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Robust Explicit MPC Based on Approximate Multi-parametric Convex Programming

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Abstract— **Many robust Model Predictive Control (MPC) schemes require the on-line solution of a convex program, which can be computationally demanding. For deterministic MPC schemes, multi-parametric programming was successfully applied to move most computations off-line. In this paper we adopt a general approximate multi-parametric algorithm recently suggested for convex problems and propose to apply it to a classical robust MPC scheme. This approach enables one to implement a robust MPC controller in real time for systems with polytopic uncertainty, ensuring robust constraint satisfaction and robust convergence to a given bounded set.**

Keywords: Model predictive control; Multi-parametric programming; Robust control; Uncertain systems.

I. INTRODUCTION

Model Predictive Control (MPC) is a control technique that is able to cope in a direct way with multi variable systems, constraints, and uncertainty. At each sampling time, a finite horizon optimal control problem is solved based on a given model of the system. One of the main drawbacks of MPC is the time needed to evaluate the solution of the posed optimization problem. For linear systems, when no uncertainty is taken into account, MPC requires the solution of a quadratic or a linear programming problem. These are well known problems and efficient tools are available for solving them. Recently, multi-parametric programming has been applied with success to solve such optimization problems off-line in order to obtain an explicit description of the control law (see [1], [2], [3], [4]).

One approach used in robust MPC is to minimize the objective function for the worst possible realization of the uncertainty. This strategy is known as min-max and was originally proposed in [5] in the context of robust receding horizon control. In robust MPC the problem was first tackled in [6]. For these schemes, however, the resulting on-line computation time is significantly larger than their deterministic counterparts.

Despite the complex nature of the problem, several different robust MPC schemes have been proposed in the literature. All of them have in common a high computational burden (see [7], [8], [9], [10], [11], [12], and references therein). However, the optimization problem associated with those schemes can often be posed as convex multi-parametric programming problems.

Parametric programming considers optimization problems where the data depends on one or more parameters. The parameter space is systematically subdivided into characteristic regions where the optimal value and an optimizer are given as explicit functions of the parameters. For linear cost functions, robust MPC controllers have been obtained in explicit form (see [13], [14]). This result has not been extended to quadratic cost functions, although the piecewise linearity of open-loop min-max MPC with quadratic cost functions has been proved by geometrical methods in [15] and an efficient off-line algorithm for parametric uncertainties was given in [16].

Recently, approximate multi-parametric convex programming solvers have been proposed in [17], [18]. The latter is based on a general approach that obtains a suboptimal explicit solution for a given convex problem with a guaranteed bound on the error. In this paper we apply the technique of [18] to a classic MPC robust scheme, namely the controller proposed by Kothare et al. in [7]. We first obtain an explicit easy-to-implement piecewise affine description of the control law with an arbitrary degree of accuracy, and then prove that for any chosen degree of accuracy constraints are handled robustly and the system robustly converges to a bounded set.

Section II introduces Kothare's controller and its properties. Section III introduces the multi-parametric convex approach and shows how to apply it to the proposed controller. Section IV presents the main results of the paper. Section V proposes different modifications to the strategy. Finally, some examples are shown in Section VI.

II. PROBLEM FORMULATION

Consider the uncertain Linear Time-Varying (LTV) system with polytopic uncertainty:

$$\begin{array}{l} x_{k+1} = A_k x_k + B_k u_k,\ y_k = C x_k, \\ [A_k\ B_k] \in \Omega, \end{array} \tag{1}$$

where $u_k \in R^{n_u}$ is the control input, $x_k \in R^{n_x}$ is the state vector, $y_k \in R^{n_y}$ is the output, and Ω is the convex hull of given matrices $\left[A^1\ B^1\right], \ldots, \left[A^L\ B^L\right]$.

System (1) is required to satisfy the input and output constraints

$$\begin{array}{lll} |e_r^T u_k| \leq u_{r,max} & k \geq 0, & r = 1, 2, \ldots, n_u \\ |e_r^T y_k| \leq y_{r,max} & k > 0, & r = 1, 2, \ldots, n_y, \end{array} \tag{2}$$

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where e_r is the r-th column of the identity matrix of appropriate dimension.

