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Applications of MPD

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Abstract

In this note, we aim to derive the MPD equations for the Schwarzschild metric. Then, we examine the conservation of several generalized quantities within the MPD system of equations, and finally, we study the evolution of the spin vector of a test particle moving in Schwarzschild spacetime.
we provide a fairly detailed calculation of the exact set of equations of motion under the influence of a specific supplementary condition, and we attempt to derive the differential equations of motion in terms of quantities such as generalized forces.
This note also discusses the generalized angular momentum and the conservation laws governing it, as well as the analysis of the spin vector evolution in terms of polar coordinates.
This note is the first part of a series on the applications of the MPD equations. The second part will cover more details about a wider variety of motions in the Schwarzschild metric, as well as an examination of the radiative effects of the test particle and the coupling of electromagnetic waves and their spin with gravity and spacetime.

Suppose a Schwarzschild field is generated by a very large and massive object.
the test particle is of macroscopic type, and its center of mass is always located within the region occupied by the particle at a given time.
It is evident from what we know about the MPD equations that the MPD equations alone are not sufficient to determine all the unknowns, and an additional constraint is required to fully specify them.
In fact, the equations for the spin components amount to 3, while the number of independent components of the spin tensor, due to its antisymmetry, is 6.
Therefore, it is necessary to reduce the number of independent parameters by introducing an additional constraint.
The constraint we use is called the SC. Note that adding a constraint corresponds to selecting a specific reference frame.
Since in special relativity all reference frames are equivalent, we can always write the transformation relation between frames and obtain the equations in any desired frame. The constraint under discussion is, in fact, valid in the rest frame of the Schwarzschild.
We begin with the MPD equations:

$$\frac{DP^\alpha}{D\tau}+\frac{1}{2}s^{\mu\nu}u^\sigma R^\alpha{}_{\sigma\mu\nu}=0 \quad (*), \quad P^\alpha=mu^\alpha+u_\beta\frac{Ds^{\alpha\beta}}{D\tau}$$

$$\frac{Ds^{\alpha\beta}}{D\tau}+\frac{u^\alpha}{u^0}\frac{Ds^{\beta 0}}{D\tau}-\frac{u^\beta}{u^0}\frac{Ds^{\alpha 0}}{D\tau}=0 \quad (**), \quad s^{i0}=0 \quad (SC)$$

$$ds^2=e^a dt^2-e^{-a}dr^2-r^2d\Omega^2, \quad e^a=1-\frac{2GM}{r}, \quad a'\equiv\frac{da}{dr}$$

The connections in spherical coordinates are as follows:

$$\Rightarrow\begin{cases}\Gamma^r_{rr}=-\frac{a'}{2}, \quad \Gamma^r_{\theta\theta}=-re^a, \quad \Gamma^r_{\phi\phi}=-re^a\sin^2\theta\\ \Gamma^r_{tt}=\frac{a'e^{2a}}{2}, \quad \Gamma^\theta_{r\theta}=\Gamma^\theta_{\theta r}=\frac{1}{r}, \quad \Gamma^\theta_{\phi\phi}=-\sin\theta\cos\theta\\ \Gamma^\phi_{r\phi}=\Gamma^\phi_{\phi r}=\frac{1}{r}, \quad \Gamma^\phi_{\theta\phi}=\Gamma^\phi_{\phi\theta}=\cot\theta, \quad \Gamma^t_{rt}=\Gamma^t_{tr}=\frac{a'}{2}\end{cases}$$

$$\frac{Ds^{\mu\nu}}{D\tau}=u^\alpha\partial_\alpha s^{\mu\nu}+u^\alpha\Gamma^\nu_{\alpha\sigma}s^{\mu\sigma}+u^\alpha\Gamma^\mu_{\alpha\sigma}s^{\sigma\nu}$$

Based on the supplementary condition (SC) introduced in the text, we have:

$$SC\Rightarrow\frac{Ds^{i0}}{D\tau}=u^\alpha\Gamma^0_{\alpha\sigma}s^{i\sigma}$$

$$(**)\Rightarrow\frac{Ds^{\alpha\beta}}{D\tau}+\frac{u^\alpha}{u^0}\Gamma^0_{\alpha'\sigma}u^{\alpha'}s^{\beta\sigma}-\frac{u^\beta}{u^0}\Gamma^0_{\beta'\sigma}u^{\beta'}s^{\alpha\sigma}=0$$

$$\frac{Ds^{\alpha\beta}}{D\tau}+\frac{\Gamma^0_{\lambda\sigma}u^\lambda}{u^0}\left[u^\alpha s^{\beta\sigma}-u^\beta s^{\alpha\sigma}\right]=0 \quad \Rightarrow$$

$$SC\Rightarrow\boxed{\frac{Ds^{ij}}{D\tau}+\frac{\Gamma^0_{k\lambda}u^\lambda}{u^0}\left(u^i s^{jk}-u^j s^{ik}\right)=0} \quad (***)$$

Now, we proceed to calculate the components of the equation (***) and note that the indices 1, 2, and 3 correspond to r, θ, and ϕ, respectively.

$$\frac{Ds^{12}}{D\tau}+\frac{\Gamma^0_{k\lambda}u^\lambda}{u^0}\left(u^1s^{2k}-u^2s^{1k}\right)=0$$

$$\frac{ds^{12}}{d\tau}+u^\alpha\Gamma^2_{\ell\alpha}s^{1\ell}+u^\alpha\Gamma^1_{\ell\alpha}s^{\ell 2}+\frac{\Gamma^0_{k\lambda}u^\lambda}{u^0}\left(u^1s^{2k}-u^2s^{1k}\right)=0$$

$$s^{ii}=0\Rightarrow\frac{ds^{12}}{d\tau}+\frac{u^1}{r}s^{12}+u^3s^{13}(-\sin\theta\cos\theta)+u^1s^{12}\left(-\frac{a'}{2}\right)-re^a\sin^2\theta u^3s^{32}+u^1s^{21}\frac{a'}{2}=0$$

$$\frac{ds^{12}}{d\tau}+\left(\frac{u^1}{r}-u^1\frac{a'}{2}-u^1\frac{a'}{2}\right)s^{12}-u^3s^{13}\sin\theta\cos\theta-re^a\sin^2\theta u^3s^{32}=0$$

$$\Rightarrow\boxed{\frac{ds^{r\theta}}{d\tau}+\left(\frac{1}{r}-a'\right)\dot{r}s^{r\theta}+re^a\sin^2\theta\dot{\phi}s^{\theta\phi}+\sin\theta\cos\theta\dot{\phi}s^{\phi r}=0} \quad (1)$$

$$\frac{Ds^{23}}{D\tau}+\frac{\Gamma^0_{k\lambda}u^\lambda}{u^0}\left(u^2s^{3k}-u^3s^{2k}\right)=0$$

