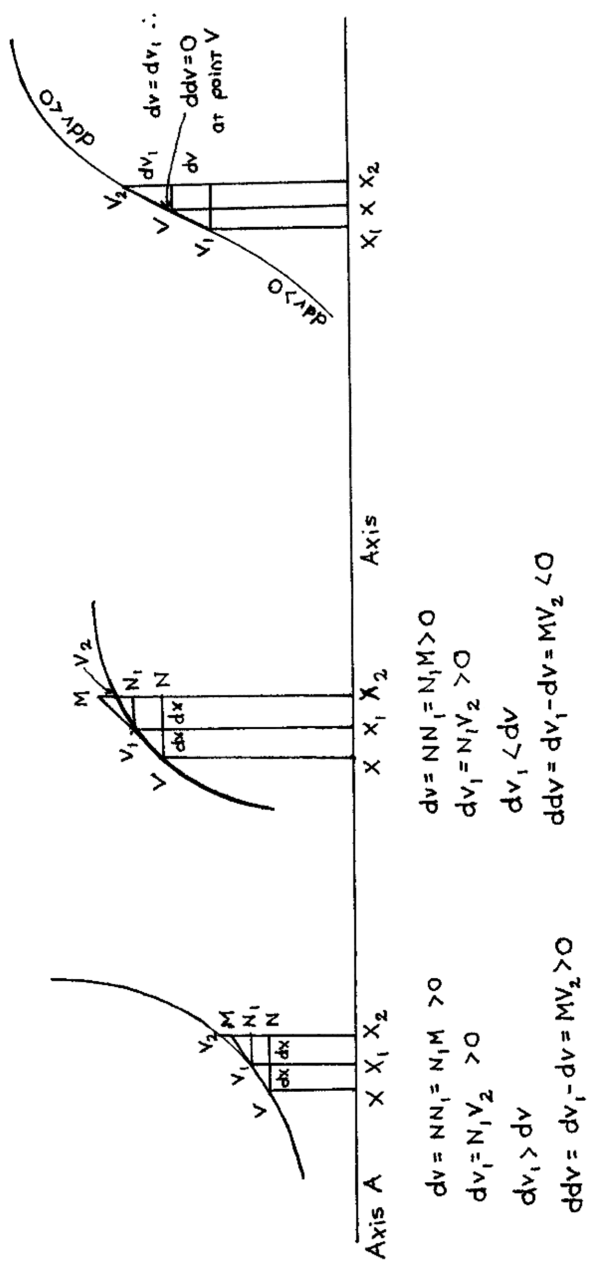



Figure 7: our figure, not Leibniz's

increasing, or conversely, there is an *inflection point*, and there is a change from concavity to convexity, or conversely [see Figure 8, page 32] provided that the ordinates do not also change from increasing to decreasing at that point, or conversely, for then the convexity or concavity would remain the same. But the increments cannot continue to increase or decrease when the ordinates go from increasing to decreasing, or conversely. And so an inflection point occurs when neither $v$ nor $dv$ is 0 but $ddv$ *is* 0. It also follows that the problem of finding an inflection point, unlike the problem of finding a greatest ordinate, has not two, but three equal roots. And all this of course depends on the correct use of signs.[6] Note 5, p. 46

*Powers*:

$$d(x^a) = ax^{(a-1)}\,dx;$$

for example, $d(x^3) = 3x^2\,dx$.

$$d\left(\frac{1}{x^a}\right) = -\frac{a\,dx}{x^{(a+1)}};$$

e.g., if

$$w = \frac{1}{x^3},$$

then

$$dw = -\frac{3\,dx}{x^4}.$$

*Roots*:

$$d\sqrt[b]{x^a} = \frac{a}{b}\,dx\,\sqrt[b]{x^{(a-b)}}$$

(hence

$$d\sqrt[2]{y} = \frac{dy}{2\sqrt[2]{y}},$$

since in this case $a$ is 1, and $b$ is 2; therefore

$$\frac{a}{b}\sqrt[b]{x^{(a-b)}} \text{ is } \frac{1}{2}\sqrt[2]{y^{-1}};$$

now $y^{-1}$ is the same as $\frac{1}{y}$, from the nature of the exponents of the geometric progression, and $\sqrt[2]{\frac{1}{y}}$ is $\frac{1}{\sqrt[2]{y}}$), and

$$d\left(\frac{1}{\sqrt[b]{x^a}}\right) = \frac{-a\,dx}{b\sqrt[b]{x^{(a+b)}}}.$$

But the rule for a whole power would have sufficed to determine both fractions and roots; for the power is a fraction when the exponent is negative, and changes

[6] Here Leibniz includes a paragraph on signs, which we again can omit if we use signs in the modern way, letting ordinates become negative when a curve crosses the axis. We have included the omitted paragraph in the accompanying notes after the fifth note (p. 47).

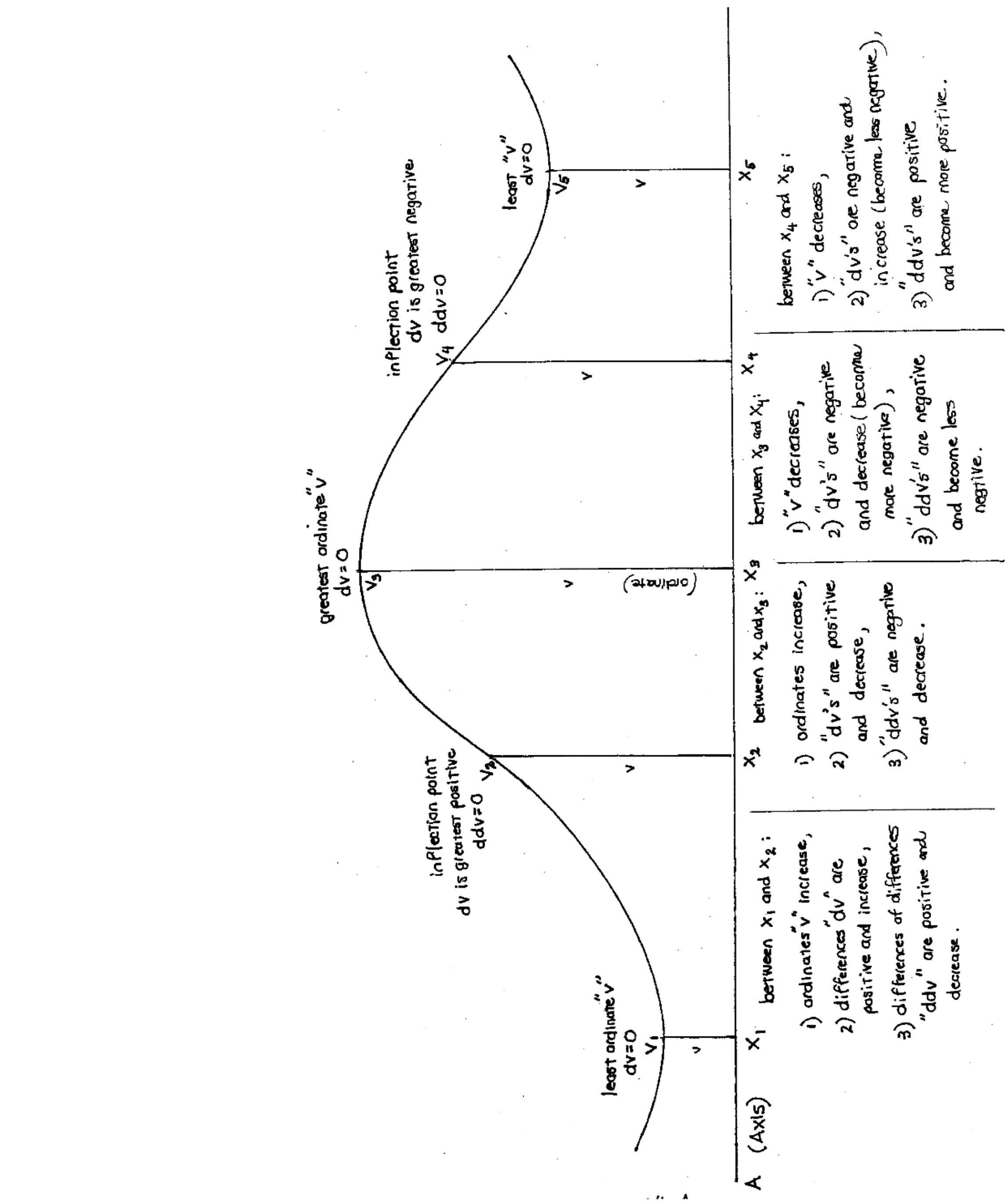



Figure 8: our figure, not Leibniz's

into a root when the exponent is a fraction. But I chose to deduce such consequences myself, rather than leave them for others to deduce, since they are quite general and occur frequently, and since when something is so inherently complicated we should try to find ways to make it easier.

Note 6, p. 47

Note 7, p. 48

Once we have learned this *Algorithm* (as I call it) of our calculus (which I call the *differential calculus*), we can find all other differential equations through the common calculus, and we can obtain least and greatest lines, as well as tangents, without needing to eliminate fractions and irrationals or other impediments, as still had to be done when using the previously published methods. Someone who is versed in these matters will easily be able to demonstrate all these things if he considers the following point (one that has not yet been given enough weight): that $dx$, $dy$, $dv$, $dw$ and $dz$ themselves can be taken as proportional to the differences (or momentary increments or decrements) of $x$, $y$, $v$, $w$, and $z$ themselves (respectively). We can use this to write down the differential equation for any given equation; we simply substitute for any *term* (that is, for any of the parts that are joined by addition or subtraction to make up the equation) the differential quantity of that term; but for any other quantity (which is not itself a term, but contributes to forming a term) we do not directly use its differential quantity when forming the differential quantity of the term to which it belongs; instead, we follow the above algorithm. In contrast, the previously published methods[7] do not have such a transition, since for the most part they use a straight line such as $DX$ [see Figure 1 or Figure 9], or another

Note 8, p. 73

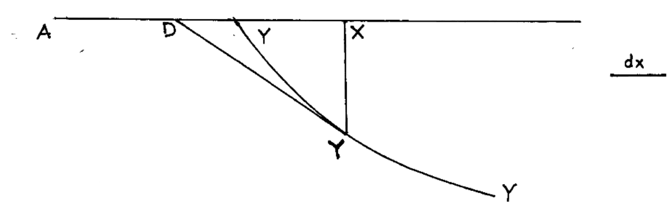



Figure 9: our figure, not Leibniz's

of the same kind, but not the line $dy$, which is a fourth proportional for $DX$, $XY$, and $dx$, and this confuses everything. Because of this confusion they make it a rule that we must first eliminate fractions and irrationals (those that the indeterminates enter into). It is clear that our method also extends to transcendent lines—lines to which the algebraic calculus cannot be applied,

[7]For an example of a previously published method for finding maxima and minima, see Pierre Fermat's "On a method for the evaluation of maxima and minima," written in 1633 and translated on pages 223–225 of Dirk Struik's *A Source Book in Mathematics, 1200–1800* (Cambridge, 1969). Fermat's $a$ corresponds to Leibniz's $x$, and Fermat's $e$ corresponds to Leibniz's $dx$. But for Fermat, the quantity $e$ is finite, not infinitely small. In each problem Fermat therefore has to engage in special algebraic manipulations to eliminate $e$ from his final equation for the maxima or minima, while Leibniz has a general method that can always find a finite equation that does not involve $dx$.

Note 9, p. 75 that is, lines which are of no definite degree —and it does this very generally, without any particular suppositions that only sometimes apply, provided we hold in general that to find a *tangent* is to draw a straight line joining two points on a curve that are an infinitely small distance apart, or to draw the side of a polygon with infinitely many angles (which is for us equivalent to the *curve*). And that infinitely small distance can always be expressed through some known difference such as $dv$, or through a relation to it, that is, through some known tangent. For example, if $y$ were a transcendent quantity, for instance the ordinate of a cycloid, and it were to enter into the calculation by means of which $z$, the ordinate of another curve, was to be determined, and if we were looking for $dz$, or through it the tangent of this latter curve, then we ought to determine $dz$ through $dy$, since we know the tangent of the cycloid. And similarly, if we were to pretend that we did not know the tangent of the cycloid, we could find
Note 10, p. 76 it from a given property of the tangents of a circle.

Let me give an example of the calculus.[8] Let the *first* or given equation be

$$\frac{x}{y} + \frac{(a+bx)(c-x^2)}{(ex+fx^2)^2} + ax\sqrt{g^2+y^2} + \frac{y^2}{\sqrt{h^2+lx+mx^2}} = 0.$$

expressing the relation between $x$ and $y$, that is, between $AX$ and $XY$ (see Figure 1 [or Figure 9]), where we suppose that $a$, $b$, $c$, $e$, $f$, $g$, $h$, $l$, and $m$ are given; we are looking for a way to draw, from a given point $Y$, the line $YD$, which touches the curve. In other words, we are looking for the ratio of the straight line $DX$ to the given straight line $XY$. For the sake of brevity let us write $n$ in place of $a+bx$, $p$ in place of $c-x^2$, $q$ in place of $ex+fx^2$, $r$ in place of $g^2+y^2$, and $s$ in place of $h^2+lx+mx^2$. Then we shall have:

$$\frac{x}{y} + \frac{np}{q^2} + ax\sqrt{r} + \frac{y^2}{\sqrt{s}} = 0,$$

which is a *second* equation. Now we know by our calculus that

$$d\left(\frac{x}{y}\right)$$

is

$$\frac{y\,dx - x\,dy}{y^2},$$

and similarly that

$$d\left(\frac{np}{q^2}\right)$$

is

$$\frac{q(n\,dp + p\,dn) - 2np\,dq}{q^3}$$

[8] Here we have omitted a parenthetical comment Leibniz makes on his notation: "Note that I designate division here in the following way: $x:y$, which is the same thing as $x$ divid. by $y$ or $\frac{x}{y}$." We always denote $x$ divided by $y$ by $\frac{x}{y}$.

and

$$d(ax\sqrt{r})$$

is

$$ax\frac{dr}{2\sqrt{r}} + a\,dx\sqrt{r};$$

and

$$d\left(\frac{y^2}{\sqrt{s}}\right)$$

is

$$\frac{4ys\,dy - y^2\,ds}{2s\sqrt{s}}.$$

All of these differential quantities, from $d(\frac{x}{y})$ to $d(\frac{y^2}{\sqrt{s}})$, will together add up to 0, and this will give us a *third* equation—the equation we get when we substitute for the terms of the second equation the quantities of their differentials.[9] Now $dn$ is $b\,dx$, and $dp$ is $-2x\,dx$, and $dq$ is $e\,dx + 2fx\,dx$, and $dr$ is $2y\,dy$, and $ds$ is $l\,dx + 2mx\,dx$. Substituting these values into the third equation we shall get a *fourth* equation, where the only remaining differential quantities—$dx$ and $dy$—are always unbound[10] and never in a denominator, and each term has either a $dx$ or a $dy$. Thus the law of homogeneity is always observed as far as these two quantities are concerned, however complicated the calculation may be. From here we can always obtain the value of $\frac{dx}{dy}$ (the ratio of $dx$ to $dy$ or of the line $DX$ we are looking for to the given line $XY$). According to our calculation[11] (changing the fourth equation into a Proportion) this ratio will be the same as the ratio of Note 11, p. 76

$$\frac{x}{y^2} - \frac{axy}{\sqrt{r}} - \frac{2y}{\sqrt{s}}$$

to

$$\frac{1}{y} - \frac{2np(e+2fx)}{q^3} + \frac{-2nx+pb}{q^2} + a\sqrt{r} - \frac{y^2(l+2mx)}{2s\sqrt{s}}.$$

But $x$ and $y$ are given, because the point $Y$ is given. The above mentioned values of the letters $n$, $p$, $q$, $r$, and $s$ are also given through $x$ and $y$. Therefore we shall have what we are looking for. I have included this rather complicated example only to help show how to use the above rules in a still more difficult calculation. Let me now give some easier examples that show how to use these rules.

Suppose we are given two points $C$ and $E$ (Figure 10), along with a straight line $SS$ in the same plane; we are looking for a point $F$ such that when we connect the lines $CF$ and $EF$, then the sum of the rectangles $CF$ times a given

[9]To be explicit, the third equation is:

$$\frac{y\,dx - x\,dy}{y^2} + \frac{q(n\,dp + p\,dn) - 2np\,dq}{q^3} + ax\frac{dr}{2\sqrt{r}} + a\,dx\sqrt{r} + \frac{4ys\,dy - y^2\,ds}{2s\sqrt{s}} = 0.$$

[10]That is, they are never contained in parentheses or under a radical.

[11]Leibniz is skipping very many algebraic steps here.

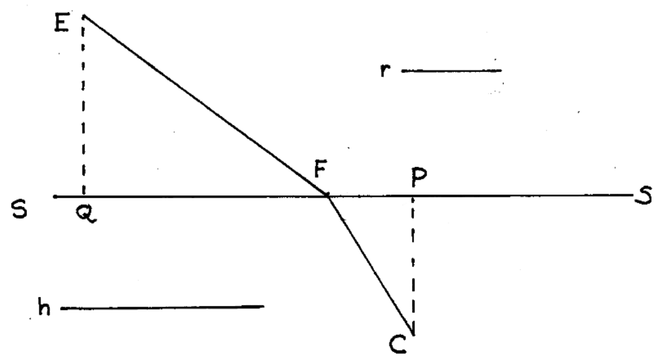



Figure 10: Leibniz's figure

line $h$ and $EF$ times a given line $r$ is as small as possible. That is, if $SS$ is a line separating two media, and $h$ represents the density of a medium (such as water) on the side where $C$ is, while $r$ represents the density of a medium (such as air) on the side where $E$ is, then we are looking for a point $F$ such that the path from $C$ to $E$ through $F$ is the easiest possible one. Let us suppose that all the possible sums of such rectangles, or all the possible difficulties of the paths, are represented by the lines $KV$ (Figure 11), that is, by the ordinates of the

Note 12, p. 77

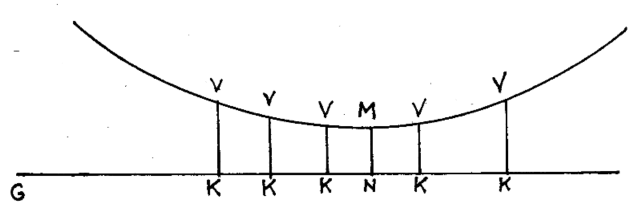



Figure 11: part of Leibniz's Figure 1, modified

Note 13, p. 78

curve $VV$ that are normal to the straight line $GK$. We shall call these ordinates $\omega$; we are looking for the least ordinate, $NM$. Because the points $C$ and $E$ [in Figure 10] are given, their perpendiculars to $SS$, namely $CP$ (which we shall call $c$) and $EQ$ (which we shall call $e$), will also be given, along with $PQ$ (which we shall call $p$). And we shall call $QF$, which is equal to $GN$ [in Figure 11], $x$, and $CF$, $f$, and $EF$, $g$. Then $FP$ will become $p - x$, and

Note 14, p. 79

$$f = \sqrt{c^2 + p^2 - 2px + x^2} \text{ or, abbreviating, } \sqrt{l},$$

and

$$g = \sqrt{e^2 + x^2} \text{ or, abbreviating, } \sqrt{m}.$$

We therefore have

$$\omega = h\sqrt{l} + r\sqrt{m},$$

and the differential of this equation (setting $d\omega$ to be 0 since $\omega$ is the least ordinate) is 0 and equal to

$$\frac{h\,dl}{2\sqrt{l}} + \frac{r\,dm}{2\sqrt{m}},$$

through the rules we have given for our calculus; now $dl$ is $-2dx(p-x)$, and Note 15, p. 80 Note 16, p. 80
$dm$ is $2x\,dx$, and therefore:

$$\frac{h(p-x)}{f} = \frac{rx}{g}.$$

Now if we apply this to dioptrics, and if we suppose that $f$ and $g$, or $CF$ and Note 17, p. 80
$EF$, are equal (for the refraction at the point $F$ remains the same, whatever length we choose for the straight line $CF$), then

$$h(p-x) = rx,$$

that is,

$$h : r :: x : p-x,$$

or $h$ is to $r$ as $QF$ is to $FP$. In other words, the sines $FP$ and $QF$ of the angles of refraction will be reciprocally as $r$ and $h$, the densities of the media in which the incidence and refraction occur. However, this density should not be understood Note 18, p. 81
in relation to ourselves, but in relation to the resistance that the rays of light meet. And thus we have a demonstration of the calculus, which we published elsewhere in these very *Acts*, when we were setting out a general foundation for optics, catoptrics and dioptrics; for other extremely learned men have pursued in Note 19, p. 81
very roundabout ways things that someone skilled in our calculus will henceforth be able to produce in three lines. I will show this with yet another example. Let the curve 133 (Figure 12) be of such a nature that if from any point on it,

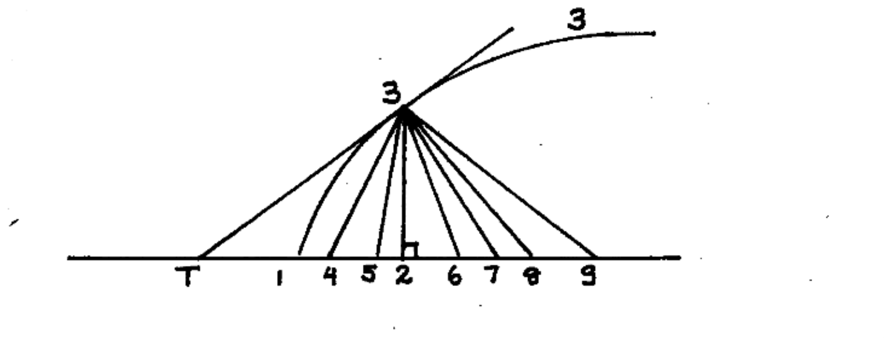



Figure 12: Leibniz's figure

such as 3, we draw six lines, 34, 35, 36, 37, 38, and 39 to six fixed points placed

on the axis, 4, 5, 6, 7, 8, and 9, then these six lines taken together are equal to a given straight line $g$. Let $T14526789$ be the axis, and let 12 be an abscissa and 23 be an ordinate. We are looking for the tangent $3T$. I say that $T2$ will
Note 20, p. 82
be to 23 as

$$\frac{23}{34}+\frac{23}{35}+\frac{23}{36}+\frac{23}{37}+\frac{23}{38}+\frac{23}{39}$$

is to

$$-\frac{24}{34}-\frac{25}{35}+\frac{26}{36}+\frac{27}{37}+\frac{28}{38}+\frac{29}{39}.$$

The same rule will apply, only with more terms, if we should suppose there are not six, but ten or more points; all such problems would be extremely tedious and sometimes even impossible to calculate by eliminating all the irrationals and using the published methods of tangents. Likewise, if the plane or solid rectangles constructed by using all possible pairs or triples of those straight lines should be equal to a given quantity, the problem would again be extemely
Note 21, p. 86
tedious or impossible using the published methods. But in all these cases, and in much more complicated ones, our method is extraordinarily easy, much more so than we might have expected. And these are only the beginnings of a certain much more elevated geometry, which also pertains to some of the most difficult and beautiful problems of mixed mathematics, problems which no one will be able to deal with easily by proceeding blindly without our differential calculus or something like it. As an appendix, let me add the solution to a problem proposed by *De Beaune* to *Descartes*. Descartes tried to solve it in Vol. 3 of his *Letters*, but failed. Here is the problem: to find a line $WW$ (Figure 1 or Figure 13) of such a nature that if a tangent $WC$ is drawn to the axis, then

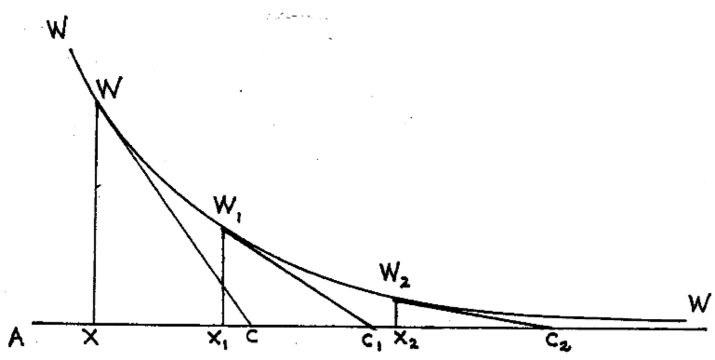



Figure 13: our figure, not Leibniz's

$XC$ is always equal to the same constant straight line $a$. Now $XW$ (or $w$) is to $XC$ (or $a$) as $dw$ is to $dx$; therefore if $dx$ (which may be taken arbitrarily) is taken to be constant or always the same (say it is $b$), that is, if the $x$'s or $AX$'s increase uniformly, then $w$ will be equal to $\frac{a}{b}dw$. These $w$'s will thus themselves be proportional to their own increments or differences; but that is to say that if the $x$'s are in an arithmetic progression, then the $w$'s will be in a geometric progression. In other words, if the $w$'s are numbers, then the $x$'s will be logarithms. $WW$ is therefore a logarithmic line.

Note 22, p. 86

# Notes on Leibniz's "A New Method"

Leibniz published this paper in the journal *Acta Eruditorum*, *Acts of the Erudite*, in October of 1684. The paper is written in Latin, and we have translated it from a text published in C. I. Gerhardt's edition of Leibniz's mathematical writings, Volume V, pages 220–226. We have slightly modernized his notation throughout the paper.

## Note 1

It may be helpful to say in general terms what Leibniz is doing in this paper. He is, as the title says, introducing a new mathematical method. The method is used for two basic kinds of problems:

1. Finding greatests and leasts. In its most general terms, the problem is to find the greatest or least possible values for a variable quantity. A simple example is finding the greatest ordinate for an ellipse with a given diameter. See Figure 14.

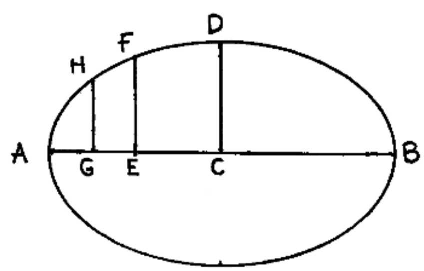
   
   

   Figure 14

   There $ADB$ is an ellipse whose major axis is $AB$ and whose center is $C$. Then the greatest ordinate is of course the ordinate $DC$ meeting the axis at the center, $C$, of the ellipse. There is no need for a new method here. But Leibniz's method will give us a way to find the greatest and least ordinates not just for an ellipse or other conic section, but for any *curve* whose Cartesian equation we have. For example, Leibniz's method can help us find the greatest ordinate $CF$ and the least ordinate $BE$ of the curve $ABCD$. (See Figure 15)

   Even if the Cartesian equation includes fractions and irrational quantities like square roots, the method still works.

2. Finding tangents to curves. Here again the method is not restricted to conic sections or other simple kinds of curves, but can be used to find tangents at any point on any curve whose Cartesian equation we have.

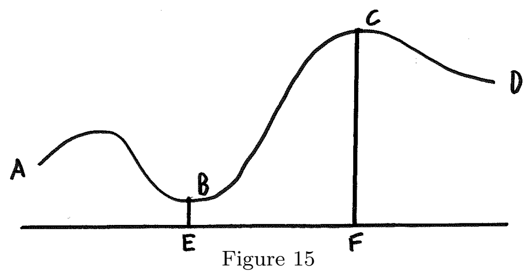


Figure 15

As the title also indicates, the method for solving these two kinds of problems depends on a "singular calculus." By a calculus, Leibniz seems to mean a way of calculating, that is, a system of symbols and a set of rules for using them. He introduces in this paper one new symbol, $d$, and gives rules for using it together with ordinary algebra. There are six rules, each showing how $d$ relates to one of six basic algebraic operations: addition, subtraction, multiplication, division, taking powers, and taking roots. This new symbol and these new rules are the key to finding greatests, leasts, and tangents.

After introducing the new calculus, Leibniz works through four examples in some detail. He shows how to find the tangents to particular curves (pages 34–35 and page 38), how to find the least value of a quantity (pages 35–37), and how to find a curve whose tangents have a given property (pages 37–38).

## Note 2

Note that in Leibniz's Figure 1 (Figure 16, below) there are two lines $VX$, namely $V_1X$ and $V_2X$, and likewise two lines $WX$, two lines $YX$, and two lines $ZX$. By drawing each of these lines twice in the diagram, Leibniz is suggesting that they should be understood as *variable* lines. (Leibniz does not use the term *variable* in this paper, but he introduces it in later writings.[12]) In other words, the line $VX$ should be understood not simply as one fixed line, but as a line that could be any of the infinitely many ordinates to the curve $VV$.

## Note 3

To understand what Leibniz means by $dv$, consider Figure 17. There we have drawn the curve $V_1V_2$ with ordinates $v$ from Figure 16, along with the axis $AX$, but without all the other curves. We have relabeled one of the two points $X$, naming it $X_1$. We have added a point $V$ on the curve to the left of $V_1$, drawn an

[12]See the papers "On the line formed by infinitely many lines drawn ordinatewise which concur with each other and touch it," published in April of 1692 in the *Acts of the Erudite*, and "A new application of the differential calculus and its use for finding multiple constructions of lines from a given condition on their tangents," published in July of 1694 in the same journal. The former is on pages 266–9 in Volume V of Gerhardt's edition of Leibniz's mathematical works, and the latter paper is on pages 301–6 of the same volume.

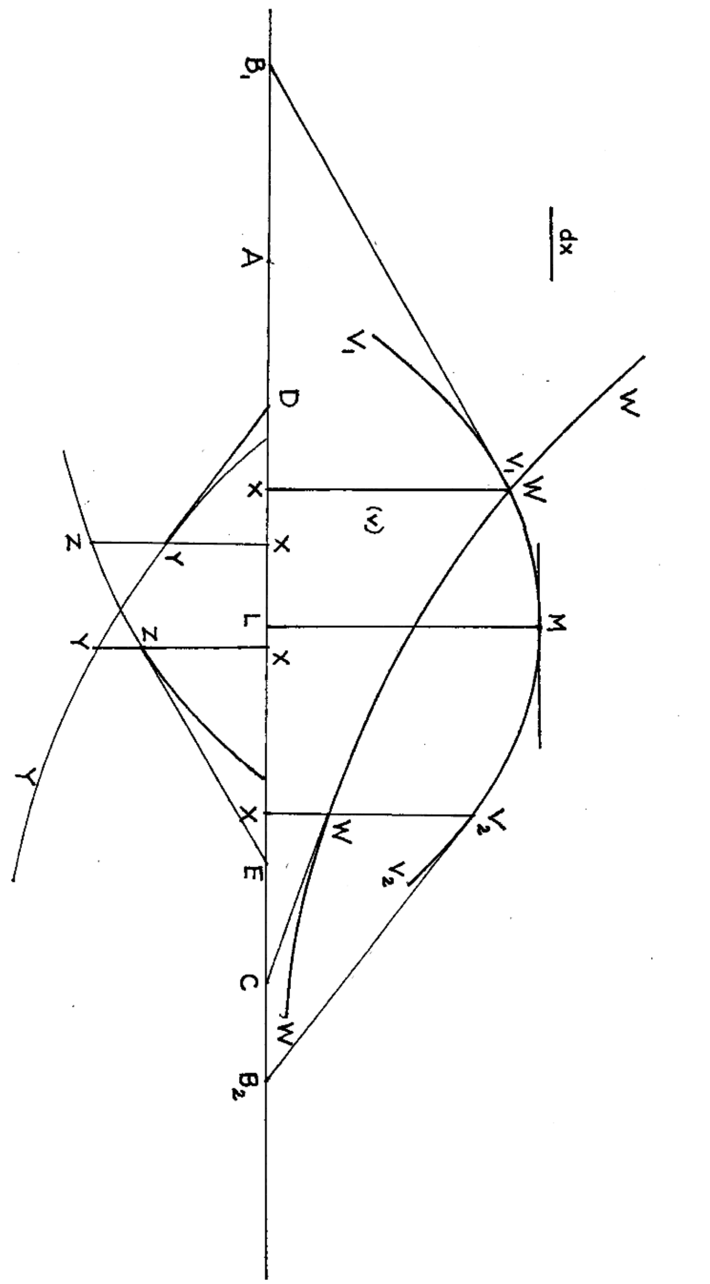



Figure 16: Leibniz's figure, slightly simplified

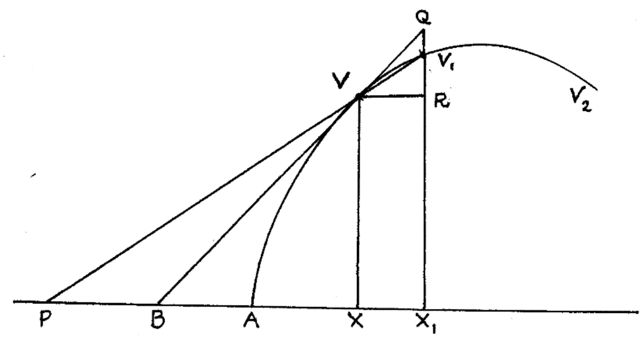



Figure 17

ordinate $VX$ and a tangent $VB$, and connected $V_1V$ and extended it to meet the axis $AX$ at $P$. We have drawn a perpendicular $VR$ from $V$ to the ordinate $V_1X_1$, and extended the tangent $BV$ and the ordinate $V_1X_1$ to meet at $Q$. Let $AX$ and $AX_1$ be called $x$, and $x_1$, respectively, and let $VX$ and $V_1X_1$ be called $v$, and $v_1$, respectively. Now $dx$ is an arbitrary line, and therefore we may take $dx$ as the difference of $x$ and $x_1$:

$$dx = x_1 - x = AX_1 - AX = XX_1 = VR.$$

Leibniz suggests here we think about $dv$ in at least two different ways:

1. As a fourth term in a proportion:

   $$dv : dx :: v : XB,$$

   that is

   $$dv : dx :: VX : XB.$$

   Because triangle $VRQ$ is similar to triangle $BXV$,

   $$VX : XB :: QR : RV.$$

   If we put these last two proportions together, it follows that

   $$dv : dx :: QR : RV.$$

   But $dx = RV$, and therefore, according to the first way of thinking about $dv$, $dv = QR$.

2. As the "difference of the $v$'s":

   $$dv = v_1 - v = V_1X_1 - VX = V_1R.$$

Now in general these two ways of thinking about $dv$ are not compatible, as $QR$ is not equal to $V_1R$. But if $V_1$ is *infinitely* close to $V$, then we may suppose that the line through $V$ and $V_1$ *is* the tangent[13] $VB$ and we may take $QR$ to be equal to $V_1R$. Thus the two ways of thinking about $dv$ are compatible when we take the arbitrary line $dx$ as an infinitely small line, so that the points $V$ and $V_1$ are infinitely close. In this case the ordinate $v_1$ only differs by an infinitely small amount from the ordinate $v$, and therefore in a certain sense these ordinates are two copies of the same ordinate $v$. The difference of $v_1$ and $v$ is thus not a difference of two different $v$'s, but a *difference of v itself.* Because the ordinate $v$ is variable, the difference $dv$ is also variable, and for the different ordinates $v$ there are different differences, all represented by the symbol $dv$.

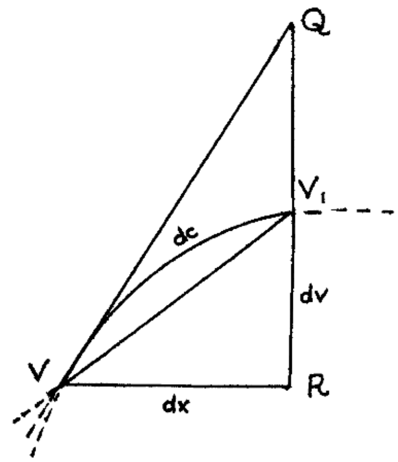



Figure 18: Magnification of Figure 17 between $V$ and $V_1$.

As Leibniz points out on page 34, taking two points as infinitely close on a curve amounts to treating it as equivalent to a polygon with infinitely many infinitely small sides: when the point $V_1$ is infinitely close to the point $V$, the straight line $V_1V$ is one of the infinitely small sides of the polygon.

Finally, we should note that in a number of later writings Leibniz calls the infinitely small triangle $VRV_1$ the *characteristic triangle* for the curve $VV_1$. (See Figure 18.) It is an infinitely small right triangle whose legs, $VR$ and $V_1R$, are equal to $dx$ and $dv$, respectively. The hypotenuse of the characteristic triangle is the chord $VV_1$. Because the points $V$ and $V_1$ are infinitely close, the chord $VV_1$ coincides with the arc $VV_1$. If we denote the length of arc $AV$ by $c$, then the arc $VV_1$ will be equal to $dc$, the difference of the $c$'s:

$$\begin{aligned} VV_1 &= \text{arc } AV_1 - \text{arc } AV \\ &= c_1 - c \\ &= dc. \end{aligned}$$

[13] Leibniz makes this explicit on page 34 of this paper.

According to Proposition I 47 in Euclid's *Elements*, the square on $VV_1$ is equal to the square on $V_1R$ and the square on $VR$, that is

$$dc^2 = dx^2 + dv^2.$$

## Note 4

In this sentence Leibniz introduces two rules for dealing with differences involving constant quantities $a$. Unlike the later rules, he does not name these. We will call the rule that $da = 0$ the *constant rule* and the rule that $d(ax) = a\,dx$ the *constant multiple rule*. For the sake of brevity we sometimes refer to these two rules together as the "constant rules."

Leibniz does not demonstrate this or any of the other rules he gives. We will give demonstrations of all the rules in the seventh note, below.

## Note 5

See Figure 19. To find a greatest ordinate, we have to find a horizontal line

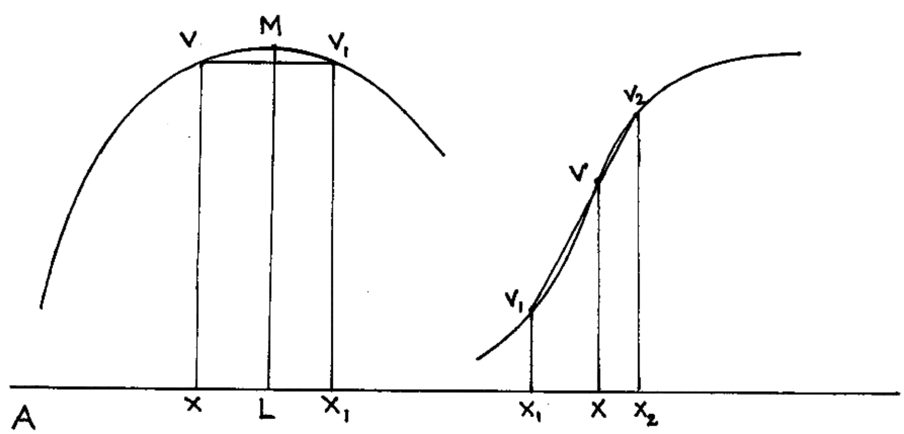



Figure 19

that meets the curve at two infinitely close points $V$ and $V_1$. Finding these two points by algebra would involve finding two roots (that is, solutions) of one equation (the details are not important here). These two roots become closer as $V$ becomes closer to $V_1$, and when $V$ and $V_1$ become infinitely close, the two roots become equal and the points $V$ and $V_1$ coincide at a point $M$ where there is a greatest ordinate $LM$. To find an inflection point, we have to find a line that meets the curve at three infinitely close points, $V$, $V_1$, and $V_2$. Finding these points by algebra would involve finding three roots to one equation, roots which would become equal as $V$, $V_1$, and $V_2$ become infinitely close to one another.

## Leibniz's paragraph on ambiguous signs (page 31)

Here is the paragraph we have omitted from the main text.

> But sometimes we must use *ambiguous Signs*, as we just did in the *Division* rule, before we know how to explicate them. And indeed, if when the $x$'s are increasing, the $\frac{v}{y}$'s are increasing (decreasing), the ambiguous signs in $d\,\frac{v}{y}$ or
>
> $$\frac{\pm v\,dy \mp y\,dv}{yy}$$
>
> should be explicated so that this fraction becomes a positive (negative) quantity. And $\mp$ signifies the opposite of $\pm$, so that if the latter is $+$, then the former is $-$, or conversely. Many ambiguities can occur in the same calculation, and I distinguish them by parentheses; for example, if $w$ were
>
> $$= \frac{v}{y} + \frac{y}{z} + \frac{x}{v},$$
>
> then $dw$ would be
>
> $$= \frac{\pm v\,dy \mp y\,dv}{yy} + \frac{(\pm)\,y\,dz\,(\mp)\,z\,dy}{zz} + \frac{((\pm))\,x\,dv\,((\mp))\,v\,dx}{vv};$$
>
> otherwise the ambiguities from different sources might be confused. Note here that an ambiguous sign when multiplied by itself gives $+$, when multiplied by its opposite gives $-$, and when multiplied by another ambiguous sign forms a new ambiguity dependent on both.

## Note 6

If Leibniz had simply given us the rule for powers,

$$d(x^a) = ax^{(a-1)}\,dx,$$

where $a$ could be any fraction, positive or negative, then the rule for fractions whose denominators are powers and the rule for roots would follow as special cases of this power rule. For if we want to find

$$d\left(\frac{1}{x^a}\right),$$

then we note that

$$\frac{1}{x^a} = x^{-a},$$

and apply the power rule, substituting $-a$ for $a$, and simplify:

$$\begin{aligned} d(x^{-a}) &= -ax^{((-a)-1)}\,dx \\ &= -ax^{-(a+1)}\,dx \\ &= -\frac{a\,dx}{x^{(a+1)}} \end{aligned}$$

This last expression is the one given by Leibniz's rule.

Likewise, if we want to find

$$d\sqrt[b]{x^a},$$

then we note that

$$\sqrt[b]{x^a} = x^{(\frac{a}{b})},$$

and apply the power rule, substituting $\frac{a}{b}$ for $a$, and simplify:

$$\begin{aligned}
d(x^{(\frac{a}{b})}) &= \frac{a}{b}x^{((\frac{a}{b})-1)}\,dx \\
&= \frac{a}{b}x^{\frac{1}{b}(a-b)}\,dx \\
&= \frac{a}{b}\sqrt[b]{x^{(a-b)}}\,dx
\end{aligned}$$

This last expression is the one given by Leibniz's rule for roots.

## Note 7

This long note contains seven parts:

1. Examples of finding differences,
2. Problems about finding differences,
3. Demonstrations of Leibniz's rules,
4. Examples of finding greatest and least ordinates,
5. Problems about finding greatest and least ordinates,
6. Examples of finding tangents, and
7. Problems about finding tangents.

### 1. Examples of finding differences

We can now use Leibniz's rules to find the differences of any algebraic expression involving $x$. Here are some examples.

1. Let $v = x^2 + 2$. (See Figure 20.) Then

$$\begin{aligned}
dv &= d(x^2+2) \\
&= d(x^2) + d(2) && \text{(by the addition rule)} \\
&= 2x\,dx + d(2) && \text{(by the power rule)} \\
&= 2x\,dx + 0 && \text{(by the constant rule)} \\
&= 2x\,dx.
\end{aligned}$$

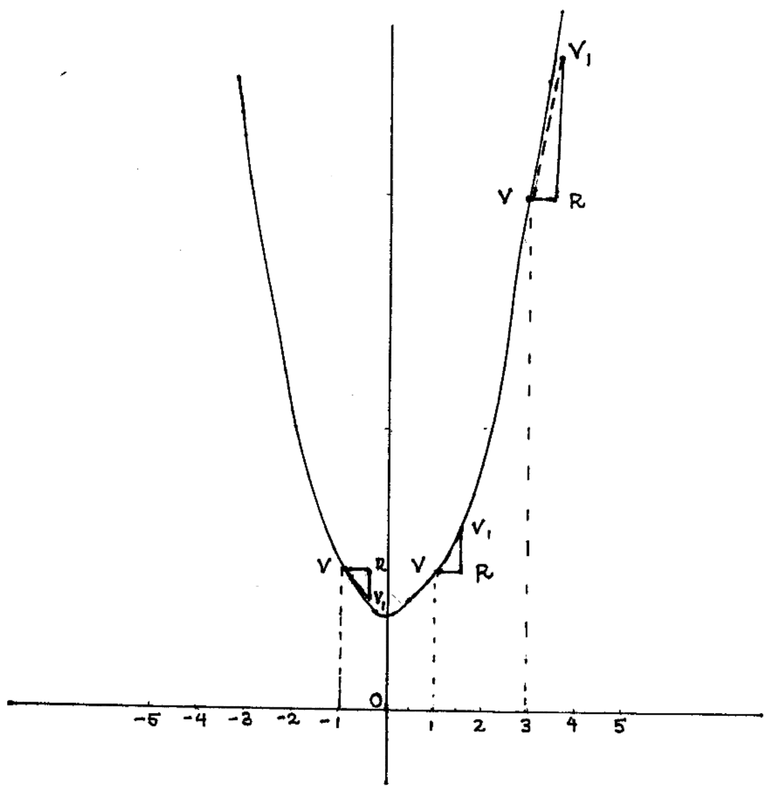



Figure 20

We can now use the equation

$$dv = 2x\,dx$$

to find the shape of the characteristic triangle for any point on the curve. For example, let the point $V$ be the point at which $x = 1$. Then

$$v = x^2 + 2 = 3.$$

Let $VV_1$ be tangent to the curve at this point, and let $VV_1R$ be the characteristic triangle. Then $VR = dx$ and $V_1R = dv$ and

$$\begin{aligned} V_1R &= dv \\ &= 2\,dx \\ &= 2\,VR. \end{aligned}$$

The characteristic triangle $VV_1R$ at this point is therefore a right triangle whose height is twice its base.

Again, if $V$ is the point at which $x = 3$, then

$$v = x^2 + 2 = 11,$$

and

$$\begin{aligned} V_1R &= dv \\ &= 2(3)\,dx \\ &= 6\,dx \\ &= 6\,VR. \end{aligned}$$

The characteristic triangle $VV_1R$ at this point is therefore a right triangle whose height is six times its base.

Finally, if $V$ is the point at which $x = -1$, then

$$v = x^2 + 2 = 3,$$

and

$$\begin{aligned} V_1R &= dv \\ &= -2\,dx \\ &= -2\,VR. \end{aligned}$$

This means that $V_1R$ is twice as long as $VR$, but now the point $V_1$ is *below* $V$, as indicated by the minus sign.

2. Let $v = x^3 - 6x^2 + 9x$. Then

$$
\begin{aligned}
dv &= d(x^3 - 6x^2 + 9x) \\
&= d(x^3) - d(6x^2) + d(9x) && \text{(addition and subtraction rule)} \\
&= d(x^3) - 6\,d(x^2) + 9\,dx && \text{(constant multiple rule)} \\
&= 3x^2\,dx - 6(2x^1\,dx) + 9\,dx && \text{(power rule)} \\
&= 3x^2\,dx - 12x\,dx + 9\,dx && \text{(ordinary algebra)} \\
&= (3x^2 - 12x + 9)\,dx. && \text{(ordinary algebra)}
\end{aligned}
$$

Again, we could use the equation we have found for $dv$, namely

$$dv = (3x^2 - 12x + 9)\,dx,$$

to find the characteristic triangle for any point $V$ on the curve. For example, if $V$ is the point at which $x = 0$, then

$$v = 0,$$

and

$$
\begin{aligned}
V_1R &= dv \\
&= (0^2 - 12(0) + 9)\,dx \\
&= 9\,dx \\
&= 9\,VR.
\end{aligned}
$$

Therefore the characteristic triangle $VV_1R$ at this point is a right triangle whose height is nine times its base.

If $x = 2$ at $V$, then

$$v = 2^3 - 6(2^2) + 9(2) = 2,$$

and

$$
\begin{aligned}
V_1R &= dv \\
&= (3(2^2) - 12(2) + 9)\,dx \\
&= -3\,dx \\
&= -3\,VR.
\end{aligned}
$$

Therefore the characteristic triangle $VV_1R$ at this point is a right triangle whose height is 3 times its base, and it is oriented so that $V_1$ is below $V$.

3. Let

$$x^2 + v^2 = 1.$$

To find $dv$ in terms of $dx$, we take differences of both sides of this equation. Now $d(1) = 0$, according to the constant rule. Therefore

$$
\begin{aligned}
0 &= d(x^2 + v^2) \\
&= d(x^2) + d(v^2) && \text{(addition rule)} \\
&= 2x\,dx + 2v\,dv. && \text{(power rule)}
\end{aligned}
$$

Therefore

$$-2x\,dx = 2v\,dv,$$

and, solving for $dv$,

$$-\frac{x}{v}\,dx = dv.$$

(If we want an expression strictly in terms of $x$, we can solve for $v$ in terms of $x$ and substitute. For, since

$$x^2 + v^2 = 1,$$

it follows that

$$v^2 = 1 - x^2$$

and therefore

$$v = \sqrt{1 - x^2}.$$

Substituting into our differential equation for $dv$ then gives

$$-\frac{x}{\sqrt{1 - x^2}}\,dx = dv.)$$

To interpret this differential equation geometrically, see Figure 21, where

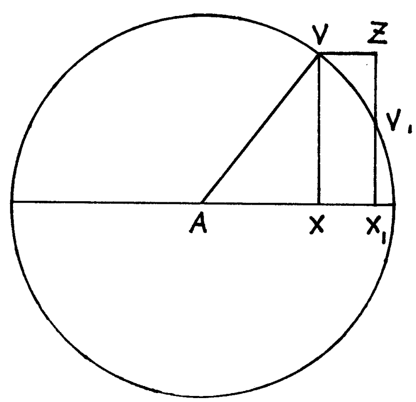



Figure 21

$AX = x$, $XV = v$, and

$$AV = x^2 + v^2 = 1.$$

The curve $VV$ is therefore a unit circle. Let $V_1$ be infinitely close to $V$, and draw the characteristic triangle $VZV_1$, where $VZ = dx$ and $dv = -ZV_1$ (where we have a minus sign because $V_1$ is below $V$). Then our differential equation

$$-\frac{x}{v}\,dx = dv$$

becomes

$$-\frac{AX}{XV}\,VZ = -ZV_1,$$

from which it follows that

$$\frac{AX}{XV} = \frac{ZV_1}{VZ},$$

that is,

$$AX : XV :: ZV_1 : VZ.$$

Since, in addition, angles $AXV$ and $VZV_1$ are both right, it follows that triangle $AXV$ is similar to triangle $V_1ZV$. Therefore

$$\angle ZVV_1 = \angle AVX,$$

and therefore

$$\begin{aligned} \text{right angle } ZVX &= \angle ZVV_1 + \angle V_1VX \\ &= \angle AVX + \angle V_1VX \\ &= \angle AVV_1. \end{aligned}$$

Therefore the tangent $VV_1$ is perpendicular to the radius $AV$. We have thus used the differential calculus to give an alternate demonstration of Proposition III 18 in Euclid's *Elements*.

4. Let

$$w = \frac{2x+3}{x^2-5}.$$

Since $w$ is a quotient of two expressions, that is,

$$w = \frac{v}{y},$$

where $v = 2x+3$ and $y = x^2-5$, we begin by using the division rule,

$$dw = d\left(\frac{v}{y}\right) = \frac{y\,dv - v\,dy}{y^2}.$$

This is an expression for $dw$ in terms of $v$, $y$, $dv$ and $dy$, but we want an expression in terms of $x$ and $dx$, so we have to substitute expressions for $v$, $y$, $dv$ and $dy$ in terms of $x$ and $dx$ into this expression for $dw$. We already have equations for $v$ and $y$ in terms of $x$, but we have to use our rules for differences to find expressions for $dv$ and $dy$ in terms of $x$ and $dx$. First, let us calculate $dv$:

$$\begin{aligned} dv &= d(2x+3) \\ &= d(2x) + d(3) && \text{(addition rule)} \\ &= 2\,dx + 0 && \text{(constant and constant multiple rules)} \\ &= 2\,dx. \end{aligned}$$

Next, let us calculate $dy$:

$$\begin{aligned}
dy &= d(x^2 - 5) \\
&= d(x^2) - d(5) && \text{(subtraction rule)} \\
&= d(x^2) - 0 && \text{(constant rule)} \\
&= 2x\,dx - 0 && \text{(power rule)} \\
&= 2x\,dx
\end{aligned}$$

We now substitute our expressions for $v$, $y$, $dv$, and $dy$ into the equation given by the division rule, and simplify:

$$\begin{aligned}
dw &= \frac{y\,dv - v\,dy}{y^2} \\
&= \frac{(x^2-5)(2\,dx) - (2x+3)(2x\,dx)}{(x^2-5)^2} && \text{(substitution)} \\
&= \frac{(2x^2-10)\,dx - (4x^2+6x)\,dx}{(x^2-5)^2} && \text{(ordinary algebra)} \\
&= \frac{(-2x^2-6x-10)}{x^4-10x^2+25}\,dx && \text{(ordinary algebra)}
\end{aligned}$$

The way we have calculated $dw$ in this example is typical: we started with a complex expression for $w$ and simplified it by rewriting it in terms of some new quantities $v$ and $y$, and then used Leibniz's rules to find $dw$, first in terms of the new quantities and their differences, and ultimately in terms of $x$ and $dx$.

5. Let

$$v = \sqrt{4x^2 - 7}.$$

Since $v$ is the square root of another expression, that is,

$$v = \sqrt[2]{y},$$

where

$$y = 4x^2 - 7,$$

we begin by using the root rule:

$$d\sqrt[b]{y^a} = \frac{a}{b}\sqrt[b]{y^{(a-b)}}\,dy,$$

setting $a = 1$ and $b = 2$. Substituting these values for $a$ and $b$ gives

$$\begin{aligned}
dv &= \frac{a}{b}\sqrt[b]{y^{(a-b)}}\,dy \\
&= \frac{1}{2}\sqrt[2]{y^{(1-2)}}\,dy \\
&= \frac{1}{2}\frac{dy}{\sqrt[2]{y}}.
\end{aligned}$$

This is an expression for $dv$ in terms of $y$ and $dy$, but we want an expression in terms of $x$ and $dx$, so we have to substitute expressions for $y$ and $dy$ in terms of $x$ and $dx$ into this expression for $dv$. We already have an equation for $y$ in terms of $x$, but we have to use our rules for differences to find an expression for $dy$ in terms of $x$ and $dx$:

$$\begin{aligned} dy &= d(4x^2 - 7) & \\ &= d(4x^2) - d(7) & \text{(addition rule)} \\ &= 4\,d(x^2) - 0 & \text{(constant and constant multiple rules)} \\ &= 4(2x\,dx) & \text{(power rule)} \\ &= 8x\,dx. & \end{aligned}$$

We now substitute for $y$ and $dy$ in the equation for $dv$:

$$\begin{aligned} dv &= \frac{1}{2}\frac{dy}{\sqrt[2]{y}} \\ &= \frac{1}{2}\frac{8x\,dx}{\sqrt[2]{4x^2 - 7}} \end{aligned}$$

We could also calculate $dv$ by using the power rule. If we again set $y = 4x^2 - 7$, then

$$v = y^{\frac{1}{2}},$$

and, as we just saw,

$$dy = 8x\,dx,$$

and therefore

$$\begin{aligned} dv &= d(y^{\frac{1}{2}}) & \\ &= \frac{1}{2}(y^{-\frac{1}{2}})\,dy & \text{(power rule)} \\ &= \frac{1}{2}\frac{1}{y^{\frac{1}{2}}}\,dy & \text{(ordinary algebra)} \\ &= \frac{1}{2}\frac{1}{(4x^2 - 7)^{\frac{1}{2}}}8x\,dx & \text{(substitution)} \\ &= \frac{4x}{\sqrt{4x^2 - 7}}\,dx. & \text{(ordinary algebra)} \end{aligned}$$

6. Let

$$v = (3x - 8)^{19}.$$

Then

$$v = y^{19},$$

where

$$y = 3x - 8.$$

According to the power rule,

$$dv = 19y^{18}\,dy.$$

To find $dy$, we use the addition rule, constant rule and constant multiple rule:

$$\begin{aligned} dy &= d(3x) - d(8) \\ &= 3\,dx. \end{aligned}$$

Substituting in for $y$ and $dy$ in the expression for $dv$, we get

$$\begin{aligned} dv &= 19(3x-8)^{18}(3\,dx) \\ &= 57(3x-8)^{18}\,dx. \end{aligned}$$

7. Let

$$v = (x+2)^5(x-7)^4.$$

Since $v$ is the product of two other expressions, we use the multiplication rule. In fact,

$$v = wy,$$

where

$$w = (x+2)^5$$

and

$$y = (x-7)^4.$$

According to the multiplication rule,

$$dv = w\,dy + y\,dw.$$

This is an expression for $dv$ in terms of $w$, $y$, $dw$, and $dy$, but we want an expression in terms of $x$ and $dx$, so we have to find $dw$ and $dy$ in terms of $x$ and $dx$.