The controller proposed in [7], that will be referred to as "Kothare's controller" from now on, is based on minimizing an upper bound of the worst case infinite time cost function

$$J_\infty(x) = \max_{[A_k\ B_k]\in\Omega, k\geq 0} \sum_{k=0}^{\infty} x_k^T Q_c x_k^T + u_k^T R_c u_k^T$$

with Q_c and R_c simetric and positive definite, while satisfying (2).

Assume that a state feedback law $u_k = Fx_k$ is used, and that there exists a quadratic, strictly convex function $x^T Px$ that satisfies

$$x_{k+1}^T P x_{k+1} - x_k^T P x_k \leq -x_k^T Q_c x_k - u_k^T R_c u_k \qquad (3)$$

for all possible $[A_k\ B_k] \in \Omega$.

By summing (3) for all $k \geq 0$ and requiring that $x_k \to 0$ as $k \to \infty$, we obtain the upper bound $J_\infty(x) \leq x^T Px$.

Kothare's controller is based on the following result.

Property 1 (cf. [7], Theorem 2): For system (1), at sampling time k with $k \geq 0$, let γ, Q, Y satisfy the LMI constraints

$$\begin{gathered}
\begin{bmatrix} 1 & x^T \\ x & Q \end{bmatrix} \geq 0, \quad Q = Q^T > 0, \\
\begin{bmatrix} Q & QA_j^T + Y^T B_j^T & QQ_c^{\frac{1}{2}} & Y^T R_c^{\frac{1}{2}} \\ A_j Q + B_j Y & Q & 0 & 0 \\ Q_c^{\frac{1}{2}} Q & 0 & \gamma I & 0 \\ R_c^{\frac{1}{2}} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0, \\
j = 1, 2, \ldots, L \\
\begin{bmatrix} X & Y \\ Y^T & Q \end{bmatrix} \geq 0, \text{ with } X_{rr} \leq u_{r,max}^2,\ r = 1, 2, \ldots, n_u \\
\begin{bmatrix} Z & C(A_j Q + B_j Y) \\ (A_j Q + B_j Y)^T C^T & Q \end{bmatrix} \geq 0, \\
\text{with } Z_{rr} \leq y_{r,max}^2,\ r = 1, 2, \ldots, n_y,\ j = 1, 2, \ldots, L
\end{gathered} \qquad (4)$$

where $M_{rr} = e_r^T M e_r$. Let $F \triangleq YQ^{-1}$ and $P \triangleq \gamma Q^{-1}$. Then $\gamma \geq x_k^T P x_k$ and the constraints (2) and (3) are satisfied for the feedback matrix F and the matrix P.

At each time step k, given the current state x_k, Kothare's control algorithm solves the following SDP problem:

$$V^*(x) = \begin{array}{c} \min_{\gamma,Q,Y,X,Z} \gamma \\ \text{s.t. } (4) \end{array} \qquad (5)$$

From the optimizer $\gamma^*(x_k)$, $Q^*(x_k)$, and $Y^*(x_k)$, the optimal feedback gain $F^*(x_k) = Y^*(x_k)Q^*(x_k)^{-1}$ is obtained as well as the matrix $P^*(x_k) = \gamma^*(x_k)Q^*(x_k)^{-1}$ which defines an upper bound on the worst case infinite time cost function for the given feedback law, that is,

$$J_\infty(x_k) \leq x_k^T P^*(x_k) x_k \leq V^*(x_k).$$

The following property will be used in the sequel.

Property 2 (cf.[7], Lemma 2): Any quintuple Y, Q, γ, Z, X satisfying (4) at time k also satisfies (4) at time $k+1$ if $u_k = YQ^{-1}x_k$ is applied.