$$\frac{ds^{23}}{d\tau}+u^{\alpha}\Gamma^{3}_{\alpha i}s^{2i}+u^{\alpha}\Gamma^{2}_{\alpha i}s^{i3}+\frac{a'}{2}\frac{u^1}{u^0}\left(u^2s^{31}-u^3s^{21}\right)=0$$
$$\frac{ds^{23}}{d\tau}+\frac{u^1}{r}s^{23}+\frac{u^3}{r}s^{21}+u^2\cot\theta s^{23}+\frac{u^1}{r}s^{23}+\frac{u^2}{r}s^{13}+\frac{a'}{2}\frac{u^1}{u^0}\left(u^2s^{31}-u^3s^{21}\right)=0$$
$$\frac{ds^{23}}{d\tau}+\left(\frac{u^1}{r}+u^2\cot\theta+\frac{u^1}{r}\right)s^{23}+\left(\frac{u^2}{r}-\frac{a'u_2^2}{2}\right)s^{13}+\left(\frac{u^3}{r}-\frac{a'u_3^2}{2}\right)s^{21}=0$$
$$\boxed{\frac{ds^{\theta\phi}}{d\tau}+\left(\frac{2\dot r}{r}+\cot\theta\dot\theta\right)s^{\theta\phi}+\left(\frac{a'}{2}-\frac{1}{r}\right)\dot\phi s^{r\theta}+\left(\frac{a'}{2}-\frac{1}{r}\right)\dot\theta s^{\phi r}=0}\quad(2)$$
$$\frac{Ds^{13}}{D\tau}+\frac{\Gamma^0_{k\lambda}u^{\lambda}}{u^0}\left(u^1s^{3k}-u^3s^{1k}\right)=0\Rightarrow$$
$$\frac{ds^{13}}{d\tau}+u^{\alpha}\Gamma^{3}_{i\alpha}s^{1i}+u^{\alpha}\Gamma^{1}_{i\alpha}s^{i3}+\frac{a'}{2}\frac{u^1}{u^0}\left(u^1s^{30}-u^3s^{10}\right)+\frac{a'}{2}u^1s^{31}=0$$
$$\frac{ds^{13}}{d\tau}+\frac{u^1}{r}s^{13}+u^3s^{12}\cot\theta+u^2s^{13}\cot\theta+u^1\left(-a'\right)s^{13}+u^2(-re^a)s^{23}=0$$
$$s^{13}u^2\cot\theta+\frac{ds^{13}}{d\tau}+u^1\left(\frac{1}{r}-a'\right)s^{13}+u^3\cot\theta s^{12}-re^au^2s^{23}=0$$
$$\boxed{\frac{ds^{r\phi}}{d\tau}+\left(\dot r\left(\frac{1}{r}-a'\right)+\dot\theta\cot\theta\right)s^{r\phi}+\cot\theta\dot\phi s^{r\theta}-re^a\dot\theta s^{\theta\phi}=0}\quad(3)$$

Now, we calculate the 0th component of the second term in equation (*), which corresponds to the Mathisson spin-curvature force:

$$\frac{1}{2}s^{\mu\nu}u^{\sigma}R^0{}_{\sigma\mu\nu}=s^{\mu\nu}u^{\sigma}(\partial_\mu\Gamma^0_{\sigma\nu}+\Gamma^0_{\mu\lambda}\Gamma^{\lambda}_{\sigma\nu})=0$$
$$\Rightarrow\frac{DP^0}{D\tau}=0\Rightarrow\frac{dP^0}{d\tau}+u^{\alpha}\Gamma^0_{\alpha\beta}P^{\beta}=0$$

According to the definition of the generalized momentum in the MPD equations and the definition of m_s, we can ultimately calculate the components of equation (*):

$$P^0=m\dot t+u_\gamma\frac{Ds^{0\gamma}}{D\tau},\qquad\frac{Ds^{0\gamma}}{D\tau}=\Gamma^0_{\alpha\beta}u^{\alpha}s^{\beta\gamma}=\Gamma^0_{01}\dot ts^{1\gamma}$$
$$u_\gamma\frac{Ds^{0\gamma}}{D\tau}=\Gamma^0_{01}\dot tu_\gamma s^{1\gamma}=\underbrace{\dot t\Gamma^0_{01}u_\gamma s^{1\gamma}}_{:=\dot tm_s}=\dot tm_s$$
$$P^0=\dot t(m+m_s)$$
$$m_s=\frac{a'}{2}\left(\dot\theta(-r^2)s^{12}-r^2\sin^2\theta\dot\phi s^{13}\right)=\frac{r^2a'}{2}\left(\sin^2\theta\dot\phi s^{31}-\dot\theta s^{12}\right)$$
$$\Rightarrow\frac{d}{d\tau}\left(\dot t(m+m_s)\right)+\left(\dot tP^1+\dot rP^0\right)\frac{a'}{2}=0$$
$$\frac{d}{d\tau}\left(\dot t(m+m_s)\right)+\frac{a'}{2}\dot t\dot r(m+m_s)+m\ddot r+u_\gamma\frac{Ds^{1\gamma}}{D\tau}=0$$
$$u_\gamma\frac{Ds^{1\gamma}}{D\tau}=u_\gamma\frac{ds^{1\gamma}}{d\tau}+u_\gamma u^{\alpha}\Gamma^{\gamma}_{\alpha\beta}s^{1\beta}+u^{\alpha}u_\gamma\Gamma^1_{\alpha\beta}s^{\beta\gamma}$$
$$=u_\gamma\frac{ds^{1\gamma}}{d\tau}+u_\gamma u^{\alpha}\Gamma^{\gamma}_{\alpha2}s^{12}+u_\gamma u^{\alpha}\Gamma^{\gamma}_{\alpha3}s^{13}+u^1u_\gamma\Gamma^1_{11}s^{1\gamma}+u^2u_\gamma\Gamma^1_{22}s^{2\gamma}+u^3u_\gamma\Gamma^1_{33}s^{3\gamma}$$
$$=u_\gamma\frac{ds^{1\gamma}}{d\tau}+u^1u^2(-re^a)s^{12}+u^2u^1\frac{1}{r}s^{12}+u^3u^3\cot\theta s^{12}$$
$$+u^1u^3(-re^a\sin^2\theta)s^{13}+u^2u^3(-\sin\theta\cos\theta)s^{13}+u^3u^1\frac{1}{r}s^{13}+u^3u^2\cot\theta s^{13}$$
$$+\left(-\frac{a'}{2}\right)u^1u^2s^{12}+u^1u^3\left(-\frac{a'}{2}\right)s^{13}+u^2u^1(-re^a)s^{21}+u^2u^3(-re^a)s^{23}$$
$$+u^3u^1(-re^a\sin^2\theta)s^{31}+u^3u^2(-re^a\sin^2\theta)s^{32}$$