First, to find $dw$, we note that $w$ is a power of another quantity; namely,

$$w = z^5,$$

where

$$z = x + 2.$$

According to the power rule,

$$dw = 5z^4\,dz.$$

According to the addition and constant rules,

$$dz = dx + d(2) = dx.$$

Substituting this expression for $dz$ into our expression for $dw$, we get

$$dw = 5z^4\,dx.$$

Substituting $(x + 2)$ for $z$ gives

$$dw = 5(x + 2)^4\,dx.$$

Next, to find $dy$, we note that

$$y = u^4,$$

where

$$u = x - 7,$$

so that, according to the power rule,

$$dy = 4u^3\,du.$$

According to the addition and constant rules,

$$du = d(x) - d(7) = dx,$$

and, substituting for $u$ and $du$ in our expression for $dy$, we get

$$dy = 4(x - 7)^3\,dx,$$

Putting it all together, we substitute these expressions for $dy$ and $dw$, along with our expressions for $y$ and $w$, into our equation for $dv$:

$$\begin{aligned} dv &= w\,dy + y\,dw \\ &= \left[(x+2)^5\right]\left[4(x-7)^3\,dx\right] + \left[(x-7)^4\right]\left[5(x+2)^4\,dx\right]. \end{aligned}$$

**2. Problems about finding differences**

For each of the following expressions for $v$, find $dv$ in terms of $x$ and $dx$.

1.
$$v = x^2 - 3x.$$

2.
$$v = x^2 + 2x - 5.$$

3.
$$v = x^3 + 4x - 6.$$

4.
$$v = 2x^3 - 3x^2 - 12x + 2.$$

5.
$$x^2 + \frac{v^2}{4} = 1.$$

6.
$$x^2 - \frac{v^2}{4} = 1.$$

7.
$$v = \frac{x}{3x+2}.$$

8.
$$v = \frac{2x-1}{x+2}.$$

9.
$$v = \frac{2x^2+1}{x^2-3x}.$$

10.
$$v = \frac{x^2}{3x^2-4}.$$

11.
$$v = \sqrt{3-4x}.$$

12.
$$v = \sqrt{3x+2}.$$

13.
$$v = \sqrt{x^2+2x-1}.$$

14.
$$v = \sqrt{x^3-x+1}.$$

15.
$$v = (x+2)^8.$$

16.
$$v = (x^2+4)^7.$$

17.
$$v = (x^2-3x+6)^5.$$

18.
$$v = (x^3-2x)^4.$$

19.
$$v = (x+1)^4(3x-2)^2.$$

20.
$$v = (x-3)^6(2x+1)^3.$$

21.
$$v = (x^2+x+1)^4(x^2-5)^7.$$

22.
$$v = (x^2+1)^3(3x^3-2x)^5.$$

23.
$$v = \frac{(x-1)^3(x+2)^5}{x^2+3}.$$

24.
$$v = \frac{(x+2)^2(x-3)^3}{2x+5}.$$

## 3. Demonstrations of Leibniz's rules

### The constant rule

To see why $da = 0$, consider Figure 22. There we have drawn the line $VV_1$

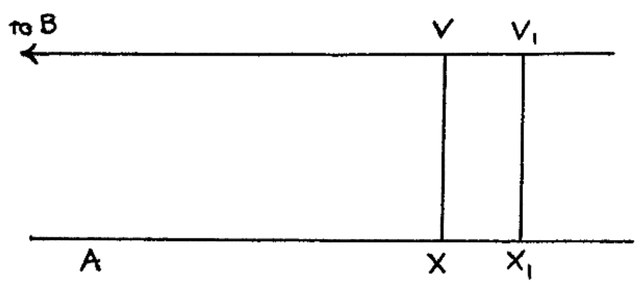



Figure 22

parallel to the axis $AX$, so that $VX = V_1X_1 = a$. If, as in Note 3,

$$da = V_1X_1 - VX,$$

then clearly $da = 0$.

Note that it is difficult to apply Leibniz's first definition of $d$ in this case. For that definition would require that

$$da : dx :: a : XB,$$

where $B$ is the point where the tangent at $V$ meets the axis, but in this case there is no such point. Even if we treat the line $VV_1$ as its own tangent, it does not meet the axis $AX$. But if we were to suppose that $VV_1$ meets the axis $AX$ at a point $B$ infinitely far to the left of the point $A$ on the axis $AX$, then the ratio $a:XB$ would be the ratio of a finite quantity to an infinite one, and therefore so would the ratio $da:dx$. In other words, $da$ would be infinitely small compared to $dx$. It would therefore be reasonable to assume that $da = 0$.

What is true in this case is true more generally: it is easier to justify Leibniz's rules by treating expressions like $dv$ as directly representing differences between two infinitely close ordinates, rather than trying to use the proportion Leibniz gives us:

$$dv:dx :: v:XB.$$

### The constant multiple rule

We can prove Leibniz's claim that $d(ax) = a\,dx$ as follows. See Figure 23. There

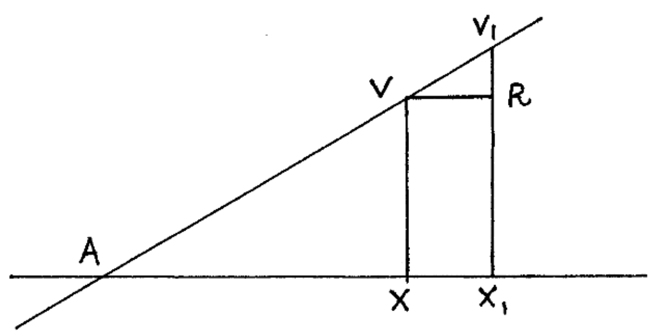



Figure 23

$AX = x$, $VX = v$, and $v = ax$. Let $V_1$ be infinitely close to $V$, $V_1X_1 = v_1$ and $AX_1 = x_1$, and let $V_1$ be on the line $AV$ extended, so that

$$v_1 = ax_1.$$

Therefore

$$dx = x_1 - x,$$

and

$$dv = v_1 - v = ax_1 - ax = a(x_1 - x) = a\,dx.$$

There is thus no need to use Leibniz's proportion, and in fact there is no need to refer to the diagram at all. We could simply have treated $dv$ as a difference of two infinitely close values of $v$ and followed the rules of algebra.

### The addition rule

Since

$$z - y + w + x = v,$$

$$\begin{aligned} dv = v_1 - v &= (z_1 - y_1 + w_1 + x_1) - (z - y + w + x) \\ &= (z_1 - z) - (y_1 - y) + (w_1 - w) + (x_1 - x) \\ &= dz - dy + dw + dx, \end{aligned}$$

just as Leibniz asserts.

### The multiplication rule

Leibniz gives the following argument for the multiplication rule in a later paper, "A Remarkable Symbolism of the Algebraic and the Infinitesimal Calculus in the Comparison of Powers and Differences, and on the Transcendental Law of Homogeneity (published in 1710 in *The Berlin Miscellany for the Growth of the Sciences*).[14]

> First,
> $$d(xv) = v\,dx + x\,dv,$$
> as we showed once, when many years ago we first published the differential calculus; from this one foundation all the rest of the calculus of differences may be demonstrated. Now this foundation may be shown as follows: $d(xv)$ is the difference between $(x+dx)(v+dv)$ and $xv$, or between the next rectangle and the given rectangle. And
> $$(x + dx)(v + dv) = xv + v\,dx + x\,dv + dx\,dv,$$
> which, if you take away $xv$, becomes $v\,dx + x\,dv + dx\,dv$; but because $dx$ or $dv$ is incomparably less than $x$ or $v$, $dx\,dv$ will also be incomparably less than $x\,dv$ and $v\,dx$, and is therefore thrown away, and finally
> $$(x + dx)(v + dv) - xv = v\,dx + x\,dv.$$

Because $dx$ is infinitely small compared to $x$, the product $dx\,dv$ will be infinitely small compared to $x\,dv$. For

$$dx : x :: dx\,dv : x\,dv.$$

Likewise, because $dv$ is infinitely small compared to $v$, the product $dx\,dv$ will be infinitely small compared to $v\,dx$. The term $dx\,dv$ is therefore infinitely small compared to the other two terms on the right side of the equation: $x\,dv$ and $v\,dx$. What $dx\,dv$ adds to these two terms is therefore negligible, so that we can leave it out of the equation and write

$$d(xv) = x\,dv + v\,dx,$$

just as Leibniz says. The general principle here is that *adding* an infinitely smaller quantity (e. g. $dw$) does not change a quantity ($w = w+dw$); multiplying

[14]We have changed the name of a quantity in this later paper to make the names correspond to those of "A New Method."

or dividing by an infinitely smaller quantity, however, does change a quantity ($w\,dw \neq w$).

This argument depends on there being two different levels of infinitely small quantities, where a quantity on one level ($dx\,dv$) is infinitely small compared to infinitely small quantities on another level ($x\,dv$ and $v\,dx$). In fact there are infinitely many different levels of infinitely small quantities, where the quantities on one level are infinitely small compared to those on the previous level. For we have an infinite geometric series:

$$1,\ dx,\ (dx)^2,\ (dx)^3,\ (dx)^4,\ \text{etc.},$$

where each term is infinitely small compared to the previous one. The difference $dx$ is an infinitely small part of the unit, $(dx)^2$ is equal to an infinitely small part of $dx$, $(dx)^3$ is equal to an infinitely small part of $(dx)^2$, and so on to infinity. We could imagine the unit divided into infinitely many infinitely small parts equal to $dx$, $dx$ in turn divided into infinitely many infinitely small parts equal to $(dx)^2$, and so on. In the *Monadology*, Leibniz writes:

> 65. And the author of nature has been able to practice this divine and infinitely marvelous art, because each portion of matter is not only divisible to infinity, as the ancients have recognized, but is also actually subdivided without end, each part divided into parts having some motion of their own; otherwise, it would be impossible for each portion of matter to express the whole universe ...
>
> 66. From this we see that there is a world of creatures, of living beings, of animals, of entelechies, of souls in the least part of matter.
>
> 67. Each portion of matter can be conceived as a garden full of plants, and as a pond full of fish. But each branch of a plant, each limb of an animal, each drop of its humors, is still another such garden or pond.[15]

### The division rule

To see why the division rule is correct, we begin with the equation Leibniz uses to define $z$, namely,

$$\frac{v}{y} = z,$$

and multiply both sides by $y$ to get

$$v = zy.$$

We will take differences of both sides of this equation, using the multiplication rule Leibniz has just given us, and solve for $dz$, as follows.

[15]The translation is by Roger Ariew and Daniel Garber in their collection of Leibniz's *Philosophical Essays* (1989, pages 221–222).

First, the multiplication rule gives us

$$\begin{aligned} dv &= d(zy) \\ &= z\,dy + y\,dz \end{aligned}$$

Subtracting $z\,dy$ from both sides, we get

$$dv - z\,dy = y\,dz.$$

Substituting $\frac{v}{y}$ for $z$ on the left side of this equation gives

$$dv - \frac{v}{y}\,dy = y\,dz.$$

Therefore, dividing both sides by $y$, we get

$$\begin{aligned} dz &= \frac{dv}{y} - \frac{v}{y^2}\,dy \\ &= \frac{y\,dv}{y^2} - \frac{v}{y^2}\,dy \\ &= \frac{y\,dv - v\,dy}{y^2}. \end{aligned}$$

This last expression is the formula for $dz$ in the division rule.

**The power rule**

This rule,

$$d(x^a) = ax^{(a-1)}\,dx,$$

is in fact a series of rules, one for every positive whole number $a$. To see why it is correct, we go through the cases in order.

To begin, if $a = 1$, then

$$d(x^a) = dx,$$

while

$$ax^{(a-1)}\,dx = 1x^{(1-1)}\,dx = dx,$$

and therefore

$$d(x^a) = ax^{(a-1)}\,dx,$$

as the rule requires.

If $a = 2$, then $x^2$ is $x$ multiplied by $x$, and so we can apply the multiplication rule that Leibniz has given us:

$$\begin{aligned} d(x^2) &= d(x \cdot x) \\ &= x\,dx + x\,dx \text{ (by the multiplication rule)} \\ &= 2x\,dx \\ &= 2x^{(2-1)}\,dx, \end{aligned}$$

as the rule requires.

If $a = 3$, then $x^3$ is $x$ multiplied by $x^2$, and so we can apply the multiplication rule:

$$\begin{aligned} d(x^3) &= d(x \cdot x^2) \\ &= x\,d(x^2) + x^2\,dx. \end{aligned}$$

We just saw that $d(x^2)$ is equal to $2x\,dx$, so we substitute the latter for the former, and therefore

$$\begin{aligned} d(x^3) &= x(2x\,dx) + x^2\,dx \\ &= 2x^2\,dx + x^2\,dx \\ &= 3x^2\,dx \\ &= 3x^{(3-1)}\,dx, \end{aligned}$$

as the rule requires.

Likewise,

$$\begin{aligned} d(x^4) &= 4x^3\,dx, \\ d(x^5) &= 5x^4\,dx, \end{aligned}$$

and so on.

We could continue in this way indefinitely. The rule for each power would follow from the rule for the previous power once we use the multiplication rule: if we have shown the rule is correct for some power $a$, so that

$$d(x^a) = ax^{(a-1)}\,dx,$$

then we can show the rule is correct for $a + 1$ by first noting that

$$x^{(a+1)} = x \cdot x^a,$$

and using the multiplication rule,

$$d(x^{(a+1)}) = x\,d(x^a) + x^a\,dx,$$

and then substituting $ax^{(a-1)}\,dx$ for $dx^a$ (we assumed we have already shown that the rule is true for $x^a$), so that we get

$$\begin{aligned} d(x^{(a+1)}) &= x(ax^{(a-1)}dx) + x^a\,dx \\ &= ax^a\,dx + x^a\,dx \\ &= (a+1)x^a\,dx, \end{aligned}$$

as the rule requires. Beginning from the rule for $a$ we would thus have shown the rule for $a + 1$. In this way we would eventually show the rule is correct for any possible (positive whole number) power.

### The power rule for negative exponents

To see why

$$d\left(\frac{1}{x^a}\right) = -\frac{a\,dx}{x^{(a+1)}},$$

we begin by using the division rule:

$$d\left(\frac{1}{x^a}\right) = \frac{x^a\,d(1) - 1\,d(x^a)}{(x^a)^2}.$$

To simplify the expression on the right, we note that $d(1) = 0$ (by the constant rule), and use the power rule to substitute $ax^{(a-1)}\,dx$ for $d(x^a)$. We get

$$\begin{aligned} d\left(\frac{1}{x^a}\right) &= \frac{x^a\,d(1) - 1\,d(x^a)}{(x^a)^2} \\ &= \frac{0 - ax^{(a-1)}\,dx}{(x^a)^2} \\ &= \frac{-ax^{(a-1)}\,dx}{x^{2a}} \\ &= \frac{-a\,dx}{x^{(2a-(a-1))}} \\ &= -\frac{a\,dx}{x^{(a+1)}} \end{aligned}$$

This last expression is the one Leibniz gives in his rule.

### Root rule

To see why

$$d(\sqrt[b]{x^a}) = \frac{a}{b}\,dx\,\sqrt[b]{x^{(a-b)}},$$

we begin by setting

$$y = \sqrt[b]{x^a}.$$

Then

$$\begin{aligned} y^b &= x^a, \text{ and} \\ d(y^b) &= d(x^a) \end{aligned}$$

Applying the power rule to both sides of this equation gives

$$by^{(b-1)}\,dy = ax^{(a-1)}\,dx.$$

Solving for $dy$, we get

$$dy = \frac{ax^{(a-1)}\,dx}{by^{(b-1)}}.$$

Since we want an expression for $dy$ purely in terms of $x$, we substitute $\sqrt[b]{x^a}$ for $y$ in this expression, and simplify. There are many steps, but we only need ordinary algebra, including especially the rules for exponents:

$$\begin{aligned}
dy &= \frac{ax^{(a-1)}\,dx}{b(\sqrt[b]{x^a})^{(b-1)}} \\
&= \frac{ax^{(a-1)}}{b(x^{(\frac{a}{b})})^{(b-1)}}\,dx \\
&= \frac{ax^{(a-1)}}{b(x^{(\frac{a}{b}(b-1))})}\,dx \\
&= \frac{ax^{(a-1)}}{b(x^{(a-\frac{a}{b})})}\,dx \\
&= \frac{a}{b}x^{((a-1)-(a-\frac{a}{b}))}\,dx \\
&= \frac{a}{b}x^{(\frac{a}{b}-1)}\,dx \\
&= \frac{a}{b}x^{(\frac{1}{b}(a-b))}\,dx \\
&= \frac{a}{b}(x^{(a-b)})^{\frac{1}{b}}\,dx \\
&= \frac{a}{b}\sqrt[b]{x^{(a-b)}}\,dx
\end{aligned}$$

This last expression is the one Leibniz gives in his rule.

A similar argument may be used to show that Leibniz's rule for

$$d\left(\frac{1}{\sqrt[b]{x^a}}\right)$$

is correct.

### 4. Examples of finding greatest and least ordinates

1. Let $v = x^2 + 2$. We want to find the points where $v$ is greatest or least. As Leibniz observed earlier (page 27), at a point where $v$ is a greatest or least ordinate, $dv = 0$; there the differences of $v$ are neither positive nor negative, and $v$ is neither increasing nor decreasing. Therefore we have to find points where $dv = 0$. In the first example of finding differences (page 48, above) we calculated $dv$, showing that

   $$dv = 2x\,dx.$$

   We have assumed that $dx$ is always positive, and therefore $dv$ is greater than 0 when $x$ is positive, less than 0 when $x$ is negative, and equal to

0 precisely when $x = 0$. Therefore when $x$ is positive and increasing, $v$ is also increasing; when $x$ is negative and increasing (that is, when $x$ is negative but is becoming less negative), then $v$ is decreasing; and when $x$ is 0, $v$ is at its least. When $x$ is 0, $v = x^2 + 2 = 2$, and therefore the least ordinate $v$ is equal to 2. See Figure 24.

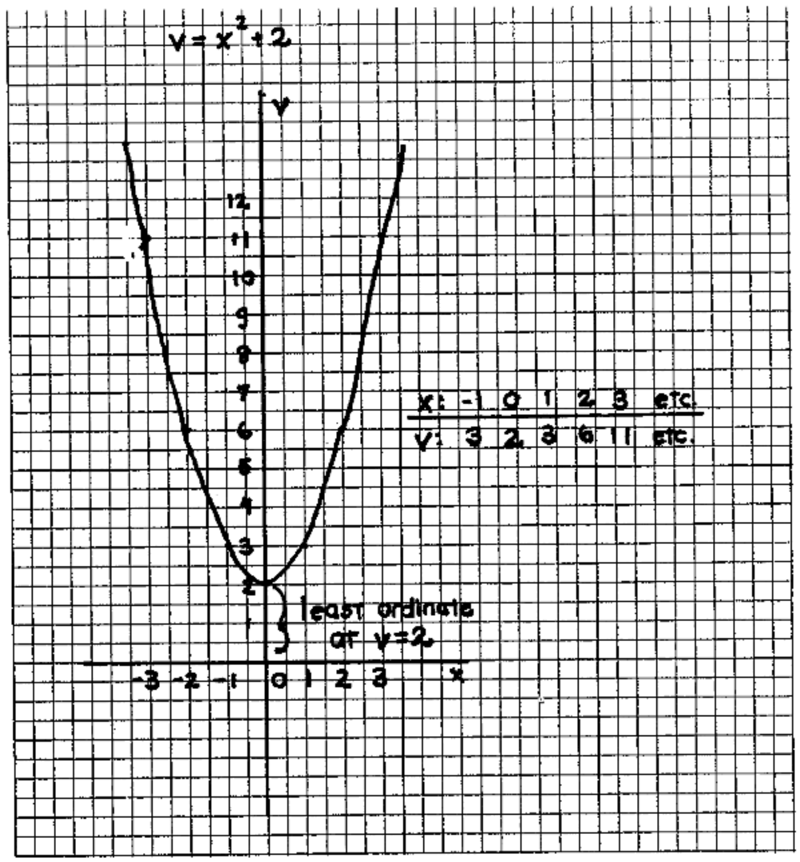



Figure 24

2. Let $v = x^3 - 6x^2 + 9x$. We want to find the points where $v$ is greatest or least, and we again begin by finding where $dv$ is positive, where it is negative, and where it is 0. As we saw in the second example (page 50, above),

$$dv = (3x^2 - 12x + 9)\, dx.$$

To find out where $dv$ is positive, negative, or zero, we factor the expression on the right:

$$dv = 3(x-1)(x-3)\, dx.$$

When $x = 1$, then $x - 1 = 0$, and therefore $dv = 0$. When $x = 3$, then $x - 3 = 0$ and $dv = 0$. When $x < 1$, then $dv$ is the product of two negative quantities (namely, $(x-1)$ and $(x-3)$) and two positive quantities (3 and $dx$), and is therefore positive. When $1 < x < 3$, then $dv$ is the product of one negative quantity ($(x-3)$) and three positive quantities (3, $(x-1)$, and $dx$), and is therefore negative. Finally, when $x > 3$, $dv$ is the product of four positive quantities, and is therefore positive. It follows from all this that $v$ is increasing when $x < 1$, at a greatest value when $x = 1$, decreasing when $1 < x < 3$, at a least value when $x = 3$, and increasing again when $x > 3$. When $x = 1$,

$$v = 1^3 - 6(1^2) + 9(1) = 4,$$

so that 4 is a greatest value for the ordinate $v$. When $x = 3$,

$$\begin{aligned} v &= 3^3 - 6(3^2) + 9(3) \\ &= 27 - 54 + 27 \\ &= 0, \end{aligned}$$

so that 0 is a least value for the ordinate $v$. See Figure 25. Note that 4 is not absolutely the greatest value of $v$, but only a value greater than all nearby values, and likewise 0 is not absolutely the least value of $v$, but only a value less than all the nearby values.

As Leibniz observes, by looking at where the differences of the differences, $ddv$, are positive, negative, and 0, we can also find where the curve whose ordinates are $v$ turns its concavity upward or downward, and where it is inflected. First, we need to find $ddv$;

$$ddv = d(dv) = d((3x^2 - 12x + 9)\,dx).$$

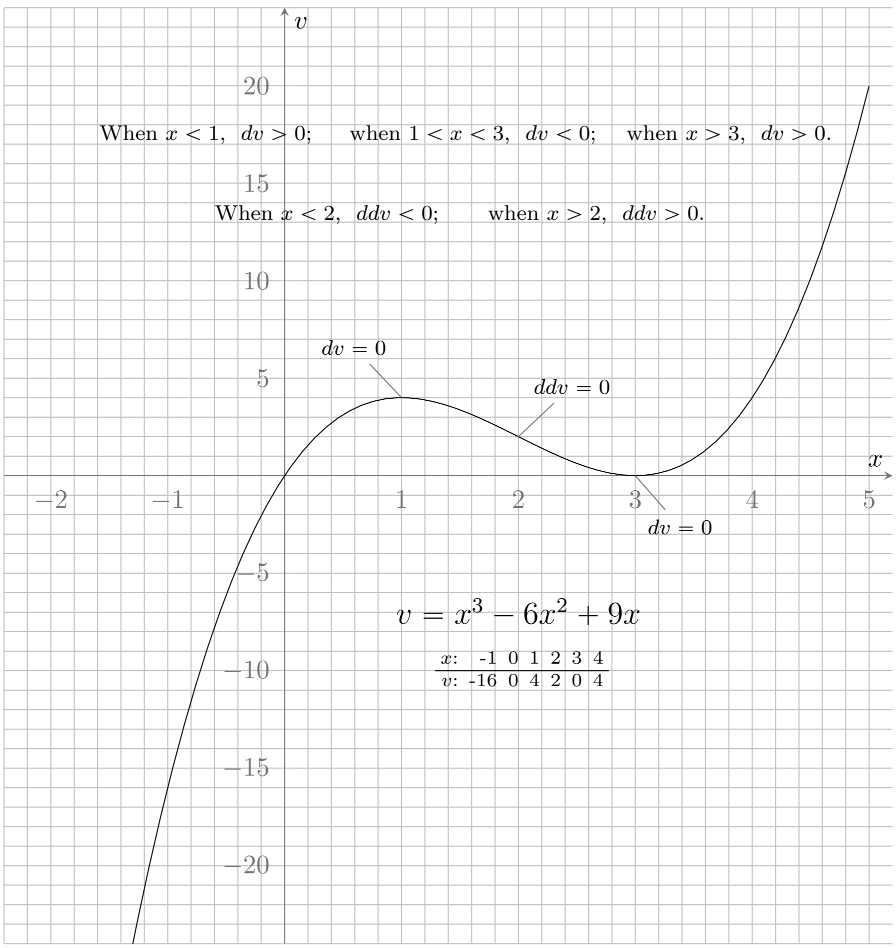
Now we may assume that $dx$ is a constant, that is, that it is the same infinitely small quantity at every point, no matter what $x$ is. Then, by the constant multiple rule,

$$\begin{aligned} ddv &= d(3x^2 - 12x + 9)\,dx && \\ &= (d(3x^2) - d(12x) + d(9))\,dx && \text{(addition rule)} \\ &= (3d(x^2) - 12\,dx + 0)\,dx && \text{(constant rules)} \\ &= (3(2x\,dx) - 12\,dx)\,dx && \text{(power rule)} \\ &= (6x - 12)\,(dx)^2 && \text{(ordinary algebra)} \\ &= 6(x - 2)\,(dx)^2 && \text{(factoring)} \end{aligned}$$

Therefore,

$$\begin{aligned} &ddv < 0 \quad \text{when } x < 2 \\ &ddv = 0 \quad \text{when } x = 2, \text{ and} \\ &ddv > 0 \quad \text{when } x > 2. \end{aligned}$$

Therefore the concavity of the curve is turned downward when $x < 2$, there is an inflection point when $x = 2$, and the concavity of the curve is turned upward when $x > 2$. See Figure 25.



When $x < 1$, $dv > 0$; when $1 < x < 3$, $dv < 0$; when $x > 3$, $dv > 0$.
When $x < 2$, $ddv < 0$; when $x > 2$, $ddv > 0$.
$dv = 0$
$ddv = 0$
$dv = 0$
$v = x^3 - 6x^2 + 9x$
$x$: -1 0 1 2 3 4
$v$: -16 0 4 2 0 4
$v$
$x$
20
15
10
5
−5
−10
−15
−20
−2
−1
1
2
3
4
5


Figure 25

**5. Problems about finding greatest and least ordinates**

For each of the curves given by the following equations, find the greatest ordinates, the least ordinates, where the ordinates are decreasing and where they are increasing. Also find where each curve turns its concavity upward, where it turns its concavity downward, and where it is inflected (if anywhere). Sketch a graph of each curve.

1. $v = x^2 - 4x + 1$.
2. $v = x^2 + 2x - 5$.
3. $v = x^3 - 3x^2 - 9x + 4$.
4. $v = 2x^3 - 3x^2 - 12x + 2$.

**6. Examples of finding tangents**

Recall that Leibniz initially defines $dv$ using a proportion depending on the tangents to the curve. In terms of Figure 26, the proportion is:

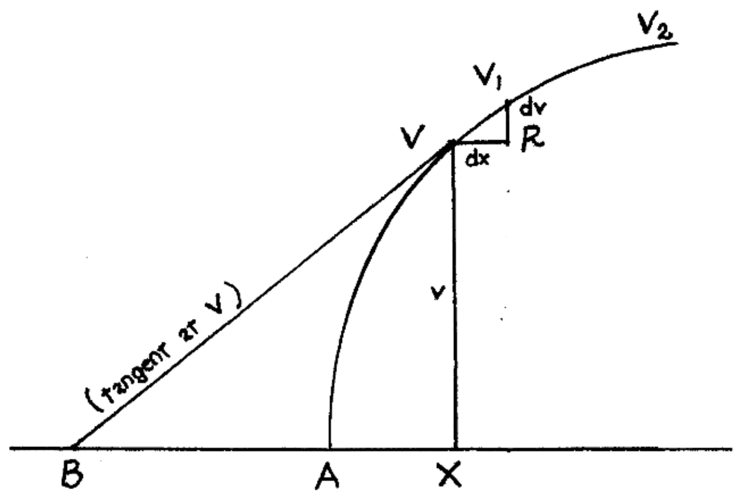



Figure 26

$$dv : dx :: VX : XB.$$

Now if $dv = 0$, then $VX$ is a greatest or least ordinate and the tangent $VB$ becomes parallel to the axis $AX$, as in Figure 27.

But if $dv$ is not equal to zero, we convert the proportion

$$dv : dx :: VX : XB \text{ (Figure 26)}$$

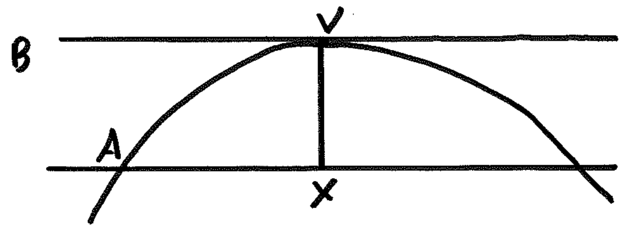



Figure 27

to an equation, and solve for $XB$:

$$\begin{aligned} \frac{dv}{dx} &= \frac{v}{XB} \\ dv \cdot XB &= v\,dx \\ XB &= \frac{v\,dx}{dv}. \end{aligned}$$

Therefore, once we have $dv$ in terms of $x$ and $dx$, we can find $XB$. But once we have found $XB$, we can draw the tangent by connecting $B$ and $V$. Leibniz's calculus thus gives us a method for finding tangents to a curve. This method is similar to the method Apollonius uses in Propositions I 33 and I 34 of the *Conics*, in that Apollonius also finds tangents by finding where they meet a diameter.

Here are two simple examples, based on the differences we have already calculated.

1. Let $v = x^2 + 2$. Then, as we saw in the first example (page 48, above),

   $$dv = 2x\,dx.$$

   Now $dv = 0$ when $x = 0$, and therefore there is horizontal tangent to the curve when $x = 0$.

   When $x \neq 0$, then $dv \neq 0$, and we can use the above equation for $XB$. Therefore

   $$\begin{aligned} XB &= \frac{v\,dx}{dv} \\ &= \frac{(x^2+2)\,dx}{2x\,dx} \\ &= \frac{x^2+2}{2x}. \end{aligned}$$

We may use this equation to find tangents to the curve. If for example, if $x = 1$, then

$$XB = \frac{1^2 + 2}{2(1)} = \frac{3}{2},$$

while if $x = -1$, then

$$XB = \frac{(-1)^2 + 2}{2(-1)} = -\frac{3}{2}.$$

The minus sign means that the point $B$ is to the right of $X$ in this case. See Figure 28.

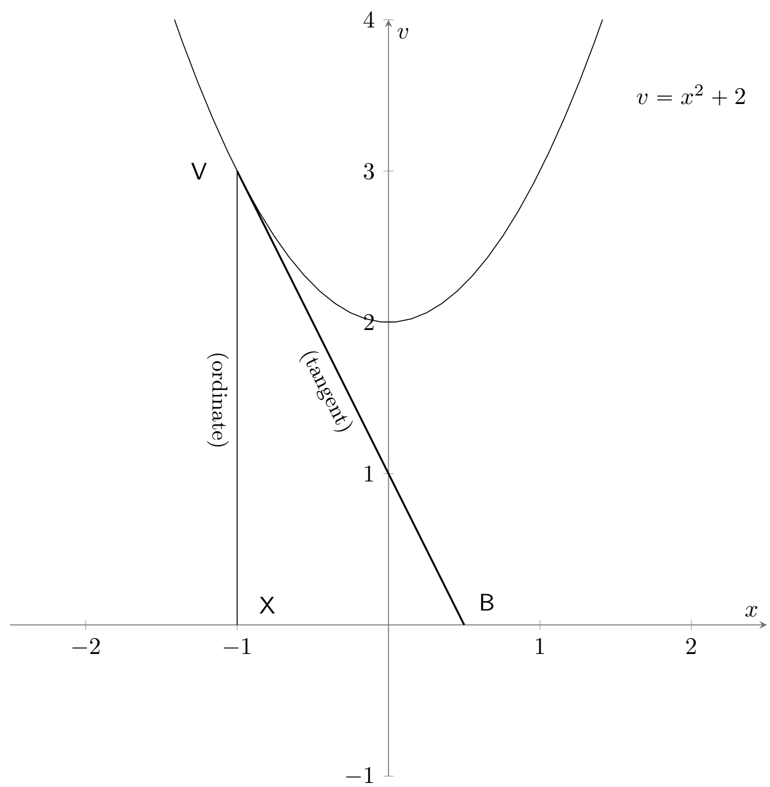



Figure 28

2. Let $v = x^3 - 6x^2 + 9x$. Then, as we saw in the second example (page 50, above),

$$dv = (3x^2 - 12x + 9)\, dx.$$

As we saw in the second example on page 67, $dv = 0$ only when $x = 1$ or $x = 3$. Therefore at these points the curve has a horizontal tangent.

But when $x \neq 1$ and $x \neq 3$, then $dv \neq 0$, and we can substitute into the above equation for $XB$, getting

$$\begin{aligned} XB &= \frac{v\,dx}{dv} \\ &= \frac{(x^3 - 6x^2 + 9x)\,dx}{(3x^2 - 12x + 9)\,dx} \\ &= \frac{x^3 - 6x^2 + 9x}{3x^2 - 12x + 9}. \end{aligned}$$

We may use this equation to find tangents to the curve. If, for example, $x = 4$, then

$$\begin{aligned} XB &= \frac{(4)^3 - 6(4)^2 + 9(4)}{3(4)^2 - 12(4) + 9} \\ &= \frac{64 - 96 + 36}{48 - 48 + 9} \\ &= \frac{4}{9}. \end{aligned}$$

See Figure 29.

**7. Problems about finding tangents**

For curves given by each of the following equations, find a general expression for the line $XB$ cut off on the axis by the ordinate and the tangent, and use this general expression to find one particular tangent.

1. $v = x^2 - 4x + 1$.
2. $v = x^2 + 2x - 5$.
3. $v = x^3 - 3x^2 - 9x + 4$.
4. $v = 2x^3 - 3x^2 - 12x + 2$.

## Note 8

Here is an example to show what Leibniz is saying here. Suppose the given equation is

$$2x^2 + 3xy = 1,$$

and we want to find its differential equation. Then the *terms* of this equation are $2x^2$, $3xy$, and 1. To write down the differential equation for this equation, we substitute for each of these terms its differential quantity; namely, we substitute $d(2x^2)$ for $2x^2$, $d(3xy)$ for $3xy$, and $d(1)$ for 1. This gives us a differential equation:

$$d(2x^2) + d(3xy) = d(1).$$

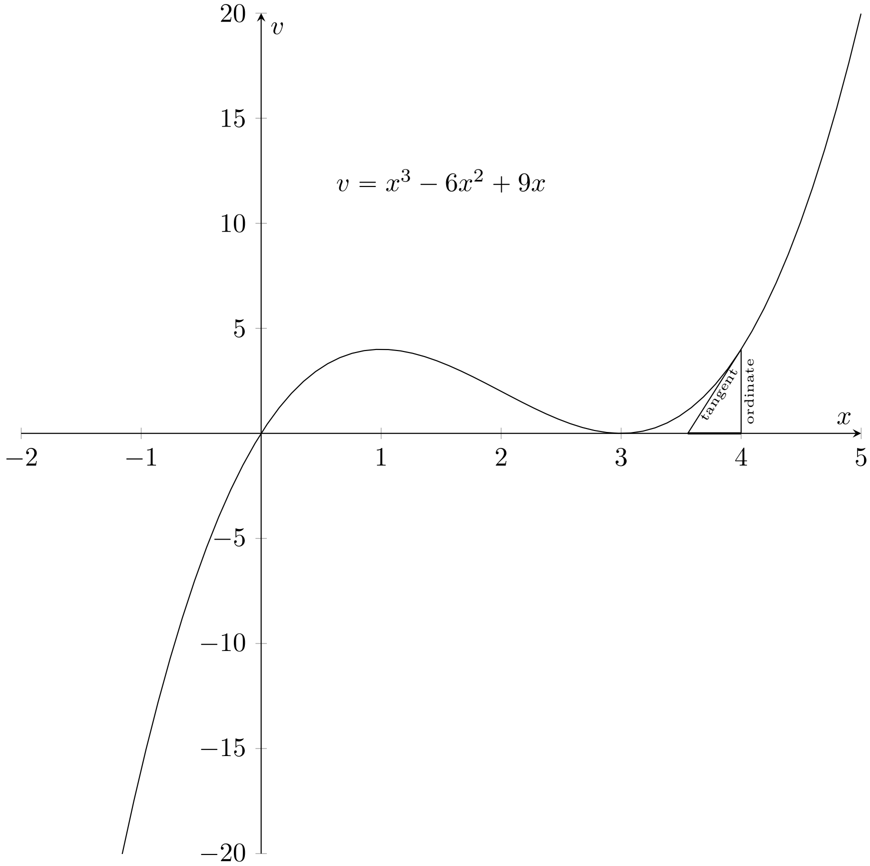



Figure 29

Next, for quantities (2, $x$, 3, $y$ and 1) which are not themselves terms, but contribute to forming terms (2 and $x$, for example, contribute to forming the term $2x^2$), we do not directly use their differential quantities. In other words, we do *not* simply substitute $dx$ for $x$, and do *not* substitute $d(2)d(x)^2$ for $d(2x^2)$. Instead, we use Leibniz's algorithm, that is, his six rules for finding differences. In this case, we use the power rule, the multiplication rule, and the constant rules to get

$$\begin{aligned} d(2x^2) &= 4x\,dx, \\ d(3xy) &= 3x\,dy + 3y\,dx \text{ and} \\ d(1) &= 0. \end{aligned}$$

We use these equations to substitute into the above differential equation, and get our final differential equation:

$$4x\,dx + 3x\,dy + 3y\,dx = 0.$$

If our original, algebraic, equation

$$2x^2 + 3xy = 1,$$

is the equation of a curve, then the differential equation

$$4x\,dx + 3x\,dy + 3y\,dx = 0$$

is another equation for the same curve, now expressing not the universal relation between two ordinary quanitities, $x$ and $y$, but the universal relation between these quantities and their differences $dx$ and $dy$. Just as we can use the algebraic equation to solve geometric problems, we may use the differential equation to solve for the differences $dx$ and $dy$ and thereby find tangents or greatest and least lines.

## Note 9

A simple example of a transcendent line is the line whose equation is $y = x^x$. It is of no definite degree because there is a variable in an exponent: $x$ could be 2, 3, 4, or any other number, and so we cannot say what the degree of the term $x^x$ is. The algebraic calculus cannot be applied to equations like this because it only includes the five basic operations Descartes mentions at the beginning of the *Geometry*: addition, subtraction, multiplication, division, and extracting of roots. The algebraic calculus does not include taking exponents with variable powers, such as $x^x$. Descartes calls all lines that can be represented by the algebraic calculus *geometric*, and all other lines *mechanical*. (See page 16 of the St. John's (1998) edition of the Geometry or page 48 of the Dover (1954) edition.) The lines Leibniz calls *transcendent* would therefore be called *mechanical* by Descartes.

## Note 10

We will come back to Leibniz's discussion of how the calculus extends to transcendent lines in the following paper, "On Recondite Geometry." We define the cycloid and find its transcendent equation in the notes to "On Recondite Geometry" (pages 125–127, below).

## Note 11

The law of homogeneity for differential quantities requires that all terms in an equation be of the same level of infinity (see the remarks after the demonstration of the multiplication rule on page 62, above). If one term of an equation is at the first level of infinitely small quantities, the level of simple differences such as $dx$, then all terms must be at that level. If one term of an equation is at the second level of infinitely small quantities, such as $(dx)^2$, then all terms must be at that level, and so on. This law thus guarantees that no term will be infinitely small or infinitely large compared to any other term in the equation.

The law of homogeneity for differential quantities would, in general, require that in each term of an equation, the sum of the exponents of the differences is always the same. For example, in the term

$$(dx)^2 dy\, dz$$

the sum of exponents is 4: $dx$ has exponent 2, while $dy$ and $dz$ each have exponent 1 ($dy = dy^1$ and $dz = dz^1$). Such a term would be at the fourth level of infinitely small quantities. It could therefore be combined with any other terms for which the sum of exponents is 4 to make a homogeneous differential equation. For example, we could combine it with $(dy)^2(dz)^2$ and $(dx)^4$ to make the homogeneous differential equation

$$(dx)^2 dy\, dz + (dy)^2(dz)^2 = (dx)^4.$$

Note that only the differential quantities matter for Leibniz's law of homogeneity; any other finite quantities that go to make up a term do not affect the level of infinity of the term. For example, in the term

$$2x^2(dx)^2 dy\, dz$$

the sum of the exponents of *differences* is still 4 (the exponent of $dx$ is 2, and the exponents of $dy$ and $dz$ are 1), and this term is therefore on the fourth level of infinitely small quantities. The exponent of the finite quantity $x$ does not affect the level of the term. This term could be combined with other terms on the fourth level of infinitely small quantities to make a homogeneous differential equation, such as

$$2x^2(dx)^2 dy\, dz + z(dy)^2(dz)^2 = (x + 2yz)(dx)^4.$$

In the case Leibniz is discussing in the text, in each term of the differential equation the sum of the exponents of the differences is always 1: each term has either $dx^1$ or $dy^1$, and no other differences.

Leibniz's law of homogeneity is analogous to Viète's algebraic law of homogeneity (see Chapter III of his *Introduction to the Analytic Art*), which requires that each term of an equation represent a magnitude of the same dimension, or what amounts to the same thing, that in each term the sum of the exponents of all the quantities in that term be the same.

## Note 12

Leibniz is thinking of something moving from the point $C$ to the point $F$ along line $CF$ (see Figure 30), and then from $F$ to $E$ along line $FE$. By *density* he

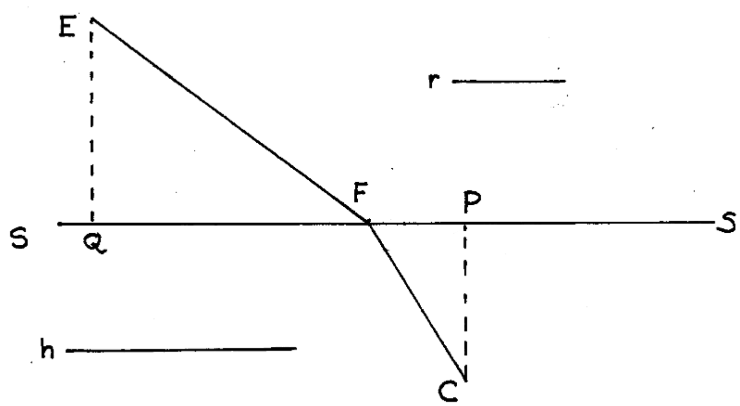



Figure 30

means the power of the medium to resist motion, so that the total difficulty of any motion in a uniform medium is equal to the product of the density of that medium and the length of the path along which the motion occurs. (Leibniz apparently is treating densities as lines, so that the product of a density and a line is a rectangle.) Therefore, $CF$ times $h$ is equal to the difficulty of the motion from $C$ to $F$, $EF$ times $r$ is equal to the difficulty of the motion from $F$ to $E$, and

$$(CF) \cdot h + (EF) \cdot r$$

is equal to the total difficulty of the motion from $C$ to $E$. Now this difficulty is variable because the point $F$ is variable: $F$ could be anywhere on the line $SS$. Leibniz wants to find where $F$ has to be for the motion to be as easy as possible, that is, he wants to find the point $F$ so that

$$(CF) \cdot h + (EF) \cdot r$$

is as small as possible.

As Leibniz suggests later, this example may be applied to optics. Suppose light moves from the point $C$ to the point $F$ along the line $CF$, and from the point $F$ to the point $E$ along the line $FE$. We take the density of a medium to be inversely proportional to the speed with which light moves in that medium, so that if the light moves with speed $v_w$ in the water below $SS$, and with speed $v_a$ in the air above $SS$, then the density $h$ of the water is equal to

$$\frac{1}{v_w},$$

while the density $r$ of the air is equal to

$$\frac{1}{v_a}.$$

If we call the time the light takes to move along $CF$ in the water $t_w$, and the time it takes the light to move along $FE$ in the air $t_a$, then

$$\begin{aligned} v_w &= \frac{CF}{t_w}, \text{ and} \\ v_a &= \frac{FE}{t_a}. \end{aligned}$$

Therefore the difficulty of the motion along $CF$,

$$\begin{aligned} (CF) \cdot h &= (CF) \cdot \frac{1}{v_w} \\ &= (CF) \cdot \frac{1}{\left(\frac{CF}{t_w}\right)} \\ &= (CF) \cdot \frac{t_w}{CF} \\ &= t_w; \end{aligned}$$

that is, the difficulty of the motion is in this case equal to the time. Likewise, the difficulty of the motion from $F$ to $E$ along the line $FE$ is equal to the time $t_a$, and therefore the difficulty of the whole motion from $C$ to $E$ is equal to the time it takes. In this case Leibniz is therefore looking for the path light could take that will take the least time to get from $C$ to $E$.

## Note 13

Leibniz is defining the curve $VV$ so that when $GK$ (in Figure 31) is equal to the line $QF$ in Figure 10, then $KV$ (in Figure 11) is equal to the difficulty of the path from $C$ to $E$ through $F$; that is, when

$$GK = QF,$$

then

$$KV = (CF) \cdot h + (EF) \cdot r.$$

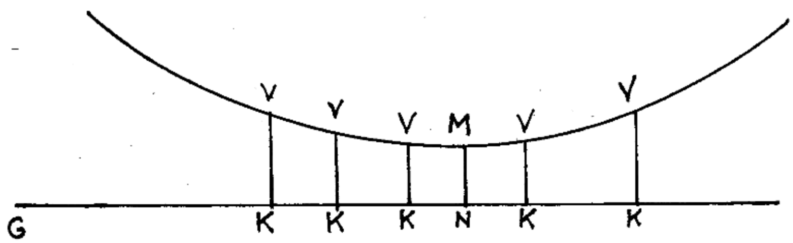



Figure 31

## Note 14

To derive Leibniz's expression for $f$, we note that it follows from Proposition I 47 in Euclid's *Elements* that

$$(CF)^2 = (PC)^2 + (FP)^2,$$

and $PC = c$, while $FP = p - x$. Substituting these expressions for $CF$, $PC$, and $FP$ into the above equation, we get

$$\begin{aligned} f^2 &= c^2 + (p-x)^2 \\ &= c^2 + p^2 - 2px + x^2. \end{aligned}$$

Taking the square root of each side of the equation, we get Leibniz's equation:

$$f = \sqrt{c^2 + p^2 - 2px + x^2}.$$

To derive Leibniz's equation for $g$, we again use Proposition I 47 in the *Elements*, according to which:

$$(EF)^2 = (EQ)^2 + (QF)^2.$$

$g$ is the line $EF$, $EQ = e$, and $QF = x$. Therefore

$$g^2 = e^2 + x^2,$$

and, after we take the square root of each side,

$$g = \sqrt{e^2 + x^2}.$$

## Note 15

Here is a derivation of Leibniz's expression for the differences of $\omega$:

$$\begin{aligned}
d\omega &= d(h\sqrt{l} + r\sqrt{m}) \\
&= d(h\sqrt{l}) + d(r\sqrt{m}) && \text{(addition rule)} \\
&= hd(\sqrt{l}) + rd(\sqrt{m}) && \text{(constant multiple rule)} \\
&= h\frac{dl}{2\sqrt{l}} + r\frac{dm}{2\sqrt{m}} && \text{(root rule)} \\
&= \frac{h\,dl}{2\sqrt{l}} + \frac{r\,dm}{2\sqrt{m}} && \text{(ordinary algebra)}
\end{aligned}$$

## Note 16

Here is a derivation of Leibniz's expression for the differences of $l$:

$$\begin{aligned}
dl &= d(c^2 + p^2 - 2px + x^2) \\
&= d(c^2) + d(p^2) - d(2px) + d(x^2) && \text{(addition rule)} \\
&= 0 + 0 - 2p\,dx + d(x^2) && \text{(constant rules)} \\
&= -2p\,dx + 2x\,dx && \text{(power rule)} \\
&= 2dx(-p + x) && \text{(ordinary algebra)} \\
&= -2dx(p - x) && \text{(ordinary algebra)}
\end{aligned}$$

## Note 17

Leibniz has just shown that

$$d\omega = \frac{h\,dl}{2\sqrt{l}} + \frac{r\,dm}{2\sqrt{m}} = 0, \tag{1}$$

and that

$$\begin{aligned}
dl &= -2dx(p - x), && (2) \\
dm &= 2x\,dx, && (3) \\
\sqrt{l} &= f, \text{ and} && (4) \\
\sqrt{m} &= g. && (5)
\end{aligned}$$

Substituting the values from equations 2 through 5 into equation 1 gives

$$\begin{aligned}
\frac{h(-2dx(p - x))}{2f} + \frac{r(2x\,dx)}{2g} &= 0, \text{ and therefore} \\
-\frac{h(p - x)}{f} + \frac{rx}{g} &= 0, \text{ and} \\
\frac{h(p - x)}{f} &= \frac{rx}{g}.
\end{aligned}$$

## Note 18

This proportion is usually called *Snell's Law*, after Willebrord Snellius, who discovered it in 1621. At the time Leibniz wrote "A New Method," Descartes and Huygens had already published derivations of it. If we draw the line $UFT$ perpendicular to the line $QFP$ in Figure 32, then the angle of incidence is $\angle CFT$

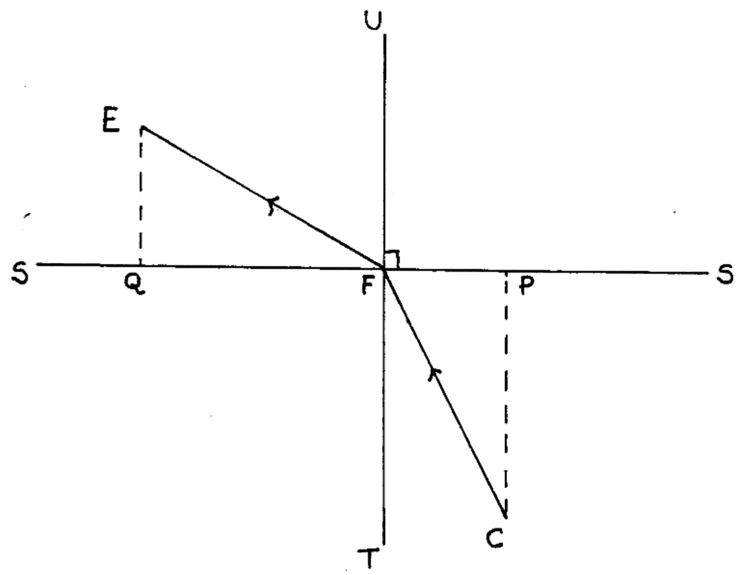



Figure 32

and the angle of refraction is $\angle EFU$. We may assume that $FC = EF = 1$. Then

$$\sin(\angle CFT) = \sin(\angle FCP) = \frac{FP}{FC} = FP,$$

and

$$\sin(\angle EFU) = \sin(\angle FEQ) = \frac{FQ}{EF} = FQ.$$

## Note 19

In June of 1682 in an article titled "The Unique Principle of Optics, Catoptrics, and Dioptrics." There he announces, without demonstration, that his "method of maxima and minima" can be used to prove Snell's law, and makes the following remark:

> We have therefore, by applying a single principle, reduced all the Laws of rays of light that have been proved by experience to pure Geometry and calculation; rightly considered, this principle comes from a final cause: for the ray of light going out from $C$ does not

consider how it might most easily arrive at $E$, nor is it carried there by itself; but the Founder of things so created light, that this most beautiful outcome might arise from light's nature. And thus those who follow *Descartes* and reject *final causes* in Physics are very much in error (not to say anything more grave), since besides admiration of divine wisdom, final causes offer us a most beautiful *principle* for *discovering* the properties of those things whose inner nature is not yet so clearly known by us that we are able to use proximate efficient causes and explicate the machines which the founder has used to produce those effects and obtain his ends. Hence we also understand that the meditations of the ancients in these matters too are not as contemptible as they seem now to some.

## Note 20

The curve 133 [Figure 33] is the solution of a locus problem: if six points (here

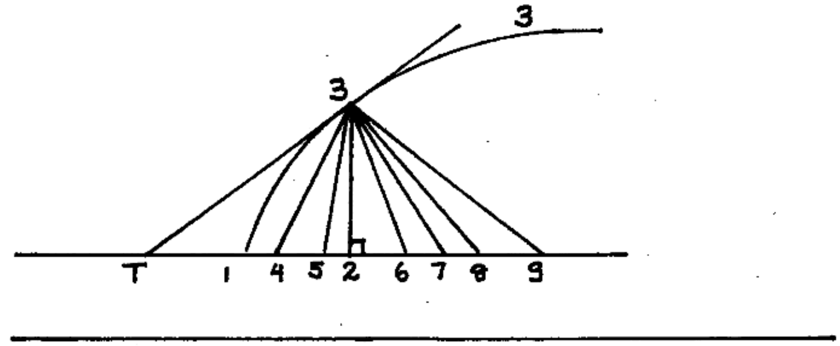



Figure 33

4, 5, 6, 7, 8, and 9) are given, all on one straight line (here the line T14526789), and if straight lines are drawn to each of these given points from one point (here we draw the straight lines 34, 35, 36, 37, 38, and 39, all from the one point 3), such that all these drawn lines are together equal to another given line (here the line $g$, so that $34+35+36+37+38+39 = g$), then the one point (3) lies on the locus in question (here the line 133). If there were only two points, 4 and 5, instead of the six points Leibniz takes as given, then the locus in question would be an ellipse whose foci are the points 4 and 5, and whose major axis is equal to $g$; for according to Proposition 52 of Book III of Apollonius' *Conics*, the sum of the two lines from any point on an ellipse to the two foci is always equal to the major axis.

Leibniz does not find an algebraic equation relating the abscissa and ordinate of his locus, as Descartes does for Pappus' locus problem. Instead, he finds a proportion giving the shape of the triangle (T23) formed by the tangent, the ordinate, and the axis for any point 3; once we have found the shape of the triangle (T23), we know where the tangent must be. Leibniz's proportion thus gives us a way to find the tangent to the curve 133 at any point.