III. MULTI-PARAMETRIC CONVEX PROGRAMMING

Problem (5) is an SDP problem and efficient tools exist for solving it. However, the computational burden may be still too high in many real applications. An efficient suboptimal off-line implementation was presented recently in [16] and is based on the computation of invariant ellipsoids. Here we take a different route and propose to use multi-parametric techniques. More precisely, we consider the algorithm suggested in [18], designed to obtain, in explicit piecewise affine form, a suboptimal solution of the multi-parametric convex optimization problem

$$W^*(x) = \min_z \{W(z, x) : g_i(z, x) \leq 0,\ i = 1, \ldots, p\} \quad (6)$$

where $z \in R^{n_z}$ are the decision variables, $x \in X \subseteq R^{n_x}$ are the parameters, and W and g_i are jointly convex functions of the optimization variables and the parameters, so that W^* is a convex function (see [19], [20]). The multi-parametric approach of [18] consists of an algorithm for defining a suboptimal solution $\hat{z}(x)$ that is a piecewise affine function of the parameters. The solution is defined for a given full dimensional polyhedron $S = \{x \in R^{n_x} | Ax \leq b\}$ of parameters for which (6) is feasible. The suboptimal solution is a piecewise affine function defined over a partition of S made out of n_r critical simplices CS_i

$$\hat{z}(x) = \hat{z}^i(x) = H_z^i x + h_z^i,\ \forall x \in CS_i,\ i = 1, 2, \ldots, n_r.$$

The algorithm proposed in [18] is divided in two phases. In the first phase, the polyhedral region S to be characterized is triangulated into a minimal set of simplices. In the second phase, the simplices are subdivided into smaller ones until an upper bound on the maximum error inside each simplex is smaller than a given accuracy threshold ϵ. Because of the recursive nature of the algorithm and of the method for subdividing each simplex, the explicit suboptimizer is a piecewise affine function of the parameters that is organized in a tree structure for evaluation (see [18] for details). Hence, the on-line computational burden depends only on the maximum tree depth T_d and on the dimension of the parameter vector n_x. The maximum number of linear inequalities that must be evaluated in order to find the solution is linear in the state dimension and in the maximum depth of the tree.

Property 3 ([18]): For all state vectors inside S, the suboptimal solution $\hat{z}(x)$, obtained by applying the approximate multi-parametric convex programming algorithm of [18] to solve Problem (6) with a fixed $\epsilon > 0$, satisfies

$$\begin{array}{l} g_i(\hat{z}(x), x) \leq 0, \quad i = 1, 2, \ldots, p, \\ W^*(x) \leq W(\hat{z}(x), x) \leq W^*(x) + \epsilon. \end{array} \qquad (7)$$

A. Convex Problem Associated with Kothare's Controller

Kothare's controller is evaluated at each time step k by solving an SDP problem that depends on the current state x_k. As SDP is a convex problem, the above multi-parametric technique can be applied. Before proceeding further, let us introduce the following notation:

- The parameter vector x is defined as the current state vector, $x = x_k$.
- The optimizer vector consists of the free variables $z = \{\gamma, Q, Y, X, Z\}$ of Problem (5).
- The objective function is linear, $W(z, x) = \gamma = c^T z$.
- The constraints $g_i(z, x)$ are defined by (4). They are convex, as they are defined as linear matrix inequality constraints.

The approximate multi-parametric convex programming algorithm defines a piecewise affine function for the suboptimizer z of Problem (5) with a fixed error bound ϵ. It is important to note that matrices X and Z are not used for defining the multi-parametric control law, so in the following they will not be taken into account in the solution. Instead, we define the following piecewise affine functions of interest

$$\begin{array}{l} \gamma(x) = \gamma^i(x) = H_\gamma^i x + h_\gamma^i,\ \forall x \in CS_i, \\ Q(x) = Q^i(x) = H_Q^i x + h_Q^i,\ \forall x \in CS_i, \\ Y(x) = Y^i(x) = H_Y^i x + h_Y^i,\ \forall x \in CS_i, \\ \quad i = 1, 2, \ldots, n_r, \end{array} \tag{8}$$

where n_r is the number of critical simplices generated by the multi-paramtetric algorithm.