$$= u_\gamma \frac{ds^{1\gamma}}{d\tau} + \dot{r}\dot{\theta}(-e^{-a})(-re^a)s^{12} + \dot{r}\dot{\theta}(-r^2)\frac{1}{r}s^{12} + \dot{\phi}^2(-r^2\sin^2\theta)\cot\theta s^{12}$$
$$+ \dot{r}\dot{\phi}(-e^{-a})(-re^a\sin^2\theta)s^{13} + \dot{\theta}\dot{\phi}(-r^2)(-\sin\theta\cos\theta)s^{13} + \dot{\phi}\dot{r}(-r^2\sin^2\theta)\frac{1}{r}s^{13}$$
$$+ \dot{\theta}\dot{\phi}(-r^2\sin^2\theta)\cot\theta s^{13} + \left(-\frac{a'}{2}\right)\dot{r}\dot{\theta}(-r^2)s^{12} + \dot{r}\dot{\phi}(-r^2\sin^2\theta)\left(-\frac{a'}{2}\right)s^{13}$$
$$+ \dot{\theta}\dot{r}(e^{-a})(re^a)s^{21} + \dot{\theta}\dot{\phi}(-r^2\sin^2\theta)(-re^a)s^{23}$$
$$+ \dot{\phi}\dot{r}(-e^{-a})(-re^a\sin^2\theta)s^{31} + \dot{\phi}\dot{\theta}(-r^2)(-re^a\sin^2\theta)s^{32}$$
$$= u_\gamma \frac{ds^{1\gamma}}{d\tau} + r\dot{r}\dot{\theta}s^{21} + \frac{a'}{2}r^2\dot{r}\left(\dot{\theta}s^{12} + \dot{\phi}\sin^2\theta s^{13}\right) - r^2\dot{\phi}^2\sin\theta\cos\theta s^{12}$$
$$= u_\gamma \frac{ds^{1\gamma}}{d\tau} + \frac{2m_s}{r^2a'}r\dot{r} - r^2\dot{r}\frac{a'}{2} - r^2\dot{\phi}^2\sin\theta\cos\theta s^{12}$$
$$u_\gamma \frac{Ds^{1\gamma}}{D\tau} = u_\gamma \frac{ds^{1\gamma}}{d\tau} + \frac{2m_s\dot{r}}{ra'}\left(1 - \frac{ra'}{2}\right) - r^2\dot{\phi}^2\sin\theta\cos\theta s^{12}$$
$$u_\gamma \frac{ds^{1\gamma}}{d\tau} = -\dot{r}e^{-a}\frac{ds^{11}}{d\tau} + \dot{\theta}(-r^2)\frac{ds^{12}}{d\tau} + \dot{\phi}(-r^2\sin^2\theta)\frac{ds^{13}}{d\tau}$$
$$u_\gamma \frac{ds^{1\gamma}}{d\tau} = -r^2\dot{\theta}\frac{ds^{12}}{d\tau} - r^2\dot{\phi}\sin^2\theta\frac{ds^{13}}{d\tau}$$
$$\Rightarrow (1),(3) = r^2\dot{\theta}\dot{r}\left(\frac{1}{r} - a'\right)s^{12} + r^3\dot{\theta}\dot{\phi}e^a\sin^2\theta s^{23} + r^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta s^{31} + r^2\dot{\phi}\sin^2\theta\left(\dot{r}\left(\frac{1}{r} - a'\right) + \cot\theta\dot{\theta}\right)s^{13}$$
$$+ r^2\dot{\phi}^2\sin\theta\cos\theta s^{12} - r^3\dot{\phi}\sin^2\theta e^a\dot{\theta}s^{23}$$
$$= r^2\dot{r}\left(\frac{1}{r} - a'\right)\underbrace{\left(\dot{\theta}s^{12} + \dot{\phi}\sin^2\theta s^{13}\right)}_{-\frac{2m_s}{r^2a'}} + s^{12}r^2\dot{\phi}^2\sin\theta\cos\theta = -\frac{2m_s\dot{r}}{a'}\left(\frac{1}{r} - a'\right) + r^2\dot{\phi}^2\sin\theta\cos\theta s^{12}$$
$$\Rightarrow \frac{d}{d\tau}\left(\dot{t}(m+m_s)\right) + a'\dot{t}\dot{r}(m+m_s) + m\ddot{r} + \frac{-2m_s\dot{r}}{a'r}(1 - a'r) + r^2\dot{\phi}^2\sin\theta\cos\theta s_{12} + \frac{2m_s\dot{r}}{a'r}\left(1 - \frac{a'r}{2}\right)$$
$$- r^2\dot{\phi}^2\sin\theta\cos\theta s_{12} = 0$$
$$\Rightarrow \frac{d}{d\tau}\left(\dot{t}(m+m_s)\right) + \dot{t}\dot{r}a'(m+m_s) = 0$$
$$Q^\alpha := \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma$$
$$Q^0 = 2\Gamma^0_{01}\dot{r}\dot{t} \quad Q^0 = \dot{t}\dot{r}a' \to \boxed{\frac{d(\dot{t}(m+m_s))}{d\tau} + Q^0(m+m_s) = 0} \quad (4)$$
$$\frac{DP^i}{D\tau} + \frac{1}{2}s^{\mu\nu}u^\sigma R^i{}_{\sigma\mu\nu} = 0$$
$$\frac{dP^i}{d\tau} + u^\alpha\Gamma^i_{\alpha\sigma}P^\sigma + s^{\mu\nu}u^\sigma(\Gamma^i_{\sigma\nu,\mu} + \Gamma^i_{\mu\lambda}\Gamma^\lambda_{\sigma\nu}) = 0$$
$$\frac{d}{d\tau}\left(mu^i + u_\beta\frac{Ds^{i\beta}}{D\tau}\right) + \Gamma^i_{\alpha\sigma}u^\alpha\left(mu^\sigma + u_\beta\frac{Ds^{\sigma\beta}}{D\tau}\right) + s^{\mu\nu}u^\sigma\Gamma^i_{\sigma\nu,\mu} + s^{\mu\nu}u^\sigma\Gamma^i_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} = 0$$
$$(**) \Rightarrow \frac{Ds^{\alpha\beta}}{D\tau} = \frac{u^\beta}{u^0}\frac{Ds^{\alpha 0}}{D\tau} - \frac{u^\alpha}{u^0}\frac{Ds^{\beta 0}}{D\tau}$$
$$\frac{Ds^{\alpha\beta}}{D\tau} = \frac{u^\beta}{u^0}\Gamma^0_{\sigma\gamma}u^\sigma s^{\alpha\gamma} - \frac{u^\alpha}{u^0}\Gamma^0_{\sigma\gamma}u^\sigma s^{\beta\gamma}$$
$$= \frac{u^\beta}{u^0}\frac{a'}{2}u^0 s^{\alpha 1} - \frac{u^\alpha}{u^0}\frac{a'}{2}u^0 s^{\beta 1} = \frac{a'}{2}(u^\beta s^{\alpha 1} - u^\alpha s^{\beta 1})$$
$$u_\beta\frac{Ds^{\alpha\beta}}{D\tau} = \frac{a'}{2}(s^{\alpha 1} - u_\beta u^\alpha s^{\beta 1}) = \frac{a'}{2}(s^{\alpha 1} - u^\alpha u^2 s^{21} - u^\alpha u^3 s^{31})$$
$$= \frac{a'}{2}\left(s^{\alpha 1} + u^\alpha(r^2\dot{\theta}s^{21} + r^2\sin^2\theta\dot{\phi}s^{31})\right) = \frac{a'}{2}s^{\alpha 1} + m_s u^\alpha$$
$$\Rightarrow u_\beta\frac{Ds^{\alpha\beta}}{D\tau} = \frac{a'}{2}s^{\alpha 1} + m_s u^\alpha \quad (II)$$