It is not necessary to go through a detailed derivation of Leibniz's proportion, but for those who are interested, in the remainder of this note we give a complete account of how to find the ratio of the line 23 to the line $T2$. We begin by noting that his ratio is equal to the ratio of $d(23)$ to $d(12)$; for the triangle $3T2$ is similar to the characteristic triangle, whose vertical side is equal to $d(23)$, and whose horizontal side is equal to $d(12)$ [See Figure 34]. Now to find the relation between

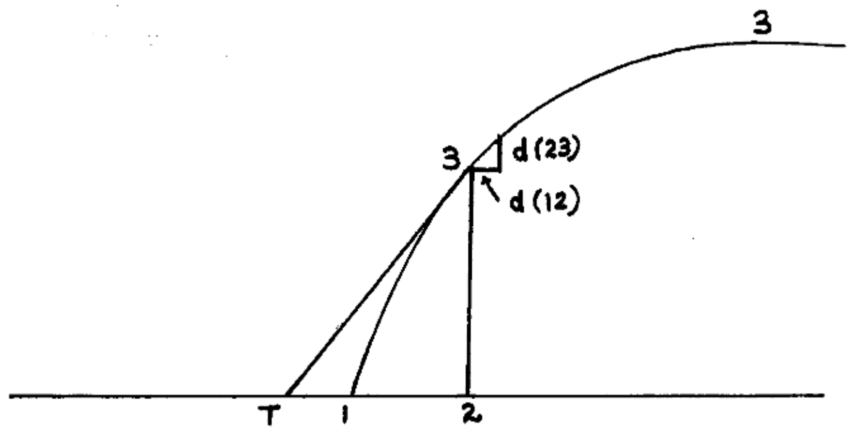



Figure 34

these two differences, we start by using the equation Leibniz uses to define the curve 13:

$$34 + 35 + 36 + 37 + 38 + 39 = g,$$

where $g$ is a constant. We then find the difference of each side of this equation, and use the addition and constant rules, to get the differential equation

$$d(34) + d(35) + d(36) + d(37) + d(38) + d(39) = dg = 0.$$

But we want an equation for $d(12)$ and $d(23)$, not $d(34)$, $d(35)$, etc. We cannot immediately find an equation for $d(12)$ and $d(23)$, but we can at least find an equation relating $d(23)$ and the differences of some other lines on the axis besides 12. To do this, we use Proposition I 47 in Euclid's *Elements* to express each of the lines 34, 35, etc., in terms of the line 23 and a line on the axis:

$$\begin{aligned}
(34)^2 &= (42)^2 + (23)^2 \\
(35)^2 &= (52)^2 + (23)^2 \\
(36)^2 &= (26)^2 + (23)^2 \\
&\text{etc.}
\end{aligned}$$

Therefore

$$\begin{aligned}
(34) &= \sqrt{(42)^2 + (23)^2} \\
(35) &= \sqrt{(52)^2 + (23)^2} \\
(36) &= \sqrt{(26)^2 + (23)^2} \\
\text{etc.}
\end{aligned}$$

Now we take the differences of all these equations, getting

$$\begin{aligned}
d(34) &= d(\sqrt{(42)^2 + (23)^2}) \\
d(35) &= d(\sqrt{(52)^2 + (23)^2}) \\
d(36) &= d(\sqrt{(26)^2 + (23)^2}) \\
\text{etc.}
\end{aligned}$$

We then use the rules for the calculus to simplify all the differences on the right sides of these equations. For example, to find $d(\sqrt{(42)^2 + (23)^2})$, we first set $a = (42)^2 + (23)^2$, so that

$$d(34) = d(\sqrt{(42)^2 + (23)^2}) = d\sqrt{a}.$$

According to the root rule,

$$d(\sqrt{a}) = \frac{da}{2\sqrt{a}}.$$

To get an expression for $d(\sqrt{a})$ in terms of the lines (42) and (23) and their differences, we need to find $da$:

$$\begin{aligned}
da &= d((42)^2 + (23)^2) \\
&= d((42)^2) + d((23)^2) && \text{(addition rule)} \\
&= 2(42)d(42) + 2(23)d(23) && \text{(power rule).}
\end{aligned}$$

In the end, since we want to find the relation between $d(12)$ and $d(23)$, we want to find an equation involving no other differences besides these. We therefore express 42 in terms of 12:

$$42 = 12 - 14,$$

and take differences

$$\begin{aligned}
d(42) &= d(12) - d(14) && \text{(addition rule)} \\
&= d(12). && \text{(constant rule)}
\end{aligned}$$

We may therefore substitute $d(12)$ for $d(42)$ in our expression for $da$:

$$\begin{aligned}
da &= 2(42)d(42) + 2(23)d(23) \\
&= 2(42)d(12) + 2(23)d(23).
\end{aligned}$$

Therefore

$$\begin{aligned}
d(34) &= d(\sqrt{a}) \\
&= \frac{da}{2\sqrt{a}} \\
&= \frac{2(42)d(12) + 2(23)d(23)}{2\sqrt{a}} \\
&= \frac{42}{\sqrt{a}}\, d(12) + \frac{23}{\sqrt{a}}\, d(23) \\
&= \frac{42}{34}\, d(12) + \frac{23}{34}\, d(23).
\end{aligned}$$

Similarly we can show that

$$\begin{aligned}
d(35) &= \frac{52}{35}\, d(12) + \frac{23}{35}\, d(23) \\
d(36) &= -\frac{62}{36}\, d(12) + \frac{23}{36}\, d(23) \\
d(37) &= -\frac{72}{37}\, d(12) + \frac{23}{37}\, d(23) \\
&\text{etc.}
\end{aligned}$$

(The minus signs in the expressions for $d(36)$, $d(37)$, etc., are there because the points 6, 7 and so on are to the right of the point 2, and therefore $62 = 16 - 12$ and $d(62) = -d(12)$, etc.)

We now return to the original differential equation for the curve, and substitute the expressions we have found for $d(34)$, $d(35)$, etc.:

$$
\begin{aligned}
0 &= d(34) + d(35) + d(36) + d(37) + d(38) + d(39) \\
&= +\frac{42}{34}\,d(12) + \frac{23}{34}\,d(23) \\
&\quad +\frac{52}{35}\,d(12) + \frac{23}{35}\,d(23) \\
&\quad -\frac{62}{36}\,d(12) + \frac{23}{36}\,d(23) \\
&\quad -\frac{72}{37}\,d(12) + \frac{23}{37}\,d(23) \\
&\quad \text{etc.} \\
&= \left(\frac{42}{34} + \frac{52}{35} - \frac{62}{36} - \frac{72}{37} - \frac{82}{38} - \frac{92}{39}\right) d(12) \\
&\quad + \left(\frac{23}{34} + \frac{23}{35} + \frac{23}{36} + \frac{23}{37} + \frac{23}{38} + \frac{23}{39}\right) d(23)
\end{aligned}
$$

Finally, we use this equation to solve for the ratio of $d(12)$ to $d(23)$ (that is, the ratio of $T2$ to 23). Subtracting the first term on the right from both sides of the equation gives:

$$\left(-\frac{42}{34} - \frac{52}{35} + \frac{62}{36} + \frac{72}{37} + \frac{82}{38} + \frac{92}{39}\right) d(12) = \left(\frac{23}{34} + \frac{23}{35} + \frac{23}{36} + \frac{23}{37} + \frac{23}{38} + \frac{23}{39}\right) d(23).$$

Dividing both sides by $d(23)$ and $\left(-\frac{42}{34} - \frac{52}{35} + \frac{62}{36} + \frac{72}{37} - \frac{82}{38} + \frac{92}{39}\right)$ gives:

$$\frac{d(12)}{d(23)} = \frac{\frac{23}{34} + \frac{23}{35} + \frac{23}{36} + \frac{23}{37} + \frac{23}{38} + \frac{23}{39}}{\left(-\frac{42}{34} - \frac{52}{35} + \frac{62}{36} + \frac{72}{37} + \frac{82}{38} + \frac{92}{39}\right)}.$$

This last equation is equivalent to the proportion Leibniz gives.

## Note 21

That the same rule applies no matter how many points are given is perhaps the reason Leibniz chooses to represent the points with numbers rather than letters: there is no limit to the number of symbols for numbers, and it may be easier to see the general pattern in the rule when it is expressed in numbers.

## Note 22

The solution to De Beaune's problem is a logarithmic line, and Leibniz seems to assume that readers are familiar with logarithmic lines. Since most of us are not familiar with logarithmic lines, in this note we give a brief introduction to them. The note is divided into six parts:

1. a definition of logarithmic lines;
2. differential equations for logarithmic lines;
3. natural logarithms;
4. the difference of $e^x$;
5. notation for logarithms; and
6. the difference of $\log x$, and finding differences of quantities involving logarithms.

### 1. A definition of logarithmic lines

An *arithmetic progression* is a series of quantities whose successive differences are always equal. The simplest example is the series of numbers

$$1,\ 2,\ 3,\ 4, \ldots;$$

the difference of any two consecutive numbers in this series is always equal to 1.

A *geometric progression* is a series of quantities whose successive ratios are always the same. A simple geometric progression is the series of numbers

$$2,\ 4,\ 8,\ 16, \ldots;$$

any two consecutive numbers in this series are always in a two to one ratio.

A *logarithmic line* may be defined as any line such that any arithmetic progression of its abscissas corresponds to a geometric progression of its ordinates. See Figure 35. There the line $AB$ is an axis for the logarithmic line $CF$. Therefore, if

$$AG_1 = G_1G_2 = G_2G_3$$

(so that the abscissas $AG_1$, $AG_2$, $AG_3$ form an arithmetic progression), then the ordinates $AC$, $G_1L_1$, $G_2L_2$, and $G_3L_3$ of $CD$ form a geometric progression, that is

$$AC : G_1L_1 :: G_1L_1 : G_2L_2 :: G_2L_2 : G_3L_3.$$

If the ordinates of a logarithmic line are *numbers*, then the abscissas of that same line are called the *logarithms* of those numbers (with respect to that line). For example, in Figure 35, if

$$AG_1 = 1,\ AG_2 = 2,\ AG_3 = 3,\ \ldots,$$

and

$$G_1L_1 = 2,\ G_2L_2 = 4,\ G_3L_3 = 8,\ \ldots,$$

then 1 would be the logarithm of the number 2, 2 would be the logarithm of the number 4, 3 would be the logarithm of the number 8, and so on. The series of logarithms,

$$1,\ 2,\ 3,\ \text{etc.}$$

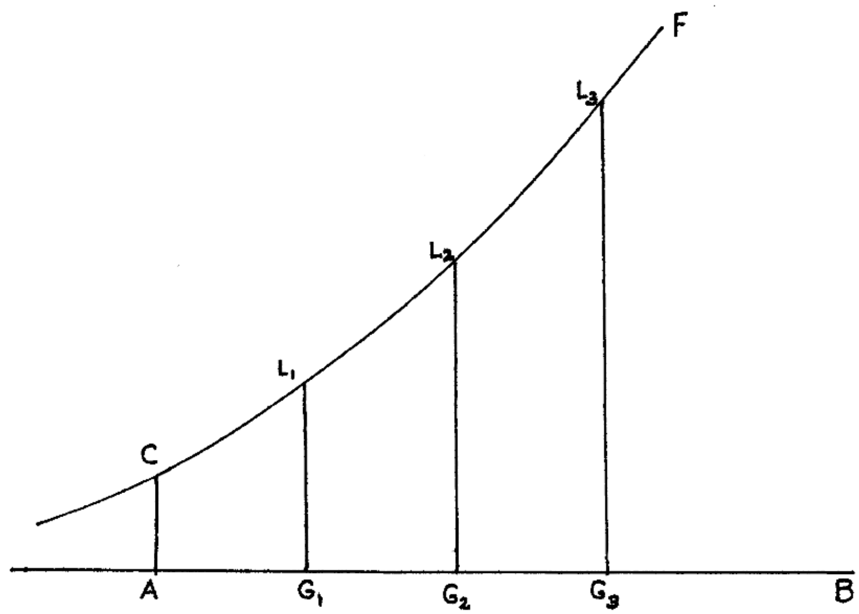



Figure 35

is an arithmetic progression, and the series of corresponding numbers,

$$2,\ 4,\ 8,\ \text{etc.}$$

is a geometric progression.

Note that when we express numbers as powers, their exponents are logarithms. For example, we can express the geometric progression

$$2,\ 4,\ 8,\ \text{etc.}$$

as a progression of powers,

$$2^1,\ 2^2,\ 2^3,\ \text{etc.}\ .$$

The power of each number is its corresponding logarithm: the logarithm of $2^1$ is 1, the logarithm of $2^2$ is 2, the logarithm of $2^3$ is 3, and so on. In general if in Figure 35 we assume that the first ordinate $AC = 1$, that $AG_1 = G_1G_2 = G_2G_3$ etc., and we let

$$k = \frac{G_1L_1}{AC},$$

then because the ordinates $AC$, $G_1L_1$, $G_2L_2$ are in a geometric progression,

$$k = \frac{G_1L_1}{AC} = \frac{G_2L_2}{G_1L_1} = \frac{G_3L_3}{G_2L_2} = \cdots .$$

Therefore

$$\begin{array}{lllll} G_1L_1 & = & k(AC) & = & k, \\ G_2L_2 & = & k(G_1L_1) & = & k^2, \\ G_3L_3 & = & k(G_2L_2) & = & k^3, \text{ etc.} \end{array}$$

If we denote the abscissa $AG$ by $x$ and the ordinates $GL$ by $w$, this means that when

$$x = 1,\ 2,\ 3,\ \text{and so on,}$$

then

$$w = k^1,\ k^2,\ k^3,\ \text{and so on;}$$

in other words, whenever $x$ is a positive integer, then

$$w = k^x.$$

The logarithm of $k^x$ is its exponent, $x$.

With a little more work we could show that whenever $x$ is a rational number (that is, a fraction whose numerator and denominator are both integers), then the same equation holds. It is therefore reasonable to assume that for *any* number $x$,

$$w = k^x. \tag{1}$$

Equation 1 is an ordinary equation for the logarithmic line $CF$. Note that it is in fact a *transcendent* equation, that is, an equation of no definite degree (see page 34 of "A New Method," and Note 9, above); for the variable $x$ is in an exponent.

The word *logarithm* comes from the Greek words λόγος (*ratio*) and ἀριθμός (*number*): a logarithm is a number of a ratio. In our example, the logarithm of 2 is 1, and 2 is to 1 in the *simple* ratio of 2 to 1. The logarithm of 4 is 2, and 4 is to 1 in the *duplicate* ratio of 2 to 1:

$$4:2 :: 2:1.$$

The logarithm of 8 is 3, and 8 is to 1 in the *triplicate* ratio of 2 to 1:

$$8:4 :: 4:2 :: 2:1.$$

In general, when the logarithm $AG$ is a whole number, it expresses the degree to which the ratio of $GL$ to $AC$ is compounded of the ratio of 2 to 1. Of course, $AG$ need not always be a whole number.

### 2. Differential equations for logarithmic lines

Leibniz has shown that the line $WW$ (in Leibniz's Figure 36) which is a solution to De Beaune's problem is a solution to the differential equation

$$w = \frac{a}{b}dw, \tag{2}$$

where $a = XC$ and $b = dx$ are constants. He then claims that to say that a line is a solution to equation 2 is equivalent to saying that $WW$ is a logarithmic

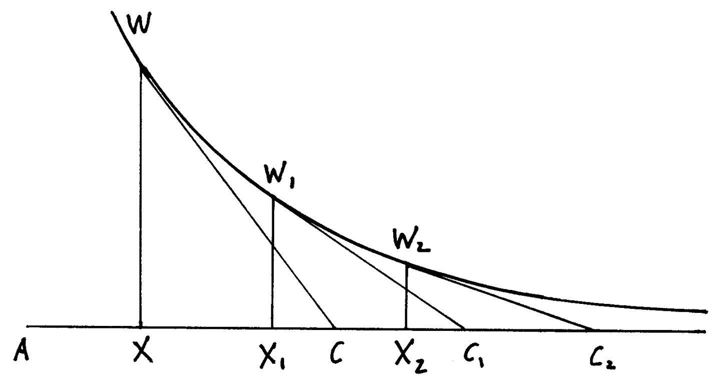



Figure 36

line; that is, that if a line is a logarithmic line then it satisfies equation 2, and, conversely, that if a line satisfies equation 2 then it is a logarithmic line. In other words, we have the following two theorems:

**Theorem 1** *If $WW$ is a logarithmic line, then*

$$w = \frac{a}{b}dw,$$

*where $a = XC$ and $b = dx$ are constants.*

**Theorem 2** *If $WW$ is a line satisfying the differential equation*

$$w = \frac{a}{b}dw,$$

*where $a = XC$ and $b = dx$ are constants, then $WW$ is a logarithmic line.*

We will give a demonstration of Theorem 1 here, showing that any logarithmic line is a solution of De Beaune's problem. Theorem 2, the converse of Theorem 1, is more difficult, and we do not demonstrate it.

**Demonstration of Theorem 1:** Let $WW$ be a logarithmic line whose ordinates $WX = w$ and whose abscissas $AX = x$. We say that $WW$ satisfies the differential equation 2:

$$w = \frac{a}{b}dw,$$

where $a$ is a constant equal to the distance $XC$ on the axis between the ordinate $WX$ and the tangent $WC$ of the line $WW$, and $b = dx$, the difference of the abscissas $AX$.

For let $AX$, $AX_1$, $AX_2$, and so on, be an arithmetic progression of infinitely close abscissas, and let the infinitely small difference between consecutive abscissas in this series be $b = dx$. Then, since $WW$ is a logarithmic line, the corresponding series of ordinates $XW$, $X_1W_1$, $X_2W_2$, and so on, is a geometric series; that is, there is one constant ratio $c$ such that

$$c = \frac{w_1}{w} = \frac{w_2}{w_1} = \frac{w_3}{w_2}, \text{ etc.}$$

(Since $w_1$ is infinitely close to $w$, the quantity $c$ is infinitely close to 1.) Therefore

$$w_1 = cw,\ w_2 = cw_1,\ w_3 = cw_2,\ etc.$$

Therefore, at the point $W$,

$$\begin{aligned} dw &= w_1 - w \\ &= cw - w \\ &= (c-1)w, \end{aligned}$$

where $c-1$ is infinitely small. Likewise, at the point $W_1$,

$$\begin{aligned} dw_1 &= w_2 - w_1 \\ &= cw_1 - w_1 \\ &= (c-1)w_1. \end{aligned}$$

A similar equation will hold for $W_2$, $W_3$, and, in fact, for *any* point on the logarithmic line $WW$. Therefore, no matter what the value of the ordinate $w$,

$$dw = (c-1)w,$$

and therefore

$$\frac{w}{dw} = \frac{1}{c-1}$$

is constant.

But, as Leibniz points out,

$$\frac{w}{dw} = \frac{XC}{b},$$

and therefore $XC$ must be constant and the line $WW$ is a solution to De Beaune's problem. Leibniz sets $XC = a$, so that

$$\frac{a}{b} = \frac{w}{dw},$$

and therefore

$$w = \frac{a}{b}dw.$$

This is equation 2. Therefore if $WW$ is a logarithmic line whose ordinates are $w$ and whose abscissas are $x$ then it satisfies the differential equation

$$w = \frac{a}{b}dw,$$

where $a$ is a constant equal to $XC$, the distance on the axis between the ordinate and the tangent, and $b$ is a constant $dx$ equal to the difference of the abscissas $AX$. **Q. E. D.**

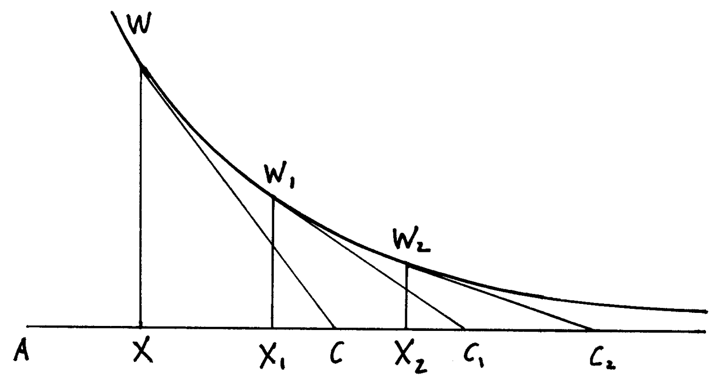



Figure 37

### 3. Natural logarithms

Both equation 1,

$$w = k^x, \tag{1}$$

and equation 2,

$$w = \frac{a}{b} dw, \tag{2}$$

contain arbitrary constants. Equation 2 depends on $a$ ($b$ is equal to $dx$, and in that sense is not arbitrary), while equation 1 depends on $k$. Different values of $a$ or of $k$ give equations corresponding to different logarithmic lines. The constant $a$ in equation 2,

$$w = \frac{a}{b} dw,$$

is a constant of proportionality between the ordinates $w$ and the differences of those ordinates, $dw$. When $a$ is positive, a positive value of $w$ corresponds to a positive value of $dw$, and therefore the logarithmic line slopes upward, while if $a$ is negative, a positive value of $w$ corresponds to a negative value of $dw$, and therefore the logarithmic line slopes downward. In Figure 38, $a$ is positive for $CE$ and $CF$, while $a$ is negative for $CD$. When $a$ is positive, as it becomes larger, for any given value of $w$, the differences $dw$ must be smaller. Therefore, larger values of $a$ correspond to more gently sloping logarithmic lines. In Figure 38, $CE$ has a larger value for $a$ than $CF$. The constant $k$ in equation 1,

$$w = k^x,$$

is the ratio of two ordinates corresponding to abscissas which are one unit apart. For example, in Figure 38, if $AC = AG_1 = 1$, then $k = G_1H_1$ for the line $CD$, $k = G_1K_1$ for the line $CE$, and $k = G_1L_1$ for the line $CF$.

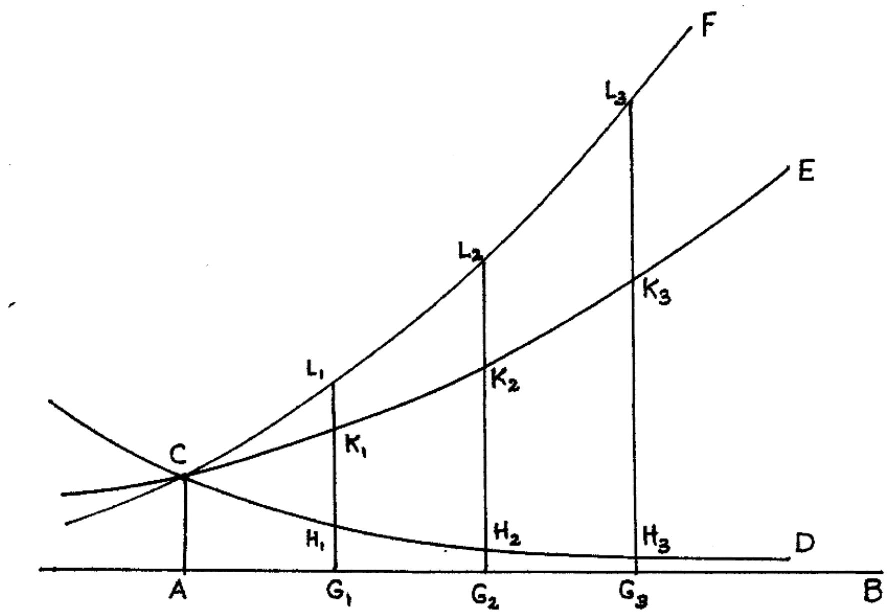



Figure 38

When the constant $a = 1$, then equation 2 becomes particularly simple:

$$w = \frac{1}{b} dw.$$

Substituting $dx$ for $b$, and solving for $dx$ gives:

$$dx = \frac{dw}{w}. \tag{3}$$

In this case the line represented by equation 3 is said to be a *natural logarithmic line*, and the abscissas $x$ are said to be the *natural logarithms* of the ordinates $w$. If we use equation 1 instead of equation 3 to represent the natural logarithmic line, then we denote by $e$ the value of $k$ that we need to use, so that equation 1 becomes

$$w = e^x$$

We have simply defined $e$ as the number such that $w = e^x$ is the equation for a natural logarithmic line; but we have as yet no notion of what the value of $e$ might be. In fact it is difficult to determine an exact value for $e$, but the property of the tangents of the logarithmic line can help us limit the range of values that $e$ might have, as shown in the following theorem.

**Theorem 3** $2 < e < 4$.

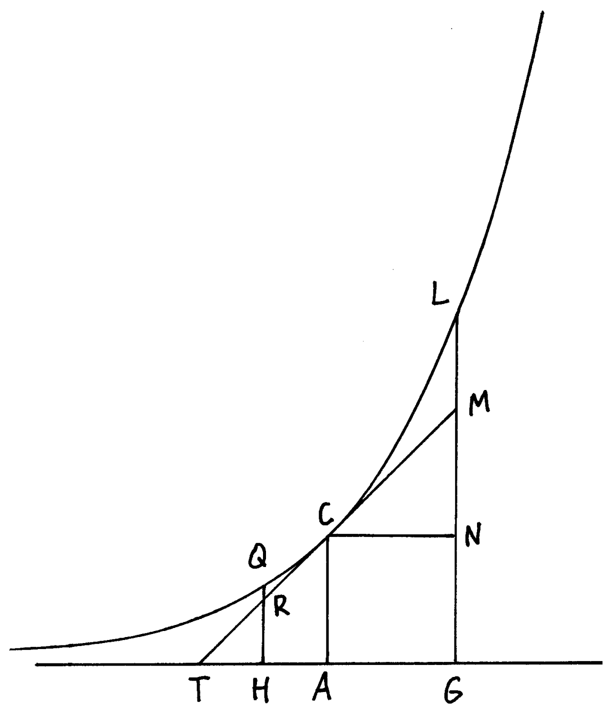



Figure 39

**Demonstration:** See Figure 39. There $CL$ is again a natural logarithmic line whose ordinates $GL$ we denote by $w$ and whose abscissas $AG$ we denote by $x$. The line $TCM$ is tangent to $CL$ at $C$, and the line $CN$ is drawn parallel to the axis $AG$. Now if $AG = 1$, then

$$GL = e^1 = e,$$

and thus $e$ appears in the diagram as the ordinate $GL$.

Since $x = 0$ at the point $A$, it follows that

$$AC = e^0 = 1.$$

The triangle $TAC$ is similar to the characteristic triangle at the point $C$. Therefore

$$\frac{AC}{AT} = \frac{dw}{dx},$$

and since $AC = 1$, it follows that

$$\frac{1}{AT} = \frac{dw}{dx}.$$

But according to equation 3, above,

$$\frac{dw}{dx} = w,$$

and at the point $C$, $w = 1$. Therefore

$$\frac{1}{AT} = 1,$$

and so $AT = 1$. Likewise, the triangle $CNM$ is similar to the characteristic triangle at the point $C$, and therefore

$$\frac{MN}{CN} = \frac{dw}{dx} = 1,$$

and since $CN = AG = 1$, it follows that $MN = 1$. Therefore $GM = GN + MN = AC + MN = 2$. But since $TM$ is tangent to $CL$ ...

$$GL > GM,$$

and therefore

$$e > 2.$$

This is half of what the theorem asserts.

To show that $e < 4$, we let $H$ be some point to the left of $A$, and draw the ordinate $HQ$, meeting the tangent $TC$ at $R$. Let

$$HA = \frac{1}{2},$$

so that

$$-\frac{1}{2}$$

is the value of the abscissa corresponding to the ordinate $HQ$; therefore

$$HQ = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}.$$

Since the tangent $TC$ falls below the logarithmic line $CL$, it follows that

$$RH < QH.$$

But

$$RH = TH$$

(since triangle $THR$ is similar to triangle $TAC$), and

$$TH = TA - HA = \frac{1}{2}.$$

Therefore, the inequality $RH < QH$ becomes

$$\frac{1}{2} < \frac{1}{\sqrt{e}},$$

and therefore

$$\sqrt{e} < 2$$

and

$$e < 4.$$

**Q. E. D.**

The methods used in Theorem 3 can be generalized to find more exact approximations for $e$.

### 4. The difference of $e^x$

Substituting the value $e^x$ for $w$ in the differential equation 3 (page 93),

$$dx = \frac{dw}{w} \tag{3}$$

gives us an equation for the differences of $e^x$:

$$\frac{d(e^x)}{e^x} = dx,$$

and therefore the differences of $e^x$

$$d(e^x) = e^x\,dx. \tag{4}$$

Equation 4 is the basic equation we use, along with the rules for the calculus, to find differences of expressions involving $e$. See the examples in part 6 of this note, below.

## 5. Notation for logarithms

If $x$ is the logarithm of $w$, so that $x$ represents the abscissa of a logarithmic line and $w$ the corresponding ordinate, that is, if $x$ and $w$ are related by equation 1,

$$w = k^x,$$

then we write

$$x = \log_k w.$$

This is simply a definition of a new symbol, $\log_k$. For example, if $k = 2$, then

$$\begin{aligned} \log_2 2 &= 1, \\ \log_2 4 &= 2, \\ \log_2 8 &= 3, \text{ and so on,} \end{aligned}$$

since

$$\begin{aligned} 2 &= 2^1, \\ 4 &= 2^2, \\ 8 &= 2^3, \text{ and so on.} \end{aligned}$$

The quantity $k$ is called the *base* of the logarithm.

When we are dealing with natural logarithms, so that equation 1 becomes,

$$w = e^x,$$

we simply write

$$x = \log w$$

or

$$x = \ln w$$

for the natural logarithm[16] of $w$.

## 6. The difference of $\log x$, and finding differences of quantities involving logarithms.

The differential equation 3,

$$dx = \frac{dw}{w},$$

where $x$ is the natural logarithm of $w$, may be used to find the differences of quantities that are expressed in terms of logarithms, as follows. When $x$ is the natural logarithm of $w$, then

$$x = \log w,$$

[16] In contexts where numerical computation is important, $\log w$ often denotes the logarithm with base 10, that is,

$$\log_{10} w.$$

Since we will not be using logarithms for computation, $\log w$ will always mean the *natural* logarithm of $w$ for us.

and equation 3 may be rewritten as

$$d(\log w) = \frac{dw}{w}, \tag{5}$$

that is, the difference of $\log w$ is equal to the difference of $w$ divided by $w$.

We can use formulas 4 (page 96) and 5 to find sums and differences of expressions involving natural logarithms and exponentials, as the following examples show.

1. Let $z = e^{(2x-3)}$. To find $dz$ in terms of $dx$. Let $v = 2x-3$, so that $z = e^v$. Therefore, by equation 4 (substituting $v$ for $x$),

   $$dz = d(e^v) = e^v\,dv;$$

   and

   $$\begin{aligned} dv &= d(2x-3) \\ &= d(2x) - d(3) \\ &= 2\,dx. \end{aligned}$$

   Therefore

   $$\begin{aligned} dz &= e^v\,dv \\ &= e^{(2x-3)}\,(2\,dx) \\ &= 2e^{(2x-3)}\,dx. \end{aligned}$$

2. Let $z = \log(5x^2+3)$. To find $dz$ in terms of $dx$. Let $v = 5x^2+3$, so that $z = \log v$. Therefore, by equation 5 (substituting $v$ for $w$),

   $$dz = \frac{dv}{v};$$

   and

   $$\begin{aligned} dv &= d(5x^2+3) \\ &= 5d(x^2) + d(3) \\ &= 5(2x\,dx) \\ &= 10x\,dx. \end{aligned}$$

   Therefore

   $$\begin{aligned} dz &= \frac{dv}{v} \\ &= \frac{10x\,dx}{5x^2+3} \\ &= \frac{10x}{5x^2+3}\,dx. \end{aligned}$$

3. Let $z = \log^3(x-4)$, that is, $z = (\log(x-4))^3$. To find $dz$ in terms of $dx$. Let $v = \log(x-4)$, so that $z = v^3$. Then, by the power rule,

$$dz = 3v^2 dv.$$

Next, let $w = x - 4$, so that $v = \log w$. Then, by equation 5,

$$dv = \frac{dw}{w}.$$

Finally, by the constant rule,

$$dw = dx.$$

Putting all these equations together, we get

$$\begin{aligned} dz &= 3v^2 dv \\ &= 3(\log(x-4))^2 \left(\frac{dw}{w}\right) \\ &= 3(\log(x-4))^2 \left(\frac{dx}{x-4}\right). \end{aligned}$$

**Problems about finding differences of exponentials and logarithms**

Using the rules of the differential calculus, and equations 4 (page 96) and 5 (page 98), find the following differences.

1. $$d(e^{(3x+1)}).$$

2. $$d(e^{(4x)} - 2e^{(2-3x)}).$$

3. $$d(\log(x^3 - 2x)).$$

4. $$d(\log(x+4) - \log^2 x).$$

5. $$d\left(\frac{\log(2x+3)}{e^x}\right).$$

6. $$d\left(\frac{e^{-x}}{\log 2x}\right).$$

# On Recondite Geometry and the Analysis of Indivisibles and Infinites Note 1, p. 111

by Gottfried Wilhelm Leibniz

I am aware that some of the things I have published in these *Acts* for the advancement of geometry have been praised in no small way by certain learned men, and indeed have been gradually put to use; nevertheless, some things, either through the writer's fault or through some other cause, have not been well enough understood, and therefore I thought it worthwhile to add some things here that may shed light on what I wrote earlier. I have of course received Mr. Craig's *On the Measurement of Figures*, published last year in London, from which it clearly appears that the author has made advances that are not contemptible into the depths of geometry. He strongly approves of a distinction I have sometimes emphasized between general and special measurements of figures, a distinction which he says on page 1 has been very well observed recently by geometers, and he rightly attributes the very many paralogisms of those who try to prove the impossibility of a quadrature to the neglect of this distinction. Note 2, p. 111
He recognizes with me that the figures that are commonly cast out of Geometry are transcendent (page 26). He has also greatly and humanely praised (pages Note 3, p. 113
27 and 29) my Method of Tangents, which I published in the *Acts* of October 1684;[1] he says that my method is most extraordinary, and by means of it the method of measurements is greatly helped, since it supplies the best remedy against irrationalities. Nevertheless, there are some things that I think it will Note 4, p. 114
not be useless or unwelcome to call to his or others' attention. Indeed, I do not know how it happened that he believes that the man who wrote the paper in the *Acts* of May, 1684 (p. 233)[2] retracted his opinion, and although he had proposed, at the beginning of the *Acts* of October, 1683, to give a demonstration that the quadrature of the circle is in no way possible, afterwards, in May of the following year, acknowledged that he had not yet demonstrated the impossibility of a special quadrature. Although the paper of October 1683 is by Mr. D. T.,[3] the paper of May 1684 is in fact by me, and on the one hand I claimed that his method is mine, so that I might not be accused of using something that belongs to someone else, while on the other hand I amicably disagreed with the use that Mr. D. T. put it to. For he thought that the impossibility of any definite quadrature followed from the impossibility of an indefinite quadrature: but my constant position (already indicated when I published the arithmetic quadrature[4] in the second month of the first year of the *Acts*, 1682) had been

[1]The article "A New Method for Greatests and Leasts, as well as for Tangents, which is not Hindered by Fractional or Irrational Quantities, and a Singular Calculus for these" (above, pages 24–39).

[2]In May of 1684 Leibniz published the article "On Finding Measurements of Figures," in which he first responded to Tschirnhaus.

[3]Ehrenfried Walther von Tschirnhaus. The authors of articles in the *Acts of the Erudite* were often only identified by their (Latin) initials.

[4]The article Leibniz refers to is "On the True Proportion of a Circle to a Circumscribed

that the inference from the latter to the former is invalid. To prove this I gave an example (in the *Acts* of May, 1684) of a certain figure that admits of a special quadrature but not a general quadrature, as I tried to show there using Mr. D. T.'s own theorems, although I erred in my calculation because I was in a hurry and certain of my method of proof. I will correct and explain this calculation later. Mr. D. T. privately responded that he did not derive his method from mine, but had come to it on his own and, as far as the objection was concerned, that he could demonstrate the inference from indefinite to definite quadratures, and this is what his method is especially good for; indeed, he said my example rested upon a bad calculation. I indeed willingly admitted (in the *Acts* of December, 1684, p. 587) that if he could demonstrate this inference, he would have done what no one else had done yet; nevertheless, I continued to have my doubts, and afterwards strengthened my example by a correct calculation, which I will get to soon. Moreover I have already had this method for more than ten years, since the time we were together in Paris and frequently spoke about geometric matters. At that time he was going in totally different directions, while I was already quite familiar with how to apply general equations to express the nature of the line sought, equations to be determined by the progress of the calculation; this is the nerve of the method, and I had never seen it published elsewhere. But nevertheless I grant so much to both his sincerity and his genius that I could easily believe that either he came upon these things himself or at least no longer remembered on what occasion the seeds of such meditations were sown, especially since I know that he has come up with even more difficult things on his own, and that we can look forward to many splendid and very important things from his genius.

However since it is admitted that I made a mistake in the calculation of the above-mentioned example, as I said, and I believe that Mr. Craig has used this error as an argument against Mr. D. T. (to whom he attributes it) in order to refute the very method of indefinites, I should correct the calculation. Look at the *Acts* of the year 1684, page 239, where, in comparing the equation $4z^2 - 8hz$ etc. with the equation $bz^2 + caz$ etc., the terms without $z$ placed outside the fraction in the latter equation should be multiplied by the denominator of the fraction before the comparison is made, so that on each side of the equation all the terms without $z$ are contained in a single fraction. And let $b = 1$, which can always be done, and because in the former equation the term $xz$ is entirely absent, $d$ becomes $= 0$ in the latter; and let the former (given) equation be divided by 4, and in the fraction of the latter (suppositional) equation let both the numerator and the denominator be divided by $g$: thus both the terms $z^2$ on each side, and the denominators of the fractions on each side, agree. When we compare the other terms, on account of the term $z$, $c$ will become $= \frac{2h}{a}$; on account of $x^4$, $g$ will become $= \frac{1}{16}$; on account of $x^3$, $f$ will become $= \frac{-1}{6a}$; and on account of $x$ in the denominator $f$ will become $\frac{-h}{8z}$. Therefore $h$ becomes $\frac{8}{6}$, that is $\frac{4}{3}$, which is absurd, because $h$ is a given quantity. Other absurdities also arise from the comparison, for both $f$ and $c$ become $= 0$, contrary to what has

Square," below, pages 183-189.

already been concluded.

Further, to say something more useful here let us *reveal a source of Transcendent Quantities*, that is, a reason why some problems are neither plane, nor solid, nor sursolid, nor of any definite degree, but transcend every algebraic equation. And at the same time we shall show how it can be demonstrated without calculation that an algebraic quadratrix for a circle or hyperbola is impossible. For if such a line were given it would follow that by means of it an angle or a ratio (or a logarithm) could be cut in a given ratio of a line to a line, and this could be done by one general construction, and consequently the problem of cutting an angle or finding an arbitrary number of mean proportionals would be of a definite degree. But for every different number of parts of an angle or every different number of mean proportionals we need an algebraic equation of a different degree, and therefore when the problem is understood to be about any possible number of parts or mean proportionals, it is of indefinite degree and transcends every algebraic equation. Because nevertheless such problems can be proposed in geometry—indeed, they should be considered to be among its leading problems—and because they are determinate, it is therefore obviously necessary to admit into geometry those lines through which alone such problems may be constructed; and since these problems can be described by an exact and continuous motion, as is obvious in the case of the cycloid and the like, they should indeed be considered to be not mechanical, but geometric, Note 5, p. 114 especially since in their usefulness they leave the lines of common geometry (if you leave out the straight line and the circle) many parasangs behind, and have properties of the greatest importance, and furthermore these properties admit of geometric demonstrations. *Therefore Descartes' error in excluding these lines from geometry was no less than that of the ancients*, who rejected solid or linear loci as less geometric. Note 6, p. 115

Moreover, because the method of investigating indefinite quadratures or their impossibilities is for me only a special case (and indeed an easier one) of a much greater problem, which I call the *inverse Method of Tangents*, in which the Note 7, p. 116 greatest part of transcendent geometry is contained, and because if this problem can be solved algebraically, all sorts of discoveries might be made, and because indeed I see nothing satisfactory about it extant; for these reasons let me show how it can be done no less than the indefinite quadrature itself can. While before now algebraists have taken letters or general numbers for the quantities sought, *I* have taken general or indefinite equations for the lines sought in such transcendent problems; e.g., if the abscissa and ordinate are $x$ and $y$, for me the equation of the line in question is

$$0 = a + bx + cy + exy + fx^2 + gy^2 \text{ etc.};$$

by means of the proposed indefinite equation, which is in fact finite (for it can always be determined how far up it has to go), I seek the tangent of the line, and comparing what I find with the given property of the tangents, I find out the value of the assumptive letters $a$, $b$, $c$, etc. and to that extent define the equation of the line sought; nevertheless sometimes some things remain arbitrary, in which Note 8, p. 117

case innumerable lines can be found that satisfy the problem, which is the reason why many, seeing it afterwards, might think that the problem is not sufficiently defined and is not in our power. The same things can also be shown through series. I have many ways to make the calculation more concise, which I save for another place. But if the comparison does not succeed, I declare that the line sought is not algebraic, but transcendent.

Note 9, p. 119

If this is the case, to find out the *species of transcendence* (for some transcendents depend on the general cutting of a ratio (or logarithm), some on the general cutting of an angle (or on the arcs of a circle), and some on other, more complex, indefinite questions) I take in addition to the letters $x$ and $y$ a third letter such as $v$, which signifies a transcendent quantity, and from these three I form a general equation for the line in question; from this equation I seek the tangent of the line by using the method of tangents I published in the *Acts* of October, 1684, a method which is not hindered by transcendents. Next, comparing what I find with the given property of the tangents of the curve, I discover not only the assumptive letters $a$, $b$, $c$, etc., but also the special nature of the transcendent. Although it can sometimes happen that we need to use more than one transcendent, whenever their natures are different from each other, and transcendents of transcendents are sometimes given, and, in general, such things can go on indefinitely, nevertheless we can be content with easier and simpler things. And it is for the most part possible to use special tricks, which I shall not give here, to make the calculation shorter and reduce the problem to simple terms as far as possible. However, when this method is applied to tetragonisms, that is, to finding quadratrices (for which a property of the tangents is of course always given), it is easy not only to find out whether an indefinite quadrature is algebraically impossible, but also how, after the impossibility has been apprehended, a transcendent quadratrix can be found. No one else has done this yet, so that it seems to me that I did not make an empty claim when I said that geometry is advanced by means of this method immeasurably far beyond the boundaries set down by *Viète* and *Descartes*, since in this way a certain and general analysis extends to problems that are of no definite degree and thus are not comprehended by means of algebraic equations.

Note 10, p. 120

Furthermore, because hardly anything can be imagined that is more useful, concise, and universal for treating transcendent problems by calculation wherever measurements and tangents occur than *my differential calculus or analysis of indivisibles and infinites*, of which only a small sample or corollary is contained in my method of tangents, published in the *Acts* of October, 1684, and so greatly praised by *Mr. Craig*; and because *Mr. Craig* himself suspected that there is something deeper hidden within this method, and consequently on page 29 of his book he tried to derive from it Barrow's theorem (that the sum of the intervals taken between the ordinates and perpendiculars of a curve and applied to the axis is equal to half the square on the last ordinate), although in carrying out the derivation he missed his mark somewhat, which I do not find surprising in a new method; for these reasons I judge that it would be very welcome to him and to others, *if I disclose here an addition to something whose usefulness extends so broadly.* For all theorems and problems of this sort, which

Note 11, p. 121

were rightly admired, flow from it with such ease that now they no more need to be learned and grasped than the many theorems of common geometry need to be learned by heart by someone who grasps specious geometry. Therefore I proceed as follows in the above-mentioned case. Let the ordinate be $x$, the abscissa $y$ [Figure 1] and let the interval I mentioned between the perpendicular Note 12, p. 122

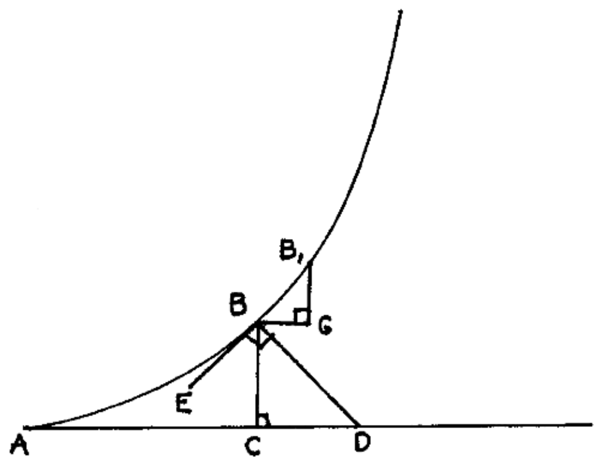



Figure 1: our figure, not Leibniz's

and the ordinate be $p$; it is immediately obvious from my method that

$$p\,dy = x\,dx,$$

which *Mr. Craig* also observed by using the same method; when this differential equation is turned into an equation of sums, we get Note 13, p. 122

$$\int p\,dy = \int x\,dx.$$

But it is obvious from the things I have set forth in the method of tangents that Note 14, p. 123

$$d\left(\frac{1}{2}x^2\right) = x\,dx;$$

and therefore correspondingly

$$\frac{1}{2}x^2 = \int x dx$$

(for sums and differences or $\int$ and $d$ are reciprocal for us just as powers and roots are in common calculations). Therefore we have Note 15, p. 123

$$\int p\,dy = \frac{1}{2}x^2,$$

which was what was to be demonstrated. Now I prefer to use '$dx$' and similar expressions rather than to substitute letters for them, because the expression '$dx$' is a certain modification of the expression '$x$,' and thus, by using this $dx$, we may ensure that, when necessary, only the letter $x$ (together with its powers and differentials) enters into the calculation, and the transcendent relations between $x$ and something else are expressed. In this way it is possible to display transcendent lines by means of an equation; for example, let $a$ be an arc and $x$ its versed sine; then

Note 16, p. 123

$$a = \int \frac{dx}{\sqrt{2x - x^2}},$$

Note 17, p. 125

and if $y$ is the ordinate of the cycloid, then

$$y = \sqrt{2x - x^2} + \int \frac{dx}{\sqrt{2x - x^2}},$$

which equation perfectly expresses the relation between the ordinate $y$ and the abscissa $x$, and from it all the properties of the cycloid can be demonstrated; and in this way analytic calculation is extended to those lines that have been thus far excluded for no greater reason than that they were believed not to admit of it; Wallis's interpolations and innumerable other things are also derived from this.

Finally, so that I may not seem to ascribe too much to myself or detract too much from others, let me say in a few words what I think that I owe to the mathematicians of our age who are distinguished in this sort of geometry. *Galileo* and *Cavalieri* first began to uncover the very involved arts of *Conon* and *Archimedes.* But Cavalieri's geometry of indivisibles belonged only to the infancy of a science in its rebirth. Three famous men helped more: *Fermat* by finding the method of greatest and least lines, *Descartes* by showing the way to express the lines of common geometry (for he excluded transcendents) through equations, and *Fr. Gregory of St. Vincent* by his many very splendid discoveries. I add to these the extraordinary rule of *Guldin* about the motion of the center of gravity. But these men also came to a stop within certain boundaries, which, after opening a new approach, the renowned geometers *Huygens* and *Wallis* crossed. Indeed it is likely enough that Huygens's discoveries gave to *Heurat*, and Wallis's gave to *Neil* and *Wren* (the men who first demonstrated that curves are equal to straight lines), the opportunity to make their own very beautiful discoveries, which nevertheless takes nothing away from their well-deserved praise. The Scot *James Gregory* and the Englishman *Isaac Barrow*, who wonderfully enriched the science with splendid theorems of this kind, followed these men. Meanwhile *Nicolas Mercator* of Holstein, a mathematician and a most outstanding one, is the first I know to have given a quadrature by means of an infinite series. But a geometer of most profound genius, *Isaac Newton*, not only made the same discovery on his own, but also solved the problem by a certain universal method; if Newton were to publish his thoughts, which I understand he has thus far suppressed, he would undoubtedly open up for us new ways to make science grow and become more concise.

It happened that while I was still a beginner in these studies, from one aspect of a certain demonstration about the magnitude of a spherical surface, a great light suddenly dawned on me. For I saw that in general the figure made from the perpendiculars to a curve, drawn ordinatewise to the axis (in the case of the circle, the figure made from the radii) is proportional to the surface of the solid that arises from the rotation of the figure about the axis. Note 18, p. 127 Wonderfully delighted with this first theorem (since I did not know that something similar had been noticed by others), I immediately invented the triangle that in any curve I called the characteristic triangle, whose sides were indivisibles (to speak more accurately, infinitely small) or differential quantities; by using it I immediately composed innumerable theorems with no trouble, part of which I afterwards found in *Gregory* and *Barrow*. And I was not yet using the true algebraic calculus; when I started to use it I soon found my arithmetic quadrature and many other things. But somehow the algebraic calculus did not satisfy me in this business, and I was forced to show many things by moving figures around that I wanted to show by analysis, until at last I found the true supplement to algebra for transcendents, namely, my calculus of the indefinitely small, which I also call differential, or summing, or tetragonistic, and quite appropriately (if I am not mistaken) the *Analysis of indivisibles and infinites*; once I had discovered this calculus, everything I myself had previously admired in this area seemed to be child's play. It not only led to marked abbreviations, but also made it possible to put together the very general method that I set forth above, whereby quadratrices or other lines, algebraic or transcendent, are determined as far as possible. Before I finish, let me give a warning: in differential equations, such as

$$a = \int \frac{dx}{\sqrt{1-x^2}},$$

let no one heedlessly neglect the $dx$ itself because it can be neglected in the case where the $x$'s themselves are assumed to grow uniformly. For most err in just this way and prevent themselves from advancing further because they do not let indivisibles like $dx$ keep their universality (so that any kind of progression might be taken for the $x$'s themselves), although innumerable transfigurations and equipotencies of figures may arise from this very thing. Note 19, p. 130

After I had already finished writing this little article, the things that Mr. D. T. shared in the *Acts* of March of this year on page 176 came into my hands, where he proposed some elegant questions that are worthy to be solved. Note 20, p. 163 And I see that the line $ACI$ (Figure 2) is one of the lines of sines, and the rectangle formed by $AH$ and $GD$ is equal to the space $ABCA$. And in Figure 3, if the solid formed by the square on $BC$ and $BD$ (or $x$) should always be equal to the given cube from $a$, I see that the paraboloid whose equation is $4a^3y^2 = 25x^5$ satisfies the problem. It is possible to determine something similar for other powers. But if $AD$, $DB$, $BC$ = the given cube, it reduces to the squaring line of the figure the value of whose ordinates is $ax^3$ divided by $\sqrt{a^6 - x^6}$; and in general the problem of finding the line with a given relation between the straight lines $AB$, $BC$, $CD$, $AD$, $DB$ in the said Figure 3 coincides with a problem of finding quadratures. But if a fixed point $L$ is taken on the line $AC$ new relations

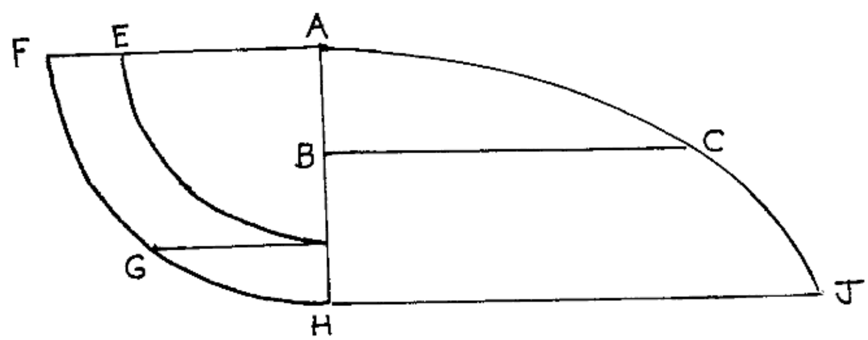



Figure 2: Leibniz's figure

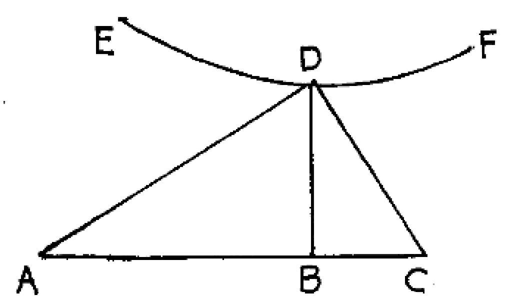



Figure 3: Leibniz's figure

of a different nature arise (for example, the ratio between $LC$ and $CD$ may be given), and this problem likewise admits of a solution.

# Notes on Leibniz's "On Recondite Geometry"

Leibniz published this paper in the *Acts of the Erudite* in June of 1686, about a year and half after "A New Method." We have translated it from the Latin text in Gerhardt's edition, Volume V, pages 226–33.

## Note 1

In "A New Method," Leibniz shows how, given a relation between quantities, we can find a *differential equation* relating their differences, and how we can use this differential equation to solve geometric problems by finding greatest and least ordinates and finding tangents (see page 33 of "A New Method" and page 73 of our notes). "On Recondite Geometry" treats the inverse problem: Leibniz shows how, given a differential equation relating the differences of some quantities, we can (in some cases) find the relation between the quantities *themselves*; and he shows how we can use this relation between the quantities to solve a geometric problem: the problem of finding the measurements of curvilinear figures (that is, lengths, areas, and volumes). To solve the problem of finding the relation between quantities with a given differential equation, Leibniz introduces into his calculus a new operation, "summing," represented by a new symbol, $\int$.

## Note 2

### The Measurement of Figures

In Proposition I 45 of the *Elements*, Euclid shows how to find a rectangle equal to any given *rectilinear* figure. To find a rectangle equal to a given *curvilinear* figure is a problem of quadrature, also called "tetragonism." For example, to find the quadrature of a circle, to "square a circle," is to find a square equal to the circle. This is not a problem new to Leibniz: Archimedes, for one, offers a demonstration that "the area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference of the circle."[5]

### General and Special Measurements of Figures

In this passage, Leibniz distinguishes a general (or indefinite) quadrature from a special (or definite) quadrature. To find a special (that is, specific) quadrature is to find the sides of the rectangle equal to a single, constant area. An example of a problem of special quadrature is shown in Figure 4: given a curve $AD$ with axis $AB$ and ordinate $EG$, and one side $L$ of a rectangle, the problem is to find the other side $M$ of the rectangle which will enclose an area equal to the curvilinear figure $AEG$.

The problem of a general (or indefinite) quadrature is to find the rectangles that enclose a varying or variable area. For example, Figure 5, given a curve $AD$

[5] See Archimedes' *Measurement of the Circle*, Proposition 1.

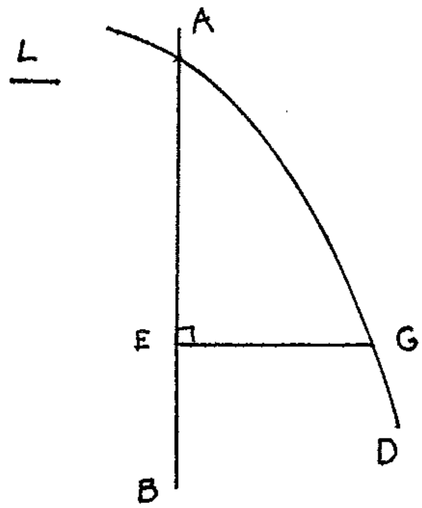



Figure 4

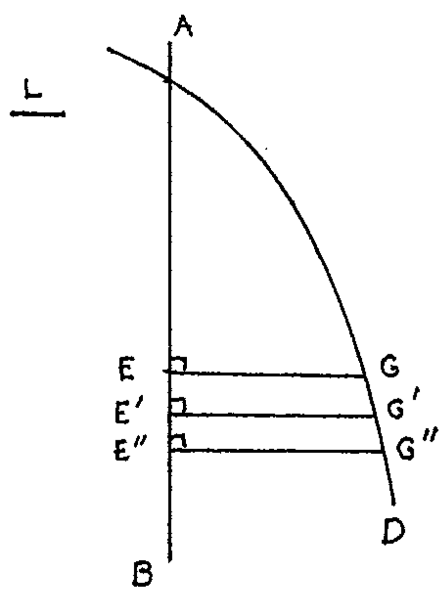



Figure 5

with axis $AB$ and ordinate $EG$, and one side $L$ of the rectangle, the problem is to find the second sides $M$, $M'$, $M''$, etc., of the rectangles equal to the curvilinear areas $AEG$, $AE'G'$, $AE''G''$, etc.

Now to represent all these lines in a single figure (Figure 6), let $EF = M$, $E'F' = M'$, $E''F'' = M''$, etc. Then if we have found all possible second sides

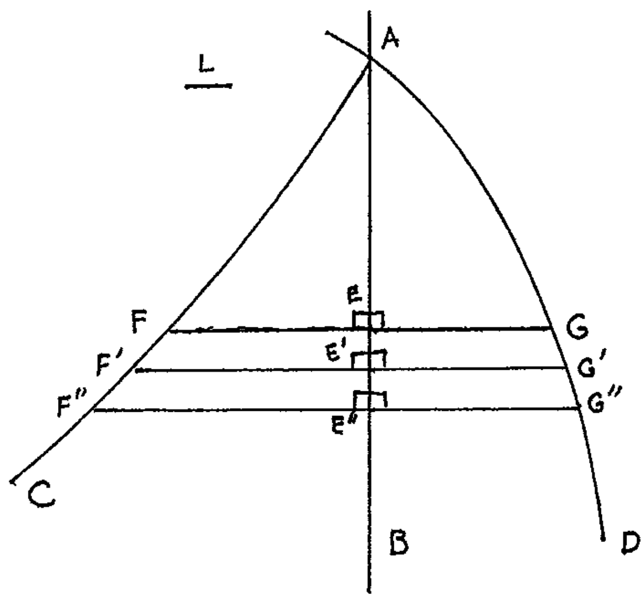



Figure 6

$M$, $M'$, etc., the points $F$, $F'$, etc., will trace out a curved line $AFC$, such that for any ordinate $EG$ the rectangle on $EF$ and $L$ is equal to the curvilinear area $AEG$. Therefore to find the line $AFC$ is equivalent to finding the quadrature of all possible areas $AEG$, that is, to finding a general quadrature. The curve $AFC$ thus gives us a way to measure not just a single area, but the variable area $AEG$, which changes as the ordinate moves. The original curve $AGD$ is called the *quadranda*, that is, the "curve to be squared", while the curve we find, $AFC$, is called the *quadratrix*, that is, the "squaring curve," because it lets us find a square equal to the area $AEG$ by finding a square equal to the rectangle on $FE$ and $L$.