Following Property 3, the suboptimizers $\gamma(x), Q(x), Y(x)$ are feasible for Problem (5), matrices $F(x) = Y(x)Q(x)^{-1}$ and $P(x) = \gamma(x)Q(x)^{-1}$ satisfy (2) and (3), and the following inequalities hold

$$V^*(x_k) \le x_k^T P(x_k) x_k \le V^*(x_k) + \epsilon. \tag{9}$$

IV. Properties of the Proposed Approach

Lemma 1: Consider a system of the form (1) and the feedback gain given by $F(x) = Y(x)Q(x)^{-1}$, where $\gamma(x)$, $Q(x)$ and $Y(x)$ are taken from a suboptimizer of (5) over a set S with a given bound on the error $\epsilon > 0$ and have the form (8). For all states $x_k \in S$, if $u_k = F(x_k)x_k$ then the following inequality holds

$$V^*(x_{k+1}) - V^*(x_k) \le -x_k^T Q_c x_k + \epsilon, \tag{10}$$

for all possible $[A_k\ B_k] \in \Omega$.

Proof: For each $x_k \in S$ the multi-parametric convex programming algorithm provides a suboptimizer $\gamma(x_k), Q(x_k), Y(x_k)$ of (5) in x_k such that (3) and (9) hold for $F(x_k)$ and $P(x_k) = \gamma(x_k)Q(x_k)^{-1}$. By Property 2 $\gamma(x_k), Q(x_k), Y(x_k)$ are also a feasible solution of (5) for all possible x_{k+1}, so that $V^*(x_{k+1}) \le x_{k+1}^T P(x_k) x_{k+1}$. As $R_c > 0$, by replacing $x_k^T P(x_k) x_k$ with $V^*(x_k) + \epsilon$ and $x_{k+1}^T P(x_k) x_{k+1}$ with $V^*(x_{k+1})$ in (3), we obtain inequality (10). ■

Theorem 1: Consider the control law

$$\begin{array}{rcl} u_k & = & \hat{F}(x_k) x_k \\ \left[\hat{F}(x_k)\ \hat{P}(x_k)\right] & = & \left\{ \begin{array}{ll} [F(x_k)\ P(x_k)] & \text{if } x_k \in S \\ \left[\hat{F}(x_{k-1})\ \hat{P}(x_{k-1})\right] & \text{otherwise,} \end{array} \right. \end{array} \tag{11}$$

where S is a full dimensional polyhedron containing the origin in its interior, $F(x) = Y(x)Q(x)^{-1}$ and $P(x) = \gamma(x)Q(x)^{-1}$ where $\gamma(x), Q(x), Y(x)$ is a suboptimizer of (5) over S with an error bound $\epsilon > 0$ for x_k. Then, if $x_0 \in S$, the controller defined by (11) robustly regulates the system to a bounded set Ω_α of the state space while satisfying (2) for all possible uncertainties, where $\Omega_\alpha = \{x \in R^{n_x} | V^*(x) \le \alpha(\epsilon)\}$, $\alpha(\epsilon) = \max_{x \in \Phi_\epsilon} \{V^*(x) + \epsilon - x^T Q_c x\}$, and $\Phi_\epsilon = \{x \in S|\ x^T Q_c x \le \epsilon\}$.

Proof: In order to prove that the closed loop system is ultimately bounded we will first prove convergence to Φ_ϵ by Lyapunov arguments. Then, we will show that once the state lands in Φ_ϵ, even if it may leave it again in no case it will go outside the set Ω_α, from which it will return again to Φ_ϵ. In this way, Ω_α is an invariant set for the system.

Let $x_k \notin \Phi_\epsilon$. By Lemma 1, if $x_k \in S$, then $V^*(x_{k+1}) < V^*(x_k)$, for all $[A_k\ B_k] \in \Omega$. If $x_k \in S$ for all $k \ge 0$ then clearly the system converges to Φ_ϵ because $V^*(x)$ acts as a Lyapunov function. Suppose instead there exists k such that $x_k \in S$ and $x_{k+1} \notin S$. For all $h \ge 1$ such that $x_{k+j} \notin S$ ($j = 1, 2, \ldots, h$), by (11) we have $\hat{F}(x_{k+h}) = \hat{F}(x_k)$ and $\hat{P}(x_{k+h}) = \hat{P}(x_k)$. Since $\hat{F}(x_k)$ and $\hat{P}(x_k)$ are defined by the suboptimizer of (5) for $x = x_k$, taking into account Property 2 and Equation (3) the following inequality hold

$$x_{k+h+1}^T \hat{P}(x_k) x_{k+h+1} < x_{k+h}^T \hat{P}(x_k) x_{k+h},$$

for all possible $[A_k\ B_k] \in \Omega$.