$$P^\alpha = mu^\alpha + m_s u^\alpha + \frac{a'}{2}s^{\alpha 1}$$

$$P^\alpha = (m+m_s)u^\alpha + \frac{a'}{2}s^{\alpha 1} \quad (III)$$

$$\frac{d}{d\tau}(\dot{r}(m+m_s)) + \Gamma^1_{\alpha\sigma}u^\alpha\left((m+m_s)u^\sigma + \frac{a'}{2}s^{\sigma 1}\right) + s^{\mu\nu}u^\sigma\Gamma^1_{\sigma\nu,\mu} + s^{\mu\nu}u^\sigma\Gamma^1_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} = 0$$

$$\frac{d}{d\tau}(\dot{r}(m+m_s)) + Q^1(m+m_s) + \frac{a'}{2}u^\alpha\Gamma^1_{\alpha\sigma}s^{\sigma 1} + \frac{1}{2}s^{\mu\nu}u^\sigma R^1{}_{\sigma\mu\nu} = 0$$

$$R^1{}_{212} = -\frac{re^a a'}{2}, \quad R^1{}_{313} = -\frac{re^a\sin^2\theta a'}{2}$$

$$\frac{1}{2}s^{\mu\nu}u^\sigma R^1{}_{\sigma\mu\nu} = -s^{12}u^2\frac{re^a a'}{2} - s^{13}u^3\frac{re^a\sin^2\theta a'}{2}$$

$$= -\frac{re^a a'}{2}\left(\dot{\theta}s^{12} + \sin^2\theta\dot{\phi}s^{13}\right) = \frac{re^a m_s}{r^2} = r^{-1}e^a m_s$$

$$\frac{d}{d\tau}(\dot{r}(m+m_s)) + Q^1(m+m_s) + \frac{a'}{2}u^\alpha\Gamma^1_{\alpha\sigma}s^{\sigma 1} + \frac{e^a m_s}{r} = 0$$

$$\Gamma^1_{\alpha\sigma}u^\alpha s^{\sigma 1} = \Gamma^1_{11}u^1 s^{11} + (re^a)\dot{\theta}s^{12} + re^a\sin^2\theta\dot{\phi}s^{13}$$

$$\frac{a'}{2}\Gamma^1_{\alpha\sigma}u^\alpha s^{\sigma 1} = \frac{a'}{2}re^a\frac{-2m_s}{r^2 a'} = -m_s e^a r^{-1}$$

$$\boxed{\frac{d}{d\tau}(\dot{r}(m+m_s)) + Q^1(m+m_s) = 0} \quad (5)$$

$$A^\alpha := u_\beta\frac{Ds^{\alpha\beta}}{D\tau}, \quad P^\alpha = mu^\alpha + A^\alpha \Rightarrow \frac{DP^\alpha}{D\tau} = \underbrace{\left[\frac{d(mu^\alpha)}{d\tau} + u_\delta\Gamma^\alpha_{\delta\sigma}mu^\sigma\right]}_{=mQ^\alpha} + \frac{dA^\alpha}{d\tau} + u_\delta\Gamma^\alpha_{\delta\sigma}A^\sigma$$

$$(II) \Rightarrow \frac{DP^\alpha}{D\tau} = \frac{d(mu^\alpha)}{d\tau} + mQ^\alpha + \frac{d}{d\tau}\left(\frac{a'}{2}s^{\alpha 1} + m_s u^\alpha\right) + u_\delta\Gamma^\alpha_{\delta\sigma}\left(\frac{a'}{2}s^{\sigma 1} + m_s u^\sigma\right)$$

$$= \frac{d}{d\tau}(u^\alpha(m+m_s)) + Q^\alpha(m+m_s) + \frac{1}{2}\frac{da'}{d\tau}s^{\alpha 1} + \frac{a'}{2}\left(\frac{Ds^{\alpha 1}}{D\tau} - u_\delta\Gamma^1_{\delta\sigma}s^{\alpha\sigma}\right)$$

$$a' = \frac{2GM}{r^2(1-2GM/r)} = \frac{2GM}{r^2 - 2GMr} \Rightarrow \frac{da'}{d\tau} = \frac{-2GM}{(r^2-2GMr)^2}(-2r\dot{r} + 2GM\dot{r})$$

$$\frac{da'}{d\tau} = \frac{4GM\dot{r}(GM-r)}{r^2(r-2GM)^2} = \frac{r(2\dot{r})a'e^a(1-e^a/2-1)}{r^2(e^{2a})}$$

$$\frac{da'}{d\tau} = \frac{2\dot{r}a'}{re^a}\left(-e^a - \frac{1}{2}\right) = -\frac{\dot{r}a'}{re^a}(1+e^a)$$

$$\frac{d(u^\alpha(m+m_s))}{d\tau} + Q^\alpha(m+m_s) + \frac{a'}{2}\left(-\frac{s^{\alpha 1}}{re^a}(1+e^a) + \frac{a'\dot{r}}{2}s^{\alpha 1} - u_\delta\Gamma^1_{\delta\sigma}s^{\alpha\sigma}\right) + \frac{1}{2}s^{\mu\nu}u^\sigma R^\alpha{}_{\sigma\mu\nu} = 0$$

$$\frac{d(u^\alpha(m+m_s))}{d\tau} + Q^\alpha(m+m_s) + \frac{a'}{2}\left(\left(1 - \frac{2}{re^a a'}(1+e^a)\right)\frac{a'\dot{r}}{2}s^{\alpha 1} - u_\delta\Gamma^1_{\delta\sigma}s^{\alpha\sigma}\right) + \frac{1}{2}s^{\mu\nu}u^\sigma R^\alpha{}_{\sigma\mu\nu} = 0$$

$$e^a a' = \frac{1}{r}\frac{2GM}{r} = \frac{1}{r}(1-e^a) \Rightarrow re^a a' = 1 - e^a$$

$$\Rightarrow 1 - \frac{2(1+e^a)}{re^a a'} = 1 - \frac{2(1+e^a)}{1-e^a} = \frac{1-e^a-2-2e^a}{1-e^a} = \frac{-1-3e^a}{1-e^a} = -\frac{1+3e^a}{1-e^a}$$

$$\frac{d(u^\alpha(m+m_s))}{d\tau} + Q^\alpha(m+m_s) + \frac{a'}{2}\left(\frac{a'\dot{r}}{2}\frac{1+3e^a}{1-e^a}s^{1\alpha} + \Gamma^1_{\delta\sigma}u^\delta s^{\sigma\alpha}\right) + \frac{1}{2}s^{\mu\nu}u^\sigma R^\alpha{}_{\sigma\mu\nu} = 0$$

$$\Downarrow$$

$$\frac{d(u^\alpha(m+m_s))}{d\tau} + Q^\alpha(m+m_s) + \frac{a'}{2}\left(\frac{\dot{r}}{2}\frac{1+3e^a}{re^a}s^{1\alpha} + \Gamma^1_{\delta\sigma}u^\delta s^{\sigma\alpha}\right) + \frac{1}{2}s^{\mu\nu}u^\sigma R^\alpha{}_{\sigma\mu\nu} = 0$$

$$\Gamma^1_{\delta\sigma}u^\delta s^{\sigma\alpha} = \Gamma^1_{11}u^1 s^{1\alpha} + \Gamma^1_{22}u^2 s^{2\alpha} + \Gamma^1_{33}u^3 s^{3\alpha}$$

$$= -\frac{a'}{2}\dot{r}s^{1\alpha} + (-re^a)\dot{\theta}s^{2\alpha} - re^a\sin^2\theta\dot{\phi}s^{3\alpha} \rightarrow$$