## Note 3

The term "transcendent" (also "transcendental") is opposed to "algebraic." It means: transcends finite algebraic operations. The term may be applied to a number or a curve. An algebraic number is one that is a solution to an algebraic equation with one variable, that is, an equation of the form

$$a_n x^n + a_{n-1} 1 x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where $n$ is a natural number and $a_0, a_1, \ldots$ are integral coefficients. For example,

$$3x^2 + 2x - 5 = x^9 - 3x^5$$

is an algebraic equation, and so its solutions are algebraic numbers.

A transcendent curve is one that cannot be expressed by an algebraic equation. While a number is defined by an equation in one variable, a curve is defined by an equation in two variables. Such an equation is algebraic when it is a polynomial in two variables with any coefficients, that is, when it is of the form

$$\begin{aligned}
0 = {} & a_{n,m} x^n y^m + a_{n-1,m} x^{n-1} y^m + \cdots + a_{1,m} x y^m + a_{0,m} y^m \\
& + a_{n,m-1} x^n y^{m-1} + a_{n-1,m-1} x^{n-1} y^{m-1} + \cdots + a_{1,m-1} x y^{m-1} + a_{0,m-1} y^{m-1} \\
& + \cdots \\
& + a_{n,1} x^n y + a_{n-1,1} x^{n-1} y + \cdots + a_{1,1} x y + a_{0,1} y \\
& + a_{n,0} x^n + a_{n-1,0} x^{n-1} + \cdots + a_{1,0} x + a_{0,0}.
\end{aligned}$$

where $n$ and $m$ are natural numbers and the numbers $a_{i,j}$ (for all values of $i$ between 0 and $n$ and all values of $j$ between 0 and $m$) could be any numbers.

For example,

$$3x^2 - 2y^2 + 2xy - 7x + 1 = 0$$

is an algebraic equation corresponding to an algebraic curve.

Descartes called transcendent curves "mechanical," as opposed to "geometrical", indicating thereby that, although one can construct mechanical devices by which such a curve may be traced, there seems to be no rigorous determinable rule that relates a given abscissa to its corresponding ordinate. Such curves thus seem to elude analysis. There are many transcendent curves: the sine curve, along with the other trigonometric curves, the logarithmic curve, the logarithmic spiral, the cycloid (the path traced by a point on the rim of a circular wheel as it rolls along a straight line), and the catenary or hanging chain.

## Note 4

By saying that his method supplies the "best remedy against irrationalities" Leibniz appears to mean simply that his differential calculus can find tangents even when the equation for a curve involves square roots and other more complicated expressions. See page 33 of "A New Method."

## Note 5

A *cycloid* is a curve traced out by a point on a wheel as the wheel rolls without slipping on level ground. See Figure 7. There we have a wheel which begins at

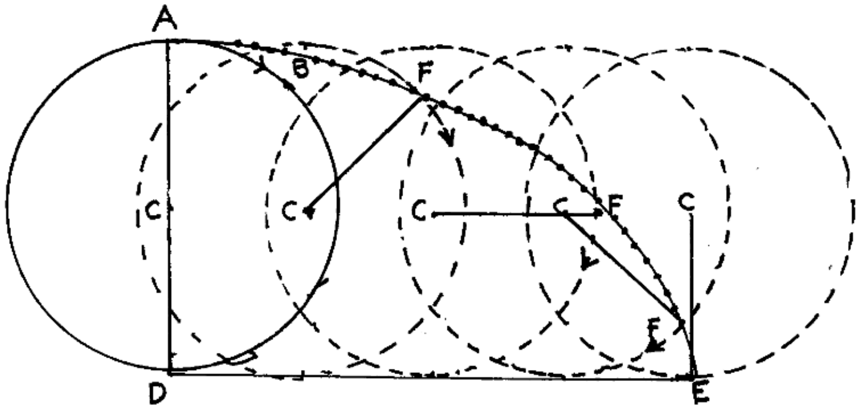



Figure 7

$ABD$ and has its center at $C$. The ground is the horizontal line $DE$. As the wheel rolls to the right along the ground the point $A$ will move to new points $F$ and trace out the *cycloid* $AFE$. We treat the cycloid in more detail below (page 125), using the differential calculus.

## Note 6

Leibniz's argument (and the sketch we give here) is far from a complete demonstration, but rather a plausible argument to cast doubt on Descartes' definition of the boundaries of geometry. Leibniz shows that there are good reasons to believe that the quadratrices for the circle and hyperbola are, in Descartes' terms, mechanical, and that Descartes would have to exclude them from geometry.

We will only treat the case of the circle, returning to the case of the hyperbola later (see below, page 156).

**Theorem:** *The quadratrix of a circle is transcendent.*

**A sketch of a demonstration:** See Figure 8.

Let circle $AGD$ be our quadranda, and let its quadratrix be the line $AFC$, so that $FE$ (or the rectangle $FE$, $L$, where $L$ is a unit) is always equal to area $AEG$. Let its abscissas $AE = x$ and its ordinates $EF = v$. Leibniz says that $AFC$ is transcendent. We argue by *reductio ad absurdum*: we (falsely) suppose that $AFC$ is not transcendent, and argue that this leads to an absurdity.

For suppose that $AFC$ were not transcendent. Then it would have a single algebraic equation of definite degree, such as

$$v^2 = x,$$

or some other algebraic equation where the exponents of $x$ and $v$ were all of definite degrees, that is, all constant whole numbers.

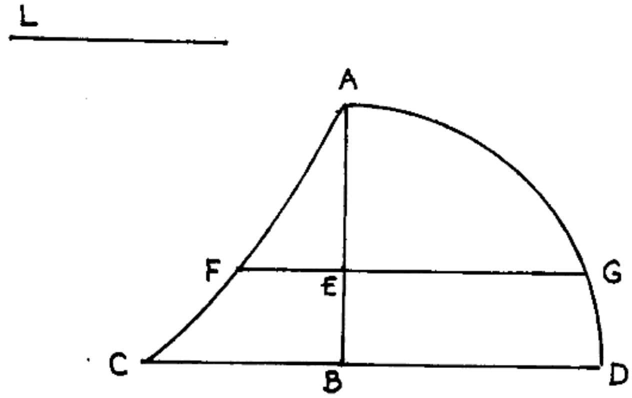



Figure 8

This algebraic equation for the quadratrix could be used to find the value of $FE$, that is, area $AEG$, for any given value of $AE$. In other words, our *reductio* assumption is that we can find all the areas $AEG$ in terms of their sides $AE$ by a *single* equation of *definite* degree.

To complete the argument we would then have to demonstrate the following two things:

1. We first would have to show that, given our *reductio* assumption, we could find a single algebraic equation of definite degree that can be used to divide any angle into any number of equal parts.

2. We then would have to show that the problem of dividing an angle into an arbitrary number of parts has no definite degree, contradicting what we just showed in the first part of the demonstration, and thus showing that our assumption that $AFC$ is not transcendent must be false.

We go through these steps in an appendix, pages 247–249, below.

## Note 7

Leibniz claims here, without proof, that finding quadratrices is a special case of an inverse tangent problem. An inverse tangent problem is a problem where we are given a property that the tangents of a curve must have, and we have to find the curve. Here, instead of immediately demonstrating that the problem of finding quadratrices is a special case of the inverse tangent problem, Leibniz first goes on to show (in this and the following paragraph) how the inverse tangent problem can be approached by a method that is closely analogous to the one presented in an earlier paper, "On finding measurements of figures," a method which in turn follows Tschirnhaus's method.

## Note 8

Here is an example of the solution to an inverse tangent problem, using the method Leibniz sketches here. Suppose we are looking for a curved line $AEB$ (Figure 9) whose ordinates are $EF$, whose abscissas are $CF$, and whose tangents

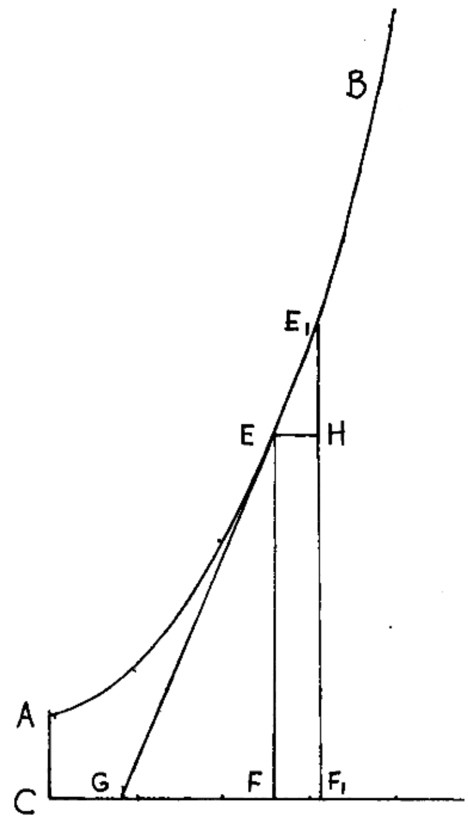



Figure 9

$EG$ have the property that

$$\frac{EF}{FG} = CF.$$

If we set $CF = x$ and $EF = y$, and we draw a characteristic triangle $EE_1H$, then $EH = dx$, and $E_1H = dy$. Since triangle $EE_1H$ is similar to triangle $GEF$,

$$\frac{EF}{FG} = \frac{dy}{dx},$$

and therefore, because of our property of tangents,

$$\frac{dy}{dx} = x. \tag{1}$$

To find the curve $AEB$ that has equation 1, we first write down a "general or indefinite equation" for it:

$$0 = a + bx + cy + exy + fx^2 + gy^2 + \text{ etc.} \tag{2}$$

This equation is general or indefinite insofar as its coefficients ($a$, $b$, $c$ etc.) are not definite numbers, but general constants, each of which could represent any number. Such an equation can represent *any* curve that has an algebraic equation, and so, in particular, it can represent the curve $AEB$ we are looking for *if* it has an algebraic equation.

We then use this general equation to find the tangent of the line, by taking its differences and solving for $\frac{dy}{dx}$, as follows.

$$\begin{aligned} 0 &= d(a + bx + cy + exy + fx^2 + gy^2 + \text{ etc.}) \\ &= b\,dx + c\,dy + e\,d(xy) + f\,d(x^2) + g\,d(y^2) + \text{ etc.} \\ &= b\,dx + c\,dy + ex\,dy + ey\,dx + 2fx\,dx + 2gy\,dy + \text{ etc.} \end{aligned}$$

(We used the multiplication rule and the power rule on the last step.) Gathering all terms involving $dy$ on the left gives

$$-c\,dy - ex\,dy - 2gy\,dy + \text{ etc.} = b\,dx + ey\,dx + 2fx\,dx + \text{ etc.},$$

or

$$dy(-c - ex - 2gy) + \text{ etc.} = dx(b + ey + 2fx + \text{ etc.}),$$

and therefore (solving for $dy$ and dividing both sides by $dx$),

$$\frac{dy}{dx} = \frac{b + ey + 2fx + \text{ etc.}}{-c - ex - 2gy + \text{ etc.}}.$$

We then "compare what [we found] with the given property of the tangents," by substituting this solution into our equation for tangents

$$\frac{dy}{dx} = x,$$

as follows:

$$\frac{b + ey + 2fx + \text{ etc.}}{-c - ex - 2gy + \text{ etc.}} = x.$$

We then use this last equation to try to find the constants $a$, $b$, $c$, etc. In this case, if $b = e = g = h = \ldots = 0$, and $f = -\frac{c}{2}$, then

$$\frac{b + ey + 2fx + \text{ etc.}}{-c - ex - 2gy + \text{ etc.}}$$

would simply equal $x$, and therefore the equation

$$\frac{dy}{dx} = x$$

would be satisfied. Note that there is no restriction on $a$ or $c$. Substituting these values back into the general equation (equation 2), gives us a *definite* equation for $AEB$, that is, it enables us to "define the equation of the line sought":

$$\begin{aligned} 0 &= a + 0x + cy + 0xy - \frac{c}{2}x^2 + 0y^2 + \text{ etc.} \\ &= a + cy - \frac{c}{2}x^2. \end{aligned}$$

Simplifying this equation to solve for $y$ gives us

$$cy = \frac{c}{2}x^2 - a, \text{ and}$$

$$y = \frac{1}{2}x^2 - \frac{a}{c}. \tag{3}$$

This is the equation for line $AEB$. "Some things remain arbitrary," namely, the constant,

$$\frac{a}{c},$$

because innumerably many lines solve the problem. In fact, the length of the line $AC$ in Figure 9 is arbitrary, and corresponds to

$$-\frac{a}{c}.$$

## Note 9

The problem at the end of "A New Method" (page 38) is an inverse tangent problem where "the comparison does not succeed" and the line is transcendent. Recall that in that problem we were looking for a line $AEB$ (Figure 10) whose

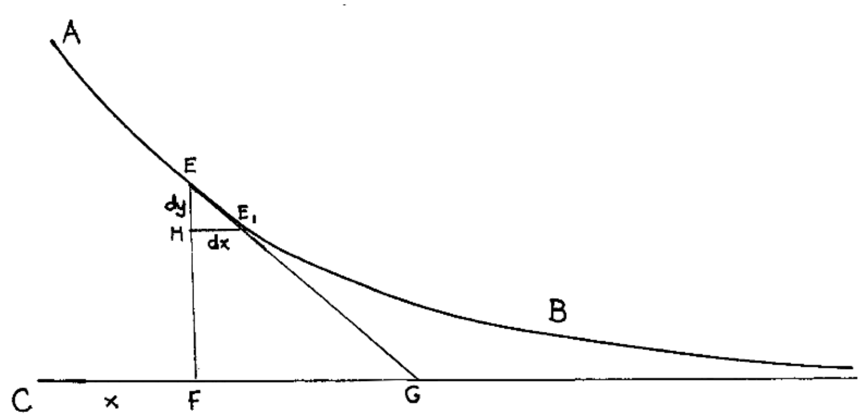



Figure 10

tangents had the property that the line $GF$ between the tangents $EG$ and ordinates $EF$ was always equal to a constant line $k$. For simplicity, let us suppose $GF = k = 1$. If we again set $CF = x$ and $EF = y$, then because the characteristic triangle $EHE_1$ is similar to triangle $GFE$,

$$\frac{EH}{E_1H} = \frac{EF}{GF},$$

and therefore

$$\frac{dy}{dx} = y. \tag{1}$$

If, following Leibniz's method, we try to find a solution by using a general or indefinite equation

$$0 = a + bx + cy + exy + fx^2 + gy^2 + \text{ etc.}, \tag{2}$$

we will not succeed. For, proceeding as in the previous note, we will find that there is no way to solve for $a$, $b$, $c$, etc., to make the equation

$$\frac{dy}{dx} = y$$

true.

## Note 10

Here Leibniz sketches a way to find more complex transcendents in terms of simpler ones. There is thus a kind of order of transcendent quantities, and the species of a transcendent is its place in this order: the basic transcendents come from circles and logarithms, and more complex transcendents can then be defined in terms of these.

To do this, Leibniz chooses a simple given transcendent $v$ and tries to express the equation for the curve he is looking for in terms of $x$, $y$, and the transcendent $v$. For example, the curve might have the equation

$$y = 3v^2 + 2x.$$

The quantity $y$ is then a new, more complex, transcendent depending on the old transcendent $v$.

The transcendent $v$ might "depend on the general cutting of a ratio;" we will see below (page 159) that the logarithm

$$v = \log x$$

is such a transcendent. Then the new transcendent $y$ would be equal to

$$3(\log(x))^2 + 2x.$$

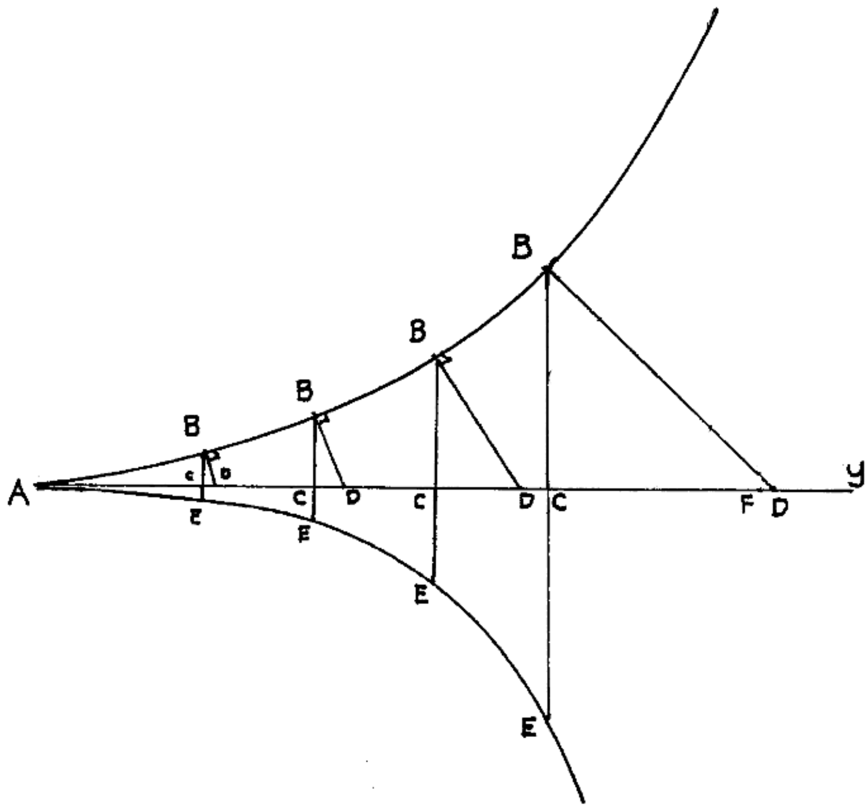



Figure 11

## Note 11

To understand what Barrow's theorem is, see Figure 11. There we are given an arbitrary curve $AB$ whose axis is $AC$. The ordinates $BC$ are equal to $x$, while the abscissas $AC$ are equal to $y$. Then for every point $B$ on the curve we extend a *perpendicular* $BD$ from the curve to the axis. This perpendicular creates an *interval* $CD = p$ between the ordinate $BC$ and the perpendicular $BD$ on the axis $AC$. We then *apply* this interval to the axis by drawing from every point $C$ on the axis a line $CE$ equal to $CD$ and perpendicular to $AC$. By doing this we construct a line $AE$ below the axis. Barrow's theorem asserts that the *sum* of all the intervals ($p = CD = CE$) applied to the $y$-axis is equal to one half the square on the final ordinate $BC$, that is, to

$$\frac{1}{2}x^2.$$

This sum is a sum of infinitely many lines $CE$. Leibniz does not immediately make clear how he understands such an infinite sum.

## Note 12

By *specious geometry* Leibniz means geometry based on algebra, as in Descartes' *Geometry*. The term *specious* comes from Viète, who calls the letters for unknown quantities *species* and calls algebra *specious arithmetic*.

## Note 13

To see where this differential equation comes from, consider Figure 12, $BD$ is perpendicular to the curve $ABB_1$, $CD$ is the interval between the ordinate and perpendicular, and $BGB_1$ is a characteristic triangle. Let the ordinate $BC = x$ and the abscissa $AC = y$, so that $B_1G = dx$ and $BG = dy$. Let $CD = p$.

Note that triangle $BGB_1$ is similar to triangle $BCD$; for angles $BGB_1$ and

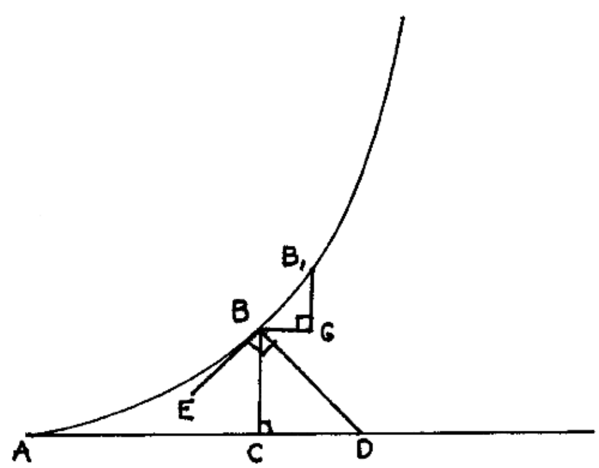



Figure 12

$BCD$ are both right, and therefore equal; and

$$\begin{aligned} \angle B_1BG + \angle GBD &= \text{the right angle } \angle B_1BD\text{, and} \\ \angle CBD + \angle GBD &= \text{the right angle } \angle CBG\text{,} \end{aligned}$$

and therefore

$$\angle B_1BG + \angle GBD = \angle CBD + \angle GBD,$$

and therefore (subtracting $\angle GBD$ from both sides of this equation)

$$\angle B_1BG = \angle CBD,$$

so that the triangles $GBB_1$ and $CBD$ share two equal angles, and are therefore similar. It follows from the similarity of these two triangles that

$$B_1G : BG :: CD : BC,$$

that is,

$$dx : dy :: p : x,$$

and therefore

$$x\,dx = p\,dy.$$

## Note 14

Leibniz denotes the *sum* of the infinitely many infinitely small quantities $p\,dy$ by

$$\int p\,dy.$$

The symbol $\int$ is an elongated letter $s$.

## Note 15

Since

$$d\left(\frac{1}{2}x^2\right) = x\,dx,$$

it follows that

$$\int x\,dx = \int d\left(\frac{1}{2}x^2\right).$$

Now, according to Leibniz, sums and differences are reciprocal like powers and roots. This means that taking sums undoes what taking differences does. Therefore, when we begin with a quantity, take its differences, and then take the sum of these differences, we get back the quantity we started with. In this case

$$\int d\left(\frac{1}{2}x^2\right) = \frac{1}{2}x^2.$$

In Note 19, below (p. 130), we demonstrate that sums and differences are reciprocal.

## Note 16

To see where Leibniz gets his equation for $a$, let $AB$ be a circle with center $C$ (Figure 13), and drop a perpendicular $BD$ (the *sine* of angle $BCA$ or arc $a$) from some point $B$ on the circle to the radius $CA$. Let the radius $CA = 1$, let $AD = x$ (so that $CD = 1 - x$) and let arc $AB = a$. (The line $AD$ in the unit circle is the *versed sine*, and the line $CD$ is the *cosine* of angle $BCA$ or arc

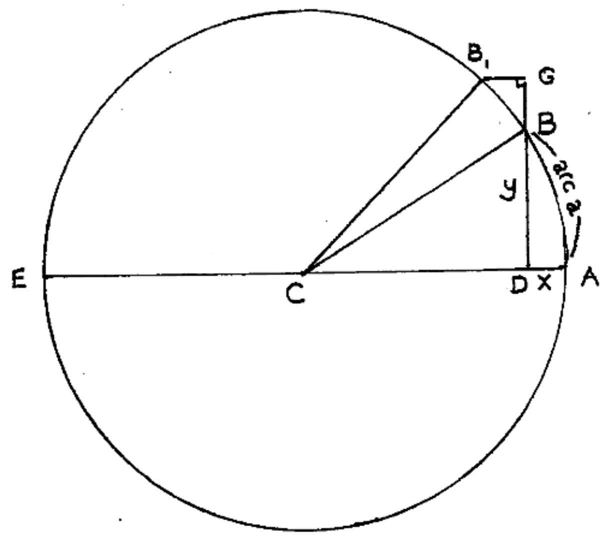



Figure 13

$a$.[6]) Then take a point $B_1$ infinitely close to $B$, and complete the characteristic triangle $B_1GB$. Then $B_1G = dx$, $BG = dy$, and $B_1B = da$.

Triangle $B_1GB$ is similar to triangle $BDC$; for

$$\angle B_1GB = \angle BDC,$$

(because they are both right), and

$$\begin{aligned} \angle GB_1B + \angle B_1GB + \angle GBB_1 &= \text{two right angles, and} \\ \angle CBD + \angle CBB_1 + \angle GBB_1 &= \text{two right angles.} \end{aligned}$$

Therefore

$$\angle GB_1B + \angle B_1GB + \angle GBB_1 = \angle CBD + \angle CBB_1 + \angle GBB_1,$$

and therefore (canceling $\angle GBB_1$ on both sides)

$$\angle GB_1B + \angle B_1GB = \angle CBD + \angle CBB_1.$$

But

$$\angle B_1GB = \angle CBB_1$$

(for they are both right), and therefore

$$\angle GB_1B = \angle CBD.$$

[6]If $AB$ is were not a unit circle, then the ratio $AD/BC$ would be the versed sine, $CD/BC$ would be the cosine, and the arc $a$ would equal $\dfrac{\text{arclength } AB}{BC}$, that is, the value of arc $AB$ measured in radians.

Triangles $B_1GB$ and $BDC$ therefore have two pairs of equal angles, and therefore must be similar.

It follows from the similarity of these triangles that

$$B_1B : B_1G :: CB : BD.$$

But

$$\begin{aligned} BD &= \sqrt{BC^2 - CD^2} \\ &= \sqrt{1 - (1-x)^2} \\ &= \sqrt{2x - x^2}. \end{aligned}$$

Therefore (substituting values into the preceding proportion and converting it into an equation)

$$\frac{da}{dx} = \frac{1}{\sqrt{2x - x^2}}.$$

Therefore

$$da = \frac{dx}{\sqrt{2x - x^2}}.$$

Taking sums of both sides of this equation gives

$$\int da = \int \frac{dx}{\sqrt{2x - x^2}}.$$

But since sums and differences are reciprocal,

$$\int da = a.$$

Therefore

$$a = \int \frac{dx}{\sqrt{2x - x^2}}.$$

This is Leibniz's equation. It is a simple equation expressing the transcendent relation between the length of the arc $AB$ of a circle and its versed sine $AD$.

## Note 17

Recall that a *cycloid* is a curve traced out by a point on a wheel as the wheel rolls without slipping on level ground. See Figure 14. There we have a wheel which begins at $ABD$ and has its center at $C$. The ground is the horizontal line $DE$. As the wheel rolls to the right along the ground the point $A$ will move to new points $F$ and trace out the *cycloid* $AFE$.

To find an equation for the cycloid, we find the length of the line $GF$ for a given position of the wheel. Let $H$ be the point that ends up on the ground at $D_1$ as the wheel rolls to the right. Draw the diameter $KCH$. Then, when $H$ has moved down to $D_1$, $K$ has moved up to $A_1$. The point $F$, where $A$ ends

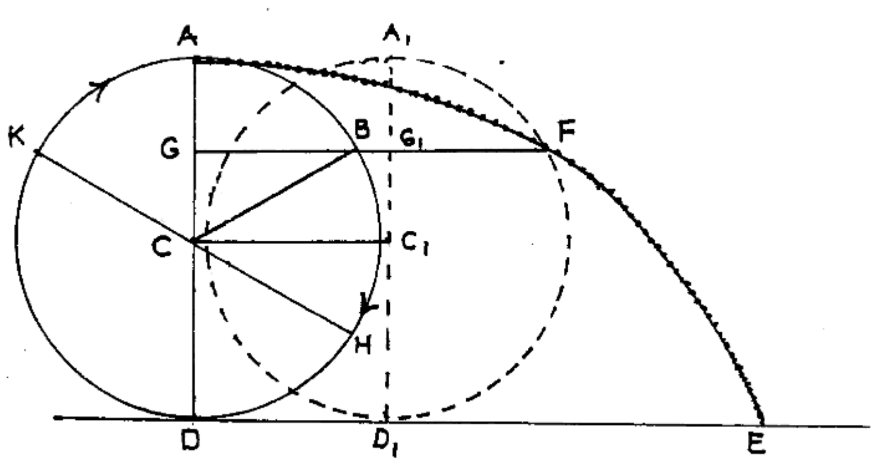



Figure 14

up, is then on the same horizontal line $KGBF$ as the point $K$. Then, because the wheel rolls without slipping,

$$\text{arc } DH = DD_1.$$

But

$$DD_1 = GG_1,$$

and

$$\text{arc } DH = \text{arc } AB,$$

and therefore

$$GG_1 = \text{arc } AB. \tag{1}$$

Moreover, because $A_1F$ and $AB$ are equal arcs on equal circles, it follows that the curvilinear triangle $AGB$ is equal and similar to the curvilinear triangle $A_1G_1F$, and therefore

$$G_1F = GB. \tag{2}$$

Putting together equations 1 and 2, we get an equation for $GF$:

$$\begin{aligned} GF &= GG_1 + G_1F && (3)\\ &= \text{arc } AB + GB. && (4) \end{aligned}$$

Now, to get Leibniz's equation, we treat $AD$ as the axis of the cycloid and $FG$ as an ordinate, so that $AG$ is the abscissa, and set

$$AG = x$$

and

$$GF = y.$$

Then, according to equation 4,

$$y = \text{arc } AB + GB,$$

and in the previous note (page 125) we showed that

$$\text{arc } AB = \int \frac{dx}{\sqrt{2x - x^2}},$$

and (page 125)

$$GB = \sqrt{2x - x^2}.$$

Therefore

$$y = FG = \int \frac{dx}{\sqrt{2x - x^2}} + \sqrt{2x - x^2}.$$

This is Leibniz's equation.

## Note 18

Let $AB$ be a curve, and let $CD$ be its axis. (See Figure 15.) Let lines $EF$ be the ordinates to the curve. From every point $E$ on the curve draw perpendicular lines $EG$ meeting the axis at $G$. From $F$, draw perpendicular lines $FL$ below the axis such that

$$FL = EG.$$

Let $HLK$ be the line going through all the points $L$. Then figure $CHKD$ is "the figure made from the perpendiculars to a curve, drawn ordinatewise to the axis."

Next, rotate the line $AB$ all the way around the axis, so that every point $E$ moves in a complete circle around the point $F$ on the axis. (See Figure 16.) As it rotates, the figure $ACDB$ then forms a solid, and let us call the surface of this solid $AMNB$. Leibniz saw the following theorem.

**Theorem:**

Figure $CHKD$ is proportional to surface $AMNB$.

**Demonstration:**

In Figure 16, consider the ring $ERPP_1R_1E_1$ formed by rotating the infinitely small line $EE_1$ about the axis. The area of the surface $AMNB$ is the sum of all these infinitely small rings. The circumference of this ring is proportional to its radius $EF$:

$$\text{circumference } ERP = 2\pi(EF).$$

(Because $E$ and $E_1$ are infinitely close, the circumference of circle $ERP$ is equal to the circumference of circle $E_1R_1P_1$.) The surface area of the ring $ERPP_1R_1E_1$ is equal to its width ($EE_1$) times its circumference ($2\pi(EF)$):

$$\text{area } ERPP_1R_1E_1 = 2\pi \times EE_1 \times EF \tag{1}$$

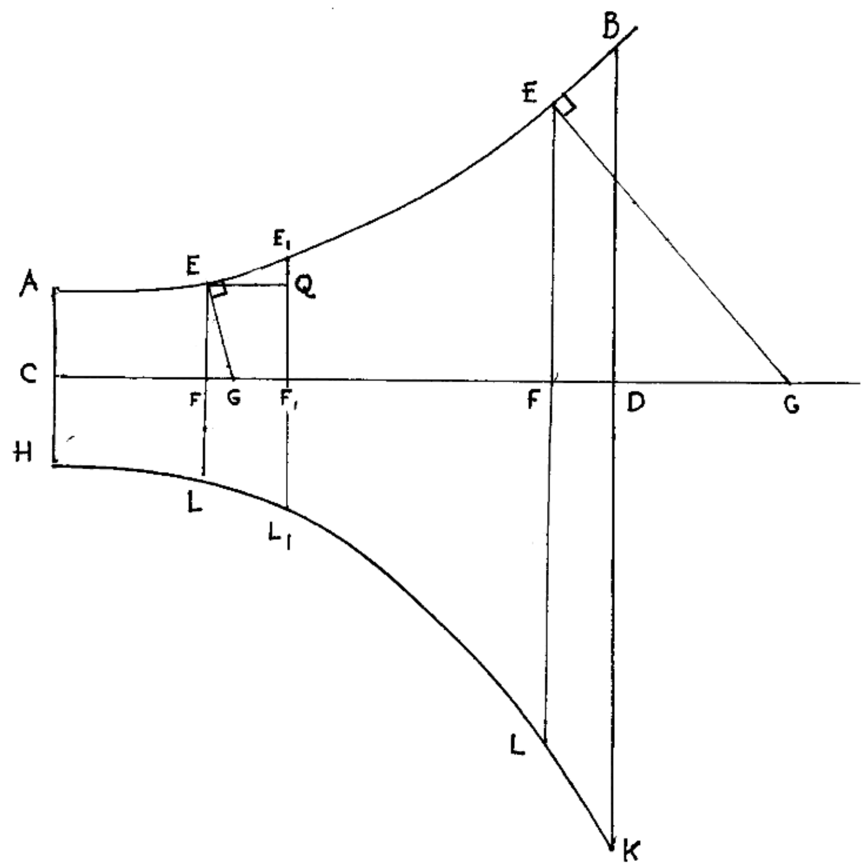



Figure 15

Now, to find $EE_1 \times EF$, turn back to Figure 15. Draw the characteristic triangle $EE_1Q$, the ordinate $E_1F_1$ and the line $F_1L_1$ equal to the the perpendicular to the curve at $E_1$. Then

$$\angle E_1EQ + \angle QEG = \angle FEG + \angle QEG = \text{a right angle.}$$

Therefore

$$\angle E_1EQ = \angle FEG,$$

and the right triangles $E_1EQ$ and $GEF$ are similar. Therefore

$$\frac{EE_1}{EG} = \frac{EQ}{EF},$$

and therefore

$$EE_1 \times EF = EQ \times EG.$$

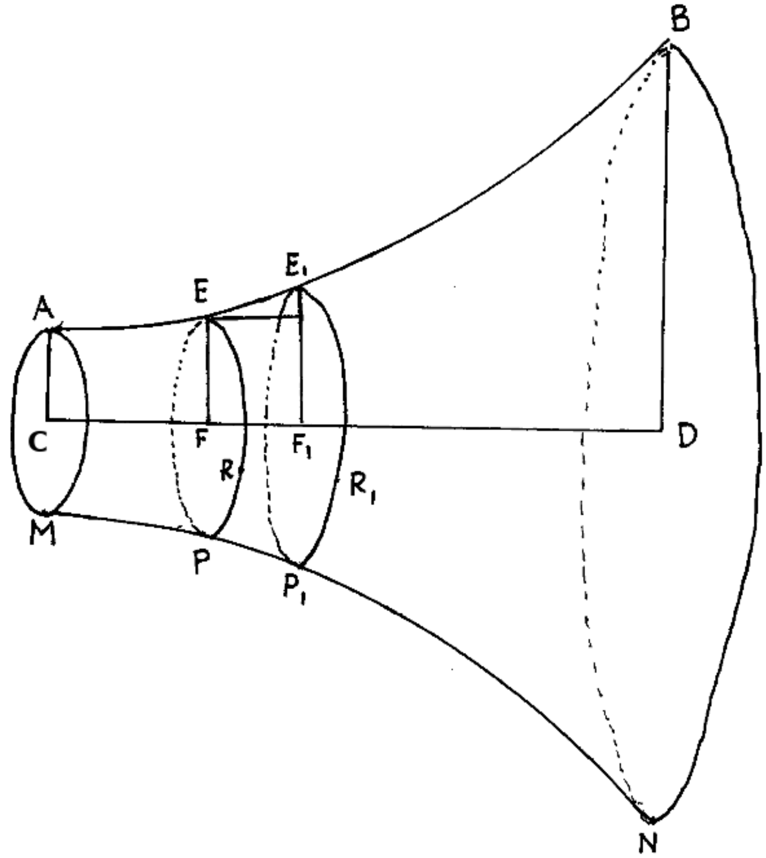



Figure 16

But $EQ = FF_1$, and $EG = FL$, and therefore

$$EE_1 \times EF = FF_1 \times FL.$$

Now $FF_1 \times FL$ is equal to the curvilinear quadrilateral $FF_1L_1L$ (since $FL = F_1L_1$, because $L$ and $L_1$ are infinitely close), and therefore

$$EE_1 \times EF = \text{quadrilateral } FF_1L_1L. \tag{2}$$

Therefore (putting together equations 1 and 2)

$$\text{area } ERPP_1R_1E_1 = 2\pi \times \text{quadrilateral } FF_1L_1L. \tag{3}$$

Equation 3 holds for every point $E$ on the curve $AEB$. We now find the sums of each side of equation 3 for all points on the curve. The sum of the left side of equation 3 is the sum of all the rings $ERPP_1R_1E_1$ in Figure 16, that is,

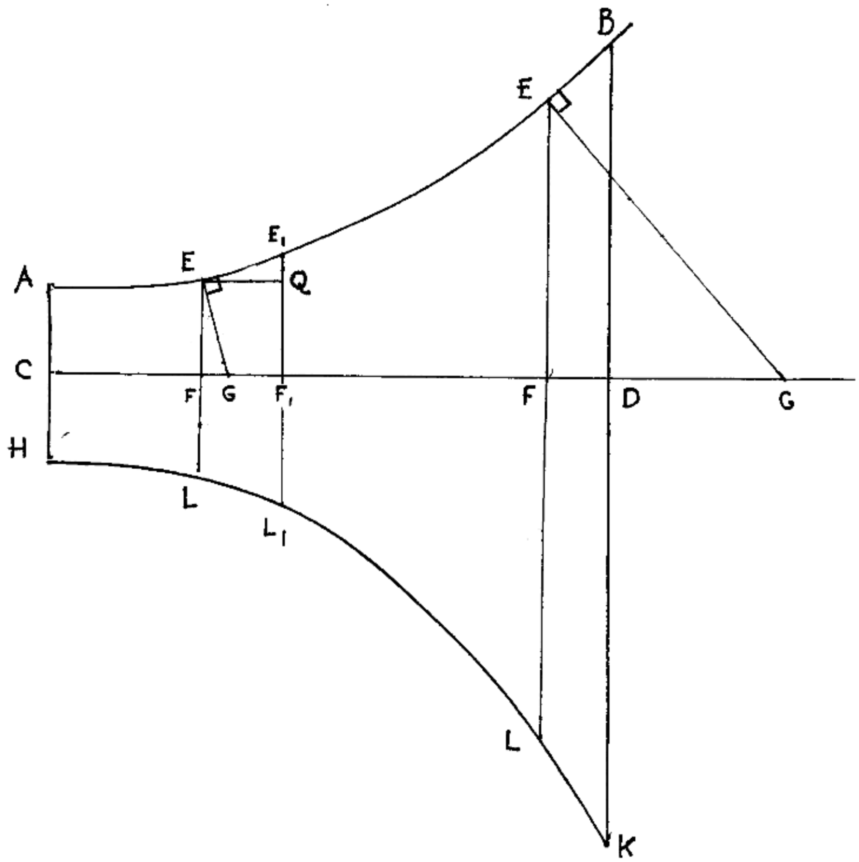



Figure 15

the surface $AMNB$. The sum of the right side is $2\pi$ times the sum of all the quadrilaterals $FLL_1F_1$ in Figure 15, that is, $2\pi$ times the whole figure $CHKD$. Therefore

$$\text{surface } AMNB = 2\pi \times \text{figure } CHKD,$$

and surface $AMNB$ is proportional to figure $CHKD$.
Q. E. D.

## Note 19

In finding measurements of figures using the calculus, we have repeatedly used the reciprocity of sums and differences. In this note, we demonstrate this reciprocity, which is traditionally called the *fundamental theorem of calculus.* Before demonstrating the fundamental theorem, we first need to spell out in greater generality how sums can be used to find the areas of curvilinear figures. After demonstrating the fundamental theorem we will give a number of examples of how the calculus can be used to determine "quadratrices or other lines, algebraic or transcendent" (p. 107).

This note is divided into five parts:

1. Finding areas of figures with sums.
2. The reciprocity of sums and differences: the fundamental theorem.
3. Examples of determining algebraic quadratrices. Here we show how the calculus can be used to find areas under curves in certain cases where both the original curve and the equation giving its area turn out to be algebraic.
4. A transcendent line: the sine curve. Here we show the calculus can be used to express a transcendent curve that arises from arcs of a circle. After showing what the sine curve looks like and demonstrating that it is transcendent, we go on to show how the calculus can be used to find sums and differences of expressions involving sines and other trigonometric quantities.
5. Another transcendent line: the logarithmic line. Here we return to the logarithmic line, which we saw at the end of "A New Method". Here we show that the logarithmic line is in fact the quadratrix of a hyperbola, expand Leibniz's argument from earlier in "On Recondite Geometry" (page 103) that the logarithm is transcendent, and go through a number of examples showing how the calculus can be used to find sums of expressions involving logarithms.

## 1. Finding areas of figures with sums

Suppose $AB$ (Figure 17) is a curved line with axis $AC$ and ordinates $EF$, and we want to find the area $AEBC$. Let the ordinates $EF = v$ and the abscissas $AF = x$. Draw an arbitrary number of ordinates $E_1F_1 = v_1$, $E_2F_2 = v_2$, etc. Let $BC = v_4$, and suppose $AF_1 = dx_1$, $F_1F_2 = dx_2$, $F_2F_3 = dx_3$, and $F_3C = dx_4$. The area $ABC$ is of course equal to the sum of the four areas $AE_1F_1$, $E_1E_2F_2F_1$, $E_2E_3F_3F_2$, and $E_3BCF_3$. Each of these four areas is still curvilinear, and it is therefore difficult to come up with a numerical value for each. But if we draw the rectangles $AGE_1F_1$, $F_1G_1E_2F_2$, $F_2G_2E_3F_3$, and $F_3G_3BC$, making a polygon $AGE_1G_1E_2G_2E_3G_3BC$, which we will call $P$, we can compute the area of this polygon as a sum of these four rectangles

$$\begin{aligned}
P &= AGE_1F_1 + F_1G_1E_2F_2 + F_2G_2E_3F_3 + F_3G_3BC \\
&= (AF_1 \times F_1E_1) + (F_1F_2 \times F_2E_2) + (F_2F_3 \times F_3E_3) + (F_3C \times CB) \\
&= v_1\,dx_1 + v_2\,dx_2 + v_3\,dx_3 + v_4\,dx_4.
\end{aligned}$$

Thus the area of a polygon, unlike a curvilinear area, can be computed by finding a finite sum.

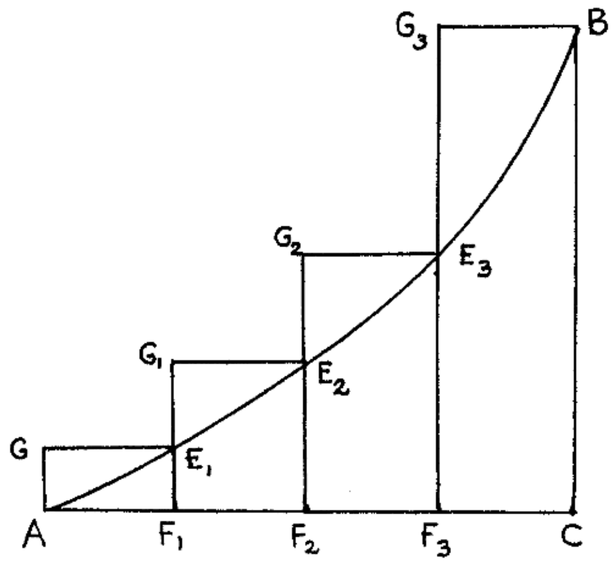



Figure 17

Now polygon $P$ is greater than the curvilinear area $ABC$ we are interested in. But if we had taken more points $E_1$, $E_2$, etc., we would have found a polygon that was closer to this area, and we could make the difference between the area of our polygon and our curvilinear area less than any given difference. Therefore if we take a polygon $P$ with infinitely many sides, we may treat it as exactly equal to the curvilinear area $ABC$. The area of the whole infinite polygon $AGE_1G_1 \ldots BC$ (that is, the curvilinear area $ABDC$) is equal to

$$v_1\,dx_1 + v_2\,dx_2 + v_3\,dx_3 + \ldots$$

We can thus compute the curvilinear area $ABC$ by finding an *infinite* sum of *infinitely small* quantities. Leibniz denotes this sum by

$$\int v\,dx;$$

the symbol $\int$ indicates that a sum is being taken, while $v\,dx$ indicates what is being summed: the products of the ordinates $v$ times the infinitely small differences of the abscissas $dx$. One could read the whole symbol $\int v\,dx$ as "the sum of the ordinates applied to the axis."

(Shortly after Leibniz introduced this new symbol for infinite sums, other mathematicians began calling them *integrals*, and that soon became the standard name. We usually read

$$\int v\,dx$$

as "the integral of $v\,dx$." Summing is called *integrating.*)

Note that a sum has to begin and end at some definite points: here we have found the sum beginning from $A$ and ending at $C$. We could have begun or ended the sum at any definite ordinates. In general, it will be clear from the problem we are considering where the sum should begin. Most often, we simply begin the sum at a point where the quantity whose differences appear in the sum is equal to zero; here we are considering

$$\int v\,dx,$$

a sum involving $dx$, and we begin the sum where $x = 0$, that is, at the point $A$. We usually let the endpoint of the sum be an undetermined variable. When we want to make explicit where the sum begins and ends, we attach quantities to the summing sign, as follows:

$$\int_0^3 v\,dx,$$

denotes the sum beginning at $x = 0$ and ending at $x = 3$ (see Figure 18), and,

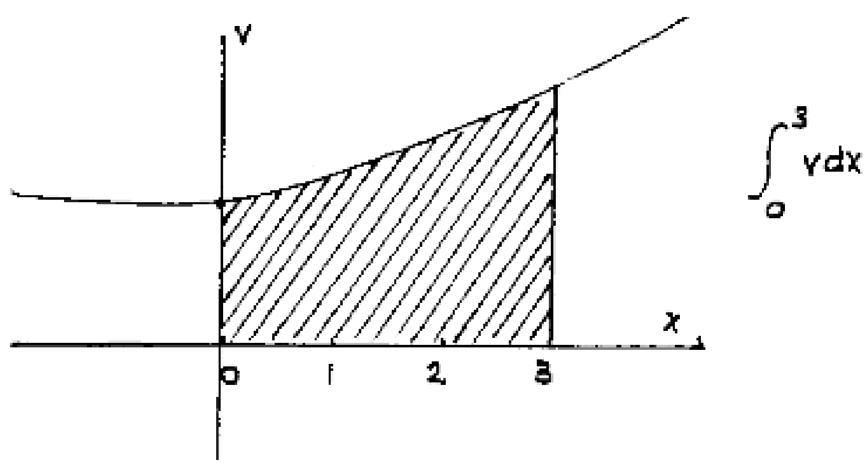



Figure 18

in general,

$$\int_a^b v\,dx$$

denotes the sum beginning at $x = a$ and ending at $x = b$ (see Figure 19).

## 2. Sums and differences as inverses

To show that sums and differences are reciprocal, we need to demonstrate two theorems:

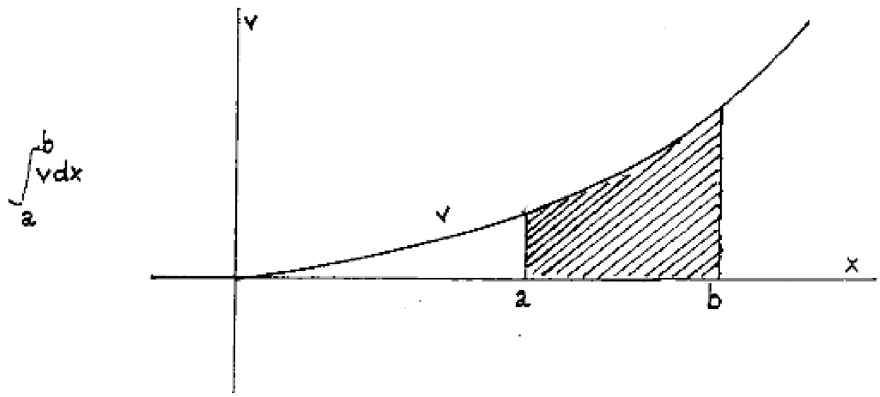



Figure 19

1. that if we find a difference and then find its sum, we get back to where we started; and
2. that if we find a sum and then find its difference, we get back to where we started.

These two theorems show that finding sums is a sort of mirror image of finding differences, just as finding square roots is a mirror image of finding squares. What finding differences does, finding sums undoes, and vice versa. Everything we know about differences is therefore reflected in sums, and, therefore also in finding measurements of figures. These theorems are thus the foundation of the application of the differential calculus to problems of finding measurements of figures, and we therefore name them the *first* and *second fundamental theorems.*

**First Fundamental Theorem**

For any variable quantity

$$\int dv = v.$$

**Demonstration:** Let the line $AB$ represent the quantity $v$. (See Figure 20.) Take infinitely many infinitely close points $C_1$, $C_2$, etc. on the line, so that $AC_1 = v_1$, $AC_2 = v_2$, $AC_3 = v_3$, etc. Therefore $AC_1 = dv_1$, $C_1C_2 = dv_2$, etc. Then

$$\begin{aligned} \int dv &= dv_1 + dv_2 + \ldots \\ &= AC_1 + C_1C_2 + C_2C_3 + \ldots \\ &= AB \\ &= v. \qquad\qquad \text{Q. E. D.} \end{aligned}$$

Note that it is important here that we begin the sum at the point where $v = 0$.

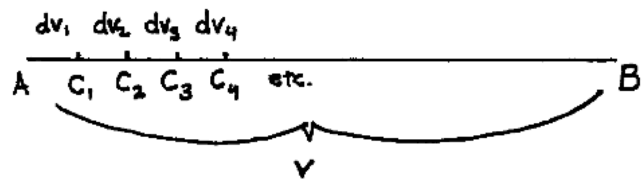



Figure 20

**Second Fundamental Theorem**

Given an infinitely small variable quantity $y\,dx$,

$$d\int y\,dx = y\,dx.$$

**Demonstration:** Let $AB$ be a curved line with axis $CD$, ordinates $EF = y$ and abscissas $CF = x$. (See Figure 21.)

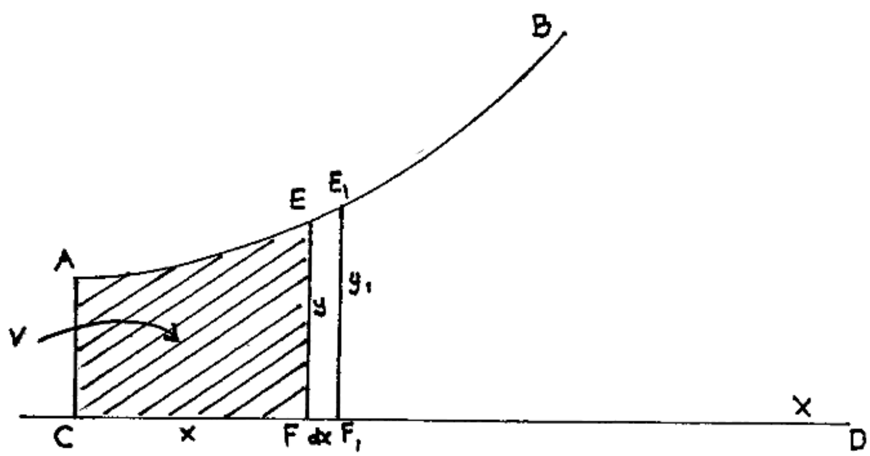



Figure 21

Let us denote the variable area $ACFE$ by $v$. Then

$$v = \int y\,dx.$$

Let $E_1$ be a point infinitely close to $E$ on the curve, and drop an ordinate $E_1F_1$ to the axis. Let $CF_1 = x_1$ and $ACF_1E_1 = v_1$. Then

$$dx = x_1 - x = FF_1,$$

and

$$dv = v_1 - v = ACF_1E_1 - ACFE = EFF_1E_1.$$

But since $E$ and $E_1$ are infinitely close, we may take $EF = E_1F_1$ and treat $EFF_1E_1$ as a rectangle. Its area is therefore equal to $EF$ times $FF_1$, that is, to $y\,dx$. Therefore

$$dv = y\,dx,$$

that is,

$$d\int y\,dx = y\,dx.$$

Q. E. D.

The second fundamental theorem says that any infinitely small variable quantity $y\,dx$, is equal to the difference of its sums.

## 3. Examples of determining algebraic quadratrices

In all the following examples we are given a curved line $ADB$ (Figure 22) whose axis is $AEC$ and whose ordinates are $DE$. Let $AE = x$ and $DE = y$. Let the curve $AFG$ be the quadratrix of the curve $ADB$ (the quadranda), that is, let the rectangle formed by the ordinate $EF$ and a unit line be always equal to the curvilinear area $ADE$. Let $v = EF$, so that $v$ times a unit line is always equal to the area $ADE$, that is, $v$ is equal to the area $ADE$. Then

$$v = \int y\,dx.$$

To find an equation for the quadratrix $AFG$ we have to find an equation relating $v$ to $x$, that is we have to find the area $\int y\,dx$ in terms of $x$.

In general, it is much more difficult to find sums, and thereby find quadratrices, than it is to find differences. To find differences we simply have to follow mechanically the rules Leibniz has given us in "A New Method." But there is no general method for finding sums. In an earlier paper Leibniz writes that

> But this is the labor, this is the task: given a Quadranda, to find some Quadratrix for it; this is especially difficult because sometimes it is impossible to find a quadratrix (at least one that can be expressed algebraically).[7]

He is alluding here to Book VI of Virgil's *Aeneid* where the Cumaean Sybil says to Aeneas

> Born of the blood
> of gods and son of Troy's Anchises, easy—
> the way that leads into Avernus: day
> and night the door of darkest Dis is open.

[7] "On Finding Measurements of Figures," published in the *Acts* in May of 1684. It is on pages 123–6 in Volume V of Gerhardt's edition.

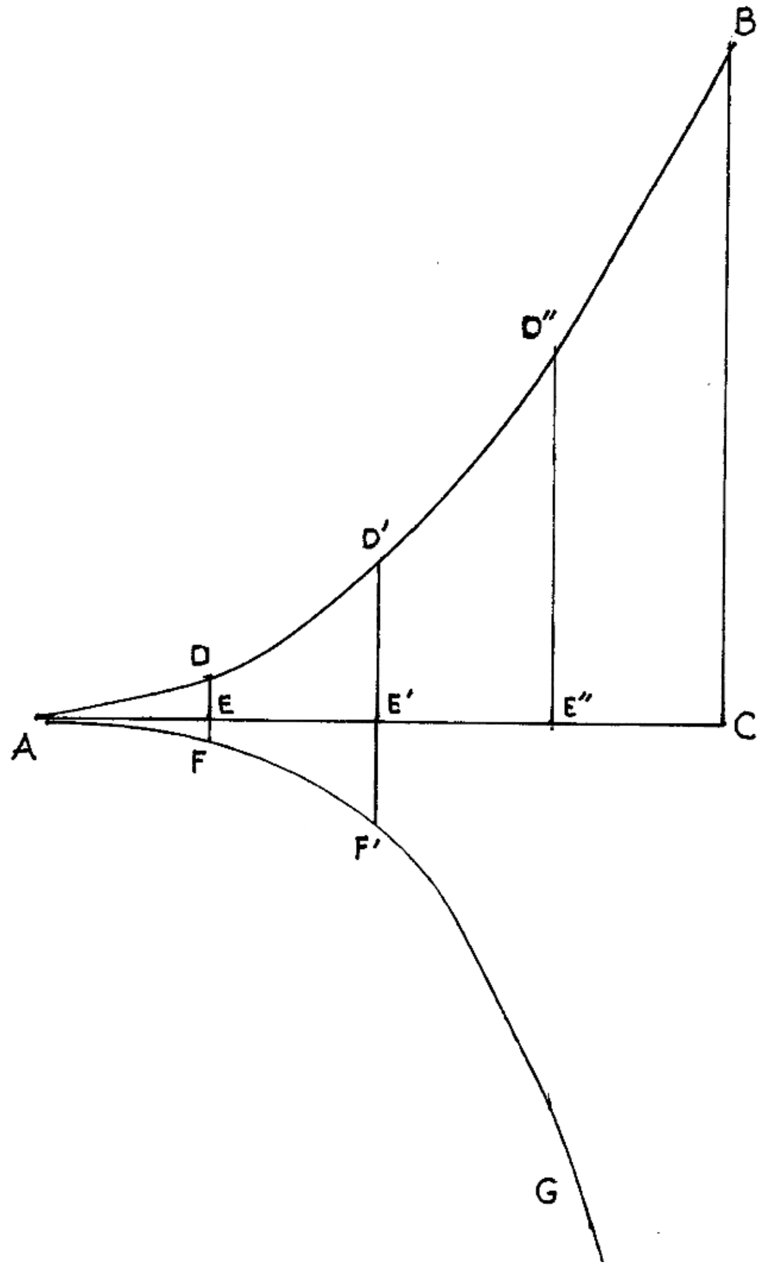


Figure 22

But to recall your steps, to rise again
into the upper air: that is the labor;
that is the task.[8]

Here are some examples.