This means that $x_{k+h}^T \hat{P}(x_k) x_{k+h}$ keeps decreasing while $x_{k+h} \notin S$. As S contains a ball centered in the origin, using Lyapunov arguments it is easy to see that there exists a finite $\bar{h}$ such that $x_{k+\bar{h}} \in S$. Then either $x_{k+\bar{h}} \in \Phi_\epsilon$ or not. In the latter case, in order to prove convergence to Φ_ϵ using Lyapunov arguments, $V^*(x_{k+\bar{h}})$ must be lower than $V^*(x_k)$. Again, taking into account that $\hat{F}(x_k)$ and $\hat{P}(x_k)$ are defined by a suboptimizer of (5) for $x = x_k$, which is also feasible for all x_{k+j} with $j \le \bar{h}$, using Property 2, (9) and (3), the following inequalities can be stated for all $j \le \bar{h}$ and uncertainty realization:

$$\begin{array}{ll} V^*(x_{k+j}) & \le x_{k+j}^T \hat{P}(x_k) x_{k+j} \\ x_{k+j}^T \hat{P}(x_k) x_{k+j} & \le x_k^T \hat{P}(x_k) x_k - x_k^T Q_c x_k \\ x_k^T \hat{P}(x_k) x_k - x_k^T Q_c x_k & \le V^*(x_k) + \epsilon - x_k^T Q_c x_k. \end{array} \tag{12}$$

By taking into account that $x_k \notin \Phi_\epsilon$, it can be seen that $V^*(x_{k+\bar{h}}) < V^*(x_k)$. As $V^*(x)$ is a convex function, $\Phi_\epsilon \subseteq \Omega_\alpha$ because $\alpha(\epsilon) \ge \max_{x \in \Phi_\epsilon} V^*(x)$. This way is also proved convergence to Ω_α. Now we will prove that once in Φ_ϵ, the state will remain inside Ω_α.

As $\Phi_\epsilon \subseteq S$, Lemma 1 holds for all $x_k \in \Phi_\epsilon$ so $V^*(x_{k+1}) \le \alpha(\epsilon)$. This means that if $x_k \in \Phi_\epsilon$ then

TABLE I

NUMERICAL RESULTS FOR SYSTEMS OF DIFFERENT ORDERS ($S = \{x : \|x\|_\infty \le x_{\max}\}$, T_D IS THE TREE DEPTH, n_r IS THE NUMBER OF REGIONS, T_{LMI} (S) IS THE AVERAGE TIME FOR SOLVING THE LMI (5), T_{mp} (S) THE TIME FOR EVALUATING THE PIECEWISE AFFINE LAW)

n_x	n_u	$x_{\max}$	T_D	n_r	T_{LMI}	T_{mp}
2	1	1	4	44	0.5	0.001
2	1	2	6	180	0.5	0.001
2	1	5	8	500	0.5	0.008
2	1	10	10	928	0.5	0.005
3	2	1	5	248	0.6	0.009
3	2	2	8	3374	0.6	0.04
3	2	5	12	25512	0.7	0.05
4	2	1	8	3056	1.0	0.04
4	2	2	12	3717	1.2	0.05

$x_{k+1} \in \Omega_\alpha$. Following the previous ideas, using (9) and (12), it is easy to see that if $x_k \in \Phi_\epsilon$ and $x_{k+1} \notin \Phi_\epsilon$ the system will enter again Φ_ϵ without leaving Ω_α.