$$\frac{d(u^\alpha(m+m_s))}{d\tau}+Q^\alpha(m+m_s)+\frac{a'}{2}\left(\frac{\dot r}{2}\left(\frac{1+3e^a}{re^a}-a'\right)s^{1\alpha}-re^a\sin^2\theta\dot\phi s^{3\alpha}-re^a\dot\theta s^{2\alpha}\right)+\frac{1}{2}s^{\mu\nu}u^\sigma R^\alpha{}_{\sigma\mu\nu}=0$$

$$\frac{1+3e^a}{re^a}-a'=\frac{1+3e^a}{re^a}-\frac{1-e^a}{re^a}=\frac{4}{r}$$

$$\Rightarrow\boxed{\frac{d(u^\alpha(m+m_s))}{d\tau}+Q^\alpha(m+m_s)+\frac{a'}{2}\left(\frac{2\dot r}{r}s^{1\alpha}-re^a\dot\theta s^{2\alpha}-re^a\sin^2\theta\dot\phi s^{3\alpha}\right)+\frac{1}{2}s^{\mu\nu}u^\sigma R^\alpha{}_{\sigma\mu\nu}=0}$$

$$\frac{1}{2}s^{\mu\nu}u^\sigma R^2{}_{\sigma\mu\nu}=u^\sigma s^{\mu\nu}(\Gamma^2_{\sigma\nu,\mu}+\Gamma^2_{\mu\lambda}\Gamma^\lambda_{\sigma\nu})=s^{\mu\nu}u^\sigma\Gamma^2_{\sigma\nu,\mu}+s^{\mu\nu}u^\sigma\Gamma^2_{\mu\lambda}\Gamma^\lambda_{\sigma\nu}$$

$$=s^{11}u^2\Gamma^2_{21,1}+s^{12}u^1\Gamma^2_{12,1}+s^{23}u^3\Gamma^2_{33,2}+s^{1\nu}u^\sigma\Gamma^2_{12}\Gamma^2_{\sigma\nu}+s^{2\nu}u^\sigma\Gamma^2_{21}\Gamma^1_{\sigma\nu}+s^{3\nu}u^\sigma\Gamma^2_{33}\Gamma^3_{\sigma\nu}$$

$$=-s^{12}\frac{\dot r}{r^2}-s^{23}\dot\phi\cos(2\theta)+s^{12}u^1\Gamma^2_{12}\Gamma^2_{12}+s^{13}u^3\Gamma^2_{12}\Gamma^2_{33}+s^{21}u^1\Gamma^2_{21}\Gamma^1_{11}+s^{23}u^3\Gamma^2_{21}\Gamma^1_{33}$$

$$+s^{31}u^3\Gamma^2_{33}\Gamma^3_{31}+s^{32}u^3\Gamma^2_{33}\Gamma^3_{32}$$

$$=s^{12}\left(-\frac{\dot r}{r^2}+\frac{\dot r}{r^2}+\frac{\dot r^2a'}{2r}\right)+s^{23}(\dot\phi)(-\cos(2\theta)+\underbrace{-e^a\sin^2\theta+\sin\theta\cos\theta\cot\theta}_{\cos^2\theta})$$

$$=\frac{\dot ra'}{2r}s^{12}+\dot\phi\sin^2\theta(1-e^a)s^{23}$$

$$\frac{du^2(m+m_s)}{d\tau}+Q^2(m+m_s)+\frac{3}{2}\frac{\dot ra'}{r}s^{12}+s^{23}\dot\phi\sin^2\theta\left(1-e^a+\frac{ra'e^a}{2}\right)=0\Rightarrow$$

$$\boxed{\frac{d\dot\theta(m+m_s)}{d\tau}+Q^2(m+m_s)+\frac{3a'}{2r}\dot rs^{12}+\frac{3a'}{2re^a}\sin^2\theta\dot\phi s^{23}=0}\quad(6)$$

$$\frac{1}{2}s^{\mu\nu}u^\sigma R^3{}_{\sigma\mu\nu}=s^{\mu\nu}u^\sigma(\Gamma^3_{\sigma\nu,\mu}+\Gamma^3_{\mu\lambda}\Gamma^\lambda_{\sigma\nu})$$

$$=s^{13}u^1\Gamma^3_{13,1}+s^{1\nu}u^\sigma\Gamma^3_{13}\Gamma^3_{\sigma\nu}+s^{3\nu}u^\sigma\Gamma^3_{31}\Gamma^1_{\sigma\nu}+s^{2\nu}u^\sigma\Gamma^3_{23}\Gamma^3_{\sigma\nu}$$

$$+s^{3\nu}u^\sigma\Gamma^3_{32}\Gamma^2_{\sigma\nu}+s^{23}u^2\Gamma^3_{23,2}+s^{22}u^3\Gamma^3_{32,2}$$

$$=-s^{13}\frac{\dot r}{r^2}+\dot\theta s^{23}\left(-\frac{1}{\sin^2\theta}\right)+s^{12}u^3\Gamma^3_{13}\Gamma^3_{32}+s^{13}u^1\Gamma^3_{13}\Gamma^3_{13}+s^{13}u^2\Gamma^3_{13}\Gamma^3_{23}$$

$$+s^{31}u^1\Gamma^3_{31}\Gamma^1_{11}+s^{32}u^2\Gamma^3_{31}\Gamma^1_{22}+s^{21}u^3\Gamma^3_{23}\Gamma^3_{31}+s^{23}u^1\Gamma^3_{23}\Gamma^3_{13}+s^{23}u^2\Gamma^3_{23}\Gamma^3_{23}$$

$$+s^{31}u^2\Gamma^3_{32}\Gamma^2_{21}+s^{32}u^1\Gamma^3_{32}\Gamma^2_{12}$$

$$=-\frac{\dot r}{r^2}s^{13}-\frac{\dot\theta s^{23}}{\sin^2\theta}+s^{12}\frac{\dot\phi\cot\theta}{r}+s^{13}\frac{\dot r}{r^2}+s^{13}\frac{\dot\theta\cot\theta}{r}-\frac{a'}{2r}\dot rs^{31}$$

$$+s^{32}\dot\theta\frac{1}{r}(-re^a)+s^{21}\frac{\dot\phi\cot\theta}{r}+s^{23}\frac{\dot r}{r\tan\theta}+s^{23}\dot\theta(\cot\theta)^2+s^{31}\frac{\dot\theta\cot\theta}{r}+s^{32}\frac{\dot r\cot\theta}{r}$$

$$=s^{23}\dot\theta\left(-\frac{1}{\sin^2\theta}+e^a+\frac{\cos^2\theta}{\sin^2\theta}\right)+\frac{a'\dot r}{2r}s^{13}=\boxed{\dot\theta(e^a-1)s^{23}+\frac{a'\dot r}{2r}s^{13}}\quad(7)$$

$$\text{other term}=\dot\theta s^{23}\left(e^a-1-\frac{ra'e^a}{2}\right)+\frac{3a'\dot r}{2r}s^{13}\quad\Rightarrow$$

$$\boxed{\frac{d\left(\dot\phi\,(m+m_s)\right)}{d\tau}+Q^3\,(m+m_s)-\frac{3a'}{2}re^a\dot\theta s^{23}+\frac{3a'}{2r}\dot rs^{13}=0}\quad(8)$$

Now, we calculate the generalized angular momentum. angular momentum consists of two parts: a spin part and an orbital part. The orbital part, similar to the classical case, is obtained from the cross product of the position vector and the linear momentum, with the difference that here the linear momentum is defined as the generalized momentum.