1. Let $y = x^2$, so that $v = \int y\,dx = \int x^2\,dx$. Suppose that the sum begins at the point $A$ where $x = 0$. We need to find an expression for $v$ such that

   (a) $dv = x^2\,dx$, and

   (b) $v = 0$ when $x = 0$, where the sum begins.

   If we find such an expression $v$, then, according to the first fundamental theorem,

   $$\text{area } ADE = \int y\,dx = \int dv = v,$$

   and $v$ will be the quantity we are looking for.

   We know from the rules of the calculus that

   $$\begin{aligned} d\left(\frac{x^3}{3}\right) &= \frac{d(x^3)}{3} \\ &= \frac{3x^2\,dx}{3} \\ &= x^2\,dx \\ &= y\,dx. \end{aligned}$$

   Since

   $$\frac{x^3}{3} = 0$$

   when $x = 0$, where the sum begins, we therefore set

   $$v = \frac{x^3}{3},$$

   and use the first fundamental theorem to conclude that

   $$\begin{aligned} \text{area } ADE &= \int y\,dx \\ &= \int dv \\ &= v \\ &= \frac{x^3}{3}. \end{aligned}$$

[8] Lines 174–180 of Allen Mandelbaum's translation. Lines 125–129 of R. A. B. Mynors's Latin text.

Therefore the equation of the quadratrix $AFG$ is in this case

$$v = \frac{x^3}{3}.$$

2. Let $y = x^3$, so that $\int y\,dx = \int x^3\,dx$. Suppose that the sum begins at the point $A$ where $x = 0$. Then according to the first fundamental theorem, if we can find a quantity $v$ such that $x^3\,dx = dv$ and $v = 0$ when $x = 0$, then

$$\text{area } ADE = \int y\,dx = \int dv = v.$$

But we know from the rules of the calculus that

$$\begin{aligned} d\left(\frac{x^4}{4}\right) &= \frac{d(x^4)}{4} \\ &= \frac{4x^3\,dx}{4} \\ &= x^3\,dx \\ &= y\,dx. \end{aligned}$$

Since

$$\frac{x^4}{4} = 0$$

when $x = 0$, where the sum begins, we therefore set

$$v = \frac{x^4}{4},$$

and use the first fundamental theorem to conclude that

$$\begin{aligned} \text{area } ADE &= \int y\,dx \\ &= \int dv \\ &= v \\ &= \frac{x^4}{4}. \end{aligned}$$

Therefore the equation of the quadratrix $AFG$ is in this case

$$v = \frac{x^4}{4}.$$

3. Let $y = x^n$, where $n$ is any nonnegative number. We proceed just as in the previous two examples, finding a quantity $v$ such that $y\,dx = dv$ and

$v = 0$ when $x = 0$, and apply the first fundamental theorem. We know from the rules of calculus that

$$\begin{aligned} d\left(\frac{x^{(n+1)}}{n+1}\right) &= \frac{d(x^{(n+1)})}{n+1} \\ &= \frac{(n+1)x^n\,dx}{n+1} \\ &= x^n\,dx \\ &= y\,dx. \end{aligned}$$

Since

$$\frac{x^{(n+1)}}{n+1} = 0$$

when $x = 0$, we therefore set

$$v = \frac{x^{(n+1)}}{n+1},$$

and use the first fundamental theorem to conclude that

$$\begin{aligned} \text{area } ADE &= \int y\,dx \\ &= \int dv \\ &= v \\ &= \frac{x^{(n+1)}}{n+1}. \end{aligned}$$

Therefore the equation of the quadratrix $AFG$ is in this case

$$v = \frac{x^{(n+1)}}{n+1}.$$

4. Let $y = x^3 + x^2$. Then, if we set

$$v = \frac{x^4}{4} + \frac{x^3}{3},$$

according to the rules of calculus,

$$dv = x^3\,dx + x^2\,dx = y\,dx.$$

Since $v = 0$ when $x = 0$, according to the first fundamental theorem,

$$\begin{aligned} \int y\,dx &= \int dv \\ &= v \\ &= \frac{x^4}{4} + \frac{x^3}{3} \end{aligned}$$

Note that here it turns out that

$$\int (x^3 + x^2)\, dx = \int x^3\, dx + \int x^2\, dx.$$

This is generally true: for any variable quantities $t$ and $u$,

$$\int (t + u) = \int t + \int u.$$

For if $t = dv$ and $u = dw$, then

$$d(v + w) = dv + dw = t + u,$$

and if we begin the sums when $v = 0$ and $w = 0$ then we will also begin the sums where $v + w = 0$, and according to the first fundamental theorem,

$$\begin{aligned} \int (t + u) &= \int d(v + w) \\ &= (v + w) \\ &= \int dv + \int dw \\ &= \int t + \int u. \end{aligned}$$

We might call this the *addition rule for sums.* We could likewise show that for any constant $a$ and any variable $t$

$$\int at = a \int t.$$

This could be called *constant multiple rule for sums.*

We can use these rules, and the rule from the third example, to find sums for many algebraic expressions, as in the following example.

5. Let $y = 3x^5 - 8x^2 + 4$. Then

$$\begin{aligned} \int y\, dx &= 3 \int x^5\, dx - 8 \int x^2\, dx + 4 \int x^0\, dx \\ &= 3\frac{x^6}{6} - 8\frac{x^3}{3} + 4\frac{x^1}{1} \\ &= \frac{x^6}{2} - \frac{8x^3}{3} + 4x. \end{aligned}$$

6. Let

$$y = 2\sqrt{x} - 8x^{\frac{5}{3}}.$$

Then

$$
\begin{aligned}
\int y\,dx &= 2\int x^{\frac{1}{2}}\,dx - 8\int x^{\frac{5}{3}}\,dx \\
&= 2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) - 8\left(\frac{x^{\frac{8}{3}}}{\frac{8}{3}}\right) \\
&= \frac{4x^{\frac{3}{2}}}{3} - 3x^{\frac{8}{3}}.
\end{aligned}
$$

Now suppose that we are interested not in the area $ADE$, but in the area $DEE_1D_1$ (see Figure 23) between two definite ordinates, $DE$ and $D_1E_1$. Let $AE = a$ and $AE_1 = b$, where $a$ and $b$ are constants. This area is equal to the sum of $y\,dx$ between $x = a$ and $x = b$, which we denote by

$$\int_a^b y\,dx.$$

This sum is called a *definite integral*, because, unlike the sums in the previous examples, it represents a single constant area, and not a variable area. To find the area $DEE_1D_1$, we take the difference of the area $AD_1E_1$ (this area is equal to the value of $v$ when we set $x = b$, which we will call $v_b$) and the area $ADE$ (this area is equal to the value of $v$ when we set $x = a$, which we will call $v_a$):

$$
\begin{aligned}
\text{area } DEE_1D_1 &= \text{area } AD_1E_1 - \text{area } ADE \\
&= v_b - v_a.
\end{aligned}
$$

7. Let $y = x^2$. Then, as we saw above (page 139),

$$v = \frac{x^3}{3}.$$

Now if $a = AE = 2$ and $b = AE_1 = 4$, then

$$
\begin{aligned}
\text{area } DEE_1D_1 &= \int_2^4 y\,dx \\
&= v_4 - v_2 \\
&= \frac{4^3}{3} - \frac{2^3}{3} \\
&= \frac{64}{3} - \frac{8}{3} \\
&= \frac{56}{3}.
\end{aligned}
$$

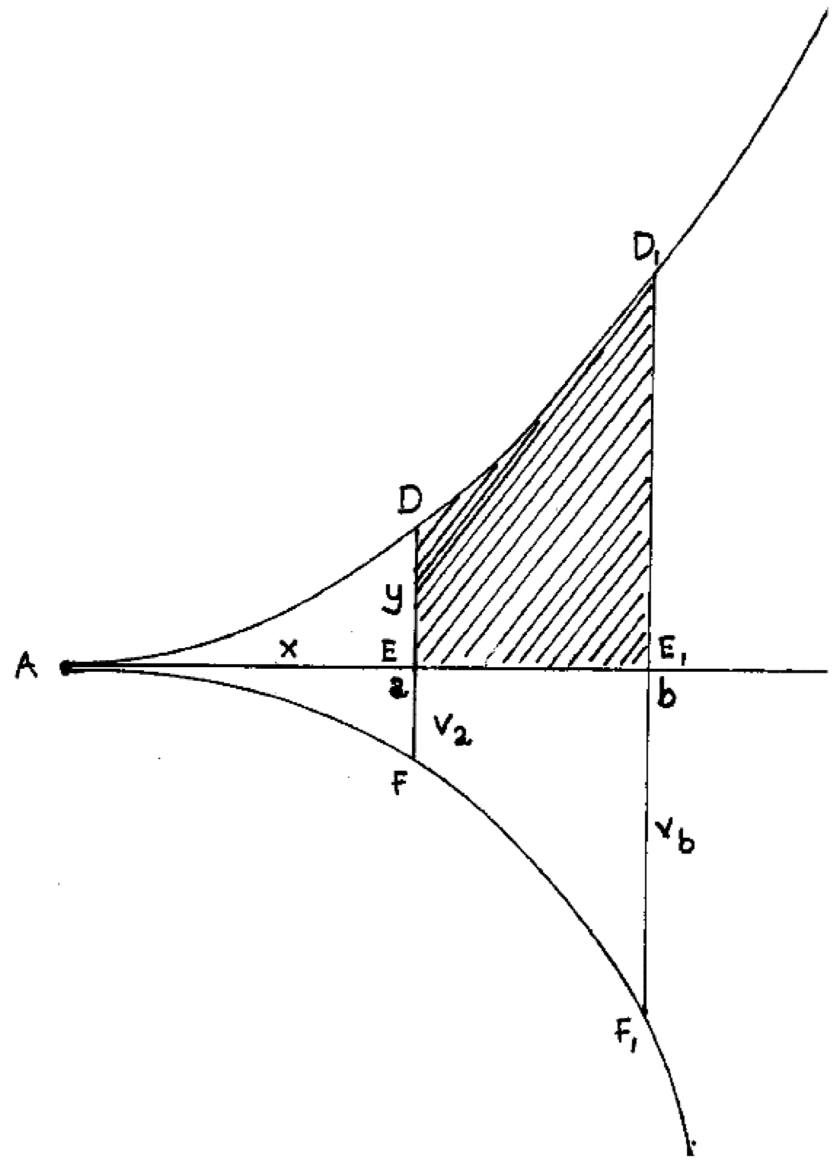



Figure 23

8. Let $y = 3x^2 + 7x$. Then

$$\begin{aligned} v &= \int y\,dx \\ &= 3\int x^2\,dx + 7\int x\,dx \\ &= 3\frac{x^3}{3} + 7\frac{x^2}{2} \\ &= x^3 + \frac{7x^2}{2}. \end{aligned}$$

Now if $a = AE = 1$ and $b = AE_1 = 5$, then

$$\begin{aligned} \text{area } DEE_1D_1 &= \int_1^5 y\,dx \\ &= v_5 - v_1 \\ &= \left(5^3 + \frac{7(5^2)}{2}\right) - \left(1^3 + \frac{7(1^2)}{2}\right) \\ &= \left(125 + \frac{175}{2}\right) - \left(1 + \frac{7}{2}\right) \\ &= \frac{425}{2} - \frac{9}{2} \\ &= \frac{416}{2} \\ &= 208. \end{aligned}$$

**Some problems on sums of algebraic quantities**

Find the following sums.

1.
$$\int (2x^3 - x + 4)\,dx.$$

2.
$$\int (3x^5 - 2x^2 + 1)\,dx.$$

3.
$$\int (\sqrt{x} + (\sqrt{x})^3)\,dx.$$

4.
$$\int (x^3 + \sqrt[3]{x})\,dx.$$

5.
$$\int_1^3 x^3\,dx.$$

6.
$$\int_{-1}^1 x^4\,dx.$$

7.
$$\int_1^2 (5x^4 - 2x)\,dx.$$

8.
$$\int_{-1}^2 (2x^2 + 1)\,dx.$$

## 4. A transcendent line: the sine curve

In the above examples we always begin with an algebraic curve and end with an algebraic quadratrix: the equation for the original curve $ADB$ (that is, the equation relating $y$ and $x$) and the final equation for the quadratrix $AFG$ (that is, the equation relating $v$ and $x$) are both ordinary algebraic equations (see Figure 22, page 137). Let us now consider some transcendent equations, beginning with Leibniz's equation for the length of an arc of a circle (on page 106 of his text—see Figure 24 here):

$$a = \int \frac{dx}{\sqrt{2x - x^2}}.$$

Leibniz has given an equation relating the *versed sine*, $x$ ($DA$ in Figure 24), to the arc $a$ ($AB$). But it will be simpler for us to use an equation relating the *sine*, $y$ ($BD$), to $a$. To get such an equation we again use the characteristic triangle $B_1GB$ (see note 16, above, page 123). Because triangle $B_1GB$ is similar to triangle $BDC$,

$$B_1B : BG :: CB : CD.$$

Now $B_1B = da$, $BG = dy$, $CB = 1$, and

$$\begin{aligned} CD &= \sqrt{CB^2 - DB^2} \quad \text{(by the Pythagorean theorem)} \\ &= \sqrt{1 - y^2}. \end{aligned}$$

Therefore (substituting into this proportion and converting it to an equation)

$$\frac{da}{dy} = \frac{1}{\sqrt{1 - y^2}}.$$

Therefore

$$da = \frac{dy}{\sqrt{1 - y^2}},$$

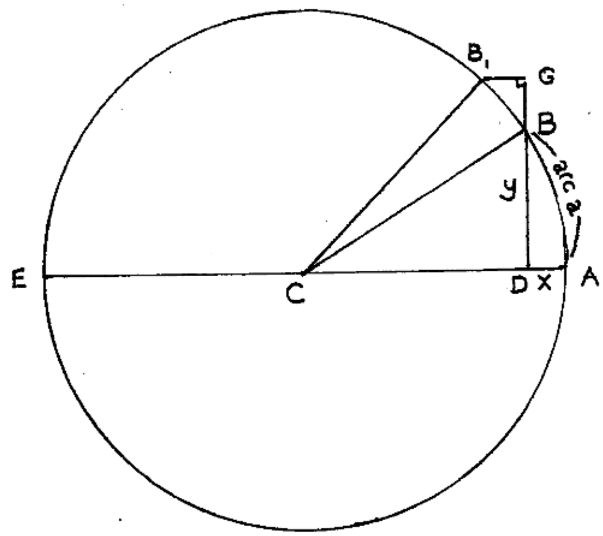



Figure 24

and (taking sums of both sides of the equation and using the first fundamental theorem)

$$a = \int \frac{dy}{\sqrt{1-y^2}}.$$

(Note that this becomes the same equation Leibniz gives on page 107 of "On Recondite Geometry," if we substitute $x$ for $y$.)

Now, in Figure 24, $y$ is a variable straight line while $a$ is a variable curved line. But we usually (following Descartes) represent an equation by taking both variables to correspond to *straight* lines. Figure 25 does this, showing the curve for the same equation when we take both $y$ and $a$ to be variable straight lines. There we draw an $a$-axis $FG$ and a $y$-axis $FK$. For any point $M$ on the axis $FG$, if $FM = a$ (= arc $AB$ in Figure 24), then we set $ML = y$ (= $BD$ in Figure 24). To see why the curve in Figure 25 has the shape it does, imagine what happens as $a$ increases uniformly. In Figure 24, as $a$ increases the point $B$ moves counterclockwise in uniform circular motion, while in Figure 25 the point $M$ moves to the right in uniform linear motion. When the point $B$ reaches the point $E$ opposite $A$ in Figure 24, $a = \pi$ (half the circumference of the circle) and $y = DB$ becomes equal to 0. Therefore in Figure 25, when $a = FG = \pi$, $y = LM$ becomes equal to 0, and the curve crosses the axis at $G$. As $a$ continues to increase to values greater than $\pi$, the point $B$ passes below $E$ in Figure 24, and so the values of $y$ become negative. Therefore in Figure 25, the curve passes below the axis beyond $G$. The values of $y$ continue to be negative until the point $B$ in Figure 24 comes back around to where it started at $A$. When $a = 2\pi$ (the whole circumference of the circle) $B$ and $A$ coincide, and $y = 0$. Therefore in

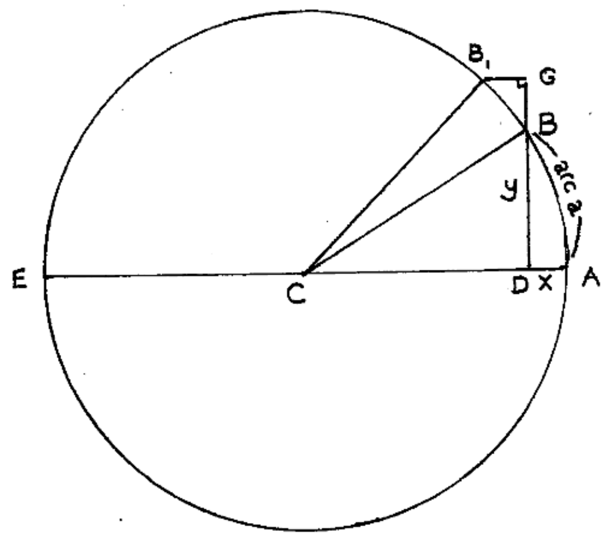



Figure 24

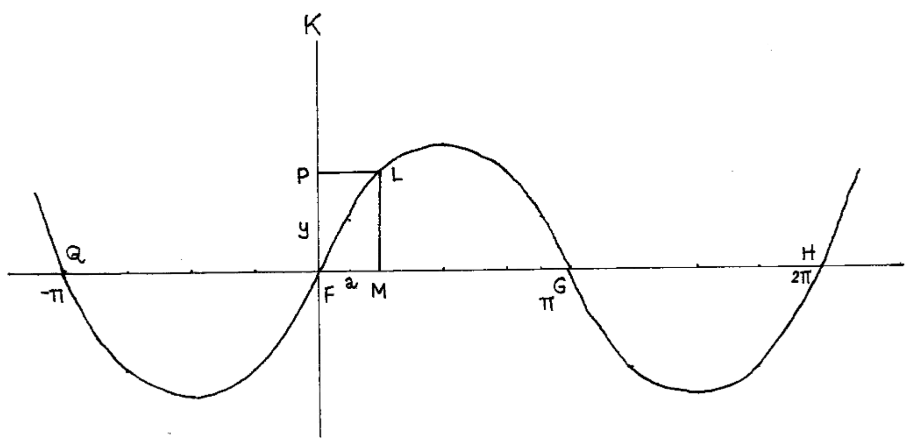



Figure 25

Figure 25, when $a = FH = 2\pi$, then $y = 0$ and the curve crosses the axis. As we move to the right of $H$ in Figure 25, the pattern repeats itself as $B$ goes around the circle again in Figure 24. Likewise, if we move to the left of $F$, we get the same pattern, as $B$ goes around the circle in the other direction in Figure 24. The resulting curve is therefore a kind of infinite wave that goes on repeating itself in both directions. Since this curve represents the relation between the sine $BD$ of the circle $ABE$ to its corresponding arc $AB$, it is called a *sine curve*.

The sine curve must be transcendent, as we can see by a simple *reductio ad absurdum* argument. For if it were not transcendent, so that the equation relating $y$ and $a$ could be expressed algebraically, then we could set $y = 0$ and get an algebraic equation for all the values of $a$ where $y = 0$. We would thereby find all the points $F$, $G$, $H$, etc., where the sine curve crosses the $a$-axis. But the equation for these points, like any algebraic equation, would have only finitely many solutions. (For example, if the equation for the sine curve were

$$y = a^3 - 3a^2 + 7,$$

we would get the equation

$$0 = a^3 - 3a^2 + 7,$$

which would have at most 3 solutions.) It would then follow that the sine curve would only cross the axis at finitely many points. But we just saw that it must cross the axis at infinitely many points: $F$, $G$, $H$, etc. Therefore the sine curve cannot be expressed by an algebraic equation; that is, there is no way to find an algebraic expression relating the abscissa of the sine curve

$$a = \int \frac{dy}{\sqrt{1 - y^2}}$$

to its ordinate $y$. In other words, there is no way to find a simply algebraic expression for

$$\int \frac{dy}{\sqrt{1 - y^2}};$$

we must use transcendent quantities to find this sum.

### Differences and sums of sines and other trigonometric quantities

Using Figures 24 and 25 we may find equations relating the differences and sums of sines and other trigonometric quantities. Here are some examples. We denote $y = BD$ (in Figure 24) by $\sin a$ and $1 - x = CD$ by $\cos a$, as is usually done in trigonometry. (Here the arc $AB$ is equal to the angle $BCA$, expressed in radians.)

1. To find $d \sin a$ in terms of $a$. In Figure 24, because of the similar triangles $B_1GB$ and $BDC$,

$$GB : B_1B :: DC : BC.$$

Substituting $d \sin a (= dy)$ for $GB$, $da$ for $B_1 B$, $\cos a$ for $DC$, and 1 for $BC$, and converting the proportion to an equation, we get

$$\frac{d \sin a}{da} = \frac{\cos a}{1}.$$

Therefore (solving for $d \sin a$),

$$d \sin a = \cos a \, da.$$

2. To find $d \cos a$ in terms of $a$. In Figure 24, because of the same similar triangles,

$$B_1 G : B_1 B :: BD : BC.$$

Note that $B_1 G$ is the amount by which $CD$ ($= \cos a$) decreases as $B$ moves to the infinitely close point $B_1$. Therefore $GB_1 = -d \cos a$. Substituting $-d \cos a$ for $GB_1$, $da$ for $B_1 B$, $\sin a$ for $BD$ and 1 for $BC$, and converting the proportion to an equation, we get

$$\frac{-d \cos a}{da} = \frac{\sin a}{1}.$$

Therefore

$$d \cos a = -\sin a \, da.$$

3. Let

$$z = \sin(3a + 4).$$

To find $dz$ in terms of $a$. Here we let $v = 3a + 4$, so that $z = \sin v$. Therefore, by the first example, above,

$$dz = \cos v \, dv.$$

And

$$\begin{aligned} dv &= d(3a + 4) \\ &= 3 \, da + d(4) \\ &= 3 \, da. \end{aligned}$$

Therefore

$$\begin{aligned} dz &= \cos v \, dv \\ &= \cos(3a + 4) \, (3 \, da) \\ &= 3 \cos(3a + 4) \, da. \end{aligned}$$

4. Let

$$z = \sin(\omega t),$$

where $\omega$ is some constant. To find $dz$ in terms of $t$. Let $v = \omega t$, so that

$$z = \sin v.$$

Therefore, by the first example, above,

$$dz = \cos v \, dv;$$

and

$$dv = \omega \, dt.$$

Therefore

$$\begin{aligned} dz &= \cos v \, dv \\ &= \cos(\omega t) \, dv \\ &= \cos(\omega t) \, \omega \, dt \\ &= \omega \cos(\omega t) \, dt. \end{aligned}$$

5. Let

$$z = \cos^2(4a).$$

To find $dz$ in terms of $a$. Here we let $v = \cos(4a)$, so that $z = v^2$ and

$$dz = 2v \, dv.$$

To find $dv$ , we let $u = 4a$, so that $v = \cos u$ and

$$dv = -\sin u \, du = -\sin(4a) \, du$$

by the second example, above. Finally, according to the constant multiple rule,

$$du = d(4a) = 4 \, da.$$

Therefore

$$\begin{aligned} dz &= 2v \, dv \\ &= 2(\cos(4a))(-\sin(4a) \, du) \\ &= 2(\cos(4a))(-\sin(4a))(4 \, da) \\ &= -8 \cos(4a) \sin(4a) \, da. \end{aligned}$$

6. To find $\int \sin a \, da$, that is, area $FLM$ in Figure 25, we would like to apply the first fundamental theorem. To do this, we need to find a quantity $v$ such that $dv = \sin a \, da$ and $v = 0$ when $a = 0$. It follows from the second example, above, that

$$d(-\cos a) = \sin a \, da.$$

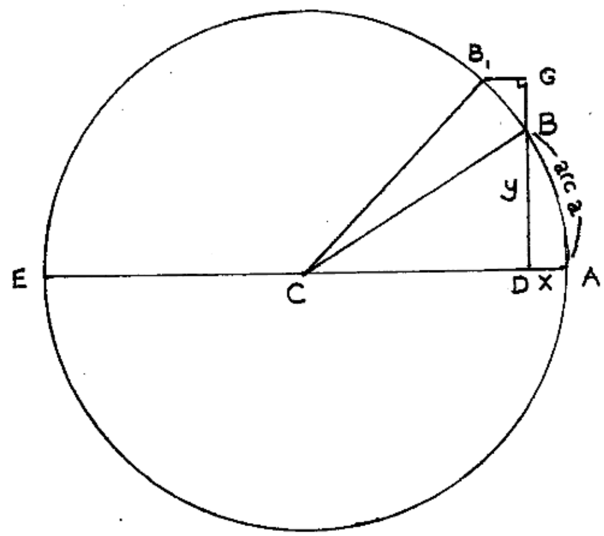



Figure 24

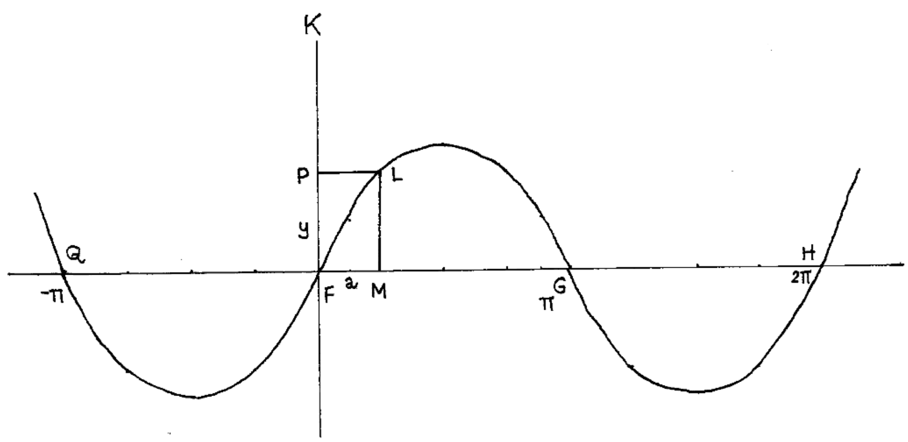



Figure 25

Therefore we might be tempted to set $v$ equal to $-\cos a$. But $-\cos a$ is not equal to 0 when $a = 0$: $-\cos 0 = -1$. Therefore instead we set

$$v = 1 - \cos a.$$

Then

$$dv = d(1) - d(\cos a) = -d(\cos a) = \sin a\, da,$$

and when $a = 0$

$$v = 1 - \cos 0 = 0.$$

Therefore we may apply the first fundamental theorem:

$$\begin{aligned} \text{area } FLM &= \int \sin a\, da \\ &= \int dv \\ &= v \\ &= 1 - \cos a. \end{aligned}$$

It follows in particular that when area $FLM$ coincides with area $FLG$, $a = \pi$ and

$$\begin{aligned} \text{area } FLG &= 1 - \cos(\pi) \\ &= 2. \end{aligned}$$

7. To find the definite integral

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin a\, da.$$

(See page 142, above, for a discussion of definite integrals.)

This sum is equal to the area $LMM_1L_1$, where

$$FM = \frac{\pi}{4}$$

and

$$FM_1 = \frac{\pi}{2}$$

(see Figure 26). This area is equal to the difference of area $FM_1L_1$ and area $FML$ . But we showed in the previous example that

$$\begin{aligned} FM_1L_1 &= v_{\frac{\pi}{2}} \\ &= 1 - \cos\left(\frac{\pi}{2}\right) \\ &= 1 - 0 \\ &= 1, \end{aligned}$$

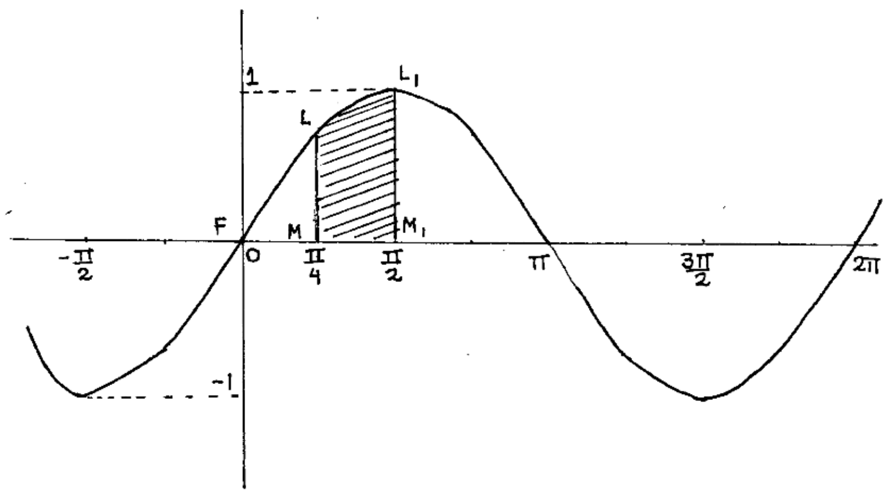



Figure 26

and

$$\begin{aligned} FML &= v_{\frac{\pi}{4}} \\ &= 1 - \cos\left(\frac{\pi}{4}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \text{area } LMM_1L &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin a \, da \\ &= v_{\frac{\pi}{2}} - v_{\frac{\pi}{4}} \\ &= 1 - \left(1 - \cos\left(\frac{\pi}{4}\right)\right) \\ &= \cos\left(\frac{\pi}{4}\right). \end{aligned}$$

(It turns out that, by a trigonometric argument that is not worth going into here,

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},$$

so that

$$\text{area } LMM_1L = \frac{\sqrt{2}}{2} \approx .7071.)$$

**Some problems on trigonometric quantities**

1. Using the rules of the differential calculus, and the above examples, find the following differences.

(a)
$$d(\sin(2a-1)+\cos(3a+2)).$$

(b)
$$d(\sin(5a)+\cos(a+3)).$$

(c)
$$d(4\sin(2a)+2\cos(a+1)).$$

(d)
$$d(2\sin(3a)-3\sin(7a)).$$

(e)
$$d(\sin^3(a)).$$

(f)
$$d(\sin^3(4a)).$$

(g)
$$d(\cos^4(1-a)).$$

(h)
$$d\left(\frac{\sin(2a)}{\cos(-a)}\right).$$

(i)
$$d\left(\frac{\cos(a^2)}{\sin(2a)}\right).$$

2. Using the first fundamental theorem and the rules of the differential calculus, find the following sums.

(a)
$$\int \cos a\, da.$$

(b)
$$\int (4\cos a - \sin a)\, da.$$

(c)
$$\int (3\sin a + 2\cos a)\, da.$$

(d)
$$\int 3\cos(a-2)\, da.$$

(e)
$$\int \sin(2a)\, da.$$

(f)

$$\int_0^{\frac{\pi}{3}} \cos a \, da.$$

(g)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2a) \, da.$$

## 5. Another transcendent line: the logarithmic line

Let the line $FD$ (Figure 27) be a logarithmic line, that is, a line such that any

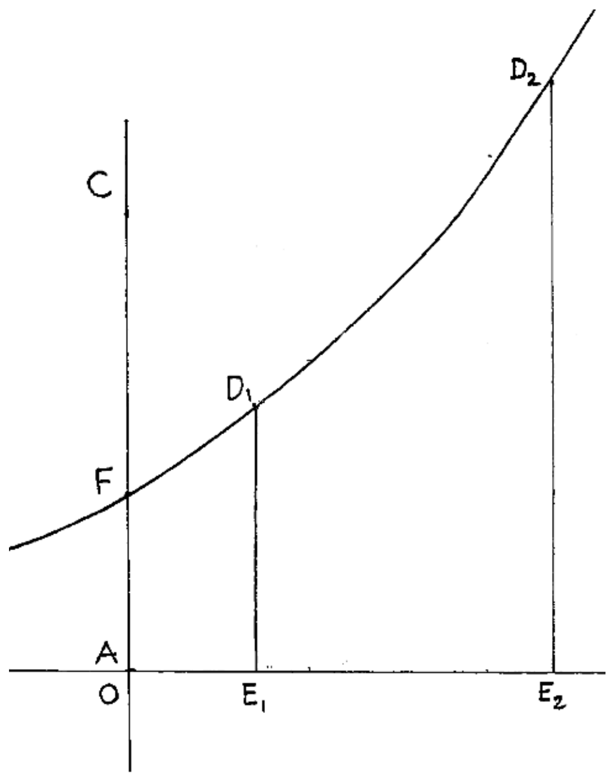



Figure 27

arithmetic progression of its abscissas $AE$ corresponds to a geometric progression of its ordinates $ED$ (see “A New Method,” page 39, and above, page 86). We will assume further that $FD$ is a *natural* logarithmic line. Let the abscissas $AE$ be denoted by $x$ and the ordinates $ED$ be denoted by $y$. Then $x$ is the natural logarithm of $y$, that is

$$x = \log y$$

(see page 93, above). Moreover,

$$y = e^x$$

(this is an equation on page 93, substituting $y$ for $w$), and $y$ and $x$ are related by the differential equation

$$\frac{dy}{y} = dx$$

(equation 3 on page 97, above, substituting $y$ for $w$).

We will show here that $FD$ is the quadratrix of a hyperbola, that it is transcendent, and give some examples of how to find sums of expressions involving logarithms.

**The logarithmic line as a quadratrix of a hyperbola**

The logarithmic line is the quadratrix of a hyperbola. For let the line $PM$ (Figure 28) be a hyperbola whose asymptotes are the perpendicular lines $LH$

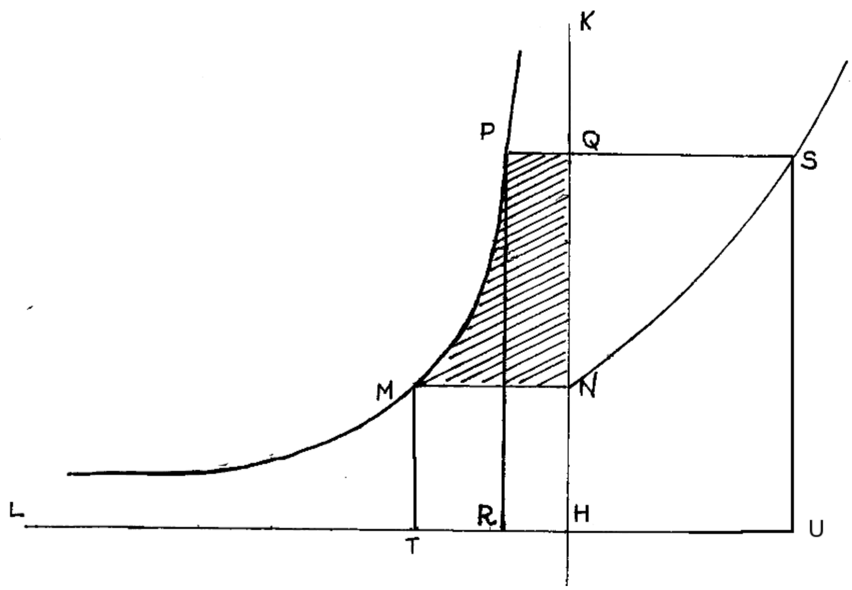



Figure 28

and $KH$. Let $M$ be the principal vertex of the hyperbola, and drop perpendicular lines $MN$ and $MT$ to $KH$ and $LH$, respectively. Let $MN$ (which is equal to $MT$) be our unit. If $P$ is any point on the hyperbola, and we drop perpendicular lines $PQ$ and $PR$ to $KH$ and $LH$, respectively, then according to Proposition II 12 in Apollonius' *Conics*,

$$\text{rectangle } PQHR = \text{square } MNHT.$$

Therefore, if we let $PQ = z$ and $QH = y$,

$$zy = \text{rectangle } PQHR = \text{square } MNHT = 1,$$

and therefore

$$z = \frac{1}{y}.$$

Let $NS$ be the quadratrix of the hyperbola $PM$, so that

$$SQ = \text{area } QPMN.$$

Now the sum

$$\int z\,dy$$

(beginning from $y = 1$) represents the area of $QPMN$ corresponding to the variable $y = QH$ (see part 1 of Note 19, above, page 131), and since

$$z = \frac{1}{y},$$

this area is also equal to

$$\int \frac{dy}{y}.$$

Therefore

$$SQ = \int \frac{dy}{y}.$$

Now let $x$ be the natural logarithm of $y$, so that

$$\frac{dy}{y} = dx.$$

(equation 3 on page 93, above, substituting $y$ for $w$). Then

$$SQ = \int dx = x,$$

according to the first fundamental theorem. Drop $SU$ perpendicular to $HU$. Then the abscissa $HU$ of the line $NS$ is equal to $SQ$, and therefore to $x$, while its ordinate $SU$ is equal to $y$. But we defined $x$ as the natural logarithm of $y$, and therefore the abscissa of $NS$ is always equal to the natural logarithm of its ordinate. Therefore $NS$ is a logarithmic line.

The calculus therefore shows us an unexpected analogy between sines and logarithms. Both sine curves and logarithmic curves arise from measurements of simple conic sections. The sine curve arises from measuring arcs of circles, while the logarithm arises from taking areas under hyperbolas with perpendicular asymptotes. Sines and the logarithms thus arise from taking measurements of two of the simplest curved lines. We might then wonder what sort of transcendent quantities would arise from more complicated curved lines. For example, what sort of transcendent quantities arise from measuring the arc lengths of ellipses? or areas under hyperbolas whose asymptotes are not perpendicular? And what sort of transcendent quantities arise from higher degree curves?

### The transcendence of the logarithmic line

We are finally in a position to expand Leibniz's argument (on page 103 of "On Recondite Geometry") that the logarithmic line is transcendent. The argument is parallel to the argument to show that the quadratrix of a circle is transcendent (see Note 6, above, pages 115–116, and the appendix, pages 247–249), and here again we give a plausible argument, and not a complete demonstration.

Suppose that $FD$ were not transcendent. Then it would have a single algebraic equation of definite degree, such as

$$y^2 = cy - \frac{cx}{b}y + ay - ac,$$

Now the logarithmic line $FD$ (which is the quadratrix of a hyperbola that Leibniz mentions on page 103) may be used to find an arbitrary number of mean proportionals for any given ratio, as follows. Let the given ratio be that of $G$ to $H$ (see Figure 29), and suppose $G$ is a unit. Note that $AF$ is also a unit, since

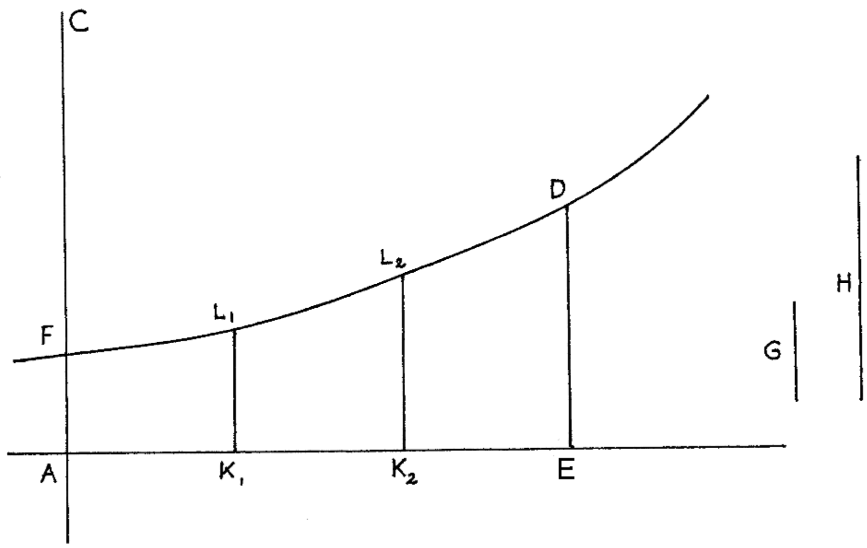



Figure 29

it represents $e^0 = 1$. Let $ED$ be the ordinate of $FD$ equal to $H$. If we want to find two mean proportionals, then let $K_1$ and $K_2$ be two points on the axis such that

$$AK_1 = K_1K_2 = K_2E.$$

Therefore the abscissas

$$0,\ AK_1,\ AK_2,\ \text{and } AE$$

form an arithmetic progression. Therefore the corresponding ordinates

$$AF,\ K_1L_1,\ K_2L_2,\ \text{and } ED$$

form a geometric progression, that is

$$AF(=G):K_1L_1 :: K_1L_1:K_2L_2 :: K_2L_2:ED(=H).$$

Therefore $K_1L_1$ and $K_2L_2$ are two mean proportionals for $G$ and $H$. If we want to find three mean proportionals for $G$ and $H$, we simply take three equally spaced points $K$ between $A$ and $E$, and so on.

Thus the line $FD$, which has an algebraic equation of a single definite degree, may be used to find an arbitrary number of mean proportionals for two given magnitudes. But the problem of finding two mean proportionals for a given ratio is of the third degree. For if we let

$$H = ED = y,$$

and

$$z^3 = y,$$

then

$$1:z :: z:z^2 :: z^2:z^3.$$

Substituting $G$ for 1 and $H$ for $z^3$ gives

$$G:z :: z:z^2 :: z^2:H.$$

Therefore $z$ and $z^2$ are the two mean proportionals for the given ratio, so that finding $z$ lets us find both these mean proportionals, and $z$ is the solution of a third degree equation, namely,

$$z^3 = y.$$

Therefore the problem of finding two mean proportionals is a problem of the third degree. Likewise, the problem of finding three mean proportionals is a problem of the fourth degree, the problem of finding four mean proportionals is of the fifth degree, and so on. The line $FD$ would then have a single definite degree but be capable of solving infinitely many problems of all possible degrees. This is absurd. Therefore our assumption that $FD$ is not transcendent must be false.

**Sums of quantities involving logarithms**

1. Let $y = e^x$. To find $\int y\,dx$, that is, area $FAED$ in Figure 27 (page 155), in terms of $x$. To do this, we need to find a quantity $v$ such that $dv = e^x\,dx$ and $v = 0$ when $x = 0$. According to equation 4 on page 96, above,

$$d(e^x) = e^x\,dx.$$

   Therefore we might be tempted to set $v$ to be equal to $e^x$. But $e^x$ is not equal to 0 when $x = 0$: $e^0 = 1$. Therefore instead we set

$$v = e^x - 1.$$

Then

$$dv = d(e^x - 1) = d(e^x) = e^x\,dx,$$

and when $x = 0$

$$v = e^0 - 1 = 0.$$

Therefore, according to the first fundamental theorem,

$$\begin{aligned}\text{area } FAED &= \int e^x\,dx \\ &= \int dv \\ &= v \\ &= e^x - 1.\end{aligned}$$

2. Let $y = e^{2x}$. To find $\int y\,dx$ in terms of $x$. To do this, let

$$u = 2x.$$

Then

$$y = e^u$$

and, by the constant multiple rule,

$$du = 2\,dx.$$

Therefore

$$dx = \frac{1}{2}\,du.$$

Therefore

$$\begin{aligned}\int y\,dx &= \int (e^u)\left(\frac{1}{2}\,du\right) \\ &= \frac{1}{2}\int e^u\,du \\ &= \frac{1}{2}(e^u - 1) \qquad \text{(previous example)} \\ &= \frac{1}{2}(e^{2x} - 1).\end{aligned}$$

The method we have used in this example is called *integration by substitution.* It is useful whenever we can find another variable $u$ such that

$$\int y\,dx$$

can more easily be found when it is expressed in terms of $u$. There are no universal rules for determining what new variable $u$ to substitute, and it can even be difficult to see in advance that substitution is a useful method for a given problem.

3. Let $y = x\sin(x^2)$. To find $\int y\,dx$ in terms of $x$. To do this, let

$$u = x^2.$$

Then

$$du = 2x\,dx$$

and

$$\begin{aligned}\frac{1}{2}\sin(u)\,du &= \frac{1}{2}\sin(x^2)(2x\,dx)\\ &= \sin(x^2)\,x\,dx\\ &= x\sin(x^2)\,dx\\ &= y\,dx.\end{aligned}$$

Therefore

$$\begin{aligned}\int y\,dx &= \int \frac{1}{2}\sin(u)\,du\\ &= \frac{1}{2}\int \sin(u)\,du\\ &= \frac{1}{2}(1-\cos(u)) && \text{(example 6, page 150)}\\ &= \frac{1}{2}(1-\cos(x^2)).\end{aligned}$$

4. Let $y = xe^x$. To find $\int y\,dx$ in terms of $x$.

Let $u = x$ and $v = e^x$. Then $dv = e^x\,dx$, and therefore

$$y\,dx = u\,dv.$$

Now, according to the multiplication rule,

$$d(uv) = u\,dv + v\,du,$$

and therefore

$$u\,dv = d(uv) - v\,du.$$

Therefore

$$\begin{aligned}\int y\,dx &= \int u\,dv\\ &= \int (d(uv) - v\,du)\\ &= \int d(uv) - \int v\,du\\ &= uv - \int v\,du && \text{(first fundamental theorem)}\\ &= xe^x - \int e^x\,dx\\ &= xe^x - (e^x - 1) && \text{(first example)}\end{aligned}$$

The method we have used in this example is called *integration by parts*: for any sum $\int y\,dx$, if we can find $u$ and $v$ such that $y\,dx = u\,dv$, then

$$\int y\,dx = uv - \int v\,du.$$

This method is useful when it is easier to find $\int v\,du$ than it is to find $\int u\,dv$.

### Some problems on sums involving logarithms

Using the first fundamental theorem and the rules of the differential calculus, find the following sums.

1. $$\int (3e^x + x^2)\,dx.$$

2. $$\int (2e^x + \sin x)\,dx.$$

3. $$\int (e^{3x})\,dx.$$

4. $$\int (e^{2x} + 3e^{-x})\,dx.$$

5. $$\int e^{(x^2)}\,x\,dx.$$

6. $$\int e^{(\sin x)}\,\cos x\,dx.$$

7. $$\int x^2 e^x\,dx.$$

8. $$\int x^3 e^x\,dx.$$

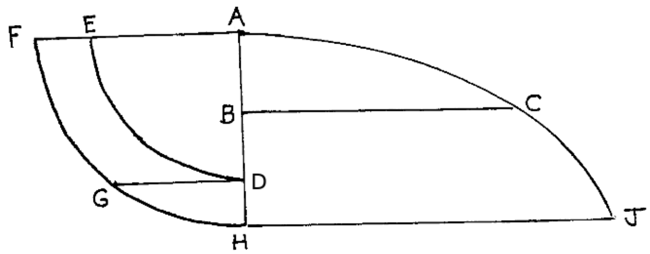



Figure 30

## Note 20

In the paper Leibniz refers to, Tschirnhaus is, like Leibniz (see the beginning of "On Recondite Geometry"), responding to Craig's recent work . According to Tschirnhaus, Craig excessively praises Barrow's discoveries. To prove that his method is superior to Barrow's, Tschirnhaus gives several problems that he claims that he can solve and Barrow cannot. These are the problems that Leibniz is speaking of here. The first problem is as follows:

> Let $FGH$ (Figure 30) be a quadrant of a circle, and suppose that
>
> $$AH : HB :: \text{arc } FGH : \text{arc } GH.$$
>
> Now for any point $G$ on the arc $FGH$, let $ED$ be a quadrant of a circle with center $A$, and draw the line $BC$ perpendicular to $AH$ such that
>
> $$BC = \text{arc } ED.$$
>
> Then as the point $G$ moves along the arc $FH$, the points $C$ will trace out a curve $ACJ$. The problem is to find the curve $ACJ$.

Leibniz claims here that $ACJ$ is a line of sines, and that

$$\text{area } ABCA = AH \times GD.$$

(He is mistaken about the area $ABCA$; in fact

$$\text{area } BHJC = AH \times GD.)$$

Tschirnhaus's other problems are obscurely posed, and it is not necessary to understand them here.

# Functional Notation and Calculus

## Functions and derivatives

The explicit objects of Leibniz's calculus are usually variable quantities. When he takes differences or sums, he takes differences or sums of variable quantities. But these variable quantities rarely stand alone. The way one variable quantity varies usually depends on the way other quantities vary. For example, the way an ordinate $y$ of a curve varies depends upon the way its abscissa, $x$, varies. If the value of $x$ is determined, then the value of $y$ is also determined. In this case the curve provides us a kind of rule relating the values of $x$ to the values of $y$. All interesting applications of the calculus depend upon relations *between* variable quantities: when we find differences or sums of a quantity $y$, we find them in terms of some other quantity $x$ on which $y$ depends.

Later mathematicians called any rule relating two variable quantities, considered as an object in its own right, a *function*. We denote a function by a single letter, such as $f$, and indicate that $f$ is the rule relating $x$ and $y$ by writing

$$y = f(x),$$

which we read as "$y$ equals $f$ of $x$." The rule represented by $f$ may or may not come from a curve or any geometric object. It simply is a law that, for any given value of $x$, determines a single value of $y$. When we write, for example,

$$f(x) = x^7 - 3x^3 + 2,$$

we need not think of a curve, but simply of the way in which we calculate $f(x)$ for any given value of $x$.

If $y$ depends on $x$, so that $y = f(x)$, then the ratio which Leibniz uses to define $dy$, namely, the ratio of $dy$ to $dx$ ("A New Method," page 25), also depends on $x$, that is, this ratio is a function of $x$. This function is called the *derivative* of $f$ with respect to $x$, and is denoted by $f'$, so that

$$f'(x) = \frac{dy}{dx}.$$

Sometimes one also denotes $f'(x)$ by

$$\frac{df}{dx} \text{ or } \frac{d}{dx} f(x).$$

Note that $f'(x)$ is always a *finite* quantity, and not infinitely small.

Since $f'(x)$ is itself a function of $x$, we can take its derivative with respect to $x$. This result is called the *second derivative* of $f$ with respect to $x$, and is denoted by $f''(x)$. That is,

$$f''(x) = \frac{d}{dx} f'(x).$$

If we want to use purely Leibnizian notation, and we assume that $dx$ is constant (as we may always do, since it is arbitrary [see also p. 68]), then we have

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d\left(\frac{dy}{dx}\right)}{dx} \\ &= \frac{\left(\frac{ddy}{dx}\right)}{dx} && \text{(constant multiple rule applied to } \frac{1}{dx}\text{)} \\ &= \frac{ddy}{(dx)^2}. \end{aligned}$$

This is usually denoted by

$$f''(x) = \frac{d^2y}{dx^2}.$$

One can continue taking derivatives indefinitely, so that the derivative of the second derivative is the third derivative, the derivative of the third derivative is the fourth derivative, and so on.

## Finding derivatives in functional notation

There are different ways to denote derivatives. Following Leibniz, we have for the most part denoted the difference of a variable quantity $y$ by $dy$ and its derivative with respect to $x$ by

$$\frac{dy}{dx}.$$

Leibniz expresses the rules for finding differences in this notation, and rules for derivatives easily follow. We will call this notation the *differential notation.* But if $y = f(x)$, we may also denote the derivative of $y$ with respect to $x$ by

$$f'(x).$$

The rules for derivatives can of course also be expressed in this notation. We will call this the *functional notation* for derivatives.

These two notations are useful in different ways. The differential notation refers more directly to the quantities $y$ and $x$, and lets us express the infinitely small quantities or differentials $dy$ and $dx$. The functional notation refers more directly to the relations $f$ between the quantities, and more clearly expresses that the derivative $f'$ of a function is the same kind of thing as the original function $f$. Later authors make frequent use of both notations. All the rules for finding derivatives are equivalent in the two notations, but since the functional notation has at least a different look and feel than the differential notation, we here try to sketch the rules for derivatives in it.

Consider, for example, the multiplication rule. In differential notation, it is

$$d(uv) = u\,dv + v\,du,$$

that is, the difference of a product of two quantities is equal to the first quantity times the difference of the second plus the second quantity times the difference of the first. If $u$ and $v$ are both functions of $x$, we can make this a rule for derivatives by dividing both sides by $dx$ to get

$$\frac{d(uv)}{dx} = u\,\frac{dv}{dx} + v\,\frac{du}{dx},$$

that is, the derivative of a product of two quantities is equal to the first quantity times the derivative of the second quantity plus the second quantity times the derivative of the first. Now if $u = f(x)$, and $v = g(x)$, then we can express this in the functional notation by

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x),$$

that is, the derivative of a product of two functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first.

**Problem 1**

Find the rules for derivatives corresponding to the constant rule, constant multiple rule, addition rule, division rule, and power rule. Express each of these in functional notation.

**Example**

Let $f(x) = \sin x$ and $g(x) = e^x$. Suppose we want to find $(fg)'(x)$. Then, according to the multiplication rule,

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x).$$

Now

$$\begin{aligned} f'(x) &= \frac{d(\sin x)}{dx} \\ &= \frac{\cos x\,dx}{dx} \\ &= \cos x, \end{aligned}$$

and

$$\begin{aligned} g'(x) &= \frac{d(e^x)}{dx} \\ &= \frac{e^x\,dx}{dx} \\ &= e^x. \end{aligned}$$

Therefore,

$$\begin{aligned} (fg)'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= (\sin x)\,e^x + e^x\,\cos x. \end{aligned}$$

## The method of substitution and the chain rule

When finding derivatives, we often have to substitute new variables. This looks somewhat different when put into the language of derivatives. For example, if $y = \sin(x^2)$ and we want to find $dy$ in terms of $x$ and $dx$. Let $x^2 = v$, then

$$\begin{aligned} dy &= (\cos v)\, dv \\ &= (\cos v)\, d(x^2) \\ &= (\cos v)\, 2x\, dx \end{aligned}$$

Now we cannot do the same thing in functional notation, since there is no way to express $dv$ in it. But if we divide both sides of the final expression by $dx$, we will have an equation for a derivative, without any infinitely small differences on their own:

$$\frac{dy}{dx} = (\cos v)\, 2x.$$

The right hand side has two factors: $\cos v$ and $2x$. The first factor, $\cos v$, is equal to $\frac{dy}{dv}$. The second factor, $2x$, is equal to $\frac{dv}{dx}$. So in this case we have

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}.$$

This is always true, no matter what $y$ and $v$ are. It is an algebraic statement of the *chain rule.*

The chain rule can be expressed in functional notation, without any $d$'s. To do this, we first need to introduce the notion of a composite function. Consider again the example above, $y = \sin(x^2)$. Then we can write $y = h(x)$, since $y$ depends on $x$. But we can also write $y = f(v)$, where $v = x^2$, and $f(v) = \sin v$, and we can let $g(x) = x^2$, so that $v = g(x)$. Then, in this notation, $y = f(v) = f(g(x))$. Thus $y$ is equal to two different expressions in functional notation:

1. $h(x)$, and
2. $f(g(x))$.

This fact is expressed by writing $h = f \circ g$. The function $h$ is then called the *composite* of the functions $f$ and $g$. We get from $x$ to $y$ by first using the function $g$ to go from $x$ to $v$, and then using the function $f$ to get from $v$ to $y$. For another example of a composite function, consider the function $h$ where

$$y = h(x) = (x^3 + 3x)^{11}.$$

Let $v = x^3 + 3x$ and $f(v) = v^{11}$. Then $y = f(v)$. Let $g(x) = x^3 + 3x$. Then $v = g(x)$ and $y = f(v) = f(g(x))$. Therefore $h(x) = f(g(x))$ and $h = f \circ g$.

To return to the chain rule, we suppose in general that $h$ is a function composed of $f$ and $g$, so that $h(x) = f(g(x))$. Then if $y = h(x)$ and $v = g(x)$, $y = f(v)$, and

$$\frac{dy}{dx} = h'(x),\ \frac{dy}{dv} = f'(v), \text{ and } \frac{dv}{dx} = g'(x),$$

and the chain rule becomes

$$h'(x) = f'(v) \cdot g'(x),$$

that is

$$h'(x) = f'(g(x)) \cdot g'(x).$$

Put into words, the chain rule says that if $y$ is a composite function, that is, if $y$ is a function of $v$ which is a function of $x$, then the derivative of $y$ with respect to $x$ equals the derivative of $y$ with respect to $v$ times the derivative of $v$ with respect to $x$. We could also say that the derivative of the whole composite function is the derivative of the "outside" times the derivative of the "inside;" in the example, the derivative of $\sin(x^2)$ is the derivative of $\sin()$ (the "outside") times the derivative of $x^2$ (the "inside").