Robust satisfaction of the constraints is assured because, by (11) and Property 1, at each time step a feedback gain obtained from a feasible solution of (5) is applied. ■

A. Complexity

In general it is not possible to bound a priori the number of regions of a multi-parametric solution given by the proposed approach (see [18] for a discussion). However, it is possible to give an upper bound of the computational burden of evaluating the parametric function organized on a tree structure. The maximum number of linear inequalities that must be evaluated in order to find the solution is linear in the state dimension and in the maximum depth of the tree ($n_x T_d$). The complexity of the controller is both measured by the number of regions (memory constraints) and the maximum depth of the tree (time constraints). Numerical results for three systems (omitted for brevity) are reported in Table I for different state constraints $\|x\|_\infty \le x_{max}$ and for a fixed error bound $\epsilon = 1$. It is apparent that the average time T_{LMI} for solving the LMI (5) with the Matlab LMI Toolbox is sensibly larger than the time T_{mp} for evaluating the piecewise affine function, despite the high number of regions.

V. EXTENSIONS

In this section, different extensions are presented to the proposed approach. These extensions are based on modifying both the multi-parametric algorithm, and the implementation of the control law.

A. Modified Error Bound

Proposition 1: Consider controller (11) based on an approximate solution (8) of the multi-parametric convex program (5) on S, where S is a full dimensional polyhedron containing the origin in its interior, such that the error inside each simplex CS_i is less than $\epsilon_{QSi} = \min_{x \in CS_i} x^T Q_c x$. If $x_0 \in S$ then (11) robustly stabilizes system (1).

Proof: For any state vector inside $CS_i \subseteq S$ Lemma 1 holds, and therefore if $x_k^T Q_c x_k > \epsilon_{QSi}$ then $V^*(x_{k+1}) - V^*(x_k) < 0$, $\forall [A_k \; B_k] \in \Omega$. Following the same ideas as in the proof of Theorem 1, it is easy to see that if the state leaves the set S, the controller assures that it will enter again with a lower value of $V^*(x)$. Following Lyapunov arguments, as S contains a ball centered in the origin, it is immediate to prove that the closed loop system is regulated to the origin. Robust satisfaction of the constraints is assured because, by (11) and Property 1, at each time step a feedback gain obtained from a feasible solution of (5) is applied. ■

The approximate multi-parametric convex programming algorithm can be modified to assure the error bound ϵ_{QSi} bymodifying the stopping criterion of the second phase of the algorithm in [18]. A given simplex is then subdivided if the upper bound on the error is greater than or equal to ϵ_{QSi} which can be evaluated solving a quadratic programming problem. The state space partition obtained is more complex around the origin (where $\epsilon_{QSi} \simeq 0$). In fact, to obtain a finite partition, an additional subdivision criterion must be added to deal with the simplex that contains the origin. In this work, a minimum volume criterion is adopted.

B. Controller with Memory

Proposition 2: Consider the control law

$$\begin{array}{rcl} u_k & = & \hat{F}(x_k)x_k \\ \left[\hat{F}(x_k)\; \hat{P}(x_k)\right] & = & \begin{cases} [F(x_k)\; P(x_k)] & \text{if } \delta = 1 \\ \left[\hat{F}(x_{k-1})\; \hat{P}(x_{k-1})\right] & \text{otherwise,} \end{cases} \\ \{\delta = 1\} & \leftrightarrow & \begin{cases} x_k \in S \text{ and} \\ x_k^T P(x_k) x_k \le x_k^T \hat{P}(x_{k-1}) x_k \end{cases} \end{array} \tag{13}$$

where S is a full dimensional polyhedron, $F(x) = Y(x)Q(x)^{-1}$ and $P(x) = \gamma(x)Q(x)^{-1}$ where $\gamma(x), Q(x), Y(x)$ is taken from a suboptimizer of (5) over S with an error bound $\epsilon > 0$ for x_k. If $x_0 \in S$ then (13) robustly stabilizes the system while satisfying (2) for all possible uncertainties.

Proof: By (13) and Property 2 at each time step $\hat{F}(x_k)$ and $\hat{P}(x_k)$ are defined by a feasible solution of (5) for $x = x_k$ so (3) holds and $x_k^T \hat{P}(x_{k-1}) x_k < x_{k-1}^T \hat{P}(x_{k-1}) x_{k-1}$.