$L_z=r^2\sin^2\theta P^3+s_z,\quad s_z=s^{xy}$

For practical applications, we want to switch to Cartesian coordinates. Since we can easily apply this coordinate transformation to the spin tensor for converting from polar to Cartesian coordinates, we can write:

$$s^{xy} = \frac{\partial x}{\partial q^\alpha}\frac{\partial y}{\partial q^\beta}s^{\alpha\beta} \quad x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$s^{xy} = \left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\right)s^{r\theta} + \left(\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \phi}\right)s^{\phi\theta} + \left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial r}\right)s^{r\phi}$$

$$s^{xy} = s^{r\theta}(\sin\theta\cos\phi(r\cos\theta\sin\phi) - r\cos\theta\cos\phi\sin\theta\sin\phi)$$

$$+ (-r^2\sin\theta\sin^2\phi\cos\theta - r^2\cos\theta\cos^2\phi\sin\theta)s^{\phi\theta} + s^{r\phi}(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi)$$

$$s^{xy} = s_z = r\sin^2\theta s^{r\phi} - r^2\cos\theta\sin\theta s^{\phi\theta}$$

$$s_z = -r^2\sin\theta\cos\theta s^{\phi\theta} + r\sin^2\theta s^{r\phi}$$

$$\boxed{L_z = r^2\sin^2\theta P^3 + r^2\sin\theta\cos\theta s^{23} + r\sin^2\theta s^{13}}$$

let's look at the time evolution of the generalized angular momentum vector:

$$\frac{dL_z}{d\tau} = P^3(2r\dot{r}\sin^2\theta + 2r^2\sin\theta\cos\theta\dot{\theta}) + r^2\sin^2\theta\frac{dP^3}{d\tau}$$

$$+ (2r\dot{r}\sin\theta\cos\theta + r^2\cos(2\theta)\dot{\theta})s^{23} + r^2\sin\theta\cos\theta\dot{s}^{23}$$

$$+ (\dot{r}\sin^2\theta + 2r\sin\theta\cos\theta\dot{\theta})s^{13} + r\sin^2\theta\dot{s}^{13}$$

$$\frac{dL_z}{d\tau} = 2r^2\sin^2\theta P^3\left(\frac{\dot{r}}{r} + \cot\theta\dot{\theta}\right) + r^2\sin^2\theta\frac{dP^3}{d\tau}$$

$$+ r^2\left(2\frac{\dot{r}}{r}\sin\theta\cos\theta + \cos(2\theta)\dot{\theta}\right)s^{23} + r\sin^2\theta(\dot{r} + 2r\cot\theta\dot{\theta})s^{13}$$

$$+ r^2\sin\theta\cos\theta\left[-\left(2\frac{\dot{r}}{r} + \cot\theta\dot{\theta}\right)s^{23} - \left(\frac{a'}{2} - \frac{1}{r}\right)\dot{\phi}s^{12} - \left(\frac{a'}{2} - \frac{1}{r}\right)\dot{\theta}s^{31}\right]$$

$$+ r\sin^2\theta\left[-\left(\left(\frac{1}{r} - a'\right)\dot{r} + \dot{\theta}\cot\theta\right)s^{13} - \cot\theta\dot{\phi}s^{12} + re^a\dot{\theta}s^{23}\right]$$

$$= r^2\sin^2\theta\left(\frac{DP^3}{D\tau} - u_\delta\Gamma^3_{\delta\sigma}P^\sigma\right) + 2\left(\frac{\dot{r}}{r} + \cot\theta\dot{\theta}\right)P^3 + r^2\sin^2\theta\dot{\theta}(e^a - 1)s^{23}$$

$$+ s^{13}\left[2r\dot{\theta}\sin\theta\cos\theta + \dot{\theta}r^2\sin\theta\cos\theta\left(a'r - \frac{1}{r}\right) - r\dot{\theta}\sin\theta\cos\theta + r\dot{r}a'\sin^2\theta\right]$$

$$+ s^{12}(-\dot{\phi})\left[r^2\sin\theta\cos\theta\left(\frac{a'}{2} - \frac{1}{r}\right) + r\sin\theta\cos\theta\right]$$

$$= r^2\sin^2\theta\left[-\dot{\theta}(e^a - 1)s^{23} - \frac{a'\dot{r}}{2r}s^{13} + P^3\left(\frac{\dot{r}}{r} + \cot\theta\dot{\theta}\right) - P^2\dot{\phi}\cot\theta - P^1\frac{\dot{\phi}}{r}!\right]$$

$$+ r^2\sin^2\theta\dot{\theta}(e^a - 1)s^{23} + \frac{a'r^2}{2}\sin^2\theta\left(2\frac{\dot{r}}{r} + \cot\theta\dot{\theta}\right)s^{13} - \frac{a'r^2}{2}\sin\theta\cos\theta\dot{\phi}s^{12}$$

$$(III) \Rightarrow= \frac{a'r^2\sin^2\theta}{2}\left(2\frac{\dot{r}}{r} + \cot\theta\dot{\theta}\right)s^{13} + r^2\sin^2\theta\left(\frac{\dot{r}}{r} + \cot\theta\dot{\theta}\right)\left((m + m_s)\dot{\phi} + \frac{a'}{2}s^{31}\right)$$

$$- r^2\dot{\phi}\sin\theta\cos\theta\left((m + m_s)\dot{\theta} + \frac{a'}{2}s^{21}\right) - r\dot{\phi}\sin^2\theta\left((m + m_s)\dot{r}\right) - \frac{a'r}{2}\dot{\phi}^2\sin\theta\cos\theta s^{12}$$

$$= r^2\sin^2\theta\left(\frac{\dot{r}}{r} + \cot\theta\dot{\theta}\right)\left(\frac{a'}{2}s^{13} + (m + m_s)\dot{\phi} + \frac{a'}{2}s^{31}\right) - r^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta(m + m_s) - r\dot{r}\dot{\phi}\sin^2\theta(m + m_s)$$

$$= \left[r^2\sin^2\theta\left(\frac{\dot{r}}{r} + \cot\theta\dot{\theta}\right)\dot{\phi} - r^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta - r\dot{r}\dot{\phi}\sin^2\theta\right](m + m_s)$$

$$= \dot{\phi}(m + m_s)\left[r\dot{r}\sin^2\theta + r^2\dot{\theta}\sin\theta\cos\theta - r^2\dot{\theta}\sin\theta\cos\theta - r\dot{r}\sin^2\theta\right] = 0$$

$$\Rightarrow \boxed{\frac{dL_z}{d\tau} = 0} \Rightarrow \boxed{L_z = \text{constant}}$$