Let us now go through the example using the chain rule instead of substitution. Again, $y = h(x) = \sin(x^2) = f(g(x))$, where $f(v) = \sin v$ and $g(x) = x^2$. We want to find $h'(x)$. According to the chain rule,

$$h'(x) = f'(g(x)) \cdot g'(x).$$

In this case $f'(v) = \cos v = \cos(g(x)) = \cos(x^2)$ and $g'(x) = 2x$, and therefore, according to the chain rule,

$$h'(x) = \cos(x^2) \cdot 2x.$$

The chain rule can be demonstrated using the functional notation, without ever using infinitely small differences. The demonstration is beyond the scope of this note, but you can find it in many introductory calculus books. Of course, if you are thinking in differences to begin with, then the chain rule is quite obviously true using simple algebra:

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx},$$

since we can simply cancel the $dv$'s on the right side of this equation to get the left side.

**Problems**

For each of the following functions $f(x)$, find $f'(x)$. Use functional notation as much as possible, and avoid infinitely small quantities

2. $f(x) = (x^2 + 3)^{17}$.

3. $f(x) = \cos(x)e^{3x}$.

4. $f(x) = 2\cos(5x + 4)$.

5. $f(x) = A\sin(kx + \alpha)$, where $A$, $k$, and $\alpha$ are arbitrary constants.

## Functions of more than one variable and partial derivatives

One variable quantity may depend on more than one other variable quantity. For example, the quantity

$$v = 2xy + 3x$$

depends on both $x$ and $y$: for any given pair of values for $x$ and $y$, there is a single value of $v$. The rule relating $v$ to $x$ and $y$ is a function of more than one variable. We denote functions of more than one variable by a single letter, such as $f$, and express the fact that a quantity $v$ depends on two other quantities $x$ and $y$ by writing

$$v = f(x, y).$$

If $v$ depends on three variables, $x$, $y$, and $z$, we would likewise write

$$v = f(x, y, z),$$

and so on.

It is often useful to find the infinitely small increments of a quantity that depends on several different variables. Each infinitely small increment of a quantity is a difference, but now, because the quantity does not depend on a single variable, there is no unambiguous way to refer to *the* difference of $v$. For if $v$ depends on $x$ and $y$, then depending on how $x$ and $y$ vary, $v$ varies in different ways. For example, if we hold $y$ constant and let only $x$ vary by an infinitely small amount $dx$, then we will get one value of $dv$, while if we hold $x$ constant and let $y$ vary by an infinitely small amount $dy$, then we will in general get a different value for $dv$. And if we hold neither $x$ nor $y$ constant but let each of them change by a different infinitely small amount, we will get still another value for $dv$. In fact, $dv$ is in general itself a function of $dx$ and $dy$ as well as $x$ and $y$. If $v = f(x, y)$ is a function of $x$ and $y$, then

$$dv = g(x, y)\, dx + h(x, y)\, dy,$$

where $g$ and $h$ are some functions of $x$ and $y$. The function $g$ is called the *partial derivative* of $f$ with respect to $x$, while the function $h$ is called the *partial derivative* of $f$ with respect to $y$. These derivatives are called partial because they each only express part of the change in $v$ as $x$ and $y$ change. Note that if we set $y$ constant and let only $x$ vary, then $dy = 0$ and

$$dv = g(x, y)\, dx,$$

so that

$$g(x, y) = \frac{dv}{dx}.$$

Since $dv$ is not any possible value of $dv$, but only that value of $dv$ which arises from varying $x$ but not $y$, there are special notations for the partial derivative:

$$g(x, y) = \frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}(x, y)$$

(so that $g = \dfrac{\partial f}{\partial x}$). Likewise

$$h(x, y) = \frac{\partial v}{\partial y} = \frac{\partial f}{\partial y}(x, y).$$

is the partial derivative of $v = f(x, y)$ with respect to $y$ (so that $h = \dfrac{\partial f}{\partial y}$). Thus the complete differential $dv$, expressed in terms of partial derivatives, is

$$dv = \frac{\partial v}{\partial x}\, dx + \frac{\partial v}{\partial y}\, dy.$$

For example, let $v = f(x, y) = 2xy + 3x$. Then

$$\begin{aligned} dv &= d(2xy + 3x) \\ &= 2\, d(xy) + 3\, dx \\ &= 2\,(x\, dy + y\, dx) + 3\, dx \\ &= (3 + 2y)\, dx + 2x\, dy. \end{aligned}$$

The partial derivative of $v$ with respect to $x$ is the coefficient of $dx$ in the complete differential $dv$, that is

$$\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}(x, y) = 3 + 2y.$$

Likewise, the coefficient of $dy$ in the complete differential $dv$ is the partial derivative of $v$ with respect $y$

$$\frac{\partial v}{\partial y} = \frac{\partial f}{\partial y}(x, y) = 2x.$$

To find partial derivatives, we do not need to find first the complete differential $dv$. For example, if $v = 2xy + 3x$, and we want to find $\dfrac{\partial v}{\partial x}$, we can treat $y$ as a constant and take the derivative with respect to $x$:

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{dv}{dx} && \text{(where } y \text{ is constant)} \\ &= y\frac{d(2x)}{dx} + 3\frac{dx}{dx} \\ &= 2y + 3. \end{aligned}$$

Likewise, to find $\dfrac{\partial v}{\partial y}$, we can treat $x$ as a constant and take a derivative with respect to $y$:

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{dv}{dy} && \text{(where } x \text{ is constant)} \\ &= 2x\frac{dy}{dy} + \frac{d(3x)}{dy} \\ &= 2x + 0. \end{aligned}$$

**Problems**

For each of the following functions $f(x, y)$, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

6. $f(x, y) = 3x^2 + 2y^2$.
7. $f(x, y) = 2y^2 + 3xy - 5x^2 + 2y$.
8. $f(x, y) = \sin(x - 2y)$
9. $f(x, y) = e^{(x-2y)}$
10. $f(x, y) = \Phi(x - 2y)$, where $\Phi$ is any function of one variable. Your answer should be expressed in terms of the derivative $\Phi'$ of $\Phi$.

# Calculus and Newtonian Physics

In the preface to his *Mathematical Principles of Natural Philosophy*, Newton writes that "the whole difficulty of philosophy seems to be to discover the forces of nature from the phenomena of motions and then to demonstrate other phenomena from these forces." The calculus can help solve this difficulty.

## Velocity and force


Newton's laws are laws of velocity and force. Velocity and force can easily be expressed by the calculus. Suppose that a body $A$ is moving along a straight line $AB$, and that $B$ is a fixed point. Let the variable distance $AB$ be denoted by $s$, and let time be denoted by $t$. The distance $s$ is a function of $t$. If in an increment of time $dt$ the body moves from $A$ to $A_1$, then $ds = AA_1$, and the average velocity of the body during this time will be the distance it travels ($ds$) divided by the time ($dt$) it takes to travel this distance. That is, the average velocity will be

$$\frac{ds}{dt}.$$

Now if the time $dt$ is very small, the velocity will not change very much during this time. And if the time is infinitely small, then the change in velocity during this time will be negligible. Therefore in this case, the average velocity will be the instantaneous velocity. If we denote this velocity by $v$, we then have

$$v = \frac{ds}{dt},$$

that is, the velocity of the body is equal to the derivative of its distance with respect to time.

Newton's second law asserts in this case that change of motion is proportional to motive force impressed. According to his second definition, quantity of motion is the product of quantity of matter and velocity. If we denote the quantity of matter (now usually called *mass*) by $m$, the quantity of motion will be $mv$. Force is therefore proportional to the change in $mv$. Now it is clear from what Newton writes later in the *Principia* that in the second law he assumes that the time is fixed. If we let time vary, then the change in motion is proportional to both the force and the time, so that, for example, if we let a constant force act for twice the time, we will get twice the change of motion. Now in general a force is not constant. But if we take a small time interval $dt$, then the force will change very little during this time, and if we take an infinitely small time interval $dt$, we can assume a force is constant. In that case, if we let $F$ denote the force, and let $dt$ denote the time in which it acts, we can express the proportionality of change of motion to force and time by

$$d(mv) \propto F dt,$$

or (since $m$ is constant)

$$m\,dv \propto F dt,$$

or (dividing by $dt$)

$$m\frac{dv}{dt} \propto F.$$

We can choose our units of force so that this proportion becomes an equation, so that

$$m\frac{dv}{dt} = F.$$

In words, the force is equal to the quantity of matter (or mass) times the derivative of velocity with respect to time. The derivative of velocity with respect to time is usually called the acceleration, and is denoted by $a$. We therefore have

$$F = ma,$$

that is, force is equal to mass times acceleration.

Note that since acceleration is the derivative of velocity, which is itself a derivative of distance, acceleration is the second derivative of distance with respect to time:

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{d^2s}{dt^2}. \end{aligned}$$

## The phenomenon and force of one falling body

To begin to understand how the calculus can help us solve Newton's problem and let us move from phenomenon to force and back from force to phenomenon, we will first consider a simple example: a single falling body. Suppose we lift the body $A$ to the fixed point $B$ and drop it. Let the time $t$ be equal to 0 at the moment we release the body, and again let $s = AB$. As we saw in lab, the distance the body falls is as the square of the time, so that

$$s = kt^2,$$

where $k$ is some constant. This is a simple physical phenomenon: the distance of fall appears to depend on time in a certain way. We will first ask what the force must be that corresponds to this phenomenon, and then, once we have found this force, try to turn around and see if we can demonstrate the original phenomenon from this force.

**From phenomenon to force**

Following Newton, we should try to "discover the forces of nature from the phenomena." To do this, we use the second law, $F = ma$. Now acceleration $a$ is equal to the derivative of $v$ with respect to $t$, so it helps to find $v$ first. In fact

$$\begin{aligned} v &= \frac{ds}{dt} \\ &= \frac{d(kt^2)}{dt} \\ &= k\frac{d(t^2)}{dt} \qquad \text{(constant multiple rule)} \\ &= k\frac{2t\,dt}{dt} \qquad \text{(power rule)} \\ &= 2kt. \end{aligned}$$

We can use this expression for $v$ to find an expression for $a$.

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{d(2kt)}{dt} \\ &= \frac{2k\,dt}{dt} \qquad \text{(constant multiple rule)} \\ &= 2k. \end{aligned}$$

Therefore acceleration is constant, and

$$F = ma = 2km$$

is also constant. We have thus gone from a phenomenon of nature to a force by taking derivatives. Q. E. F.

**From force to phenomenon**

Conversely, suppose we are not given a phenomenon, but are given a force. In particular, suppose that we are given that the force acting on the body $B$ is constant and acts straight down. Then, since $F = ma$, and $m$ is constant, we know that the acceleration $a$ must also be equal to a constant $g$. This constant force is not a phenomenon: we cannot directly measure the force of a falling body. But we can ask what phenomena can be demonstrated from this constant force. In particular, we might ask where the body will be at a given time $t$. Here we begin with the differential equation

$$\frac{dv}{dt} = g.$$

We can multiply both sides by $dt$ to get $dv = g\,dt$. This equation expresses the fact that an infinitely small increment in velocity $v$ is equal to the rate of change of velocity $g$ times the infinitely small time $dt$. We then integrate (take sums of) both sides of this equation to get an equation for $v$:

$$\int dv = \int g\,dt.$$

The integral of the left side is $v$ by the first fundamental theorem: the sum of all the infinitely small increments of velocity is equal to the final velocity. (Note that to apply the fundamental theorem in this form we have used the assumption that $v = 0$ when $t = 0$: the body starts out with no velocity.) Therefore

$$\begin{aligned} v &= \int g\,dt \\ &= g\int dt \\ &= gt \end{aligned}$$

In the last step we are again using the first fundamental theorem: the sum of all the infinitely small increments of time $dt$ is equal to the final time $t$ (assuming again we start at time 0).

We have thus shown that if the force is constant, then the velocity is proportional to the time. But velocity is also hard to directly measure, so we have not yet gone all the way from the force to a phenomenon. To do this, we have to integrate our equation one more time:

$$\int v\,dt = \int gt\,dt.$$

The quantity $v\,dt$ is the product of the velocity $v$ and the infinitely small time $dt$. During the infinitely small time $dt$ we may assume the velocity $v$ is constant. When a body moves with constant speed over a time, the distance it moves is the product of its speed and the time in which it moves. In this case, therefore, the increment $ds$ of distance $s$ over the time $dt$ is equal to the product of the velocity $v$ and the time $dt$. Therefore

$$ds = v\,dt = gt\,dt.$$

Integrating gives

$$\int ds = \int v\,dt = \int gt\,dt.$$

But according to the first fundamental theorem, $\int ds = s$: the sum of the infinitely small increments of distance is equal to the total distance traveled.

Therefore

$$\begin{aligned} s &= \int gt\,dt \\ &= g\int t\,dt \\ &= \frac{1}{2}gt^2 \qquad\qquad \text{(by the power rule for integration).} \end{aligned}$$

We are finally back to the phenomenon we started with: distance is proportional to the square of time. We have thus moved from a force to a phenomenon by taking integrals. Q. E. D.

The above example of the falling body is relatively simple, and indeed Galileo treats it carefully without using calculus or Newtonian physics. But in this simple example we have a model for how to use calculus on more complicated problems. We may begin with a phenomenon, expressed in ordinary equations like $s = kt^2$. We then can take derivatives to come to an equation for force like $F = 2km$. Or, conversely, we may begin with an equation for force, like $F = mg$, where $g$ is constant. Then we integrate to get back to a phenomenon $s = kt^2$.

**Problems**

1. Suppose now the body $A$ does not begin at rest, but instead starts with an initial velocity of ten meters per second downward at the point $B$ at time $t = 0$. Again let $s$ be the distance $AB$. Use the calculus to show that

   (a) if the acceleration is equal to a constant $g = 9.8$ meters per second per second, then
   $$s = 10t + 4.9t^2, \text{ and}$$

   (b) if $s = 10t + 4.9t^2$, then the acceleration is equal to 9.8 meters per second per second and the velocity
   $$v = 10 + 9.8t.$$

2. Suppose that the body begins at a position 1 meter above the point $B$ with an upward velocity of 15 meters per second. Find an equation for $s$ in terms of $t$.

3. Suppose that the body begins at rest at the point $B$, as in the original example, but now suppose that the force is not constant. Instead, suppose that the body is acted on by a force causing a downward acceleration
   $$a = 2t + 1.$$
   Find an equation for $s$ in terms of $t$.

4. Suppose that the acceleration is given by the equation

$$a = \sin t.$$

   (a) If the body begins at the point $B$ at rest at time $t = 0$, find an equation for $s$ in terms of $t$.
   (b) If the body begins at the point $B$ with a downward velocity of 5 meters per second, find an equation for $s$ in terms of $t$.

5. Make and solve a problem involving a body moving along a line and acted on by a force.

## Projectile motion

All of our examples so far involve a single body moving along a straight line. Such problems are not very useful by themselves. But we can build on them to make more complicated and interesting problems. Suppose, for example, a body $A$ begins at a point $B$ with a horizontal velocity $v_0$.

If, as in the case of a falling body, the force is constant and acts in a downward direction, we can use the calculus to demonstrate the phenomenon in this new case, as follows.

We begin by drawing a horizontal line $BC$ and a vertical line $BD$. Then we drop perpendiculars $AC$ from $A$ to $BC$ and $AD$ from $A$ to $BD$. Let $BC = x$ and $BD = y$. Then the position of $A$ is determined by the values of $x$ and $y$. These quantities are functions of time: $x = f(t)$ and $y = h(t)$. To find out how the body moves we have to find both functions $f$ and $h$.

Velocity, like position, must now be expressed by two variables. For we cannot simply say how fast the body is going, but must say what direction it is going. Another way to express the velocity is by looking separately at how fast it is moving horizontally and how fast it is moving vertically. To get the horizontal velocity, we take the the derivative of $x$ with respect to $t$,

$$\frac{dx}{dt}$$

which we call $v_x$ or the $x$-component of the velocity. Since $dx$ expresses a change in $x$ and $dt$ an interval of time, $\frac{dx}{dt}$ expresses the change in $x$ divided by the time this change takes. We could call this quotient the *rate of change* of $x$. Since $dx$ and $dt$ are infinitely small, we can call it an *instantaneous rate of change*. In fact, this is true of any derivative with respect to time: for any variable u, the derivative $\frac{du}{dt}$ is the instantaneous rate of change of $u$. In this case, then, $v_x$, the horizontal velocity, is the instantaneous rate of change of $x$. To get the vertical velocity, we take the derivative of $y$ with respect to $t$,

$$\frac{dy}{dt},$$

which we call $v_y$ or the $y$-component of the velocity, and which is the instantaneous rate of change of $y$. If we know both $v_x$ and $v_y$, we can find the speed and

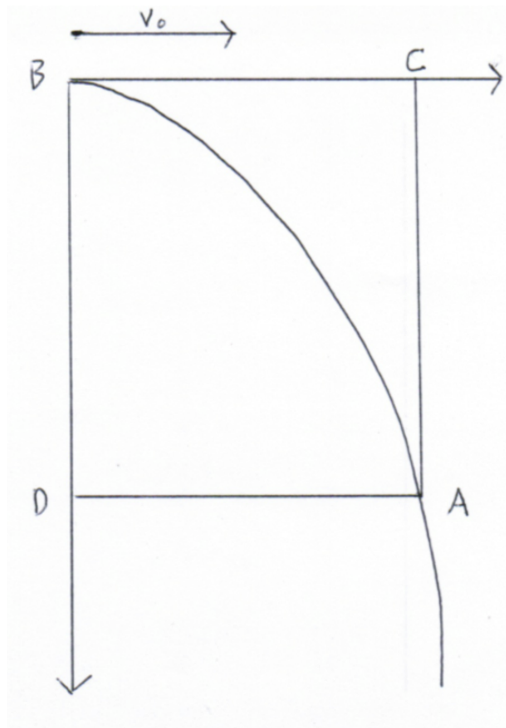



Figure 1

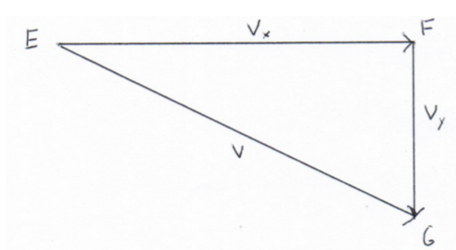



Figure 2

direction the body is moving. Let the body move a horizontal distance $EF$ per unit time and a vertical distance $FG$ per unit time, so that $EF$ is proportional to $v_x$ and $FG$ to $v_y$. Then the body must move the total distance $EG$ per unit time, so that the body moves in the direction of $EG$ and $EG$ is proportional to the speed of the body.

Acceleration, like velocity, must also be expressed by two variables. Let

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = f''(t),$$

and

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = h''(t).$$

Then the direction and magnitude of the acceleration of the body is given by $a_x$ and $a_y$.

The force of gravity acts only vertically, not horizontally, and therefore only changes the vertical component $v_y$ of the velocity of the body. If we assume that this force is constant, as for falling bodies, then it follows that the vertical component $a_y$ of the acceleration is equal to a constant $g$, and therefore

$$\frac{d^2y}{dt^2} = g.$$

We can integrate this equation twice, as for a falling body, to get

$$y = \frac{1}{2}gt^2.$$

To find the $x$ component, we note that the component $v_x$ of the velocity in the $x$-direction does not change, and therefore $v_x$ is always equal to $v_0$, the initial horizontal velocity. Therefore

$$\frac{dx}{dt} = v_0.$$

Integrating this equation gives

$$\int \frac{dx}{dt} dt = \int v_0 \, dt.$$

The integral on the left hand side is equal to

$$\int dx = x,$$

by the fundamental theorem (since $x = 0$ at time $t = 0$). The integral on the right-hand side is equal to $v_0 t$. Therefore

$$x = v_0 t.$$

We have thus found equations for $x$ and $y$ in terms of $t$ which describe the phenomenon of projectile motion.

To summarize, we have treated a case where the position of the body is determined by two variables by choosing coordinate axes and breaking down position, velocity, acceleration, and force each into their components with respect to these coordinate axes. Once we have done this, we can treat each dimension separately as a one-dimensional problem: in the vertical dimension, we have a falling body, while in the horizontal dimension we have a body moving with a fixed velocity with no forces at all acting on it. Both of these problems can be solved analytically by integrating the given equations.

**Problems**

1. Suppose that a body $A$ begins at a point $B$ with a horizontal velocity of 10 meters per second, and a vertical velocity downward of 5 meters per second. Assume the acceleration of gravity is 9.8 meters per second per second. Where will the body be after $t$ seconds?

2. Suppose a cannonball $A$ (Figure 3) is fired from a gun at point $B$ with a speed of 200 meters per second and at an angle of 30 degrees above horizontal. If we neglect air resistance, where will the cannonball be at time $t$? How far away will the cannonball land?

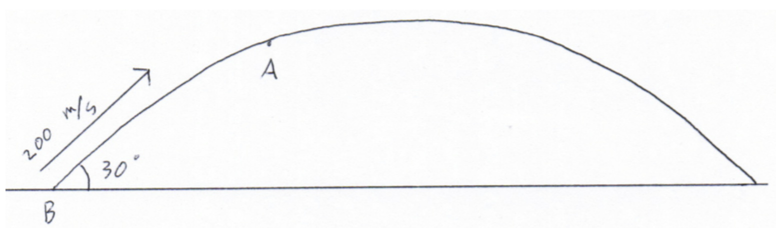



Figure 3

3. Suppose now that a body $A$ begins at the point $B$ with a horizontal velocity of 10 meters per second and no vertical velocity, but that a force acts on $A$ that has both a vertical and a horizontal component. Let the vertical component of force give a constant acceleration of 9.8 meters per second per second downward, and let the horizontal component of force give a constant acceleration of $-\sin t$. Where is the body at time $t$?

4. Make and solve a problem involving a body whose motion is determined by two variables.

# On the True Proportion, Expressed in Rational Numbers, of a Circle to a Circumscribed Square. Note 1, p. 191

by Gottfried Wilhelm Leibniz

Geometers have always tried to investigate the proportions of curvilinear figures to rectilinear figures, but even now, after applying algebra, they still have not succeeded—at least with the methods they have published: for these problems cannot be reduced to algebraic equations, but they have very beautiful uses, especially in reducing mechanics to terms of pure geometry. Those who have looked more deeply into such things know this; few have done so, but they are among the most outstanding mathematicians. Archimedes was the first, as far we know, to find the ratio between a cone, a sphere and a cylinder with the same height and base—this ratio is the ratio that the numbers 1, 2, and 3 have to each other, so that the cylinder is triple the cone and one and a half times the sphere—and this is why he had a sphere and a cylinder inscribed on his tomb.[1] He also found the quadrature of the parabola.[2] In our time a way

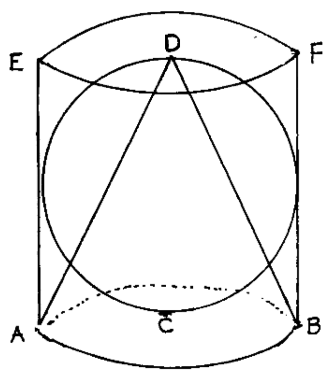



Figure 1: our figure, not Leibniz's

of measuring innumerable curvilinear figures has been discovered, in the first place when the ratio of the ordinates $BC$ (Figure 2) is obtained by multiplying or submultiplying, directly or reciprocally, the ratio of the abscissas $AB$ or $DC$; for the figure $ABCA$ will be to the circumscribed rectangle $ABCD$ as a unit is to the number expressing the multiplicity of the ratio, increased by a unit. For example, because in the parabola when the abscissas $AB$ or $DC$ are as the Note 2, p. 191
natural numbers, 1, 2, 3, etc., the ordinates $BC$ are as their squares, 1, 4, 9, Note 3, p. 192
etc., that is, they are in the duplicate ratio of the numbers, it follows that the number expressing the multiplicity of the ratio will be two; therefore the figure

[1]In *On the Sphere and the Cylinder.* See Figure 1.
[2]In *On the Quadrature of the Parabola.*

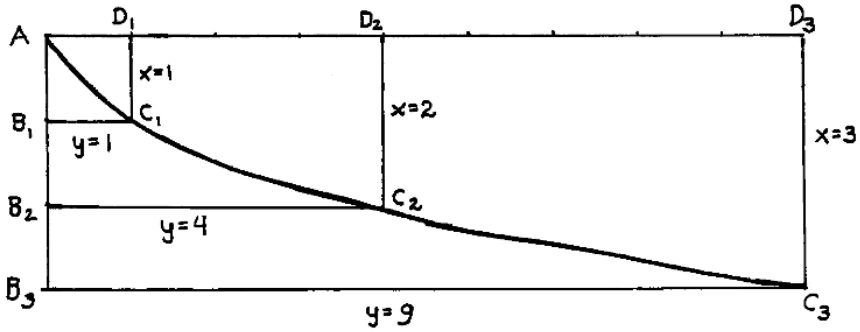



Figure 2: Leibniz's figure

$ABCA$ will be to the circumscribed rectangle $ABCD$ as 1 is to $2 + 1$, or as 1 is to 3; in other words the figure will be a third of the rectangle. If $AB$ or $CD$ are still natural numbers, and the $BC$'s become the cubes 1, 8, 27, etc. (in the cubic paraboloid), the ratio of the ordinates will be the triplicate ratio of the abscissas; therefore the figure will be to the rectangle as 1 is to $3 + 1$ or 4; in other words the figure will be one fourth of the rectangle. But if the $DC$'s are squares and the $BC$'s are cubes, that is, if the ratio of the $BC$'s is the triplicate ratio of the subduplicate ratio of the $DC$s, the figure (a cubico-subquadratic paraboloid) $ABCA$ will be to the rectangle $ABCD$ as 1 is to $\frac{3}{2} + 1$; in other words it will make up two fifths of the rectangle. In reciprocals we prefix the sign "$-$" (minus) to the number expressing the multiplicity of the ratio.

Note 4, p. 192

Note 5, p. 193

Note 6, p. 193

But until now no one has found a way to bring the circle under such laws, although geometers have tried for as long as anyone can remember. For no one has yet been able to find a number expressing the ratio of a circle $A$ to a circumscribed square $BC$ (the square on the diameter $DE$) (Figure 3). Nor has

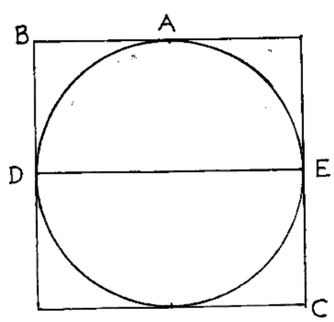



Figure 3: Leibniz's figure

anyone been able to find the ratio of the circumference to the diameter, a ratio that is is four times the ratio of the circle to the square. To be sure, Archimedes,[3] by inscribing and circumscribing polygons (since the circle is greater than the inscribed polygons and less than the circumscribed) showed a way to set out limits in which the circle falls or to find approximations; in fact he showed that the ratio of the circumference to the diameter is greater than 3 to 1 or than 21 to 7, and less than 22 to 7. Others have pursued this method: Ptolemy, Viète, Metius, and especially Ludolph van Ceulen, who showed the circumference is to the diameter as 3.14159265358979323846 to 1.00000000000000000000.

However, approximations of this sort, although useful in practical geometry will not satisfy a mind greedy for truth until we have found the *progression* of such indefinitely continuing numbers. To be sure, many have announced a completed tetragonism, such as Cardinal Cusa, Orontius Finaeus, Joseph Scaliger, Thomas Gephyrander, and Thomas Hobbes, but they were all wrong: for they were refuted by the calculations of Archimedes or those today of Ludolph.

But since I see that many have not fully understood what they are looking for, we should note that a tetragonism, that is, a conversion of a circle into an equal square or another equal rectilineal figure (a conversion which depends on the ratio of a circle to the square on its diameter, or of its circumference to its diameter) can be understood in four ways: Note 7, p. 193 either through a calculation or through a construction of lines, and each of these can be either accurate or nearly so. An accurate quadrature through calculation I call *analytic*; but one done through an accurate construction I call *geometric*; we call one done through a nearly accurate calculation an *approximation*, while one done through a nearly accurate construction is a *mechanism*. Ludolph produced a very long approximation; Viète, Huygens and others have given outstanding mechanisms.

We can obtain an *accurate geometric construction*, whereby we measure not only an entire circle, but also an arbitrary sector or arc, by a motion that is ordered and exact but follows transcendent curves. Note 8, p. 194 (Some have erroneously considered transcendent curves to be mechanical, although in fact they are as geometric as the common curves, even though they are not algebraic and cannot be reduced to equations that are algebraic or of a definite degree; for they have their own equations which, although they are not algebraic, are nevertheless analytic. But we cannot explain these things here in the way they deserve.) *Analytic quadrature*, that is, quadrature done through accurate calculation, can again be divided into three parts: into transcendent analytic quadrature, algebraic quadrature, and arithmetic quadrature. We get a *transcendent* analytic quadrature by means of, among other things, equations of indefinite degree, equations which no one has yet considered. For example, if $x^x + x$ is equal to 30, and we are looking for $x$, we will find that it is 3, because $3^3 + 3$ is $27 + 3$ or 30; I will give such equations for a circle in the proper place. An *algebraic* expression is one made using common numbers (possibly irrational) or roots of common equations: such an expression is in fact impossible for a general quadrature of a circle or sector. There remains *arithmetic quadrature*, Note 9, p. 196

[3]In *On the Measurement of the Circle*.

which is done using series, by exhibiting the exact value of the circle through a progression of terms, especially rational terms. This is the kind of quadrature I am presenting here.

Accordingly I found that (Figure 4), when the diameter of the circle is 1,

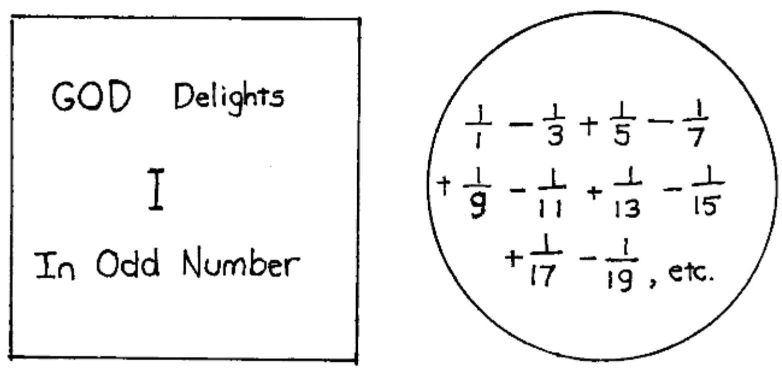



Figure 4: Leibniz's figure, including the text from Virgil's *Eclogues*, VIII 75

the area of the circle will be

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} \text{ etc.,}$$

Note 10, p. 197

namely the entire square of the diameter, after a third of it is taken away (so that the value does not become too large), and a fifth is added back (because we took too much away), and a seventh is again taken away (because we re-added too much), and so on; and in relation to the correct value

| | | | | |
|---|---|---|---|---|
| 1 | will be | greater, | yet with an error less than | $\frac{1}{3}$ |
| $\frac{1}{1} - \frac{1}{3}$ | ...... | less, | .......................... | $\frac{1}{5}$ |
| $\frac{1}{1} - \frac{1}{3} + \frac{1}{5}$ | ...... | greater, | .......................... | $\frac{1}{7}$ |
| $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ | ...... | less, | .......................... | $\frac{1}{9}$ |
| etc. | | | | etc. |

Therefore the whole series simultaneously contains all the approximations or values greater than or less than the correct one: for if it is continued far enough the error will be less than a given fraction, and thus also less than any given quantity. The whole series therefore expresses the exact value. And although we cannot express the sum of this series by one number, and the series may be produced indefinitely, nevertheless, because the series keeps to a single law of progression, we sufficiently perceive the whole with our minds. For since the circle is indeed incommensurable with the square, it cannot be expressed by one number, but it must be expressed in rationals through a series, just as we express the diagonal of a square, the section made by an extreme and mean ratio (which some call divine) and many other quantities that are irrational. And indeed, if Ludolph had been able to give a rule by which the numbers 3.14159 etc. might be continued indefinitely, he would have given us an exact arithmetic quadrature in integers, a quadrature which we are presenting in fractions.

However to prevent anyone who is little versed in these things from thinking that a series composed of infinitely many terms cannot be equal to a circle, which is a finite quantity, we should note that many series that are infinite with respect to their number of terms are finite quantities with respect to their sum. A very easy example is the series of the double geometric progression beginning from the unit and decreasing indefinitely,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64} \text{ etc.}$$

Even though it goes on indefinitely it nevertheless makes no more than 1. For let the attached straight line $AB$ (Figure 5) be 1. Then $AC$ will be $\frac{1}{2}$; bisecting

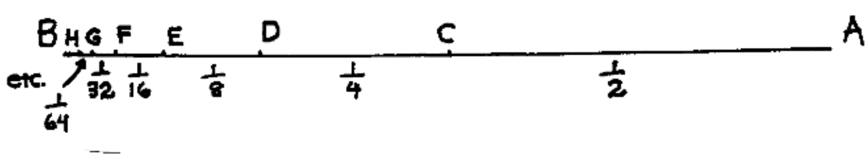



Figure 5: Leibniz's figure

the remainder $(CB)$ at $D$, you will have $CD=\frac{1}{4}$; bisecting the remainder $(DB)$ at $E$, you will have $DE=\frac{1}{8}$; bisecting the remainder $(EB)$ at $F$, you will have $EF=\frac{1}{16}$; and by continuing endlessly in this way you will never go beyond the boundary $B$. I showed elsewhere that the same thing happens with the fractions of figurate numbers or the harmonic triangle.[4]

We do not have time to say everything that could be said about this quadrature, but we should not fail to mention the fact that the terms of our series $\frac{1}{1}$, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, etc. belong to a harmonic progression, that is, are in continued harmonic proportion—this will be clear to anyone who tries it out; indeed, the series we get by skipping, Note 11, p. 201

$$\frac{1}{1}, \frac{1}{5}, \frac{1}{9}, \frac{1}{13}, \frac{1}{17} \text{ etc.}$$

is also the series of a harmonic progression, and

$$\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \frac{1}{19} \text{ etc.}$$

is likewise a series of harmonic proportionals. And so, since the circle is

$$\frac{1}{1}+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\frac{1}{17} \text{ etc.} -\frac{1}{3}-\frac{1}{7}-\frac{1}{11}-\frac{1}{15}-\frac{1}{19} \text{ etc.}$$

[4]In the paper "An Approach to the Arithmetic of Infinites," written in late 1672. See Leibniz's *Sämtliche Schriften und Briefe* (Collected Writings and Letters), in Series III, Volume 1, on pages 1–20.

(subtracting the latter partial series from the former) the magnitude of the circle will be the difference of two series of a harmonic progression. And because there are shortcuts to compute the sum of a harmonic progression with an arbitrary number of finite terms, it follows that we could calculate concise approximations (if there were any need of them after Ludolph).

If anyone should wish to remove from our series the terms affected by the sign $-$, then by adding together the two nearest terms, $+\frac{1}{1}$ and $-\frac{1}{3}$, and also $+\frac{1}{5}$ and $-\frac{1}{7}$, $+\frac{1}{9}$ and $-\frac{1}{11}$, $+\frac{1}{13}$ and $-\frac{1}{15}$, $+\frac{1}{17}$ and $-\frac{1}{19}$, and so on, he will get a new series for the magnitude of the circle, namely $\frac{2}{3}$ (that is $+\frac{1}{1}-\frac{1}{3}$) $+\frac{2}{35}$ (that is $\frac{1}{5}-\frac{1}{7}$) $+\frac{2}{99}$ (that is $\frac{1}{9}-\frac{1}{11}$), and so:

Note 12, p. 201 *If the area of the inscribed square is $\frac{1}{4}$, then the area of the circle will be $\frac{1}{3}+\frac{1}{35}+\frac{1}{99}+\frac{1}{195}+\frac{1}{323}$ etc.*

Now the numbers 3, 35, 99, 195, 323, etc. are taken by skipping terms in the series that results when the square numbers (4, 9, 16, 25, etc.) are all decreased by a unit—namely, in the series 3, 8, 15, 24, 35, 48, 63, 80, 99, 120, 143, 168, 195, 224, 255, 288, 323, 360, 399, etc. Every fourth number in this latter series belongs to our series. Now I have found (a remarkable fact) that the sum of the infinite series

$$\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{35}+\frac{1}{48}+\frac{1}{63}+\frac{1}{80}+\frac{1}{99}\text{ etc.}$$

Note 13, p. 202 is $\frac{3}{4}$; But if we take terms by a simple skipping, that is, if we take

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99}\text{ etc.,}$$

Note 14, p. 203 then the sum of this infinite series makes $\frac{2}{4}$ or $\frac{1}{2}$. And if we again take terms from the latter series by simple skipping, that is, if we take

$$\frac{1}{3}+\frac{1}{35}+\frac{1}{99}\text{ etc.,}$$

then the sum of this series will be a semicircle, if we suppose that the square of its diameter is 1. Moreover, since we grasp the *arithmetic quadrature of the hyperbola* in the same way, it seems fitting that the whole harmony be placed before our eyes in the following table:

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | | |
| 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 | 121 | 144 | 169 | 196 | 225 | 256 | | |
| 0 | 3 | 8 | 15 | 24 | 35 | 48 | 63 | 80 | 99 | 120 | 143 | 168 | 195 | 224 | 255 | | |
| | $\frac{1}{3}$ | $\frac{1}{8}$ | $\frac{1}{15}$ | $\frac{1}{24}$ | $\frac{1}{35}$ | $\frac{1}{48}$ | $\frac{1}{63}$ | $\frac{1}{80}$ | $\frac{1}{99}$ | $\frac{1}{120}$ | $\frac{1}{143}$ | $\frac{1}{168}$ | $\frac{1}{195}$ | $\frac{1}{224}$ | $\frac{1}{255}$ | etc. equals | $\frac{3}{4}$ |
| | $\frac{1}{3}$ | . | $\frac{1}{15}$ | . | $\frac{1}{35}$ | . | $\frac{1}{63}$ | . | $\frac{1}{99}$ | . | $\frac{1}{143}$ | . | $\frac{1}{195}$ | . | $\frac{1}{255}$ | etc. equals | $\frac{2}{4}$ |
| | . | $\frac{1}{8}$ | . | $\frac{1}{24}$ | . | $\frac{1}{48}$ | . | $\frac{1}{80}$ | . | $\frac{1}{120}$ | . | $\frac{1}{168}$ | . | $\frac{1}{224}$ | . | etc. equals | $\frac{1}{4}$ |
| | $\frac{1}{3}$ | . | . | . | $\frac{1}{35}$ | . | . | . | $\frac{1}{99}$ | . | . | . | $\frac{1}{195}$ | . | . | etc. equals | circle $ABCD$ |
| | . | $\frac{1}{8}$ | . | . | . | $\frac{1}{48}$ | . | . | . | $\frac{1}{120}$ | . | . | . | $\frac{1}{224}$ | . | etc. equals | figure $CBEHC$ |

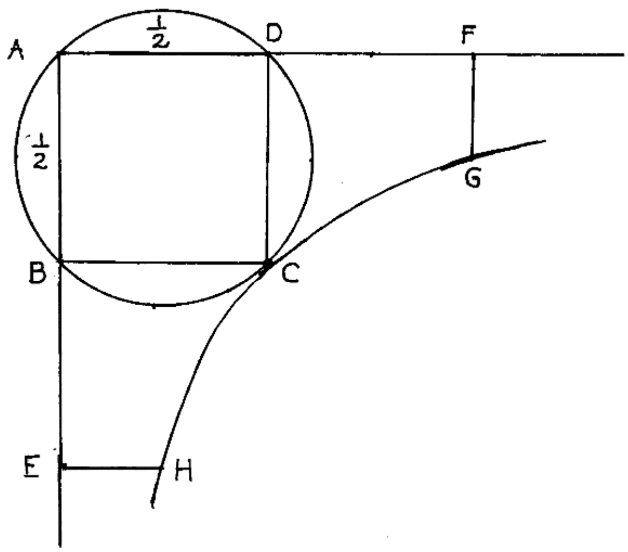



Figure 6: Leibniz's figure

In Figure 6 let a hyperbolic curve $GCH$ be described, having perpendicular asymptotes $AF$ and $AE$, and vertex $C$; let the inscribed power of the hyperbola (that is, the square that is always equal to the rectangle formed by any ordinate $EH$ on its own abscissa $AE$) be $ABCD$; let a circle be described about this square, and suppose that the hyperbola has been continued far enough, from $C$ to $H$, so that $AE$ is twice $AB$. Then if we suppose that $AE$ is a unit, $AB$ will be $\frac{1}{2}$, its square $ABCD$ will be $\frac{1}{4}$, and the circle whose inscribed power is $ABCD$ will be

Note 15, p. 204

$$\frac{1}{3} + \frac{1}{35} + \frac{1}{99} \text{ etc.,}$$

while the portion $CBEHC$ of the hyperbola (whose inscribed power is the same square, $\frac{1}{4}$), which represents the logarithm of the ratio of $AE$ to $AB$ (a two to one ratio), will be

$$\frac{1}{8} + \frac{1}{48} + \frac{1}{120} \text{ etc.}$$

Note 16, p. 204

# Notes on Leibniz's "On the True Proportion."

Leibniz published this paper in the *Acts of the Erudite* in February of 1682, more than two years before "A New Method." The paper is written in Latin, and we have translated it from Gerhardt's edition, Volume V, pages 118–122.

## Note 1

This is the first paper Leibniz published in the *Acts*; it appeared before "On Finding Measurements of Figures", "A New Method", and "On Recondite Geometry". In it, Leibniz gives a way to square a circle, that is, to find the area of a circle, but he does not give a demonstration. We have placed "On the True Proportion" here, after "On Recondite Geometry", because the methods of "On Recondite Geometry" allow us to demonstrate easily Leibniz's squaring of the circle in "On the True Proportion".

Leibniz almost entirely avoids the calculus, and even algebra, in this paper. He had already intensively developed the calculus, and the new theorem he announces here follows naturally from his work on the calculus, but he apparently thought it would be better to first announce his work by trying to use the language of classical geometry and arithmetic as far as possible. For us, who have studied algebra and the calculus, it will be easier at times to express what he says in terms of algebra, and the notes try to do that when it is appropriate.

## Note 2

In terms the calculus, this amounts to saying that, for any $n$,

$$\int x^n \, dx = \frac{x^{(n+1)}}{n+1},$$

a formula we already demonstrated (at least for positive $n$) in example 3 on page 139 of our notes on "On Recondite Geometry".

Leibniz expresses himself obscurely here because he avoids using algebra. We will translate what he has to say into algebra and the calculus. Let the abscissas $DC$ (Figure 7) equal $x$, and the ordinates $AD (= BC)$ equal $y$. Then to say that "the ratio of the ordinates $BC$ is obtained by multiplying or submultiplying, directly or reciprocally, the ratio of the abscissas $AB$ or $DC$" amounts to saying that there is an equation

$$y = x^n$$

for some $n$.

Now figure $ABCA$ is equal to

$$\int y \, dx = \int x^n \, dx,$$

and as we saw in example 3 on page 139 of our notes on "On Recondite Geometry",

$$\int x^n \, dx = \frac{x^{(n+1)}}{n+1}.$$

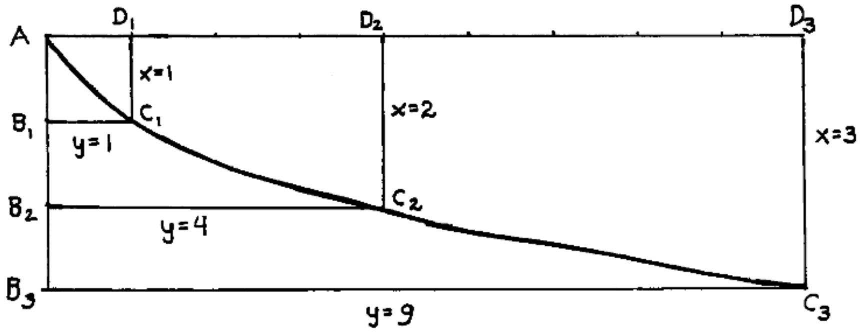



Figure 7

The circumscribed rectangle $ABCD$ is equal to

$$AB \times BC = xy = x\,x^n = x^{(n+1)}.$$

Therefore the ratio of figure $ABCA$ to the circumscribed rectangle $ABCD$ is

$$\begin{aligned}\frac{ABCA}{ABCD} &= \frac{\left(\frac{x^{(n+1)}}{n+1}\right)}{x^{(n+1)}} \\ &= \frac{1}{n+1};\end{aligned}$$

but $n$ is "the number expressing the multiplicity of the ratio" (that is, the multiplicity of the ratio of the abscissas with respect to the ratio of the ordinates), so that $n+1$ is "the number expressing the multiplicity of the ratio, increased by a unit," and therefore the ratio of figure $ABCA$ to rectangle $ABCD$ is the same as the ratio of the unit (1) to "the number expressing the multiplicity of the ratio, increased by a unit" $(n+1)$.

## Note 3

Leibniz here uses his terms differently from Apollonius. In Apollonius' *Conics* (Prop. I 20), the abscissas of a parabola are in the duplicate ratio of the ordinates, while for Leibniz the ordinates are in the duplicate ratio of the abscissas.

## Note 4

In terms of the calculus, if

$$y = x^2,$$

then

$$\text{figure } ABCA\!:\!\text{rectangle ABCD} :: \left(\int x^2\,dx\right)\!:\!x^3 :: 1\!:\!3.$$

## Note 5

In terms of the calculus, if

$$y = x^3,$$

then

$$\text{figure } ABCA : \text{rectangle ABCD} :: \left(\int x^3\, dx\right) : x^4 :: 1:4.$$

## Note 6

In terms of the calculus, if

$$y = x^{\frac{3}{2}},$$

then

$$\text{figure } ABCA : \text{rectangle ABCD} :: \left(\int x^{\frac{3}{2}}\, dx\right) : x^{\frac{5}{2}} :: 1 : \frac{5}{2} :: 2:5.$$

## Note 7

Leibniz says that we are looking for a conversion of a circle into an equal square. Let $ADE$ (Figure 8) be our given circle, and suppose we are looking for a square

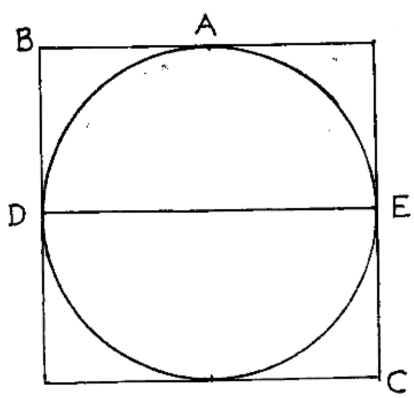



Figure 8

$FG$ equal to $ADE$. Suppose that the diameter $DE$ of the given circle is a unit. Let us denote by $\pi$ the ratio of the circumference of a circle to its diameter, so that in this case the circumference of $ADE$ is equal to $\pi$. (Here we are simply using the symbol $\pi$ as a shorthand for the ratio of the circumference of a circle to its radius, and are not assuming that we have found this ratio.) It follows from Archimedes' *Measurement of a Circle* (included as section 12) that the area of the circle $ADE$ is equal to

$$\frac{\pi}{4}.$$

Therefore the area of the square $FG$ must also be

$$\frac{\pi}{4}.$$

Therefore, if we can find $\pi$, the ratio of a circumference of a circle to its diameter, we can find $FG$ and square the circle.

## Note 8

The transcendent curve in question is the sine curve. As we saw in the notes (pages 145–148) to "On Recondite Geometry", if we are given a circle $ABE$ (see Figure 9), and we let arc $AB$ be equal to $a$ and $DB$ be equal to $y$, then the sine

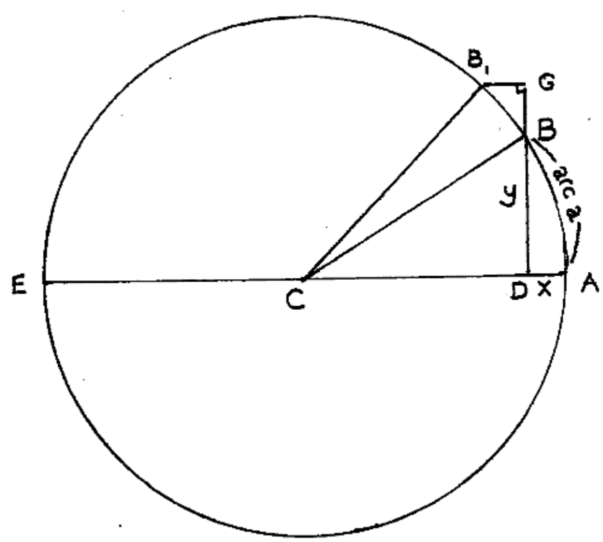



Figure 9

curve is the curve (see Figure 25, page 151) whose abscissas $FM$ are equal to $a$ and whose ordinates $LM$ are equal to $y$.

Given a point $B$ on the circle, we can use the sine curve to find the length of the arc $AB$ (in Figure 9), as follows. First, drop a perpendicular $BD$ from $B$ to $CA$. Choose a point $P$ on $KF$ so that

$$FP = BD.$$

Extend $PL$ parallel to $FH$ until it meets the sine curve at $L$. Finally, drop $LM$ perpendicular to $FH$. The line $FM = a$ will then be equal to arc $AB$, since $LM = FP$ is equal to the corresponding sine $DB = y$. The sine curve thus gives us an "accurate geometric construction" of an arbitrary *arc* of a circle. In particular, when $B$ moves all the way around the circle and back to $A$, the whole circumference of the circle becomes equal to $FH$.

To see how the sine curve also gives us a construction to measure an arbitrary *sector* of a circle, we need the following theorem, which relates arcs and sectors:

**Theorem** For any given arc $AB = a$ (see Figure 10), if the radius

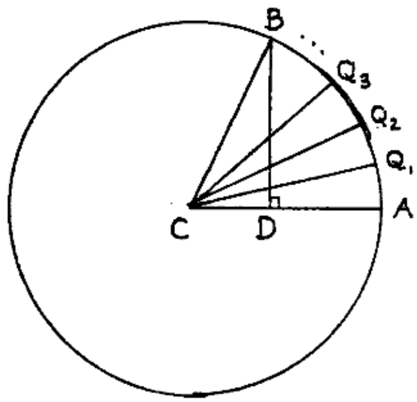



Figure 10

of the circle $CA = r$, then

$$\text{sector } BCA = \frac{1}{2}ra.$$

**Demonstration** We choose infinitely many points $Q_1$, $Q_2$, etc., along arc $AB$, then the whole area $AB$ is equal to

$$\text{sector } ACQ_1 + \text{sector } Q_1CQ_2 + \text{sector } Q_2CQ_3 + \text{etc.}$$

Now since the points $A$ and $Q_1$ are infinitely close to each other,

$$\begin{aligned} \text{sector } ACQ_1 &= \text{triangle } ACQ_1, \\ \text{sector } Q_1CQ_2 &= \text{triangle } Q_1CQ_2, \end{aligned}$$

etc., and the height of each of these triangles is equal to the radius of the circle. Therefore

$$\begin{aligned} \text{sector } ACQ_1 = \text{tri } ACQ_1 &= \frac{1}{2}(AC \times AQ_1) = \frac{1}{2}r(AQ_1) \\ \text{sector } Q_1CQ_2 = \text{tri } Q_1CQ_2 &= \frac{1}{2}(Q_1C \times Q_1Q_2) = \frac{1}{2}r(Q_1Q_2), \end{aligned}$$

etc. Therefore

$$\begin{aligned}
\text{sector ACB} &= \text{tri } ACQ_1 + \text{tri } Q_1CQ_2 + \text{tri } Q_2CQ_3 + \text{etc.} \\
&= \frac{1}{2}r(AQ_1) + \frac{1}{2}r(Q_1Q_2) + \frac{1}{2}r(Q_2Q_3) + \text{etc.} \\
&= \frac{1}{2}r(AQ_1 + Q_1Q_2 + Q_2Q_3 + \text{etc.}) \\
&= \frac{1}{2}r(\text{arc } AB) \\
&= \frac{1}{2}ra.
\end{aligned}$$

Q. E. D.

It follows from the theorem that once we have found a measurement $a$ for arc $AB$, we can find a measurement for the area of sector $ACB$ by multiplying $a$ by $\frac{1}{2}r$, that is, by taking the rectangle on $a$ and $\frac{1}{2}r$. Therefore, because the sine curve gives us an accurate geometric construction to measure $a$, it also gives us an accurate geometric construction to measure the sector $ACB$. In particular, the area of the whole circle is equal to the whole circumference ($FH$) times one half the radius.

## Note 9

Here is a reductio argument to show that a general algebraic quadrature of circle or sector is impossible. Suppose there were such a quadrature. Then there would be an algebraic equation relating the area

$$\frac{1}{2}ra$$

of an arbitrary sector $BCA$ of a circle $AB$ (see Figure 10) to the ordinate

$$BD$$

of the point $B$. Now

$$BD = \sin a,$$

and therefore an algebraic equation relating $BD$ and

$$\frac{1}{2}ra$$

would also give us an algebraic equation relating $a$ and $\sin a$, that is, an algebraic equation for the sine curve. But we showed above, in the notes to "On Recondite Geometry" (page 148), that the sine curve is transcendent, that is, that it has no algebraic equation. Therefore there is no general algebraic quadrature of a circle or sector, just as Leibniz says.

## Note 10

We will find the area of the whole circle by finding its circumference, and using the theorem in Note 8 (page 195). To find its circumference we will find a general expression relating the length of any arc to a corresponding tangent. See Figure 11. $ABD$ is a circle whose center is $C$ and whose diameter $DA$ is 2.

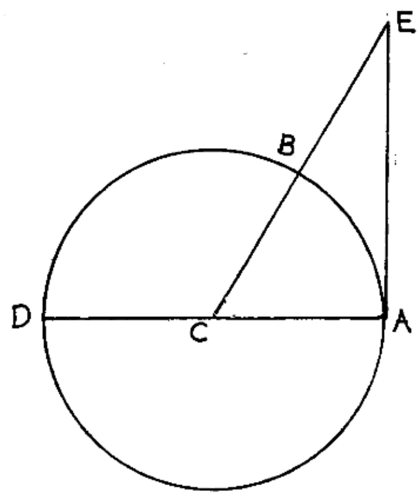



Figure 11

(We have doubled the diameter to make the calculations easier. The resulting circle is four times as large as the one Leibniz considers.) $CB$ is an arbitrary radius of the circle, and $AE$ is a tangent that meets $CB$ extended at $E$. Let $EA = z$ and

$$\text{arc } AB = \angle BCA = a.$$

Then

$$\begin{aligned} z &= EA \\ &= \frac{EA}{CA} \\ &= \tan a \end{aligned}$$

This is an expression for $z$ in terms of $a$. But we want to find an expression for $a$ in terms of $z$, so that we may find the area in terms of the tangent. We will do this in two steps:

1. we first find differences of both sides of this equation, giving us a differential equation, and solve this equation for $da$; and
2. we simplify this equation further until we get it into a form in which we can easily find sums and solve for $a$ by using the first fundamental theorem;

the solution will be an ordinary (not differential) equation for $a$, that is, one that does not explicitly involve differences or sums.

Here is the demonstration:

1. *Finding $da$ using our rules for the calculus:*

$$\begin{aligned}
dz &= d\tan a \\
&= d\left(\frac{\sin a}{\cos a}\right) \\
&= \frac{(\cos a)(d\sin a) - (\sin a)(d\cos a)}{\cos^2 a} && \text{(division rule)} \\
&= \frac{(\cos a)(\cos a\, da) - (\sin a)(-\sin a\, da)}{\cos^2 a} && \text{(Notes, page 148)} \\
&= \frac{(\cos^2 a + \sin^2 a)\, da}{\cos^2 a} \\
&= \frac{1}{\cos^2 a}\, da \\
&= \frac{1}{\left(\frac{CA}{CE}\right)^2}\, da \\
&= \left(\frac{CE}{CA}\right)^2 da \\
&= \frac{EA^2 + AC^2}{CA^2}\, da \\
&= \frac{1 + z^2}{1^2}\, da \\
&= (1 + z^2)\, da
\end{aligned}$$

Therefore

$$da = \frac{1}{1 + z^2}\, dz.$$

2. *Solving for $a$.* If we find sums of both sides of this differential equation and use the first fundamental theorem, we get an equation for $a$:

$$a = \int \frac{1}{1 + z^2}\, dz. \tag{1}$$

But this is still a differential equation (it involves $dz$) until we can evaluate the sum on the right. To do this we need to simplify it further. We use the following Lemma:

**Lemma:** For any positive quantity $u$, if $u < 1$, then

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + u^4 \text{ etc.}$$

**Demonstration:**

$$\begin{aligned}
1 &= 1 + u - u + u^2 - u^2 + u^3 - u^3 + u^4 \text{ etc.} \\
&= (1+u) - (u+u^2) + (u^2+u^3) - (u^3+u^4) \text{ etc.} \\
&= (1+u)(1) - (1+u)u + (1+u)u^2 - (1+u)u^3 \text{ etc.} \\
&= (1+u)(1 - u + u^2 - u^3 \text{ etc.})
\end{aligned}$$

Therefore

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 \text{ etc.}$$

Q.E.D.