By (13) if $x_k^T P(x_k) x_k > x_k^T \hat{P}(x_{k-1}) x_k$ or the state leaves S, no update is made. This assures that $x_k^T \hat{P}(x_k) x_k < x_{k-1}^T \hat{P}(x_{k-1}) x_{k-1}$ so following Lyapunov arguments, it is easy to prove that the closed loop system converges to the origin. Robust satisfaction of the constraints is assured because, by (13) and Property 1, at each time step a feedback gain obtained from a feasible solution of (5) is applied. ■

When using the introduced control law with memory (13), robust stability and constraint handling is assured for any given error bound of the suboptimizer. The error bound then affects only the performance of the controller, not the stabilizing properties.

C. Relative Error Bounds

The control strategies mentioned in the previous section are based on absolute error bounds. In alternative, a controller can be constructed using an approximate multi-parametric convex programming algorithm that assures a bound ϵ_R on the maximum relative error evaluated as $(\hat{V}(x) - V^*(x))/V^*(x)$. This is easily obtained by modifying the stopping criterion of the second step of the algorithm, so that a given simplex CS_i is subdivided if the upper bound on the relative error is greater than ϵ_R. For all simplices that do not contain the origin, the maximum relative error can be overestimated as

$$\max_{x \in CS_i} \epsilon_R(x) \leq \frac{\epsilon_i}{\min\limits_{x \in CS_i} V^*(x)},$$

where ϵ_i is the maximum absolute error on CS_i and the minimum of $V^*(x)$ is evaluated by solving Problem (5) with the state treated as an additional optimization vector constrained in CS_i.

As in the modified error bound case, the state space partition obtained when a relative error bound is used is more complex around the origin (where $V^*(x) \simeq 0$). In fact, to obtain a finite partition, an additional subdivision criterion must be added to deal with the simplex that contains the origin. In this work, a minimum volume criterion is adopted. It is important to note, that using the introduced control law with memory (13), robust stability and constraint handling is assured.

VI. Numerical Examples

In this section we exemplify the ideas developed above on the following simple LTV second order uncertain system:

$$A_1 = \begin{bmatrix} 0.9 & 0.9 \\ 0 & 0.9 \end{bmatrix}, A_2 = \begin{bmatrix} 0.9 & 0.5 \\ 0 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{14}$$

with $\|x\|_\infty \leq 2$, $\|u\|_\infty \leq 1$, $Q_c = I$ and $R_c = 1$.

For this system, Figures 1(a) and 1(b) respectively show the optimal upper bound $V^*(x)$ defined by Kothare's controller and the corresponding optimal control law $u^*(x)$. Note that the value of $V^*(x)$ goes up to 30. Table II shows the number of regions in the state partition for different ϵ. Figure 2(a) shows the state partition obtained with an absolute error bound $\epsilon = 0.1$. The state partition is more complex near the boundary of the feasible region. This is due to the fact that towards the boundaries the optimal cost function to be approximated has a larger gradient. In order to lower the complexity towards the boundary, an approximate solution with a bound on the relative error can be used, but in that case the state space partition obtained is more complex around the origin (where $V^*(x) \simeq 0$). Figure 2(b) shows the state partition obtained with a relative error bound $\epsilon_R = 0.1$.

A way to obtain low complexity partitions is to modify the stopping criterion in the second phase of the multi-parametric algorithm, in order to assure that *either* the absolute error is below a given bound *or* the relative error is below a given bound. Figure 2(c) shows the state partition that assures a maximum relative error $\epsilon_R = 0.1$ or a maximum error $\epsilon = 0.1$.

TABLE II

NUMBER OF REGIONS n_r OF THE STATE PARTITION FOR DIFFERENT VALUES OF THE ERROR BOUND ϵ FOR SYSTEM 14.

ϵ	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
n_r	70	86	124	148	170	176	248	379	766

Using the introduced control law with memory (13), robust stability and constraint handling is assured. It can be noticed that the partition of Figure 2(c) is less complex than the ones of Figures 2(a) and 2(b).

Figure 2(d) shows the state partition of a suboptimizer which assures a bound on the error on each simplex lower than ϵ_{QSi} as in Section V-A. It can be noticed how the partition is rather complex around the origin (as for the relative bound case) but less towards the boundary.