Remarkably, we see that the angular momentum in the z-direction remains constant over time!
Therefore, we have the conservation of generalized angular momentum.
It is evident that since the coordinate axes are not preferred over each other, the conservation of angular momentum holds in all three directions: x, y, and z.
Based on equation (4), we have:

$$\frac{d(\dot{t}(m+m_s))}{d\tau} + a'\dot{r}\dot{t}(m+m_s) = 0 \Rightarrow \frac{d(\dot{t}(m+m_s))}{\dot{t}(m+m_s)} = -da \Rightarrow \ln|\dot{t}(m+m_s)| = -a + \text{constant}$$

$$\Rightarrow e^a\dot{t}(m_s+m) = \text{constant} := E \Rightarrow \boxed{E = e^a\dot{t}(m+m_s)}$$

Thus, with the definition of the aforementioned constant as energy, we also have energy conservation!
Now, let's go a step further and explicitly calculate the generalized linear momentum vector and the generalized angular momentum vector in Cartesian coordinates:

$$\vec{P} = (P_r\sin\theta\cos\phi + rP_\theta\cos\theta\cos\phi - rP_\phi\sin\phi\sin\theta)\hat{x}$$
$$+ (P_r\sin\theta\sin\phi + rP_\theta\cos\theta\sin\phi + rP_\phi\sin\theta\cos\phi)\hat{y}$$
$$+ (P_r\cos\theta - rP_\theta\sin\theta)\hat{z}$$
$$\vec{L} = \vec{r}\times\vec{P} + \vec{s}$$
$$\vec{L} = (r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z})\times\vec{P} + \vec{s}$$
$$= \hat{z}(P_r r\sin^2\theta\sin\phi\cos\phi + r^2P_\theta\sin\theta\cos\theta\sin\phi\cos\phi + r^2P_\phi\sin^2\theta\cos^2\phi$$
$$- P_r r\sin^2\theta\sin\phi\cos\phi - r^2P_\theta\sin\theta\cos\theta\sin\phi\cos\phi + r^2P_\phi\sin^2\theta\sin^2\phi)$$
$$+ \hat{y}(-P_r r\sin\theta\cos\theta\cos\phi + r^2P_\theta\sin^2\theta\cos\phi + rP_r\sin\theta\cos\theta\cos\phi + r^2P_\theta\cos^2\theta\cos\phi$$
$$- r^2P_\phi\sin\phi\sin\theta\cos\theta)$$
$$+ \hat{x}(rP_r\sin\theta\cos\theta\sin\phi - r^2P_\theta\sin^2\theta\sin\phi - rP_r\sin\theta\cos\theta\sin\phi - r^2P_\theta\cos^2\theta\sin\phi$$
$$- r^2P_\phi\sin\theta\cos\theta\cos\phi) + \vec{s}$$
$$- r^2\cos^2\theta\sin\phi - r^2\cos^2\theta\sin\theta\cos\theta\cos\phi + \vec{s}$$
$$\vec{L} = r^2P_\phi\sin^2\theta\hat{z} + (r^2P_\theta\cos\phi - r^2P_\phi\sin\phi\sin\theta\cos\theta)\hat{y} - \hat{x}(r^2P_\theta\sin\phi + r^2P_\phi\sin\theta\cos\theta\cos\phi)$$
$$\vec{L} = -r^2(P_\theta\sin\phi + P_\phi\sin\theta\cos\theta\cos\phi)\hat{x} + r^2(P_\theta\cos\phi - P_\phi\sin\theta\cos\theta\sin\phi)\hat{y} + r^2P_\phi\sin^2\theta\hat{z} + \vec{s}$$

Once again, as we did earlier, we convert the components of the spin tensor from polar coordinates to Cartesian coordinates for practical applications:

$$s_x = s^{yz} = \frac{\partial y}{\partial q^\alpha}\frac{\partial z}{\partial q^\beta}s^{\alpha\beta} \Rightarrow s_x = \left(\frac{\partial y}{\partial r}\frac{\partial z}{\partial\theta} - \frac{\partial y}{\partial\theta}\frac{\partial z}{\partial r}\right)s^{r\theta} + \left(\frac{\partial y}{\partial\theta}\frac{\partial z}{\partial\phi} - \frac{\partial y}{\partial\phi}\frac{\partial z}{\partial\theta}\right)s^{\theta\phi} + \left(\frac{\partial y}{\partial r}\frac{\partial z}{\partial\phi} - \frac{\partial y}{\partial\phi}\frac{\partial z}{\partial r}\right)s^{r\phi}$$
$$= (-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)s^{r\theta} + (r\cos\theta\sin\phi(0)) + r^2\sin\theta\cos\phi s^{\theta\phi}$$
$$+ (\sin\theta\sin\phi(0) - \cos\theta(r\sin\theta\cos\phi))s^{r\phi}$$
$$s_x = -r\sin\phi s^{r\theta} + r^2\sin\theta\cos\phi s^{\theta\phi} - r\sin\theta\cos\theta\cos\phi s^{r\phi}$$
$$s_y = s^{zx} = \frac{\partial z}{\partial q^\alpha}\frac{\partial x}{\partial q^\beta}s^{\alpha\beta} = \left(\frac{\partial z}{\partial r}\frac{\partial x}{\partial\theta} - \frac{\partial z}{\partial\theta}\frac{\partial x}{\partial r}\right)s^{r\theta} + \left(\frac{\partial z}{\partial r}\frac{\partial x}{\partial\phi} - \frac{\partial z}{\partial\phi}\frac{\partial x}{\partial r}\right)s^{r\phi} + \left(\frac{\partial z}{\partial\theta}\frac{\partial x}{\partial\phi} - \frac{\partial z}{\partial\phi}\frac{\partial x}{\partial\theta}\right)s^{\theta\phi}$$
$$= (r\cos^2\theta\cos\phi + r\sin^2\theta\cos\phi)s^{r\theta} + (-r\sin\theta\sin\phi\cos\theta)s^{r\phi} + r^2\sin^2\theta\sin\phi s^{\theta\phi}$$
$$s_y = r\cos\phi s^{r\theta} - r\sin\theta\sin\phi\cos\theta s^{r\phi} + r^2\sin^2\theta\sin\phi s^{\theta\phi}$$

Now, let's calculate the x and y components of the angular momentum:

$$L_x = -r^2(P_2\sin\phi + P_3\sin\theta\cos\theta\cos\phi) + s_x$$
$$\Rightarrow L_x = -r^2\left(\left((m+m_s)\dot{\theta} + \frac{a'}{2}s^{21}\right)\sin\phi + \left((m+m_s)\dot{\phi} + \frac{a'}{2}s^{31}\right)\sin\theta\cos\theta\cos\phi\right) + s_x$$

$$\boxed{L_x = -r^2(m+m_s)(\dot{\theta}\sin\phi + \dot{\phi}\sin\theta\cos\theta\cos\phi) + \underbrace{\left[s_x - \frac{r^2a'}{2}(s^{21}\sin\phi + s^{31}\sin\theta\cos\theta\cos\phi)\right]}_{\equiv D_x}}$$