(We assumed that $u < 1$ so that $1 - u + u^2 - u^3$ etc. would be a finite sum. If $u$ is equal to or greater than 1, then the sum is undefined.)

Now if we assume that arc $a$ is less than one eighth of the circumference of the circle, that is, that

$$a < \frac{\pi}{4},$$

then it follows that

$$AE < CA,$$

and therefore (since $AE = z$ and $CA = 1$)

$$z < 1,$$

and therefore

$$z^2 < 1.$$

We may therefore use the Lemma, substituting $z^2$ for $u$, to transform the sum on the right of equation 1 (page 198) into a sum we can easily evaluate:

$$\begin{aligned}
a &= \int\left(\frac{1}{1+z^2}\,dz\right)\\
&= \int\left((1-(z^2)+(z^2)^2-(z^2)^3+\text{ etc.})\,dz\right)\\
&= \int\left(1-z^2+z^4-z^6\text{ etc.}\,dz\right)\\
&= \int 1\,dz-\int z^2\,dz+\int z^4\,dz-\int z^6\,dz+\text{ etc.}\\
&= z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}\text{ etc.}
\end{aligned}$$

This equation holds for any value of $a$ less than $\frac{\pi}{4}$, that is, one eighth of the circumference of the circle; and as $a$ approaches $\frac{\pi}{4}$, $z$ approaches 1, and the equation ultimately becomes

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\text{ etc.}$$

Therefore, according to the theorem on page 195,

$$\begin{aligned}
\text{sector } ACB &= \frac{1}{2}ra\\
&= \frac{1}{2}(1)\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\text{ etc.}\right).
\end{aligned}$$

Therefore the total area of the circle is eight times the area of this sector, that is

$$8\left(\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\text{ etc.}\right)\right).$$

or

$$4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\text{ etc.}\right).$$

Therefore the area of a circle whose diameter is 1 instead of 2 will be one fourth of this area, namely,

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\text{ etc.}$$

This is Leibniz's formula.

## Note 11

Three quantities $A$, $B$, and $C$ are in continued harmonic proportion if

$$(A - B) : A :: (B - C) : C.$$

Equivalently (as one can see by doing some algebra), $A$, $B$, and $C$ are in continued harmonic proportion if

$$\frac{1}{A}, \frac{1}{B}, \text{ and } \frac{1}{C},$$

are in continued arithmetic progression, that is, if

$$\frac{1}{B} - \frac{1}{A} = \frac{1}{C} - \frac{1}{B}.$$

The simplest harmonic proportion is constituted by

$$A = 1,\ B = \frac{1}{2}, \text{ and } C = \frac{1}{3}.$$

For here

$$\frac{1}{A}, \frac{1}{B}, \text{ and } \frac{1}{C},$$

form the simplest possible arithmetic progression, namely 1, 2, 3.

These proportions are called *harmonic* because of their relations to simple musical intervals. If, for example, we have three strings of uniform material, thickness and tension whose lengths are in the harmonic proportion of

$$\frac{1}{2}, \frac{1}{3}, \text{ and } \frac{1}{4},$$

then the first two strings together will make a perfect fifth, the last two strings together will make a perfect fourth, and the first and third strings together will make an octave.

## Note 12

See Figure 12. There $ABCD$ is a circle with an inscribed square $ABCD$ whose area is $\frac{1}{4}$. Therefore $AD = \frac{1}{2}$ and

$$AC = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}}.$$

If $EF$ is a circle whose diameter $EF$ is a unit, then

$$\begin{aligned} \text{circle } ABCD : \text{circle } EF \quad &:: \quad AC^2 : EF^2 \\ &:: \quad \frac{1}{2} : 1 \end{aligned}$$

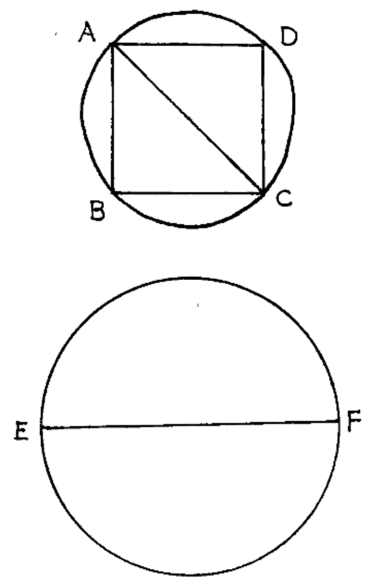



Figure 12

Since, as Leibniz has just shown,

$$\text{circle } EF = \frac{2}{3} + \frac{2}{35} + \frac{2}{99} + \text{etc.}$$

it follows that

$$\begin{aligned} \text{circle } ABCD &= \frac{1}{2} \text{ circle } EF \\ &= \frac{1}{3} + \frac{1}{35} + \frac{1}{99} \text{ etc.} \end{aligned}$$

## Note 13

To see why the sum

$$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \frac{1}{48} + \frac{1}{63} + \frac{1}{80} + \frac{1}{99} \text{ etc.}$$

is $\frac{3}{4}$ we note that

$$\begin{aligned}
\frac{1}{3} &= \frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right),\\
\frac{1}{8} &= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right),\\
\frac{1}{15} &= \frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right),\\
\frac{1}{24} &= \frac{1}{2}\left(\frac{1}{4}-\frac{1}{6}\right), \text{ etc.}
\end{aligned}$$

Therefore (summing all these equations),

$$\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24} \text{ etc. } = \frac{1}{2}\left(\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)\text{etc.}\right)$$

But all the terms inside the parentheses on the right side of this last equation cancel except $\frac{1}{1}$ and $\frac{1}{2}$. Therefore

$$\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24} \text{ etc. } = \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right)=\frac{3}{4}.$$

## Note 14

To see why the sum

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99} \text{ etc.},$$

is in fact $\frac{1}{2}$, we note that

$$\begin{aligned}
\frac{1}{3} &= \frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right),\\
\frac{1}{15} &= \frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right),\\
\frac{1}{35} &= \frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right),\\
\frac{1}{63} &= \frac{1}{2}\left(\frac{1}{7}-\frac{1}{9}\right), \text{ etc.}
\end{aligned}$$

Therefore (summing all these equations),

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{64} \text{ etc.} = \frac{1}{2}\left(\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)\text{etc.}\right)$$

But all the terms on the right side of this last equation cancel except $\frac{1}{1}$ (again, by the "principle of sums of differences"). Therefore

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63} \text{ etc. } = \frac{1}{2}\left(\frac{1}{1}\right)=\frac{1}{2}.$$

## Note 15

It follows from Proposition II 12 in Apollonius' *Conics* that the rectangle on $AE$ and $EH$ is equal to the square $ABCD$.

## Note 16

In Figure 13 we suppose at first that $BE$ is a variable quantity $x$ and let the

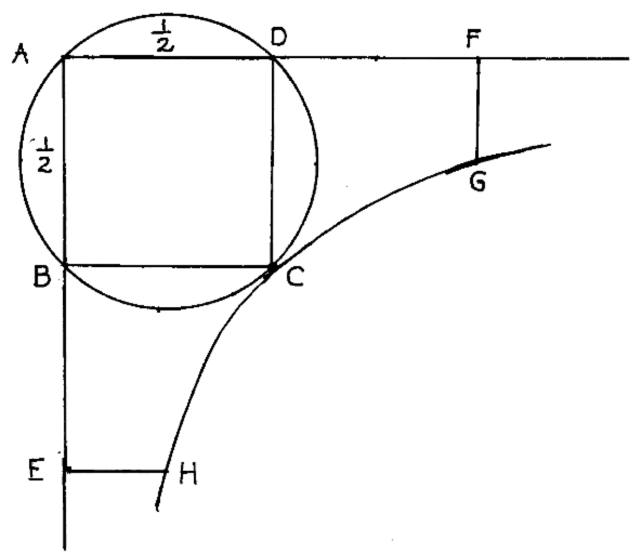



Figure 13

corresponding $EH$ be $y$. Then

$$\text{area } CBEH = \int y\,dx.$$

Now according to Proposition II 12 in Apollonius' *Conics*,

$$\text{rectangle } AE, EH = \text{square } ABCD = \frac{1}{4},$$

and therefore

$$(x + \frac{1}{2})y = \frac{1}{4}$$

and

$$\begin{aligned}
y &= \frac{1}{4(x+\frac{1}{2})} \\
&= \frac{1}{2+4x} \\
&= \frac{1}{2}\left(\frac{1}{1+2x}\right) \\
&= \frac{1}{2}\big(1-(2x)+(2x)^2-(2x)^3 \text{ etc.}\big) && \text{(Lemma, page 199)} \\
&= \frac{1}{2}-x+2x^2-4x^3+8x^4 \text{ etc.}
\end{aligned}$$

Therefore

$$\begin{aligned}
\text{area } CBEH &= \int y\,dx \\
&= \int\left((\frac{1}{2}-x+2x^2-4x^3+8x^4 \text{ etc.})\,dx\right) \\
&= \int\frac{1}{2}\,dx-\int x\,dx+\int 2x^2\,dx-\int 4x^3\,dx+\int 8x^4 \text{ etc.} \\
&= \frac{1}{2}x-\frac{x^2}{2}+2\frac{x^3}{3}-4\frac{x^4}{4}+8\frac{x^5}{5} \text{ etc.}
\end{aligned}$$

Finally, since Leibniz has constructed the figure so that $AE$ is twice $AB$, and $AB=\frac{1}{2}$, it follows that

$$BE = x = \frac{1}{2}.$$

We therefore substitute $\frac{1}{2}$ for $x$ in the above expression for area $CBEH$, and simplify by adding together the terms with opposite signs:

$$\begin{aligned}
\text{area } CBEH &= \frac{1}{2}\left(\frac{1}{2}\right)-\frac{\left(\frac{1}{4}\right)}{2}+2\frac{\left(\frac{1}{8}\right)}{3}-4\frac{\left(\frac{1}{16}\right)}{4}+8\frac{\left(\frac{1}{32}\right)}{5}-16\frac{\left(\frac{1}{64}\right)}{6} \text{ etc.} \\
&= \left(\frac{1}{4}-\frac{1}{8}\right)+\left(\frac{1}{12}-\frac{1}{16}\right)+\left(\frac{1}{20}-\frac{1}{24}\right) \text{ etc.} \\
&= \frac{1}{8}+\frac{1}{48}+\frac{1}{120} \text{ etc.}
\end{aligned}$$

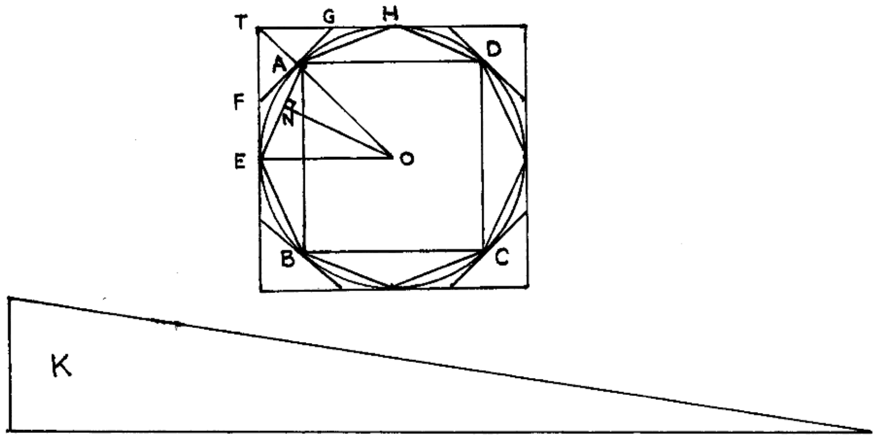
For the argument that the area $CBEHC$ "represents the logarithm of the ratio of $AE$ to $AB$," see above, pages 156–157.

# On the Measurement of the Circle[1]

by Archimedes

Proposition 1

***The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.***




Let $ABCD$ be the given circle, $K$ the triangle described.

Then, if the circle is not equal to $K$, it must be either greater or less.

I. If possible, let the circle be greater than $K$.

Inscribe a square $ABCD$, bisect the arcs $AB$, $BC$, $CD$, $DA$, then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the area of the circle over $K$.

Thus the area of the polygon is greater than $K$.

Let $AE$ be any side of it, and $ON$ the perpendicular on $AE$ from the centre $O$.

Then $ON$ is less than the radius of the circle and therefore less than one of the sides about the right angle in $K$. Also the perimeter of the polygon is less than the circumference of the circle, i.e. less than the other side about the right angle in $K$.

Therefore the area of the polygon is less than $K$; which is inconsistent with the hypothesis.

Thus the area of the circle is not greater than $K$.

II. If possible, let the circle be less than $K$.

[1]Trans. by T. L. Heath

Circumscribe a square, and let two adjacent sides, touching the circle in $E$, $H$, meet in $T$. Bisect the arcs between adjacent points of contact and draw the tangents at the points of bisection. Let $A$ be the middle point of the arc $EH$, and $FAG$ the tangent at $A$.

Then the angle $TAG$ is a right angle.

Therefore

$$\begin{aligned} TG &> GA \\ &> GH. \end{aligned}$$

It follows that the triangle $FTG$ is greater than half the area $TEAH$.

Similarly, if the arc $AH$ be bisected and the tangent at the point of bisection be drawn, it will cut off from the area $GAH$ more than one-half.

Thus, by continuing the process, we shall ultimately arrive at a circumscribed polygon such that the spaces intercepted between it and the circle are together less than the excess of $K$ over the area of the circle.

Thus the area of the polygon will be less than $K$.

Now, since the perpendicular from $O$ on any side of the polygon is equal to the radius of the circle, while the perimeter of the polygon is greater than the circumference of the circle, it follows that the area of the polygon is greater than the triangle $K$; which is impossible.

Therefore the area of the circle is not less than $K$.

Since then the area of the circle is neither greater nor less than $K$, it is equal to it.

# CONTINUITY AND IRRATIONAL NUMBERS[2]

by Richard Dedekind

My attention was first directed toward the considerations which form the subject of this pamphlet in the autumn of 1858. As professor in the Polytechnic School in Zürich I found myself for the first time obliged to lecture upon the elements of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic. In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. Even now such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis. The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given; even the most rigorous expositions of the differential calculus do not base their proofs upon continuity but, with more or less consciousness of the fact, they either appeal to geometric notions or those suggested by geometry, or depend upon theorems which are never established in a purely arithmetic manner. Among these, for example, belongs the above-mentioned theorem, and a more careful investigation convinced me that this theorem, or any one equivalent to it, can be regarded in some way as a sufficient basis for infinitesimal analysis. It then only remained to discover its true origin in the elements of arithmetic and thus at the same time to secure a real definition of the essence of continuity. I succeeded Nov. 24, 1858, and a few days afterward I communicated the results of my meditations to my dear friend Durège with whom I had a long and lively discussion. Later I explained these views of a scientific basis of arithmetic to a few of my pupils, and here in Braunschweig read a paper upon the subject before the scientific club of professors, but I could not make up my mind to its publication, because, in the first place, the presentation did not seem altogether simple, and further, the theory itself had little promise. Nevertheless I had already half determined to select this theme as subject for this occasion, when a few days ago, March 14, by the kindness of the author, the paper *Die Elemente der Funktionenlehre* by E. Heine (*Crelle's Journal*, Vol. 74) came into my hands and confirmed me in my decision. In the main I fully agree with the substance of this memoir, and

Note 1, p. 223

[2]1872. English translation by W. W. Beman (1901). Subsequent footnotes are in the original text.

indeed I could hardly do otherwise, but I will frankly acknowledge that my own presentation seems to me to be simpler in form and to bring out the vital point more clearly. While writing this preface (March 20, 1872), I am just in receipt of the interesting paper *Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen*, by G. Cantor (*Math. Annalen*, Vol. 5), for which I owe the ingenious author my hearty thanks. As I find on a hasty perusal, the axiom given in Section II. of that paper, aside from the form of presentation, agrees with what I designate in Section III. as the essence of continuity. But what advantage will be gained by even a purely abstract definition of real numbers of a higher type, I am as yet unable to see, conceiving as I do of the domain of real numbers as complete in itself.

## I. Properties of Rational Numbers

The development of the arithmetic of rational numbers is here presupposed, but still I think it worth while to call attention to certain important matters without discussion, so as to show at the outset the standpoint assumed in what follows. I regard the whole of arithmetic as a necessary, or at least natural, consequence of the simplest arithmetic act, that of counting, and counting itself as nothing else than the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding; the simplest act is the passing from an already-formed individual to the consecutive new one to be formed. The chain of these numbers forms in itself an exceedingly useful instrument for the human mind; it presents an inexhaustible wealth of remarkable laws obtained by the introduction of the four fundamental operations of arithmetic. Addition is the combination of any arbitrary repetitions of the above-mentioned simplest act into a single act; from it in a similar way arises multiplication. While the performance of these two operations is always possible, that of the inverse operations, subtraction and division, proves to be limited. Whatever the immediate occasion may have been, whatever comparisons or analogies with experience, or intuition, may have led thereto; it is certainly true that just this limitation in performing the indirect operations has in each case been the real motive for a new creative act; thus negative and fractional numbers have been created by the human mind; and in the system of all rational numbers there has been gained an instrument of infinitely greater perfection. This system, which I shall denote by $R$, possesses first of all a completeness and self-containedness which I have designated in another place[3] as characteristic of a *body of numbers* [Zahlkörper] and which consists in this that the four fundamental operations are always performable with any two individuals in $R$, i. e., the result is always an individual of $R$, the single case of division by the number zero being excepted.

For our immediate purpose, however, another property of the system $R$ is still more important; it may be expressed by saying that the system $R$ forms a well-arranged domain of one dimension extending to infinity on two opposite

[3] *Vorlesungen über Zahlentheorie*, by P. G. Lejeune Dirichlet. 2d ed. §159.

sides. What is meant by this is sufficiently indicated by my use of expressions borrowed from geometric ideas; but just for this reason it will be necessary to bring out clearly the corresponding purely arithmetic properties in order to avoid even the appearance as if arithmetic were in need of ideas foreign to it.

To express that the symbols $a$ and $b$ represent one and the same rational number we put $a = b$ as well as $b = a$. The fact that two rational numbers $a$, $b$ are different appears in this that the difference $a - b$ has either a positive or negative value. In the former case $a$ is said to be *greater* than $b$, $b$ *less* than $a$; this is also indicated by the symbols $a > b$, $b < a$.[4] As in the latter case $b - a$ has a positive value it follows that $b > a$, $a < b$. In regard to these two ways in which two numbers may differ the following laws will hold:

I. If $a > b$, and $b > c$, then $a > c$. Whenever $a$, $c$ are two different (or unequal) numbers, and $b$ is greater than the one and less than the other, we shall, without hesitation because of the suggestion of geometric ideas, express this briefly by saying: $b$ lies between the two numbers $a$, $c$.

II. If $a$, $c$ are two different numbers, there are infinitely many different numbers lying between $a$, $c$.

III. If $a$ is any definite number, then all numbers of the system $R$ fall into two classes, $A_1$ and $A_2$, each of which contains infinitely many individuals; the first class $A_1$ comprises all numbers $a_1$ that are $< a$, the second class $A_2$ comprises all numbers $a_2$ that are $> a$; the number $a$ itself may be assigned at pleasure to the first or second class, being respectively the greatest number of the first class or the least of the second. In every case the separation of the system $R$ into the two classes $A_1$, $A_2$ is such that every number of the first class $A_1$ is less than every number of the second class $A_2$.

## II. Comparison of the Rational Numbers with the Points of a Straight Line

The above-mentioned properties of rational numbers recall the corresponding relations of position of the points of a straight line $L$. If the two opposite directions existing upon it are distinguished by "right" and "left," and $p$, $q$ are two different points, then either $p$ lies to the right of $q$, and at the same time $q$ to the left of $p$, or conversely $q$ lies to the right of $p$ and at the same time $p$ to the left of $q$. A third case is impossible, if $p$, $q$ are actually different points. In regard to this difference in position the following laws hold:

I. If $p$ lies to the right of $q$, and $q$ to the right of $r$, then $p$ lies to the right of $r$; and we say that $q$ lies between the points $p$ and $r$.

II. If $p$, $r$ are two different points, then there always exist infinitely many points that lie between $p$ and $r$.

III. If $p$ is a definite point in $L$, then all points in $L$ fall into two classes, $P_1$, $P_2$, each of which contains infinitely many individuals; the first class $P_1$ contains all the points $p_1$, that lie to the left of $p$, and the second class $P_2$ contains all

[4]Hence in what follows the so-called "algebraic" greater and less are understood unless the word "absolute" is added.

the points $p_2$ that lie to the right of $p$; the point $p$ itself may be assigned at pleasure to the first or second class. In every case the separation of the straight line $L$ into the two classes or portions $P_1$, $P_2$, is of such a character that every point of the first class $P_1$ lies to the left of every point of the second class $P_2$.

This analogy between rational numbers and the points of a straight line, as is well known, becomes a real correspondence when we select upon the straight line a definite origin or zero-point $o$ and a definite unit of length for the measurement of segments. With the aid of the latter to every rational number $a$ a corresponding length can be constructed and if we lay this off upon the straight line to the right or left of $o$ according as $a$ is positive or negative, we obtain a definite end-point $p$, which may be regarded as the point corresponding to the number $a$; to the rational number zero corresponds the point $o$. In this way to every rational number $a$, i. e., to every individual in $R$, corresponds one and only one point $p$, i. e., an individual in $L$. To the two numbers $a$, $b$ respectively correspond the two points, $p$, $q$, and if $a > b$, then $p$ lies to the right of $q$. To the laws I, II, III of the previous Section correspond completely the laws I, II, III of the present.

## III. Continuity of the Straight Line

Of the greatest importance, however, is the fact that in the straight line $L$ there are infinitely many points which correspond to no rational number. If the point $p$ corresponds to the rational number $a$, then, as is well known, the length $op$ is commensurable with the invariable unit of measure used in the construction, i. e., there exists a third length, a so-called common measure, of which these two lengths are integral multiples. But the ancient Greeks already knew and had demonstrated that there are lengths incommensurable with a given unit of length, e. g., the diagonal of the square whose side is the unit of length. If we lay off such a length from the point $o$ upon the line we obtain an end-point which corresponds to no rational number. Since further it can be easily shown that there are infinitely many lengths which are incommensurable with the unit of length, we may affirm: The straight line $L$ is infinitely richer in point-individuals than the domain $R$ of rational numbers in number-individuals.

If now, as is our desire, we try to follow up arithmetically all phenomena in the straight line, the domain of rational numbers is insufficient and it becomes absolutely necessary that the instrument $R$ constructed by the creation of the rational numbers be essentially improved by the creation of new numbers such that the domain of numbers shall gain the same completeness, or as we may say at once, the same *continuity*, as the straight line.

The previous considerations are so familiar and well known to all that many will regard their repetition quite superfluous. Still I regarded this recapitulation as necessary to prepare properly for the main question. For, the way in which the irrational numbers are usually introduced is based directly upon the conception of extensive magnitudes—which itself is nowhere carefully defined—and explains number as the result of measuring such a magnitude by another of the same kind.[5] Instead of this I demand that arithmetic shall be developed out of itself.

That such comparisons with non-arithmetic notions have furnished the immediate occasion for the extension of the number-concept may, in a general way, be granted (though this was certainly not the case in the introduction of complex numbers); but this surely is no sufficient ground for introducing these foreign notions into arithmetic, the science of numbers. Just as negative and fractional rational numbers are formed by a new creation, and as the laws of operating with these numbers must and can be reduced to the laws of operating with positive integers, so we must endeavor completely to define irrational numbers by means of the rational numbers alone. The question only remains how to do this.

The above comparison of the domain $R$ of rational numbers with a straight line has led to the recognition of the existence of gaps, of a certain incompleteness or discontinuity of the former, while we ascribe to the straight line completeness, absence of gaps, or continuity. In what then does this continuity consist? Everything must depend on the answer to this question, and only through it shall we obtain a scientific basis for the investigation of *all* continuous domains. By vague remarks upon the unbroken connection in the smallest parts obviously nothing is gained; the problem is to indicate a precise characteristic of continuity that can serve as the basis for valid deductions. For a long time I pondered over this in vain, but finally I found what I was seeking. This discovery will, perhaps, be differently estimated by different people; the majority may find its substance very commonplace. It consists of the following. In the preceding section attention was called to the fact that every point $p$ of the straight line produces a separation of the same into two portions such that every point of one portion lies to the left of every point of the other. I find the essence of continuity in the converse, i. e., in the following principle:

"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."

As already said I think I shall not err in assuming that every one will at once grant the truth of this statement; the majority of my readers will be very much disappointed in learning that by this commonplace remark the secret of continuity is to be revealed. To this I may say that I am glad if every one finds the above principle so obvious and so in harmony with his own ideas of a line; for I am utterly unable to adduce any proof of its correctness, nor has any one the power. The assumption of this property of the line is nothing else than an axiom by which we attribute to the line its continuity, by which we find continuity in the line. If space has at all a real existence it is *not* necessary for it to be continuous; many of its properties would remain the same even were it discontinuous. And if we knew for certain that space was discontinuous there

[5] The apparent advantage of the generality of this definition of number disappears as soon as we consider complex numbers. According to my view, on the other hand, the notion of the ratio between two magnitudes of the same kind can be clearly developed only after the introduction of irrational numbers.

would be nothing to prevent us, in case we so desired, from filling up its gaps, in thought, and thus making it continuous; this filling up would consist in a creation of new point-individuals and would have to be effected in accordance with the above principle.

## IV. Creation of Irrational Numbers

From the last remarks it is sufficiently obvious how the discontinuous domain $R$ of rational numbers may be rendered complete so as to form a continuous domain. In Section I it was pointed out that every rational number $a$ effects a separation of the system $R$ into two classes such that every number $a_1$ of the first class $A_1$ is less than every number $a_2$ of the second class $A_2$; the number $a$ is either the greatest number of the class $A_1$ or the least number of the class $A_2$. If now any separation of the system $R$ into two classes $A_1$, $A_2$ is given which possesses only *this* characteristic property that every number $a_1$ in $A_1$ is less than every number $a_2$ in $A_2$, then for brevity we shall call such a separation a *cut* [Schnitt] and designate it by $(A_1, A_2)$. We can then say that every rational number $a$ produces one cut or, strictly speaking, two cuts, which, however, we shall not look upon as essentially different; this cut possesses, *besides*, the property that either among the numbers of the first class there exists a greatest or among the numbers of the second class a least number. And conversely, if a cut possesses this property, then it is produced by this greatest or least rational number.

But it is easy to show that there exist infinitely many cuts not produced by rational numbers. The following example suggests itself most readily.

Let $D$ be a positive integer but not the square of an integer, then there exists a positive integer $\lambda$ such that

$$\lambda^2 < D < (\lambda + 1)^2.$$

If we assign to the second class $A_2$, every positive rational number $a_2$ whose square is $> D$, to the first class $A_1$ all other rational numbers $a_1$, this separation forms a cut $(A_1, A_2)$, i. e., every number $a_1$ is less than every number $a_2$. For if $a_1 = 0$, or is negative, then on that ground $a_1$ is less than any number $a_2$, because, by definition, this last is positive; if $a_1$ is positive, then is its square $\leqq D$, and hence $a_1$ is less than any positive number $a_2$ whose square is $> D$.

But this cut is produced by no rational number. To demonstrate this it must be shown first of all that there exists no rational number whose square $= D$. Although this is known from the first elements of the theory of numbers, still the following indirect proof may find place here. If there exist a rational number whose square $= D$, then there exist two positive integers $t$, $u$, that satisfy the equation

Note 2, p. 228

$$t^2 - Du^2 = 0,$$

and we may assume that $u$ is the *least* positive integer possessing the property that its square, by multiplication by $D$, may be converted into the square of an integer $t$. Since evidently

$$\lambda u < t < (\lambda + 1)u,$$

the number $u' = t - \lambda u$ is a positive integer certainly *less* than $u$. If further we put

$$t' = Du - \lambda t,$$

$t'$ is likewise a positive integer, and we have

$$t'^2 - Du'^2 = (\lambda^2 - D)(t^2 - Du^2) = 0,$$

which is contrary to the assumption respecting $u$.

Hence the square of every rational number $x$ is either $< D$ or $> D$. From this it easily follows that there is neither in the class $A_1$ a greatest, nor in the class $A_2$ a least number. For if we put

$$y = \frac{x(x^2 + 3D)}{3x^2 + D},$$

we have

$$y - x = \frac{2x(D - x^2)}{3x^2 + D}$$

and

$$y^2 - D = \frac{(x^2 - D)^3}{(3x^2 + D)^2}.$$

If in this we assume $x$ to be a positive number from the class $A_1$, then $x^2 < D$, and hence $y > x$ and $y^2 < D$. Therefore $y$ likewise belongs to the class $A_1$. But if we assume $x$ to be a number from the class $A_2$, then $x^2 > D$, and hence $y < x$, $y > 0$, and $y^2 > D$. Therefore $y$ likewise belongs to the class $A_2$. This cut is therefore produced by no rational number.

In this property that not all cuts are produced by rational numbers consists the incompleteness or discontinuity of the domain $R$ of all rational numbers.

Whenever, then, we have to do with a cut $(A_1, A_2)$ produced by no rational number, we create a new, an *irrational* number $\alpha$, which we regard as completely defined by this cut $(A_1, A_2)$; we shall say that the number $\alpha$ corresponds to this cut, or that it produces this cut. From now on, therefore, to every definite cut there corresponds a definite rational or irrational number, and we regard two numbers as *different* or *unequal* always and only when they correspond to essentially different cuts.

In order to obtain a basis for the orderly arrangement of all *real*, i. e., of all rational and irrational numbers we must investigate the relation between any two cuts $(A_1, A_2)$ and $(B_1, B_2)$ produced by any two numbers $\alpha$ and $\beta$. Obviously a cut $(A_1, A_2)$ is given completely when one of the two classes, e. g., the first $A_1$ is known, because the second $A_2$ consists of all rational numbers not contained in $A_1$, and the characteristic property of such a first class lies in this that if the number $a_1$ is contained in it, it also contains all numbers less than $a_1$. If now we compare two such first classes $A_1$, $B_1$ with each other, it may happen

1. That they are perfectly identical, i. e., that every number contained in $A_1$ is also contained in $B_1$, and that every number contained in $B_1$ is also contained in $A_1$. In this case $A_2$ is necessarily identical with $B_2$, and the two cuts are perfectly identical, which we denote in symbols by $\alpha = \beta$ or $\beta = \alpha$.

But if the two classes $A_1$, $B_1$ are not identical, then there exists in the one, e. g., in $A_1$, a number $a'_1 = b'_2$ not contained in the other $B_1$ and consequently found in $B_2$; hence all numbers $b_1$ contained in $B_1$ are certainly less than this number $a'_1 = b'_2$ and therefore all numbers $b_1$ are contained in $A_1$.

2. If now this number $a'_1$ is the only one in $A_1$ that is not contained in $B_1$, then is every other number $a_1$ contained in $A_1$ also contained in $B_1$ and is consequently $< a'_1$, i. e., $a'_1$ is the greatest among all the numbers $a_1$, hence the cut $(A_1, A_2)$ is produced by the rational number $a = a'_1 = b'_2$. Concerning the other cut $(B_1, B_2)$ we know already that all numbers $b_1$ in $B_1$ are also contained in $A_1$ and are less than the number $a'_1 = b'_2$ which is contained in $B_2$; every other number $b_2$ contained in $B_2$ must, however, be greater than $b'_2$, for otherwise it would be less than $a'_1$, therefore contained in $A_1$ and hence in $B_1$; hence $b'_2$ is the least among all numbers contained in $B_2$, and consequently the cut $(B_1, B_2)$ is produced by the same rational number $\beta = b'_2 = a'_1 = \alpha$. The two cuts are then only unessentially different.

3. If, however, there exist in $A_1$ at least two different numbers $a'_1 = b'_2$ and $a''_1 = b''_2$, which are not contained in $B_1$, then there exist infinitely many of them, because all the infinitely many numbers lying between $a'_1$ and $a''_1$ are obviously contained in $A_1$ (Section I, II) but not in $B_1$. In this case we say that the numbers $\alpha$ and $\beta$ corresponding to these two essentially different cuts $(A_1, A_2)$ and $(B_1, B_2)$ are *different*, and further that $\alpha$ is *greater* than $\beta$, that $\beta$ is *less* than $\alpha$, which we express in symbols by $\alpha > \beta$ as well as $\beta < \alpha$. It is to be noticed that this definition coincides completely with the one given earlier, when $\alpha$, $\beta$ are rational.

The remaining possible cases are these:

4. If there exists in $B_1$ one and only one number $b'_1 = a'_2$, that is not contained in $A_1$ then the two cuts $(A_1, A_2)$ and $(B_1, B_2)$ are only unessentially different and they are produced by one and the same rational number $\alpha = a'_2 = b'_1 = \beta$.

5. But if there are in $B_1$ at least two numbers which are not contained in $A_1$, then $\beta > \alpha$, $\alpha < \beta$.

As this exhausts the possible cases, it follows that of two different numbers one is necessarily the greater, the other the less, which gives two possibilities. A third case is impossible. This was indeed involved in the use of the *comparative* (greater, less) to designate the relation between $\alpha$, $\beta$; but this use has only now been justified. In just such investigations one needs to exercise the greatest care so that even with the best intention to be honest he shall not, through a hasty choice of expressions borrowed from other notions already developed, allow himself to be led into the use of inadmissible transfers from one domain to the other.

If now we consider again somewhat carefully the case $\alpha > \beta$ it is obvious that the less number $\beta$, if rational, certainly belongs to the class $A_1$; for since there is in $A_1$ a number $a'_1 = b'_2$ which belongs to the class $B_2$, it follows that the number $\beta$, whether the greatest number in $B_1$ or the least in $B_2$ is certainly $\leqq a'_1$ and hence contained in $A_1$. Likewise it is obvious from $\alpha > \beta$ that the greater number $\alpha$, if rational, certainly belongs to the class $B_2$, because $\alpha \geqq a'_1$. Combining these two considerations we get the following result: If a cut is produced by the number $\alpha$ then any rational number belongs to the class $A_1$ or to the class $A_2$ according as it is less or greater than $\alpha$; if the number $\alpha$ is itself rational it may belong to either class.

From this we obtain finally the following: If $\alpha > \beta$, i. e., if there are infinitely many numbers in $A_1$ not contained in $B_1$ then there are infinitely many such numbers that at the same time are different from $\alpha$ and from $\beta$; every such rational number $c$ is $< \alpha$, because it is contained in $A_1$ and at the same time it is $> \beta$ because contained in $B_2$.

## V. Continuity of the Domain of Real Numbers

In consequence of the distinctions just established the system $\mathfrak{R}$ of all real numbers forms a well-arranged domain of one dimension; this is to mean merely that the following laws prevail: Note 3, p. 230

I. If $\alpha > \beta$, and $\beta > \gamma$, then is also $\alpha > \gamma$. We shall say that the number $\beta$ lies between $\alpha$ and $\gamma$.

II. If $\alpha$, $\gamma$ are any two different numbers, then there exist infinitely many different numbers $\beta$ lying between $\alpha$, $\gamma$.

III. If $\alpha$ is any definite number then all numbers of the system $\mathfrak{R}$ fall into two classes $\mathfrak{A}_1$ and $\mathfrak{A}_2$ each of which contains infinitely many individuals; the first class $\mathfrak{A}_1$ comprises all the numbers $\alpha_1$ that are less than $\alpha$, the second $\mathfrak{A}_2$ comprises all the numbers $\alpha_2$ that are greater than $\alpha$; the number $\alpha$ itself may be assigned at pleasure to the first class or to the second, and it is respectively the greatest of the first or the least of the second class. In each case the separation of the system $\mathfrak{R}$ into the two classes $\mathfrak{A}_1$, $\mathfrak{A}_2$ is such that every number of the first class $\mathfrak{A}_1$ is smaller than every number of the second class $\mathfrak{A}_2$ and we say that this separation is produced by the number $\alpha$.

For brevity and in order not to weary the reader I suppress the proofs of these theorems which follow immediately from the definitions of the previous section.

Beside these properties, however, the domain $\mathfrak{R}$ possesses also *continuity*; i. e., the following theorem is true:

IV. If the system $\mathfrak{R}$ of all real numbers breaks up into two classes $\mathfrak{A}_1$, $\mathfrak{A}_2$ such that every number $\alpha_1$ of the class $\mathfrak{A}_1$ is less than every number $\alpha_2$ of the class $\mathfrak{A}_2$ then there exists one and only one number $\alpha$ by which this separation is produced.

*Proof.* By the separation or the cut of $\mathfrak{R}$ into $\mathfrak{A}_1$ and $\mathfrak{A}_2$ we obtain at the same time a cut $(A_1, A_2)$ of the system $R$ of all rational numbers which is defined by this that $A_1$ contains all rational numbers of the class $\mathfrak{A}_1$ and $A_2$ all other rational numbers, i. e., all rational numbers of the class $\mathfrak{A}_2$. Let $\alpha$ be the perfectly definite number which produces this cut $(A_1, A_2)$. If $\beta$ is any number different from $\alpha$, there are always infinitely many rational numbers $c$ lying between $\alpha$ and $\beta$. If $\beta < \alpha$, then $c < \alpha$; hence $c$ belongs to the class $A_1$ and consequently also to the class $\mathfrak{A}_1$, and since at the same time $\beta < c$ then $\beta$ also belongs to the same class $\mathfrak{A}_1$, because every number in $\mathfrak{A}_2$ is greater than every number $c$ in $\mathfrak{A}_1$. But if $\beta > \alpha$, then is $c > \alpha$; hence $c$ belongs to the class $A_2$ and consequently also to the class $\mathfrak{A}_2$, and since at the same time $\beta > c$, then $\beta$ also belongs to the same class $\mathfrak{A}_2$, because every number in $\mathfrak{A}_1$ is less than every number $c$ in $\mathfrak{A}_2$. Hence every number $\beta$ different from $\alpha$ belongs to the class $\mathfrak{A}_1$ or to the class $\mathfrak{A}_2$ according as $\beta < \alpha$ or $\beta > \alpha$; consequently $\alpha$ itself is either the greatest number in $\mathfrak{A}_1$ or the least number in $\mathfrak{A}_2$, i. e., $\alpha$ is one and obviously the only number by which the separation of $\mathfrak{R}$ into the classes $\mathfrak{A}_1$, $\mathfrak{A}_2$ is produced. Which was to be proved.

## VI. Operations with Real Numbers

To reduce any operation with two real numbers $\alpha$, $\beta$ to operations with rational numbers, it is only necessary from the cuts $(A_1, A_2)$, $(B_1, B_2)$ produced by the numbers $\alpha$ and $\beta$ in the system $R$ to define the cut $(C_1, C_2)$ which is to correspond to the result of the operation, $\gamma$. I confine myself here to the discussion of the simplest case, that of addition.

If $c$ is any rational number, we put it into the class $C_1$, provided there are two numbers one $a_1$ in $A_1$ and one $b_1$ in $B_1$ such that their sum $a_1 + b_1 \geqq c$; all other rational numbers shall be put into the class $C_2$. This separation of all rational numbers into the two classes $C_1$, $C_2$ evidently forms a cut, since every number $c_1$ in $C_1$ is less than every number $c_2$ in $C_2$. If both $\alpha$ and $\beta$ are rational, then every number $c_1$ contained in $C_1$ is $\leqq \alpha + \beta$, because $a_1 \leqq \alpha$, $b_1 \leqq \beta$, and therefore $a_1 + b_1 \leqq \alpha + \beta$; further, if there were contained in $C_2$ a number $c_2 < \alpha + \beta$, hence $\alpha + \beta = c_2 + p$, where $p$ is a positive rational number, then we should have

$$c_2 = (\alpha - \frac{1}{2}p) + (\beta - \frac{1}{2}p),$$

which contradicts the definition of the number $c_2$, because $\alpha - \frac{1}{2}p$ is a number in $A_1$, and $\beta - \frac{1}{2}p$ a number in $B_1$; consequently every number $c_2$ contained in $C_2$ is $\geqq \alpha + \beta$. Therefore in this case the cut $(C_1, C_2)$ is produced by the sum $\alpha + \beta$. Thus we shall not violate the definition which holds in the arithmetic of rational numbers if in all cases we understand by the sum $\alpha + \beta$ of any two real numbers $\alpha$, $\beta$ that number $\gamma$ by which the cut $(C_1, C_2)$ is produced. Further, if only one of the two numbers $\alpha$, $\beta$ is rational, e. g., $\alpha$, it is easy to see that it makes no difference with the sum $\gamma = \alpha + \beta$ whether the number $\alpha$ is put into the class $A_1$ or into the class $A_2$.

Just as addition is defined, so can the other operations of the so-called elementary arithmetic be defined, viz., the formation of differences, products, quotients, powers, roots, logarithms, and in this way we arrive at real proofs of theorems (as, e. g., $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$), which to the best of my knowledge have never been established before. The excessive length that is to be feared in the definitions of the more complicated operations is partly inherent in the nature of the subject but can for the most part be avoided. Very useful in this connection is the notion of an *interval*, i. e., a system $A$ of rational numbers possessing the following characteristic property: if $a$ and $a'$ are numbers of the system $A$, then are all rational numbers lying between $a$ and $a'$ contained in $A$. The system $R$ of all rational numbers, and also the two classes of any cut are intervals. If there exist a rational number $a_1$ which is less and a rational number $a_2$ which is greater than every number of the interval $A$, then $A$ is called a finite interval; there then exist infinitely many numbers in the same condition as $a_1$ and infinitely many in the same condition as $a_2$; the whole domain $R$ breaks up into three parts $A_1$, $A$, $A_2$ and there enter two perfectly definite rational or irrational numbers $\alpha_1$, $\alpha_2$ which may be called respectively the lower and upper (or the less and greater) *limits* of the interval; the lower limit $\alpha_1$ is determined by the cut for which the system $A_1$ forms the first class and the upper $\alpha_2$ by the cut for which the system $A_2$ forms the second class. Of every rational or irrational number $\alpha$ lying between $\alpha_1$ and $\alpha_2$ it may be said that it lies *within* the interval $A$. If all numbers of an interval $A$ are also numbers of an interval $B$, then $A$ is called a portion of $B$.

Still lengthier considerations seem to loom up when we attempt to adapt the numerous theorems of the arithmetic of rational numbers (as, e. g., the theorem $(a + b)c = ac + bc$) to any real numbers. This, however, is not the case. It is easy to see that it all reduces to showing that the arithmetic operations possess a certain continuity. What I mean by this statement may be expressed in the form of a general theorem:

"If the number $\lambda$ is the result of an operation performed on the numbers $\alpha$, $\beta$, $\gamma$, … and $\lambda$ lies within the interval $L$, then intervals $A$, $B$, $C$, … can be taken within which lie the numbers $\alpha$, $\beta$, $\gamma$, … such that the result of the same operation in which the numbers $\alpha$, $\beta$, $\gamma$, … are replaced by arbitrary numbers of the intervals $A$, $B$, $C$, … is always a number lying within the interval $L$." The forbidding clumsiness, however, which marks the statement of such a theorem convinces us that something must be brought in as an aid to expression; this is, in fact, attained in the most satisfactory way by introducing the ideas of *variable magnitudes*, *functions*, *limiting values*, and it would be best to base the definitions of even the simplest arithmetic operations upon these ideas, a matter which, however, cannot be carried further here.

## VII. Infinitesimal Analysis

Here at the close we ought to explain the connection between the preceding investigations and certain fundamental theorems of infinitesimal analysis.

We say that a variable magnitude $x$ which passes through successive definite numerical values approaches a fixed limiting value $\alpha$ when in the course of the process $x$ lies finally between two numbers between which $\alpha$ itself lies, or, what amounts to the same, when the difference $x - \alpha$ taken absolutely becomes finally less than any given value different from zero.

One of the most important theorems may be stated in the following manner: "If a magnitude $x$ grows continually but not beyond all limits it approaches a limiting value."

I prove it in the following way. By hypothesis there exists one and hence there exist infinitely many numbers $\alpha_2$ such that $x$ remains continually $< \alpha_2$; I designate by $\mathfrak{A}_2$ the system of all these numbers $\alpha_2$, by $\mathfrak{A}_1$ the system of all other numbers $\alpha_1$; each of the latter possesses the property that in the course of the process $x$ becomes finally $\geqq \alpha_1$, hence every number $\alpha_1$ is less than every number $\alpha_2$ and consequently there exists a number $\alpha$ which is either the greatest in $\mathfrak{A}_1$ or the least in $\mathfrak{A}_2$ (V, IV). The former cannot be the case since $x$ never ceases to grow, hence $\alpha$ is the least number in $\mathfrak{A}_2$. Whatever number $\alpha_1$ be taken we shall have finally $\alpha_1 < x < \alpha$, i. e., $x$ approaches the limiting value $\alpha$.

This theorem is equivalent to the principle of continuity, i. e., it loses its validity as soon as we assume a single real number not to be contained in the domain $\mathfrak{R}$; or otherwise expressed: if this theorem is correct, then is also theorem IV. in V. correct.

Another theorem of infinitesimal analysis, likewise equivalent to this, which is still oftener employed, may be stated as follows: "If in the variation of a magnitude $x$ we can for every given positive magnitude $\delta$ assign a corresponding position from and after which $x$ changes by less than $\delta$ then $x$ approaches a limiting value."

This converse of the easily demonstrated theorem that every variable magnitude which approaches a limiting value finally changes by less than any given positive magnitude can be derived as well from the preceding theorem as directly from the principle of continuity. I take the latter course. Let $\delta$ be any positive magnitude (i. e., $\delta > 0$), then by hypothesis a time will come after which $x$ will change by less than $\delta$, i. e., if at this time $x$ has the value $a$, then afterwards we shall continually have $x > a-\delta$ and $x < a+\delta$. I now for a moment lay aside the original hypothesis and make use only of the theorem just demonstrated that all later values of the variable $x$ lie between two assignable finite values. Upon this I base a double separation of all real numbers. To the system $\mathfrak{A}_2$ I assign a number $\alpha_2$ (e.g., $a+\delta$) when in the course of the process $x$ becomes finally $\leqq \alpha_2$; to the system $\mathfrak{A}_1$ I assign every number not contained in $\mathfrak{A}_2$; if $\alpha_1$ is such a number, then, however far the process may have advanced, it will still happen infinitely many times that $x > \alpha_1$. Since every number $\alpha_1$ is less than every number $\alpha_2$ there exists a perfectly definite number $\alpha$ which produces this cut $(\mathfrak{A}_1, \mathfrak{A}_2)$ of the system $\mathfrak{R}$ and which I will call the upper limit of the variable $x$ which always remains finite. Likewise as a result of the behavior of the variable $x$ a second cut $(\mathfrak{B}_1, \mathfrak{B}_2)$ of the system $\mathfrak{R}$ is produced; a number $\beta_1$ (e. g., $a-\delta$) is assigned to $\mathfrak{B}_1$ when in the course of the process $x$ becomes finally $\geqq \beta_1$; every other number $\beta_2$, to be assigned to $\mathfrak{B}_2$, has the property that $x$ is never finally $\geqq \beta_2$; therefore infinitely many times $x$ becomes $< \beta_2$; the number $\beta$ by which this cut is produced I call the lower limiting value of the variable $x$. The two numbers $\alpha$, $\beta$ are obviously characterised by the following property: if $\epsilon$ is an arbitrarily small positive magnitude then we have always finally $x < \alpha + \epsilon$ and $x > \beta - \epsilon$, but never finally $x < \alpha - \epsilon$ and never finally $x > \beta + \epsilon$. Now two cases are possible. If $\alpha$ and $\beta$ are different from each other, then necessarily $\alpha > \beta$, since continually $\alpha_2 \geqq \beta_1$; the variable $x$ oscillates, and, however far the process advances, always undergoes changes whose amount surpasses the value $(\alpha - \beta) - 2\epsilon$ where $\epsilon$ is an arbitrarily small positive magnitude. The original hypothesis to which I now return contradicts this consequence; there remains only the second case $\alpha = \beta$ since it has already been shown that, however small be the positive magnitude $\epsilon$, we always have finally $x < \alpha + \epsilon$ and $x > \beta - \epsilon$, $x$ approaches the limiting value $\alpha$, which was to be proved.

These examples may suffice to bring out the connection between the principle of continuity and infinitesimal analysis.

# Notes on Dedekind's "Continuity and Irrational Numbers"

## Note 1

Dedekind is concerned with finding a scientific foundation for the differential calculus. According to him, previous accounts of the foundations of the differential calculus all depend on geometry, and this makes them unscientific. He would like to remedy this defect by providing a purely arithmetic foundation. He asserts that a certain theorem "can be regarded in some way as a sufficient basis for infinitesimal analysis,"[1] namely the theorem that

> every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value.

He will demonstrate this theorem in a purely arithmetic way in the final section of "Continuity and Irrational Numbers." But he never explains why he thinks this theorem is a sufficient basis for infinitesimal analysis or the differential calculus. We will try to sketch here why one might think this. We will first discuss the notion of a "limiting value," and then show how derivatives can be seen as based on limits.

### Limits

When Leibniz first speaks of how the calculus is used ("A New Method," page 34) he writes that

> to find a *tangent* is to draw a straight line joining two points on a curve that are an infinitely small distance apart, or to draw the side of a polygon with infinitely many angles (which is for us equivalent to the *curve*).

In a paper published later that year[2] Leibniz elaborates:

> I think that this method and all the other methods [for finding measurements of figures] that have been used thus far can be deduced from a certain general principle of mine for measuring curvilinear areas, namely, *that a curvilinear figure should be considered as equivalent to a polygon with infinitely many sides*; it follows that whatever can be demonstrated about such a Polygon—whether in such a way that we pay no attention to the number of sides, or in such a way that it is made more and more true the greater number of sides we take, so that the error finally becomes less than any given number—can be declared to be true about the curve.

[1] "Infinitesimal calculus" seems to be a synonym for "differential calculus" for Dedekind.

[2] "An Addendum to the Paper on Finding Measurements of Figures," published in December of 1684 in the *Acts*. See page 126 of Volume V of Gerhardt's Edition of Leibniz's mathematical works.

Here Leibniz is thinking of the polygon not so much as actually having an infinite number of sides, but as having an indefinitely large number of sides, a number that can always be increased. For example, if the curvilinear figure is a circle, we could begin by inscribing a triangle, which has an area much less than that of the circle. Next, we could inscribe a square, which has a larger area, but still less than that of the circle. We could go on to inscribe a pentagon, a hexagon, and so on. The magnitude of the inscribed polygon would grow continually. Moreover, as we increase the number of sides the polygon's area would differ from the circle's by less than any given number; that is, we could find a polygon that is as close as we would like to being equal to a circle. For Leibniz, a circle is thus equivalent to a polygon with infinitely many sides.

Note that for Dedekind, the polygon with infinitely many sides inscribed in a circle is a "magnitude which grows continually." But it does not go "beyond all limits," since all the inscribed polygons are less than the circle. Therefore, according to the theorem Dedekind aims to prove, the polygon with infinitely many sides "must certainly approach a limiting value."

If a polygon has infinitely many sides, all these sides must be infinitely small. Leibniz is thus inviting us here to think of infinitely small differences like $dx$ as *indefinitely* small, and his differential equations as approximate equations between indefinitely small quantities that become "more and more true" the smaller the quantities become, "so that the error finally becomes less than any given number." If we think about the calculus in this way, we thus avoid, to some extent, the paradoxes of the infinite and the infinitely small.

Later mathematicians call a curvilinear area a *limit* of polygons. For while the curvilinear area is not itself a polygon, we can find polygons whose areas are as close as we please to the curvilinear area. In general, suppose that a quantity $y$ depends on a quantity $x$, so that $y$ is a function of $x$, that is,

$$y = f(x),$$

for some function $f$. Then we say that the *limit* of $f(x)$ as $x$ approaches infinity is equal to $L$ if, as $x$ increases, the difference between $f(x)$ and $L$ becomes less than any given quantity. We denote this by

$$\lim_{x \to \infty} f(x) = L.$$

The limit of $f(x)$ as $x$ goes to infinity is $L$ if, for any given quantity $\epsilon$, there is a quantity $N$ such that when $x > N$, the difference between $f(x)$ and $L$ is less than the given quantity $\epsilon$. For example, if $f(x) = \dfrac{1}{x}$, then

$$\lim_{x \to \infty} f(x) = 0.$$

For if $\epsilon$ is any given quantity, and if $N = \dfrac{1}{\epsilon}$, then when $x$ is greater than $N$,

$$f(x) = \frac{1}{x} < \frac{1}{\left(\frac{1}{\epsilon}\right)} = \epsilon,$$

and therefore the difference between $f(x)$ and 0 is less than the arbitrary given quantity $\epsilon$.

In Leibniz's example, $f(x)$ could denote the area of a polygon inscribed within a curvilinear figure, $x$ the number of its sides, and $L$ the area of the curvilinear figure. For for any given $\epsilon$, there is a number $N$ such that any polygon with more than $N$ sides will always differ from the curvilinear area by less than the given quantity $\epsilon$.

We can also speak of the limit of $f(x)$ as $x$ approaches a finite value. We say that the limit of $f(x)$ as $x$ approaches some finite value $a$ is equal to $L$ if, as $x$ approaches $a$, the difference between $f(x)$ and $L$ becomes less than any given quantity. In other words, if $\epsilon$ is any given quantity, then there is a quantity $\delta$ such that when the difference between $x$ and $a$ is less than $\delta$, then the difference between $f(x)$ and $L$ is less than $\epsilon$. We denote this by

$$\lim_{x \to a} f(x) = L.$$

For example, if $f(x) = x^2$, then

$$\lim_{x \to 0} f(x) = 0.$$

For if $\epsilon$ is any given quantity, and if $\delta = \sqrt{\epsilon}$, then if $x < \delta$,

$$f(x) = x^2 < (\sqrt{\epsilon})^2 = \epsilon,$$

and therefore the difference between $f(x)$ and 0 is less than the arbitrary given quantity $\epsilon$.

### Limits and derivatives

If we are going to found the calculus on Dedekind's theorem, we first need to see how we can think of differences or derivatives in terms of it. To do this, we express a ratio of differences

$$\frac{dv}{dx},$$

that is the derivative of $v$ with respect to $x$, as a "limiting value" of a magnitude that "grows continually."

First, following Leibniz's suggestion, we take the tangent to a curve as a line connecting two indefinitely close points, so that the tangent will be a limit of lines that cut that curve at two points as those two points come together. See Figure 1. There, as $V_1$ approaches $V$, the cutting line (or *secant*) $PVV_1$ approaches the tangent $BVQ$. If we let $VX = v$, $AX = x$, $VR = \Delta x$, and $V_1R = \Delta v$, and suppose now that $\Delta x$ and $\Delta v$ are indefinitely small variable quantities, then the slope of the cutting line $PVV_1$ is equal to

$$\frac{\Delta v}{\Delta x},$$

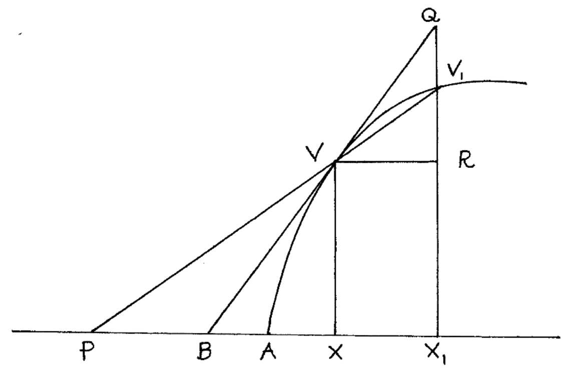



Figure 1

while the slope of the tangent line is

$$\frac{dv}{dx}.$$

The slope of the secant $PVV_1$ continually grows, and as $\Delta x$ approaches 0, the slope of the cutting line approaches the slope of the tangent line; that is, the limit of the slope of the cutting line as $\Delta x$ approaches 0 is equal to the slope of the tangent line, that is,

$$\frac{dv}{dx} = \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}.$$

One could say that, in Leibniz's words, the approximate equation

$$\frac{dv}{dx} \approx \frac{\Delta v}{\Delta x}$$

becomes "more and more true" the smaller we take $\Delta x$, and "the error finally becomes less than any given number," and therefore that if we take $\Delta x$ as infinitely small the equation "can be declared to be true."