VII. Conclusions

Multi-parametric quadratic and linear programming theory has been applied with success for implementing deterministic MPC controllers. In this note we have proposed to apply the approximate multi-parametric convex programming solver of [18] to the robust MPC control scheme proposed in [7] and have analized the effecs of the approximation errors on robust stability. An explicit description of the control law is obtained for ease of implementation. The control law assures robust constraint handling and robust convergence to a given bounded set. Also, alternative approaches were given in order to assure convergence to the origin and explicit descriptions with reduced complexity.

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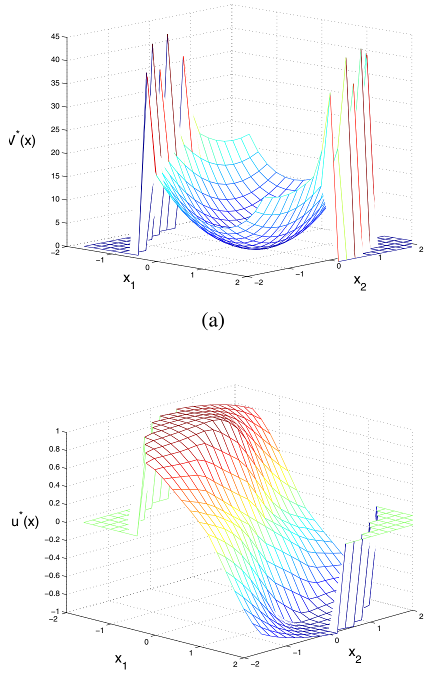

Fig. 1. Optimal cost function $V^*(x)$ (a), and control input $u^*(x) = F^*(x)x$ (b).

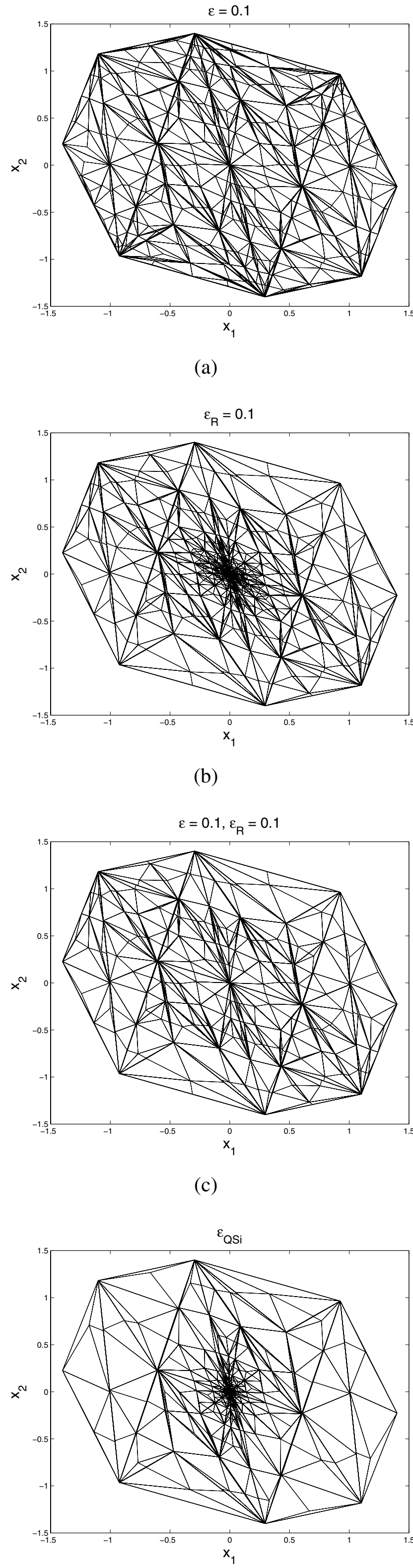

Fig. 2. State space partition corresponding to the approximate solution with absolute error bound $\epsilon = 1$ (a), with relative error bound $\epsilon_R = 0.1$ (b), with absolute error bound $\epsilon = 0.1$ or relative error bound $\epsilon_R = 0.1$ (c), and to the stabilizing criterion described in Section V-A (d).

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