$$L_x = -r^2\frac{Ee^{-a}}{\dot t}(\dot\theta\sin\phi + \dot\phi\sin\theta\cos\theta\cos\phi) + D_x$$
$$L_x = -Ee^{-a}r^2\left(\frac{d\theta}{dt}\sin\phi + \frac{d\phi}{dt}\sin\theta\cos\theta\cos\phi\right) + D_x$$
$$L_y = r^2(P_2\cos\phi - P_3\sin\theta\cos\theta\sin\phi) + s_y$$
$$\Rightarrow L_y = r^2\left(\left((m+m_s)\dot\theta + \frac{a'}{2}s^{21}\right)\cos\phi - \left((m+m_s)\dot\phi + \frac{a'}{2}s^{31}\right)\sin\theta\cos\theta\sin\phi\right) + s_y$$
$$\boxed{L_y = r^2\frac{Ee^{-a}}{\dot t}(\dot\theta\cos\phi - \dot\phi\sin\theta\cos\theta\sin\phi) + \underbrace{\left[\frac{r^2a'}{2}(s^{21}\cos\phi - s^{31}\sin\theta\cos\theta\sin\phi) + s_y\right]}_{\equiv D_y}}$$
$$L_z = r^2\sin^2\theta\left((m+m_s)\dot\phi + \frac{a'}{2}s^{31}\right) + s_z$$
$$\boxed{L_z = r^2\sin^2\theta e^{-a}E\frac{d\phi}{dt} + \underbrace{\frac{a'}{2}r^2\sin^2\theta s^{31} + s_z}_{\equiv D_z}}$$
$$\Rightarrow L_z = r^2\sin^2\theta e^{-a}E\frac{d\phi}{dt} + D_z$$
based on our intuition, we know that at each moment, we can pass a tangential plane through the test particle and the central body, which produces a quantity with conserved properties:
$$L_x\cos\phi + L_y\sin\phi = -Ee^{-a}r^2\frac{d\phi}{dt}\sin\theta\cos\theta + (D_x\cos\phi + D_y\sin\phi)$$
$$L_x\cos\phi + L_y\sin\phi = (L_z - D_z)\cot\theta + (D_x\cos\phi + D_y\sin\phi)$$
$$[L_x\cos\phi + L_y\sin\phi + L_z\cot\theta] = D_x\cos\phi + D_y\sin\phi + D_z\cot\theta$$
$$D_x = s_x\left(1 - \frac{ra'}{2}\right) + \frac{r^3a'}{2}\sin^2\theta\cos\phi s^{23}$$
$$D_y = s_y\left(1 - \frac{ra'}{2}\right) + \frac{r^3a'}{2}\sin^2\theta\sin\phi s^{23}$$
$$D_z = s_z + \frac{r^2a'}{2}\sin^2\theta s^{31}$$
$$D_x\cos\phi + D_y\sin\phi = (s_x\cos\phi + s_y\sin\phi)\left(1 - \frac{ra'}{2}\right) + \frac{ra'}{2}\sin^2\theta s^{23}$$
$$D_z\cot\theta = s_z\cot\theta + \frac{r^2a'}{2}\sin\theta\cos\theta s^{31}$$
$$D_x\cos\phi + D_y\sin\phi + D_z\cot\theta = (s_x\cos\phi + s_y\sin\phi)\left(1 - \frac{ra'}{2}\right) + s_z\cot\theta + \frac{a'r^2}{2}\left(r\sin^2\theta s^{23} + \sin\theta\cos\theta s^{31}\right)$$
$$= (s_x\cos\phi + s_y\sin\phi)\left(1 - \frac{ra'}{2}\right) + s_z\cot\theta\left(1 - \frac{ra'}{2}\right) + \frac{a'r^3}{2}s^{23}$$
$$s^{23} = \frac{\partial\theta}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta}s^{\alpha\beta} = \left(\frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial\theta}{\partial y}\frac{\partial\phi}{\partial x}\right)s^{xy} + \left(\frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial z} - \frac{\partial\theta}{\partial z}\frac{\partial\phi}{\partial x}\right)s^{xz} + \left(\frac{\partial\theta}{\partial y}\frac{\partial\phi}{\partial z} - \frac{\partial\theta}{\partial z}\frac{\partial\phi}{\partial y}\right)s^{yz}$$
$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z}, \quad \tan\phi = \frac{y}{x}$$
$$\frac{\partial\theta}{\partial x} = \frac{\cos\phi\cos\theta}{r}, \quad \frac{\partial\theta}{\partial y} = \frac{\sin\phi\cos\theta}{r}, \quad \frac{\partial\theta}{\partial z} = -\frac{\sin\theta}{r}$$
$$\frac{\partial\phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}, \quad \frac{\partial\phi}{\partial y} = \frac{\cos\phi}{r\sin\theta}$$
$$s^{23} = \left(\frac{\cos^2\phi\cot\theta}{r^2} + \frac{\sin^2\phi\cot\theta}{r^2}\right)s_z + \frac{\sin\phi}{r^2}s_y + \frac{\cos\phi}{r^2}s_x$$
$$\Rightarrow s^{23} = \frac{\cot\theta}{r^2}s_z + \frac{\sin\phi}{r^2}s_y + \frac{\cos\phi}{r^2}s_x$$

$$L_x \cos\phi + L_y \sin\phi + L_z \cot\theta = \left(1 - \frac{ra'}{2}\right)(s_x \cos\phi + s_y \sin\phi + s_z \cot\theta) + \frac{ra'}{2}(\cos\phi s_x + \sin\phi s_y + \cot\theta s_z)$$

$$\Rightarrow \boxed{L_x \cos\phi + L_y \sin\phi + L_z \cot\theta = s_x \cos\phi + s_y \sin\phi + s_z \cot\theta}$$

Since the angular momentum is conserved and constant, this equation provides a constraint on the evolution of the spin vector in terms of the two angular position coordinates of the body!

Or in other words:

$$\boxed{(L_x - s_x)\sin\theta\cos\phi + (L_y - s_y)\sin\theta\sin\phi + (L_z - s_z)\cos\theta = 0}$$

Which is clearly the equation of a moving plane passing through the origin of the coordinate system (the central body)! From the beginning, we intuitively expected such a phenomenon, but now, based on the calculations, we see that everything is correct, aligns with our intuition, and works properly.

let's examine the motion constrained to remain on a fixed plane passing through the central body and the test body:

Plane motion: $\theta = \pi/2 \Rightarrow (L_x - s_x)\cos\phi + (L_y - s_y)\sin\phi = 0$

$P_2 = 0 \Rightarrow s^{12} = 0 \Rightarrow s^{23} = 0$

$$\Rightarrow \begin{cases} s_x = 0 \\ s_y = 0 \qquad\qquad (I) \\ s_z = rs^{13} \end{cases}$$

$$\frac{ds^{13}}{d\tau} + \dot{r}\left(\frac{1}{r} - a'\right)s^{13} = 0 \Rightarrow \frac{ds^{13}}{s^{13}} = dr\left(a' - \frac{1}{r}\right) = da - \frac{dr}{r}$$

$$\Rightarrow \ln s^{13} = a - \ln(r) + \text{constant} \to e^{-a} r s^{13} = \text{constant} \Rightarrow \boxed{e^{-a} s_z = \text{constant}}$$

This formula has a remarkable, though qualitative analogy with the gravitational red shift of spectral lines. In fact, let us assume that S_z be proportional to the angular velocity of the internal rotation of the test particle; the period of such a rotation increases with decreasing r according to a law similar to that describing the influence of the gravitational potential on the wave-length of light.

Besides the strict plane motion based on the condition (I), the motion will be very nearly plane in all cases when the spin is much smaller than the orbital angular momentum. This condition is fulfilled in the planetary motions.