We have used "geometric notions" to see that the slope of the tangent line $BVQ$ is the limit of the slope of the cutting lines $PVV_1$. But for Dedekind this is not sufficient. For him, only an arithmetic demonstration can be truly rigorous and scientific. To get a purely arithmetic foundation for the differential calculus, we need to translate our diagram into arithmetic terms. Therefore we have to assume we are given a function $f(x)$ arithmetically, for example, by an algebraic equation. To translate our diagram into arithmetic terms, let $VX = f(x)$. Then, since $AX_1 = x + \Delta x$, $V_1X_1 = f(x + \Delta x)$ and

$$\Delta v = V_1X_1 - VX = f(x + \Delta x) - f(x).$$

Then

$$\begin{aligned} f'(x) &= \frac{dv}{dx} \\ &= \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \end{aligned}$$

This last equation may be considered the *definition* of the derivative in functional notation. It would be meaningful even if $f(x)$ is not defined geometrically.

We could use this equation to demonstrate the basic rules of the differential calculus. For example, if $f(x) = x^2$, then, according to this definition,

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x^2 + 2x\Delta x + (\Delta x)^2) - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2x + \Delta x) \\ &= 2x. \end{aligned}$$

We have thus demonstrated the power rule for the exponent 2 purely arithmetically:

$$\frac{d(x^2)}{dx} = 2x.$$

We could likewise demonstrate all the other rules.

In this example it is clear that the limit exists. But in general, it is not at all clear that for an arithmetically defined function $f(x)$,

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

will approach a limiting value as $\Delta x$ approaches 0. If we want a foundation of the calculus that works for as many functions as possible, we therefore have to be able to distinguish those functions for which derivatives exist from those that do not. Dedekind's theorem provides a powerful tool to do this. It says that whenever

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

increases continually, but does not go beyond all limits, as $\Delta x$ approaches zero, then the derivative

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

always exists.

## Note 2

If there *were* a rational number whose square = $D$, then it would consist of positive integers $t$ and $u$, $u$ being least, such that

$$\frac{t^2}{u^2} = D. \tag{1}$$

Simple algebraic manipulation gives us the equivalent forms in Dedekind:[3]

$$t^2 = Du^2 \quad \text{and} \quad t^2 - Du^2 = 0.^* \tag{2}$$

On the previous page (p. 214) Dedekind has already shown that if $D$ is a positive integer but not the square of an integer, then there exists a positive integer $\lambda$ such that

$$\lambda^2 < D < (\lambda + 1)^2.^* \tag{3}$$

We can substitute for $D$ from (1) like so,

$$\lambda^2 < \frac{t^2}{u^2} < (\lambda + 1)^2. \tag{4}$$

Taking the square root to simplify yields

$$\lambda < \frac{t}{u} < (\lambda + 1). \tag{5}$$

And multiplying through by $u$ gives us

$$\lambda u < t < (\lambda + 1)u.^* \tag{6}$$

We can also multiply $u$ through the last term of the inequality to get

$$\lambda u < t < \lambda u + u. \tag{7}$$

And by subtracting $\lambda u$ from all terms, we get

$$0 < t - \lambda u < u. \tag{8}$$

Since $u' = t - \lambda u$,* it follows from (8) that $u'$ is a positive integer less than $u$.

Going back to (5), we now multiply through by $t$ instead of $u$:

$$\lambda t < \frac{t^2}{u} < \lambda t + t. \tag{9}$$

[3]Steps explicit in Dedekind are asterisked.

Then we multiply the middle term by $u/u$, yielding

$$\lambda t < \frac{t^2 u}{u^2} < \lambda t + t. \tag{10}$$

Since $D = t^2/u^2$, we can substitute $D$ in the middle term, getting

$$\lambda t < Du < \lambda t + t. \tag{11}$$

We then subtract $\lambda t$ from all terms, getting

$$0 < Du - \lambda t < t. \tag{12}$$

Letting $t' = Du - \lambda t$*, we can see from (12) that $t'$ is likewise a positive integer, as Dedekind says.

We now show that $t'$ and $u' < u$ also satisfy equation (2) above if $t$ and $u$ do, contrary to our assumption about $u$. Replacing $t$ and $u$ in (2) with $t'$ and $u'$, we get:

$$\begin{align}
t'^2 - Du'^2 &= (Du - \lambda t)^2 - D(t - \lambda u)^2 \tag{13}\\
&= D^2u^2 - 2Du\lambda t + \lambda^2 t^2 - Dt^2 + 2Du\lambda t - D\lambda^2 u^2 \tag{14}\\
&= \lambda^2 t^2 - Dt^2 - D\lambda^2 u^2 + D^2 u^2 \tag{15}\\
&= (\lambda^2 - D)(t^2 - Du^2).^* \tag{16}
\end{align}$$

From (16), we can see that $t'^2 - Du'^2 = 0$ if $t^2 - Du^2 = 0$. But then $t'$ and $u'$ satisfy this equation even though $u' < u$, contrary to our assumption about $u$, and thus about any positive integers $t$ and $u$ able to satisfy equation (2). Hence the square of every rational number $x$ is either $< D$ or $> D$.

Now consider:

$$y = \frac{x(x^2 + 3D)}{3x^2 + D}.^* \tag{17}$$

If we subtract $x$ from both sides of (17) like so,

$$y - x = \frac{x(x^2 + 3D)}{3x^2 + D} - \frac{x(3x^2 + D)}{3x^2 + D}, \tag{18}$$

we can get to Dedekind's second form of the equation:

$$y - x = \frac{2x(D - x^2)}{3x^2 + D}.^* \tag{19}$$

And by squaring both sides of (17) and subtracting D like so,

$$y^2 - D = \frac{[x(x^2 + 3D)]^2}{(3x^2 + D)^2} - \frac{D(3x^2 + D)^2}{(3x^2 + D)^2}, \tag{20}$$

we can get to Dedekind's third form of the equation:

$$y^2 - D = \frac{(x^2 - D)^3}{(3x^2 + D)^2}.^* \tag{21}$$

For every positive rational $x$, the equation (17) will return a positive rational $y$ ($y > 0$). And in its second form (19), the equation clarifies by sign that this positive rational $y$ is greater than $x$ for every positive rational $x$ in the class $A_1$ (where $x^2 < D$), and less than $x$ for every positive rational $x$ in the class $A_2$ (where $x^2 > D$). And in its third form (21), the equation clarifies again by sign that this $y$ belongs to $A_1$ ($y^2 < D$) if $x$ does ($x^2 < D$), and to $A_2$ ($y^2 > D$) if $x$ does ($x^2 > D$).

The following note by Michael Comenetz provides a way to understand the origin of the expression for $y$ in (17).

**The Origin of Dedekind's Expression for the Number $y$ (p. 215)**

The problem is this: given positive rational numbers $x$ and $D$, $D$ not the square of a rational number, to find a rational number $y$ between $x$ and $\sqrt{D}$. Let $s = x - \sqrt{D}$, $t = y - \sqrt{D}$; then $t$ is to have the same sign as $s$ and to be less than $s$ in absolute value. What quantity has the same sign as $s$? An odd power of $s$, the simplest (other than $s$ itself) being $s^3$. This may not be less than $s$ in absolute value; perhaps then $t = us^3$, for some $u > 0$, will do. That is

$$\begin{aligned} y - \sqrt{D} = u(x - \sqrt{D})^3 &= u(x^3 - 3x^2\sqrt{D} + 3xD - D\sqrt{D}) \\ &= u(x^3 + 3xD) - u(3x^2 + D)\sqrt{D}, \end{aligned}$$

which suggests $u = \frac{1}{3x^2+D}$ (to make the coefficient of $\sqrt{D}$ on the right equal to that on the left) and therefore $y = \frac{x^3+3xD}{3x^2+D}$. This $y$ fulfills the conditions of the problem.

## Note 3

"the system $\mathfrak{R}$ of all real numbers ..."

Dedekind consistently uses letters of the Fraktur typeface to denote classes of real numbers, and roman letters to denote classes of rational numbers. $\mathfrak{R}$ is a Fraktur $R$, $\mathfrak{A}$ is a Fraktur $A$, and $\mathfrak{B}$ is a Fraktur $B$.

# Cantor's Transfinite Set Theory Informally Introduced[1]

1. What is a set?

   Here are some examples of sets: the collection of all chairs now in the Great Hall; the collection of all chairs now in the Great Hall and in the Santa Fe campus cafeteria; the collection of all things now in your closet; the collection of all prime numbers between 1 and 10; the collection consisting in the number 5 and you; the collection of all collections just now described.

   DEFINITION: A set is any collection into a whole $S$ of definite and separate objects $m$ of our intuition or our thought.

   > We represent it thus: $S = \{m\}$
   >
   > For example, $\{2, 3, 5, 7\}$ is the set of all prime numbers between 1 and 10.

   *Questions*: What difference, if any, is there between 5 and 7, and $\{5, 7\}$? Between $\{2, 3\}$ and $\{\{2\}, \{3\}\}$? Between 2 and $\{2\}$? Between $\{2\}$ and $\{\{2\}\}$?

2. Some technical terms:

   A. We say that $a$, $b$, and $c$ are *members* of $\{a, b, c\}$.

   B. We say that $S_1$ is a *subset* of $S_2$ just in case every member of $S_1$ is a member of $S_2$.[2]

   C. We say that $S_1$ is a *proper subset* of $S_2$ just in case every member of $S_1$ is a member of $S_2$ and there is at least one member of $S_2$ that is not a member of $S_1$.

   > Thus $\{a, b\}$ is a subset, and a proper subset, of $\{a, b, c\}$.
   >
   > And $\{a, b, c\}$ is a subset, but not a proper subset, of $\{b, a, c\}$.

3. DEFINITION: Two sets are *identical* just in case all the members of one are members of the other, and vice versa.

   > Thus $\{2, 3\} = \{3, 2\}$, but $\{2, 3, 5\} \neq \{3, 5, 7\}$.

   Compare a set with an ordered $n$-tuple—for example, $\{2, 3\}$ with the coordinates $(2, 3)$ of a point in the Cartesian plane.

   Compare a set with a society—for example, a family $F$ with the set $S$ whose members are the members of $F$.

   *Question*: What is the metaphysical status of a set?

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[1] The original of this text was written by Stewart Umphrey. Later additions include the definition of "greater cardinality", "lesser cardinality", the footnote on "just in case", and this footnote.

[2] "Just in case", like "if and only if", is an expression of logical equivalence.

4. Definition: Two sets are *equivalent* just in case all the members of one can be put into a one-to-one correspondence with all the members of the other.

   Take, for example, the set of all prime numbers between 1 and 10—$\{2, 3, 5, 7\}$—and the set whose members are the letters of the word 'four'—{f, o, u, r}. We can establish the following 1–1 correspondence:

$$\begin{array}{cccc} 2, & 3, & 5, & 7 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \text{f,} & \text{o,} & \text{u,} & \text{r} \end{array}$$

   We see that each member of one set can be paired with one member of the other, and vice versa, so that there are no unpaired or multiply-paired members.

   Hence the two sets are equivalent.

   Therefore, if two sets are identical, they are equivalent. But if two sets are equivalent, they may or may not be identical.

   At this point it may seem that no set could be equivalent to a proper subset of itself. But let us not rush to judgment.

5. Ordinal numbers include: 1st, 2nd, 3rd, .... Cardinal numbers include: 1, 2, 3, .... What a cardinal number is has been much debated in the last century. For our present purposes it will suffice to say that *how many* members there are in a set is the cardinal number associated with that set.

   Thus, for $\{a\}$ the cardinal number is 1. For $\{a, b\}$ it is 2. And so on.

   We say that the *cardinality* of a set is the cardinal number associated with that set.

   Proposition: Two sets have the same cardinality just in case they are equivalent.

   Thus $\{2, 3\}$ and {Kant, George Eliot} have the same cardinality.

   Propositions: One set has greater cardinality than another just in case the latter is equivalent to a proper subset, but not the whole, of the former. One set has lesser cardinality than another just in case the latter has greater cardinality than the former.

   Thus $\{2, 3\}$ has greater cardinality than {Kant}, and greater cardinality again than {George Eliot}; while $\{2\}$, or $\{3\}$, has lesser cardinality than {Kant, George Eliot}.

   Compare the cardinality of a set with number as defined by Euclid (*Elements* VII.Df 2 with Df 1).

*Question*: Is zero a cardinal number? And if so, is there a set having this cardinality? If we answer 'yes' we'll have to modify our definition in §1 above, so as to admit the set having no members at all. Cantor and others admitted just such a set. It is called the *null set*, and is represented by '$\varnothing$'. Thus, the cardinality of $\varnothing$ is zero, that of $\{\varnothing\}$ is 1, that of $\{\varnothing, \{\varnothing\}\}$ is 2, and so on. We ourselves will admit the null set in §14 below.

---

6. Let $S$ be the set of the first ten 'natural' numbers: $\{1, 2, 3, \ldots, 10\}$. The cardinality of $S$ is 10. Its proper subsets are $\{1\}$, $\{2\}$, $\{1, 2\}$, and so on. Each proper subset of $S$ has fewer members than $S$ itself, hence none is equivalent to $S$.

   Let $S'$ be the set of all natural numbers: $\{1, 2, 3, \ldots, n, \ldots\}$. Among its proper subsets is $S''$: $\{2, 4, 6, \ldots, 2n, \ldots\}$. We can establish a 1–1 correspondence between the members of $S''$ and the members of $S'$.

$$\begin{array}{cccccc} 2, & 4, & 6, & \ldots, & 2n, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 1, & 2, & 3, & \ldots, & n, & \ldots \end{array}$$

   Hence $S''$ and $S'$ are equivalent even though $S''$ is a proper subset of $S'$. (Recall Galileo, *Two New Sciences* [National Ed.], pp. 78–79.)

   Definition: A set $S$ is *finite* just in case no proper subset of $S$ is equivalent to $S$.

   Definition: A set $S$ is *infinite* just in case at least one proper subset of $S$ is equivalent to $S$.

   Notice, we suppose the set of all natural numbers to be complete, to have infinitely many members in actuality, not merely in potentiality. Aristotle would have said that there is no such entity. Kant would have said that it could not be an object of experience. Galileo, on the other hand, seems to have admitted actual infinities. So too did Spinoza, Newton, and Leibniz. And so does Cantor.

7. Let $S$ be the set of all natural numbers. What is the cardinality of $S$? It cannot be any finite cardinal $n$. It must then be some *transfinite* cardinal, if indeed $S$ has any cardinality at all. There is such a cardinal number, Cantor held, and to name it he used the first letter of the Hebrew alphabet, combined with zero-subscript: $\aleph_0$ (pronounced 'aleph-null'). The reason for the subscript will soon become evident.

   Let $S'$ be the set of all even natural numbers. Since $S'$ and $S$ (the set of all natural numbers) are equivalent, the cardinality of $S'$ too is $\aleph_0$.

   Let $S''$ be the infinite set $1^1, 2^2, 3^3, \ldots, n^n, \ldots$. Since this proper subset of $S$, too, is equivalent to $S$, its cardinality too is $\aleph_0$.

8. Some transfinite arithmetic.

(1) $\aleph_0 \pm n = \aleph_0$

(2) $\aleph_0 + \aleph_0 = \aleph_0$

(3) $n \cdot \aleph_0 = \aleph_0$ and $\frac{1}{n} \cdot \aleph_0 = \aleph_0$

(4) $\aleph_0 \cdot \aleph_0 = \aleph_0$

Try to sketch proofs for each of these equalities. The key, in each case, is to establish a 1–1 correspondence between sets having the cardinal numbers indicated on each side of the '=' sign. And to establish this correspondence, some ingenuity is needed.

PROPOSITION: The set of all the integers (positive, negative, and zero) has cardinality $\aleph_0$.

Consider the set of all points in the Cartesian plane having integral coordinates. Is its cardinality, too, only $\aleph_0$?

DEFINITION: An infinite set is *denumerable* just in case it is equivalent to the set of all natural numbers, i.e., just in case it has cardinality $\aleph_0$.

> The set of all integers is denumerable.
>
> Also denumerable is the set of all possible names in a language having a finite alphabet. (Pause a moment to assure yourself that this is so.)

Is there even one *non*denumerable set?

9. Consider the set of all rational numbers $> 0$. Is it denumerably or nondenumerably infinite?

Between any two rationals there are infinitely many rationals; i.e., the rationals are *dense* (Dedekind [Dover ed.], p. 6). Are they not then infinitely infinite in quantity, and so nondenumerable? However, Cantor found a way of putting them into a 1–1 correspondence with the natural numbers, and from this he concluded that the cardinality of all the rationals $> 0$ is only $\aleph_0$. Here's a sketch of his proof:

(1) We set out all rationals $> 0$ in the array below (see Figure 1). Assure yourself that there is no rational $> 0$ not mentioned at least once in this infinite array.

(2) Draw and follow the dotted-line arrows, as shown, writing down each fraction as you go. Thus:

$$\tfrac{1}{1}, \tfrac{2}{1}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{2}, \tfrac{3}{1}, \tfrac{4}{1}, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}, \ldots.$$

(3) Reduce each fraction to its lowest terms, and cancel out all redundancies. This yields

$$1, 2, \tfrac{1}{2}, \tfrac{1}{3}, 3, 4, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}, \ldots.$$

(4) There is a 1–1 correspondence between all these rationals and the natural numbers.

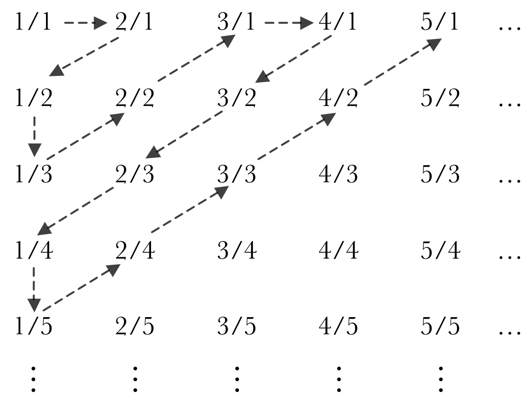

Figure 1

(5) Therefore, the set of all rationals $> 0$ has the same cardinality as the set of all natural numbers.

(6) Therefore, the set of all rationals $> 0$ is denumerable.

10. Proposition: The set of all rational numbers (positive, negative, and zero) is denumerable.

    Sketch a proof of this proposition.

    Porism: Between any two rational numbers there are $\aleph_0$ rational numbers.

    Consider the set of all points in the Cartesian plane having rational coordinates. Is it denumerable?

11. An ordinary algebraic equation has the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

where $n$ is a natural number and $a_0, a_1, \ldots$ are integral coefficients, and $a_0 > 0$.

Definition: A number is *algebraic* just in case it is a real (and not imaginary) root of an ordinary algebraic equation.

> Thus every rational number, we may say, is an algebraic number. To see as much, consider the roots of $2x - 6 = 0$, $6x + 2 = 0$, $x^2 - 4 = 0$, and so on.
>
> Many irrational numbers, too, are algebraic. To see as much, consider the roots of $x^2 - 2 = 0$, $x^3 - 2 = 0$, and so on.
>
> But some irrationals are not algebraic. Two examples are $\pi$ and $e$. Such numbers Euler called *transcendental*, because they appeared to

transcend algebraic methods. That $\pi$ and $e$ are not roots of any ordinary algebraic equations was demonstrated (!) in the decades preceding Cantor's investigation of infinite sets.

Proposition: The set of all algebraic numbers is denumerable.

Here's a sketch of Cantor's proof:

(1) For any given algebraic equation we can pick out the coefficients, together with the degree $n$, and construct the sum

$$a_0 + |a_1| + \cdots + |a_n| + n.$$

Call this sum the *index* of the given equation. Every algebraic equation has some definite index. There are denumerably many such indices.

(2) Since $a_0 \geqslant 1$ and $n \geqslant 1$, there is no index 1. For index 2, the only equation is $x = 0$, and the only root of this equation is 0. For index 3, the only equations are $2x = 0$, $x + 1 = 0$, $x - 1 = 0$, and $x^2 = 0$, and the only roots of these equations are 0, $-1$, 1. And so on to infinity.

(3) Evidently, for each index there are finitely many equations having that index, and finitely many roots of those equations.

(4) We can arrange the real roots according to the size of the index. Doing so we get

$$0, -1, 1, -\tfrac{1}{2}, \tfrac{1}{2}, -2, 2, \ldots.$$

(5) This set we can put into a 1–1 correspondence with all the natural numbers.

(6) Therefore, the set of all algebraic numbers is denumerable.

Consider the set of all points in the Cartesian plane having algebraic coordinates. Is it, too, denumerable?

---

12. Proposition: The set of all real numbers between 0 and 1 is nondenumerable.

Here's a sketch of Cantor's famous *reductio* proof:

(1) Suppose the set of all reals between 0 and 1 to be denumerable.

(2) Then there is a 1–1 correspondence between them and all the natural numbers. We may represent it thus:

$$\begin{array}{l} 1 \leftrightarrow 0.\, a_{1,1}\, a_{1,2}\, a_{1,3}\, a_{1,4} \ldots \\ 2 \leftrightarrow 0.\, a_{2,1}\, a_{2,2}\, a_{2,3}\, a_{2,4} \ldots \\ 3 \leftrightarrow 0.\, a_{3,1}\, a_{3,2}\, a_{3,3}\, a_{3,4} \ldots \\ 4 \leftrightarrow 0.\, a_{4,1}\, a_{4,2}\, a_{4,3}\, a_{4,4} \ldots \\ \vdots \quad\ \ \vdots \end{array}$$

In the left-hand column are listed all the natural numbers. In the right-hand column are listed all the reals between 0 and 1, in no particular order. We represent them by their decimal expansions. So for example in the expression '$a_{3,2}$' the '$a$' stands for some digit $\geqslant 0$ and $\leqslant 9$, while the subscript indicates that this digit occupies the second position in the third row. Our supposition is that all the reals between 0 and 1 are represented in this list.

> In steps (2) and (3) of this proof, bear in mind that $0.999999\ldots = 1.000000\ldots$. To see as much, let $N = 0.999999\ldots$. Then $10N = 9.99999\ldots$. Hence $9N = 10N - N = 9.00000\ldots$. Hence $N = 1.00000\ldots$. Similarly, $0.499999\ldots = 0.500000\ldots$, and so on.

(3) Pick out the diagonal array of digits $a_{1,1}\, a_{2,2}\, a_{3,3} \ldots a_{n,n} \ldots$. By looking to this array, and invoking some well-chosen transformation rule, we can produce a new array of digits $b_1\, b_2\, b_3 \ldots b_n \ldots$. One such rule is this: if $a_{i,j} = 1$, write '2', and if $a_{i,j} \neq 1$, write '1'. Thus, if our diagonal array should be $37155\ldots$, we shall write '$11211\ldots$'.

(4) Construct the decimal expansion of a real number between 0 and 1 in accordance with this new array:

$$0.\, b_1\, b_2\, b_3 \ldots.$$

For example, if the given diagonal array should be $37155\ldots$, and if our transformation rule be the one described in step (3), then the constructed decimal expansion will be $0.11211\ldots$.

(5) This number cannot be the first one mentioned in our list, since $b_1 \neq a_{1,1}$. Nor can it be the second mentioned in our list, since $b_2 \neq a_{2,2}$. And so on. Hence the number $0.\, b_1\, b_2\, b_3\, b_4 \ldots$ cannot be among those mentioned in our list. And yet it is a real number between 0 and 1.

(6) Therefore our list of reals between 0 and 1 is incomplete. Yet by hypothesis it is complete. Our list, then, is both complete and incomplete, which is absurd.

(7) Therefore the set of all reals between 0 and 1 is nondenumerable.

Steps (3)–(4) exemplify Cantor's *diagonal procedure.* Acquaint yourself with it by constructing yet another real number not mentioned anywhere in our list. Show, moreover, that the set of all reals between 0 and 1 not mentioned in our original list cannot itself be denumerable.

> *Hint*: Suppose this set is denumerable, and then reduce this supposition to absurdity.

PORISM: It is impossible, even in principle, to mention all the reals between 0 and 1 *seriatim*, in the form of a list.

13. Does the set of all reals between 0 and 1 have some cardinality? Yes, says Cantor. And what is it? For the time being let us call it $\aleph_c$, taking

the subscript from the assumption that the reals between 0 and 1 form a *continuum.*

Other symbols for this set's cardinality include $C$, $c$, and $\mathfrak{c}$.

We have found that $\aleph_c > \aleph_0$. So there are at least two transfinite cardinals, $\aleph_0$ and $\aleph_c$.

Since $1 < 2 < 3 < \cdots < n < \cdots < \aleph_0$, $\frac{1}{1} > \frac{1}{2} > \frac{1}{3} > \cdots > \frac{1}{n} > \cdots > \frac{1}{\aleph_0}$. And evidently, since $\aleph_c > \aleph_0$, $\frac{1}{\aleph_c} < \frac{1}{\aleph_0}$. $\frac{1}{\aleph_0}$ is infinitely smaller than $\frac{1}{n}$. How much smaller is $\frac{1}{\aleph_c}$ than $\frac{1}{\aleph_0}$? (Recall Newton, *Principia* I, Scholium after Lemma XI, first paragraph.)

PROPOSITION: The cardinality of all the real numbers (positive, negative, and zero) is $\aleph_c$.

To see as much, replace the '0.'s in step (2) of the forgoing *reductio* proof with '$N_1$.', '$N_2$.', and so on, where '$N$' stands for some integer. The rest of the proof is very similar.

To see this geometrically (see Figure 2), take a straight line of unit-length, bend it into a semicircle with center at point 0, and place it over a straight line L infinite in both directions. Let the line determined by the endpoints of the semicircle be parallel to L. We notice

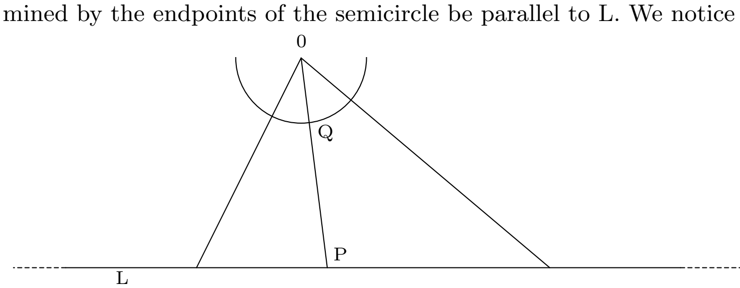


Figure 2

that for every point P 'in' L there is a corresponding point Q 'in' the unit-line, and vice versa. There are, we assume, $\aleph_c$ points in the unit-line. Hence there are $\aleph_c$ points in the line infinite in both directions.

PROPOSITION: $\aleph_c \cdot \aleph_0 = \aleph_c$.

Let the real-number 'line' be divided into unit-intervals. Each interval, we'll say, consists in all the real numbers $r$ such that $n \leqslant r < n + 1$, where $n$ is some integer. There are $\aleph_c$ real numbers in each interval. There are $\aleph_0$ intervals in the real-number line. And there are $\aleph_c$ real numbers altogether. Hence in this case, and generally, $\aleph_c \cdot \aleph_0 = \aleph_c$.

Three more transfinite equalities:

(1) $\aleph_c + \aleph_c = \aleph_c$

(2) $\aleph_c + \aleph_0 = \aleph_c$

(3) $\aleph_c - \aleph_0 = \aleph_c$

The cardinality of all the rational numbers is $\aleph_0$ [see §10 above]. The cardinality of all the reals is $\aleph_c$. Hence the cardinality of all the irrational numbers is $\aleph_c$. In other words, most real numbers are irrational.

Further, the cardinality of all the algebraic numbers is $\aleph_0$ [see §11 above]. Hence the cardinality of all the transcendental numbers is $\aleph_c$. In other words, most of the real numbers are irrationals that we haven't even begun to study. Indeed we've scarcely begun to name them. (Pause a moment to let this consequence sink in.)

Proposition: In any language having a finite alphabet it is impossible, even in principle, to list or in any way name all the transcendental numbers.

> (Supposing each and every real number to be an object of God's thought, what could such divine thought be like?)

Proposition: Between any two real numbers there are $\aleph_c$ real numbers. Proposition: The cardinality of all real-number coordinates in the Cartesian plane is $\aleph_c$. Here's a sketch of Cantor's proof:

(1) Consider first the area bounded by the axes and the lines $y = 1$, $x = 1$ (see Figure 3). Let $S$ be the set of reals $x$ such that $0 < x \leqslant 1$. And

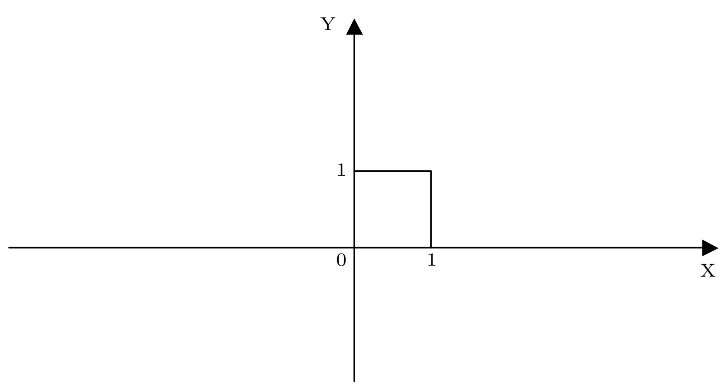


Figure 3

let $S'$ be the set of all real-number coordinates $(x, y)$ where $0 < x \leqslant 1$, $0 < y \leqslant 1$.

(2) The cardinality of $S$ is $\aleph_c$.

(3) Take any coordinates $(x, y)$ in the given area and represent them by their decimal expansions. So, for example, we have

$$(x, y) = (0.6993\ldots, 0.7128\ldots).$$

(4) Interlace the two arrays of digits, beginning with the first digit of the $x$-term. To use the forgoing example, we get

$$0.67919238\ldots.$$

We now have the digital expansion of a single real number, between 0 and 1, to represent the given real-number coordinates.

(5) Since every real-number coordinate in the given area has some such representative, and there are $\aleph_c$ such representatives, the cardinality of $S'$ is the same as the cardinality of $S$, namely $\aleph_c$.

(6) There are $\aleph_0 \cdot \aleph_0 = \aleph_0$ such areas in the Cartesian plane.

(7) Therefore, the cardinality of all real-number coordinates in the Cartesian plane is $\aleph_0 \cdot \aleph_c = \aleph_c$.

It can now easily be shown that $\aleph_c \cdot \aleph_c = \aleph_c$, and that $(\aleph_c)^n = \aleph_c$. So there are only $\aleph_c$ real-number coordinates in an $n$-dimensional Cartesian space. When Cantor reached this conclusion, he wrote to Dedekind: "I see it, but I don't believe it!"

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14. We shall now define $\aleph_c$ more precisely. To do so, we must first learn something about power sets.

DEFINITION: The *power set* of $S$ is the set whose members are all the subsets of $S$.

We represent the power set of $S$ by '$\mathscr{P}(S)$'.

Let $S = \{a, b\}$. Its subsets include $\{a\}, \{b\}, \{a, b\}$. We now admit the null set $\varnothing$ (see §5 above) and say that $\varnothing$ is a subset of any other set. Accordingly $\mathscr{P}(S) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$.

In this example, the cardinality of $\mathscr{P}(S)$ is $4 = 2^2$. If $S = \varnothing$, $\mathscr{P}(S)$ has cardinality $1 = 2^0$. And if $S = \{a, b, c\}$, then $\mathscr{P}(S)$ has cardinality $8 = 2^3$. (Assure yourself that this is so.)

PROPOSITION: If $S$ has $n$ members, $\mathscr{P}(S)$ has $2^n$ members.

PROPOSITION: For all $n \geqslant 0$, $2^n > n$.

We now return to infinite sets.

POSTULATE: For any set $S$, there is a power set $\mathscr{P}(S)$.

Cantor can now demonstrate the following propositions:

(1) For any set $S$, the cardinality of $\mathscr{P}(S)$ is greater than the cardinality of $S$.

This is often called *Cantor's Theorem*.

(2) If $S$ is the set of all natural numbers, then the cardinality of $\mathscr{P}(S)$ is $2^{\aleph_0}$.

(3) $\aleph_c = 2^{\aleph_0}$.

Sketches of his proofs are omitted here.

15. To generalize: If an infinite set $S$ has cardinality $\aleph_n$, there is a $\mathscr{P}(S)$ having cardinality $2^{\aleph_n} > \aleph_n$.

There are then infinitely many transfinite cardinals, as well as infinitely many finite cardinals.

$$0, 1, 2, \ldots, n, \ldots, \aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots$$

Cantor asks two questions:

(i) Is there a transfinite cardinal greater than $\aleph_0$ but less than $2^{\aleph_0}$? In other words, does $2^{\aleph_0} = \aleph_1$?

(ii) Is there no largest transfinite cardinal?

HYPOTHESIS: There is no cardinal between $\aleph_0$ and $2^{\aleph_0}$.

This hypothesis Cantor tried to prove true. He failed. Gödel and Cohen later proved (!) that it could be neither proved nor disproved within the now-standard set theory. Today some accept Cantor's *Continuum Hypothesis*, as it came to be called, while others reject it.

PROPOSITION: There is no set of all sets. For

(1) Suppose there is. Call it '$S$'.

(2) Then there is a $\mathscr{P}(S)$, and $\mathscr{P}(S)$ has a cardinality greater than the cardinality of $S$.

(3) There are then sets in $\mathscr{P}(S)$ that are not sets in $S$.

(4) Hence $S$ both is and is not the set of *all* sets, which is absurd.

(5) Hence there is no set of all sets.

PORISM: There is no largest set.

PORISM: There is no largest cardinal number.

If this proof is sound, and if Cantor's Continuum Hypothesis is true, we have determined the following series :

$$0, 1, 2, \ldots, n, \ldots, \aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_n, \ldots, \aleph_{\aleph_0}, \aleph_{\aleph_1}, \aleph_{\aleph_2}, \ldots.$$

But we shouldn't think that we could list them all, even in principle. Nor should we think that all the cardinal numbers constitute a set. (Pause a moment to let this consequence sink in.)

Cantor now distinguishes between a collection that is one set, and not just many objects (see §1 above), and a "collection" that is many objects and cannot possibly be one set (for example, all the sets there are). The latter he calls "inconsistent", on the ground that it could not be comprehended, held together, or unified in thought.

*Question*: Under what conditions, exactly, are objects too many to constitute a set?

> About this there is ongoing discussion. Some have doubted that the postulate introduced in §14 is true.

*Question*: Supposing the real existence of space apart from our conceptual thought (cf. Dedekind, p. 12, with Newton or Kant), and letting there be a line segment in that space, is there any way of telling how many points there are 'in' that line?

> Cantor assumed that there are $2^{\aleph_0}$ points in the line. But Abraham Robinson has established the system of *hyperreal* numbers, whose cardinality is $2^{2^{\aleph_0}}$. Conceivably, then, there are this many points in the line, in which case analytic geometry should admit all the hyperreals. And in any case we should re-think the concept of continuity.

---

16. Transfinite Sets and Nature.

   A. Is it not true *a priori* that there could never be more than $\aleph_0$ stars, or bodies of any finite size, in the universe? For stimulating reflections of this sort, cf. J. Benardete, *Infinity*.

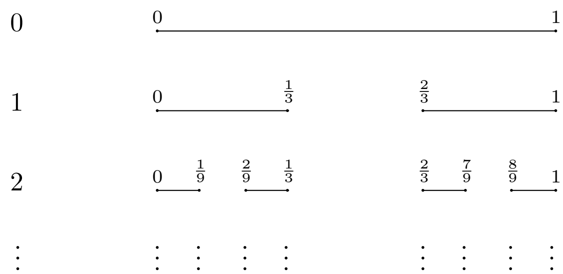
   B. The Cantor Set.

   Let there be a line of unit length whose point-coordinates are all $x$ such that $0 \leqslant x \leqslant 1$. Remove the middle third of the line (or all $x$ such that $\frac{1}{3} < x < \frac{2}{3}$). Then, from each of the two remaining lines, remove the middle thirds (or all $x$ such that $\frac{1}{9} < x < \frac{2}{9}$ and $\frac{7}{9} < x < \frac{8}{9}$). Continue so augmenting and diminishing *in infinitum*.

   The resulting set of points Cantor called the *ternary set*. It is now called the *Cantor Set*. And its members collectively are called *Cantor Dust*.

   For the role of this set in fractal geometry, and its possible applications in physics, cf. H.–O. Peitgen *et al*, *Chaos and Fractals*, and B. Mandelbrot, *The Fractal Geometry of Nature*.

   The Cantor Set has features of immediate interest to us. Its members collectively have the properties of transitivity and orderedness (Dedekind, p. 7). But they are not dense, since there are finite gaps




between $\frac{1}{3}$ and $\frac{2}{3}$, $\frac{1}{9}$ and $\frac{2}{9}$, and so on. Indeed, are they dense anywhere?

At step 0 the linear magnitude is 1. At step 1 the remaining magnitude is $\frac{2}{3} = (1 - \frac{1}{3})$. At step 2 it is $\frac{4}{9} = (1 - \frac{5}{9})$. In general, at step $n$ there is a magnitude $L$ such that

$$L = 1 - \frac{3^n - 2^n}{3^n}.$$

We can show that

$$\lim_{n\to\infty} L = 0$$

Thus Cantor Dust has no magnitude. (Pause a moment to let this sink in.)

At step 0 the cardinality of the endpoints is 2. At step 1 the cardinality of the endpoints is 4. At step 2 it is 8. In general, at step $n$ it is $2^{n+1}$. It seems, then, that the cardinality of the Cantor Set is $2^{\aleph_0+1} = 2^{\aleph_0}$.

On the other hand, at step 0 the endpoints all have rational coordinates, and so too at steps $1, 2, \ldots, n, \ldots$. The cardinality of all rational numbers between 0 and 1 is $\aleph_0$. It seems, then, that the cardinality of the Cantor Set is only $\aleph_0$. Compare the Zeno set, as we may call it (after Zeno's bisection paradox):

$$\left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots, \frac{2^n - 1}{2^n}, \ldots\right\}$$

Clearly this set is denumerable. How could the Cantor Set be otherwise?

Proposition: The Cantor Set is nondenumerable. For

(1) Consider the difference between any remaining point-coordinate and the endpoint 0. We may represent it by an infinite sum of

*ternary* rather than decimal fractions. Thus $\frac{2}{3}$, which ordinarily we would represent by $0 + \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots$, we now represent by $0 + \frac{2}{3} + \frac{0}{9} + \frac{0}{27} + \cdots$. In other words, whereas we ordinarily represent $\frac{2}{3}$ by the decimal expansion $0.666\ldots$, we now represent it by the ternary expansion $0.200\ldots$.

(2) Any point-coordinate in the Cantor Set can be represented by

$$0.\, a_1\, a_2\, a_3 \ldots a_n \ldots,$$

where the digit $a_i$ is always 0 or 2, never 1.

Cantor's proof of this is beyond us. But we can perhaps assure ourselves by means of some examples. The included coordinate $\frac{2}{3} = 0.200\ldots$, as we just now saw. The included coordinate $1 = 1.000\ldots = 0.222\ldots$ (recall our having found, in §12 above, that $1.000\ldots = 0.999\ldots$). And the included coordinate $\frac{1}{3} = 0.100\ldots = 0.022\ldots$. On the other hand, $\frac{1}{2} = 0.111\ldots$ is a coordinate excluded at step 1.

(3) The Cantor Set is denumerable only if it can be put into a 1–1 correspondence with the set of all natural numbers. Suppose it can, and proceed as in §12 above. First we give ourselves a list of all point-coordinates, in no particular order.

$$\begin{array}{l} 1 \leftrightarrow 0.02000220\ldots \\ 2 \leftrightarrow 0.02202020\ldots \\ 3 \leftrightarrow 0.00222222\ldots \\ \vdots \quad\ \ \vdots \end{array}$$

(4) By Cantor's diagonal procedure we determine a ternary expansion of a point-coordinate not mentioned in our list. So our list is both complete and incomplete, which is absurd.

(5) Therefore the Cantor Set is nondenumerable.

*Question*: Where then did we go wrong in our reasoning to the conclusion that the Cantor Set has cardinality $\aleph_0$?

C. Cantor was interested in the following argument:

(i) Necessarily, motion is continuous only if space is continuous.

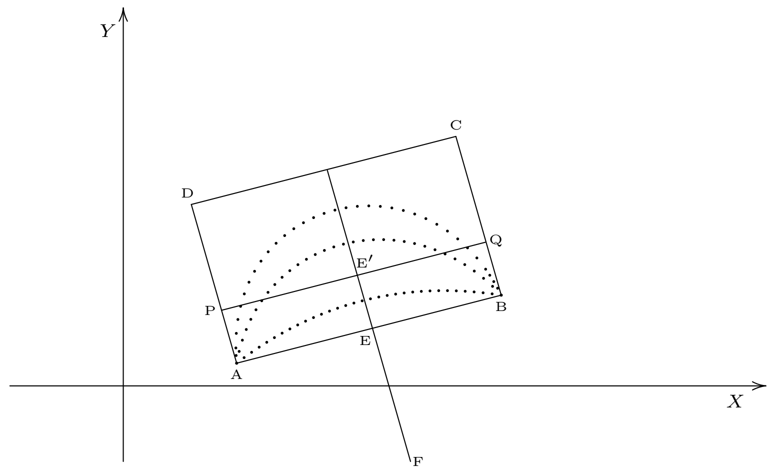
(ii) Motion is continuous.

(iii) Therefore space is continuous.

The conclusion follows necessarily from the premisses. But are the premisses both true? Cantor proved that premiss (i) is false. Here's a sketch of his proof (modified in a couple of respects):

(1) Suppose a plane that is Cartesian in all respects but one: wherever the coordinates are both rational, there is no corresponding point. This space is everywhere discontinuous; no region of it, however small, is without infinitely many punctual 'gaps'.

Henceforth regard something as a point only if it has at least one irrational coordinate. But regard something as a line or arc even if it is everywhere punctuated by such 'gaps'.




(2) Pick any two points, A and B, and join them by the straight line AB. AB may be continuous. (Consider, for example, the line whose equation is $y = \pi x$, in the interval $1 \leqslant x \leqslant 2$.) If so, it is a possible path for a point-mass moving from A to B.

(3) Or AB may be discontinuous. If so, notice that many rectangles can be constructed on AB as a base. Let one of them be the rectangle ABCD.

(4) Either AB can be bisected or it cannot. If it can, let it be bisected at E. If it cannot, notice that within the rectangle ABCD there can be many lines parallel to AB, some of which can be bisected. Let one such line be PQ, and let it be bisected at E′.

(5) At E (or E′) draw EF (or E′F) at right angles to AB (or PQ).

(6) On line EF produced there are $2^{\aleph_0}$ points which can be centers of circles passing through points A and B such that the arc $\overset{\frown}{\mathrm{AB}}$ falls within the rectangle ABCD. Hence there are $2^{\aleph_0}$ such possible arcs.

(7) Of these possible arcs, no more than $\aleph_0$ can be discontinuous, since there are only $\aleph_0$ punctual 'gaps' within the rectangle ABCD. Hence, within this rectangle, the cardinality of those possible arcs which are continuous is $2^{\aleph_0} - \aleph_0 = 2^{\aleph_0}$.

(8) Being continuous, every one of these arcs is a possible path for a point-mass moving continuously from A to B.

(9) Therefore continuous motion is possible in a plane everywhere discontinuous.

Porism: Of all the possible paths from any point to any other point in this punctuated plane, most are continuous.

We can make the height of the rectangle ABCD as small as we want. Therefore, even if rectilinear motion from A to B is impossible, we can make the path so close to rectilinear that no human being could tell the difference. And even if an elliptical path through A and B were impossible, we could make the path so close to elliptical that again no human being could tell the difference. Could we not then preserve Newton's physics even if we gave up the assumption that space is everywhere continuous?

*Question*: Can we now prove that motion could be continuous in a space so 'punctuated' that in it no rectilinear motion at all could be continuous?

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17. Transfinite Sets and God.

In Cantor's writings there are scattered references to Plato's *Philebus*, Augustine's *City of God*, Spinoza's *Ethics*, and the writings of Leibniz and of several Thomists. He seems to have made no radical distinction between the teachings he found there and his own set-theoretical mathematics.

According to Dedekind, numbers and number systems are human creations. According to his friend Cantor, on the other hand, numbers themselves exist eternally in the divine intellect—*all* of them. Cantor thought that he had discovered the transfinite domain; had *discovered* that some infinite sets are larger than others, and that some "collections" are too large even to be sets. True, his reasonings involve much invention—necessarily, it seems—and yet, if he is right, all these constructions or creations belong to the human way of coming to notice certain eternal entities, not at all to the entities one thus comes to notice.

Following Spinoza, apparently, Cantor sometimes identified God with *natura naturans* (nature naturing), and everything created with *natura naturata* (nature natured). God is the absolute infinite. The transfinite is the eternal yet created infinite, which mediates, in being and in cognition, the finite and its incomprehensibly infinite source. No wonder Cantor took the "inconsistent collection" to be a symbol of God. No wonder he sometimes spoke as if transfinite set theory were the royal road to trans-mathematical theology or first philosophy.

This summary of Cantor's metaphysical and epistemological views is based on scattered remarks most of which have not been translated into English. Should you wish to read more about them, see J. W. Dauben, *Georg Cantor*, esp. chapters 6, 10, and 12.

# An Appendix on the Transcendence of the Circle's Quadratrix

Here is the rest of the argument that the quadratrix of the circle is transcendent, continued from Note 6 to "Recondite Geometry" (page 115).

### The first part of the argument

Given our (hypothetical) algebraic equation for the quadratrix $AFC$, we can find a single algebraic equation that lets us divide angles into equal parts, as follows. We first construct a new curve $AMC$ to the left of $AFC$ such that the rectangle on $MF$ and $L$ is always equal to the triangle $GEB$ (see Figure 1).

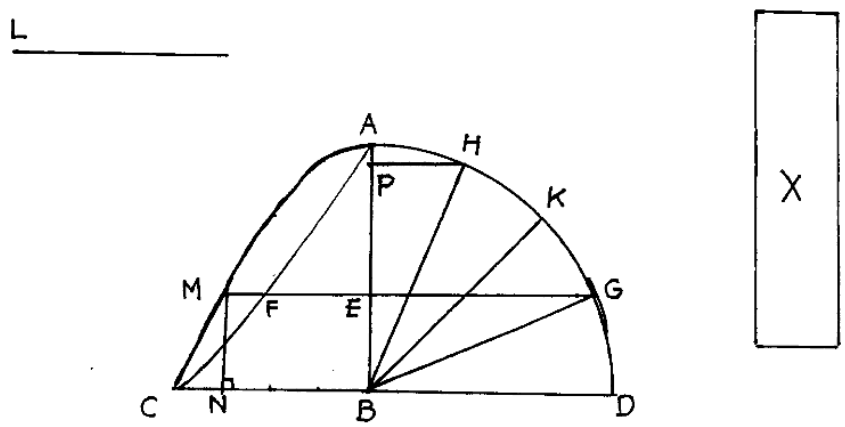


Figure 1

Then

$$\begin{aligned}\text{rectangle } ME, L &= \text{rect. } MF, L + \text{rect. } FE, L\\ &= \text{triangle } GEB + \text{area } AEG\\ &= \text{sector } ABG.\end{aligned}$$

Thus the rectangle on $ME$ and $L$ is always equal to the area of the sector $ABG$. $AMC$, like $AFC$, has an algebraic equation of a definite degree, since it is constructed, using a finite number of algebraic steps, from the circle $AGD$ (which has a second degree algebraic equation) and the quadratrix $AFC$ (which, according to our *reductio* assumption, has a single algebraic equation of a definite degree).

(Note that the line $AMC$ is a cosine curve. For if we let $BD = L = 1$, $\angle ABG = \theta$, and $EB = y$, then $ME = \theta$ and $BE = y = \cos\theta$.)

We can use the line $AMC$ both to find a rectilineal area equal to a given sector and to find a sector equal to a given rectilineal area, as follows:

1. *To find the area of a given sector.* Let the given sector be $ABG$. Then draw line $GM$ parallel to $BD$ and meeting $AMC$ at $M$. Then the rectangle on $ME$ and $L$ is equal to the area of $ABG$.

2. *To find a sector of a given area.* Let the given rectilineal area be $X$. Then let $N$ be a point on $BC$ such that the rectangle on $NB$ and $L$ is equal to $X$ (see Euclid's *Elements*, I 45). Draw the line $MN$ perpendicular to $CB$ until it meets $AMC$ at $M$. Draw the line $MEG$ parallel to $CB$ until it meets $AGD$ at $G$. Then sector $ABG$ is equal to the rectangle on $ME$ and $L$, that is, the rectangle on $NB$ and $L$, that is, $X$.

We can use these two constructions to trisect any given angle $ABG$: we use the first construction to find the area $X$ of sector $ABG$, then divide this area by 3 to get $\frac{X}{3}$, and finally use second construction to find an angle $ABH$ such that the area of sector $ABH$ is equal to $\frac{X}{3}$. The angle $ABH$ must then be one third of the original angle. Likewise, we could divide the angle $ABG$ into 4 parts by using $\frac{X}{4}$ in the second construction, and into five parts by using $\frac{X}{5}$, and so on. Thus the line $AMC$, which is represented by an algebraic equation of a single definite degree, may be used to cut an angle into *any* number of (equal) parts. Therefore, if our *reductio* assumption that the quadratrix $AFC$ is algebraic were true, there would have to be a single algebraic equation of one definite degree that would let us divide any given angle into any number of equal parts.

### The second part of the argument

But this is absurd, for the problem of cutting the angle into three equal parts is of third degree, the problem of cutting it into four equal parts is of fourth degree, and so on, and thus there is no one definite degree for such problems. For suppose angle $ABG$ is given, and we want to find an angle $ABH$ that is equal to one third of $ABG$. If we drop a perpendicular $HP$ from $H$ to $AB$, then to find the angle $ABH$ it suffices to find the line $PB$. Let $EB = a$ and $PB = z$. We may take the radius $BD$ of the circle as our unit. If we denote the angle $ABH$ by $\theta$, then

$$\cos\theta = \frac{PB}{HB} = PB = z$$

and

$$\begin{aligned} \cos(3\theta) &= \cos(\angle ABG) \\ &= \frac{EB}{GB} \\ &= a. \end{aligned}$$

Now according to elementary trigonometry, for any angles $\zeta$ and $\eta$,

$$\cos(\zeta+\eta) = \cos(\zeta)\cos(\eta) - \sin(\zeta)\sin(\eta) \text{ and} \quad (1)$$

$$\sin(\zeta+\eta) = \sin(\zeta)\cos(\eta) + \cos(\zeta)\sin(\eta). \quad (2)$$

Therefore, taking $\zeta = 2\theta$ and $\eta = \theta$ and using equation 1, we have

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta + \theta) \\
&= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)
\end{aligned}$$

Now according to equation 1, when $\zeta = \eta = \theta$, then

$$\begin{aligned}
\cos(2\theta) &= \cos(\theta + \theta) \\
&= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) \\
&= \cos^2(\theta) - \sin^2(\theta);
\end{aligned}$$

and according to equation 2,

$$\begin{aligned}
\sin(2\theta) &= \sin(\theta + \theta) \\
&= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) \\
&= 2\sin(\theta)\cos(\theta).
\end{aligned}$$

Therefore

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \\
&= (\cos^2(\theta) - \sin^2(\theta))\cos(\theta) - (2\sin(\theta)\cos(\theta))\sin(\theta) \\
&= \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta).
\end{aligned}$$

But according to elementary trigonometry,

$$\sin^2(\theta) = 1 - \cos^2(\theta),$$

and therefore

$$\begin{aligned}
\cos(3\theta) &= \cos^3(\theta) - 3(1 - \cos^2(\theta))\cos(\theta) && (3) \\
&= 4\cos^3(\theta) - 3\cos(\theta). && (4)
\end{aligned}$$

Substituting $a$ for $\cos(3\theta)$ and $z$ for $\cos(\theta)$ into equation 4 gives

$$a = 4z^3 - 3z. \tag{5}$$

Equation 5 is a third degree equation for the unknown quantity $z$, and therefore the problem of finding the angle $ABH$ which is one third of the given angle is a third degree problem. A similar argument shows that finding an angle equal to one fourth of the given angle is a fourth degree problem, and so on.

Now we showed above that the line $AMC$ can be used to cut an angle into any number of equal parts. Therefore the line $AMC$, which has a single definite degree, would be capable of solving infinitely many problems of all possible degrees. This is absurd. Therefore our *reductio* assumption that $AFC$ is not transcendent must be false. **Q. E. D.**

# Solutions to Odd-numbered Problems

## Problems about finding differences, page 57

1.
$$dv = (2x - 3)\,dx.$$

3.
$$dv = (3x^2 + 4)\,dx.$$

5.
$$dv = -\frac{4x}{v}\,dx,$$

or, if we want an expression strictly in terms of $x$,

$$dv = -\frac{4x}{\sqrt{4 - 4x^2}}\,dx.$$

7.
$$dv = \frac{2}{(3x + 2)^2}\,dx.$$

9.
$$dv = \frac{-6x^2 - 2x + 3}{(x^2 - 3x)^2}\,dx.$$

11.
$$dv = \frac{-2}{\sqrt{3 - 4x}}\,dx.$$

13.
$$dv = \frac{x + 1}{\sqrt{x^2 + 2x - 1}}\,dx.$$

15.
$$dv = 8(x + 2)^7\,dx.$$

17.
$$dv = 5(x^2 - 3x + 6)^4(2x - 3)\,dx.$$

19.
$$dv = 6(x + 1)^4(3x - 2)\,dx + 4(x + 1)^3(3x - 2)^2\,dx.$$

21.
$$\begin{aligned} dv =& 4(x^2 + x + 1)^3(2x + 1)(x^2 - 5)^7\,dx \\ &+ 14(x^2 + x + 1)^4(x^2 - 5)^6 x\,dx \end{aligned}$$

23.
$$\begin{aligned} dv = \{&(x^2 + 3)\left[3(x - 1)^2(x + 2)^5 + 5(x - 1)^3(x + 2)^4\right] \\ &- 2x(x - 1)^3(x + 2)^5\} \cdot \frac{dx}{(x^2 + 3)^2}. \end{aligned}$$

## Problems about finding greatest and least ordinates, page 70

1. Least ordinate at $x = 2$, where $v = -3$. No greatest ordinate, no inflection points. Ordinates increasing when $x > 2$, decreasing when $x < 2$. Curve turns its concavity up everywhere.

3. Greatest ordinate at $x = -1$, where $v = 9$. Least ordinate at $x = 3$, where $v = -23$. Ordinates increasing when $x < -1$ or $x > 3$. Ordinates decreasing when $-1 < x < 3$. Inflection point at $x = 1$. Concave down when $x < 1$ and concave up when $x > 1$.

## Problems about finding tangents, page 73

1. $$XB = \frac{x^2 - 4x + 1}{2x - 4}.$$

3. $$XB = \frac{x^3 - 3x^2 - 9x + 4}{3x^2 - 6x - 9}.$$

## Problems about finding differences of exponentials and logarithms, page 99

1. $$3e^{(3x+1)}\,dx.$$

3. $$\frac{3x^2 - 2}{x^3 - 2x}\,dx.$$

5. $$\frac{2e^{-x}\,dx}{2x+3} - e^{-x}\log(2x+3)\,dx.$$

## Problems on sums of algebraic quantities, page 144

1. $$\frac{x^4}{2} - \frac{x^2}{2} + 4x.$$

3. $$\frac{2x^{\frac{3}{2}}}{3} + \frac{2x^{\frac{5}{2}}}{5}.$$

5. $$20.$$

7. $$28.$$

## Problems on trigonometric quantities, page 153

1. Problems on differences.

   (a)
   $$[2\cos(2a-1) - 3\sin(3a+2)]\ da.$$

   (c)
   $$[8\cos(2a) - 2\sin(a+1)]\ da.$$

   (e)
   $$3\sin^2(a)\cos(a)\,da.$$

   (g)
   $$4\cos^3(1-a)\sin(1-a)\,da.$$

   (i)
   $$\frac{-2a\sin(2a)\sin(a^2) - 2\cos(a^2)\cos(2a)}{\sin^2(2a)}\,da.$$

2. Problems on sums

   (a)
   $$\sin a.$$

   (c)
   $$3 - 3\cos a + 2\sin a.$$

   (e)
   $$\frac{1}{2} - \frac{1}{2}\cos(2a).$$

   (g)
   $$\frac{1}{2}.$$

## Problems on sums involving logarithms, page 162

1.
$$3e^x + \frac{x^3}{3} - 3.$$

3.
$$\frac{e^{3x}}{3} - \frac{1}{3}.$$

5.
$$\frac{1}{2}e^{x^2} - \frac{1}{2}.$$

7.
$$x^2e^x - 2xe^x + 2e^x - 2.$$