Modelling non-commuting trips

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1 Theoretical Model

To model non-commuting trips, we will assume that residents try to maximize a leisure index that features love for variety given a time endowment. This seeks to capture the fact that residents enjoy spending their time in multiple locations of a city but they must prioritize some locations over others because they have a finite amount of time.

Utility of a resident who commutes i to j is

$$U_{ij} = L_{ij}^{1-\beta} C_{ij}^{\beta} s_{ij}$$

where L_{ij} is a leisure index and C_{ij} is a consumption index.

The consumption index C_{ij} takes the standard Cobb-Douglas form in consumption of a numeraire good and housing

$$C_{ij} = \left(\frac{g_{ij}}{\gamma}\right)^{\gamma} \left(\frac{h_{ij}}{1-\gamma}\right)^{1-\gamma}$$

while the leisure index L_{ij} takes a CES form[1] in the total number of time spent at multiple destinations

$$L_{ij} = \left(\sum_k a_k^{1-\rho} (n_{ijk} l_{ijk}^{\alpha})^{\rho}\right)^{1/\rho}$$

The term n_{ijk} is the (endogenous) number of trips to location k that an i to j commuter makes over a period of time while the term l_{ijk}^{α} is the utility of the (endogenous) time spent in location k. I will assume that $0 < \alpha < 1$ to capture the fact that the time spent at a location has diminishing marginal utility. Finally, the term a_k captures the relative attractiveness of locations.

[1] An alternative to the CES is the Melitz-Ottaviano framework. This framework may allow for 0 trips to destinations.

The total time endowment of a resident can be spent either working or in leisure activities, which both require spending some time traveling

$$x_{ij}(W + t_{ij}) + \sum_k n_{ijk}(l_{ijk} + t_{ik}) = T$$

where x_{ij} is the number of commutes, W is the length of a working day, t_{ij} is the length of the commute and t_{ik} is the length of the trip to location k. The implicit assumption in this constraint is that all trips originate from home.

Similarly, the monetary budget constraint for an i to j commuter is

$$g_{ij} + r_i h_{ij} + \sum_k n_{ijk} c_{ik} = x_{ij}(w_j W - c_{ij})$$

where net of commuting income is spent in consumption or traveling to multiple destination.

This maximization problem has a nested structure that I will use to solve it. Given a distribution of commuting and non-commuting trips, the consumption allocation problem is independent of the time allocation problem. In a first step, I will use this fact to solve for the optimal consumption allocation and optimal leisure time allocation. Next, in a second step, I will solve for the optimal number commuting trips conditional on a distribution for non-commuting trips. Finally, I will solve for the optimal number of non-commuting trips.

2 Solution to the maximization problem

2.1 Optimal consumption allocation

Conditional on a distribution of commuting and non-commuting trips, the monetary budget constraint can be written as

$$g_{ij} + r_i h_{ij} = x_{ij}(w_j W - c_{ij}) - \sum_k n_{ijk} c_{ik} \equiv M_{ij}$$

where the right-hand side term can be seen as an exogenous quantity that represents the available money that can be spent in the consumption of the numéraire and housing.

Due to the Cobb-Douglas structure, the optimal consumption allocation is

$$g_{ij} = \gamma M_{ij}$$
$$h_{ij} = (1 - \gamma)\frac{M_{ij}}{r_i}$$

which implies that the optimal value of the consumption index is

$$C_{ij} = r_i^{\gamma-1} M_{ij}$$

2.2 Optimal leisure time allocation

Similarly to the monetary budget constraint, the temporal budget constraint can also be written in terms of a component that depends on the maximization variables and an exogenous component

$$\sum_k n_{ijk} l_{ijk} = T - x_{ij}(W + t_{ij}) - \sum_k n_{ijk} t_{ik} \equiv T_{ij}$$

where T_{ij} can be seen as a the time endowment of an i to j commuter conditional on a number of commuting and non-commuting trips.

The lagrangian of this problem is

$$\mathcal{L} = \left(\sum_k a_k^{1-\rho} (n_{ijk} l_{ijk}^{\alpha})^{\rho} \right)^{1/\rho} - \lambda \left(\sum_k n_{ijk} l_{ijk} - T_{ij} \right)$$

where λ is the Lagrange multiplier associated to the constraint.

The first order condition with respect to l_{ijk} is

$$\frac{1}{\rho} \left(\sum_k a_k^{1-\rho} (n_{ijk} l_{ijk}^{\alpha})^{\rho} \right)^{\frac{1}{\rho}-1} a_k^{1-\rho} \rho (n_{ijk} l_{ijk}^{\alpha})^{\rho-1} a_k n_{ijk} \alpha l_{ijk}^{\alpha-1} = \lambda n_{ijk}$$

The ratio of two FOC is

$$\frac{a_k^{1-\rho} n_{ijk}^{\rho-1} l_{ijk}^{\alpha\rho-1}}{a_t^{1-\rho} n_{ijt}^{\rho-1} l_{ijt}^{\alpha\rho-1}} = 1$$

Rearranging and multiplying by n_{ijt}:

$$a_k^{\frac{1-\rho}{\alpha\rho-1}} n_{ijk}^{\frac{\rho-1}{\alpha\rho-1}} l_{ijk} a_t^{\frac{\rho-1}{\alpha\rho-1}} n_{ijt}^{\frac{\rho(\alpha-1)}{\alpha\rho-1}} = n_{ijt} l_{ijt}$$

Summing across all destinations

$$a_k^{\frac{1-\rho}{\alpha\rho-1}} n_{ijk}^{\frac{\rho-1}{\alpha\rho-1}} l_{ijk} \sum_t a_t^{\frac{\rho-1}{\alpha\rho-1}} n_{ijt}^{\frac{\rho(\alpha-1)}{\alpha\rho-1}} = \sum_t n_{ijt} l_{ijt}$$

and solving for l_{ijk} yields

$$l_{ijk} = \frac{a_k^{\frac{1-\rho}{1-\alpha\rho}} n_{ijk}^{-\frac{1-\rho}{1-\alpha\rho}}}{\sum_t a_t^{\frac{1-\rho}{1-\alpha\rho}} n_{ijt}^{\frac{\rho(1-\alpha)}{1-\alpha\rho}}} \sum_t n_{ijt} l_{ijt}$$

which can be further simplified using the time constraint

$$l_{ijk} = \frac{a_k^{\frac{1-\rho}{1-\alpha\rho}} n_{ijk}^{-\frac{1-\rho}{1-\alpha\rho}}}{\sum_t a_t^{\frac{1-\rho}{1-\alpha\rho}} n_{ijt}^{\frac{\rho(1-\alpha)}{1-\alpha\rho}}} T_{ij}$$

Using this result, I can find the optimal value for the non-commuting trips index as

$$L_{ij} = T_{ij}^{\alpha} \left(\sum_t a_t^{\frac{1-\rho}{1-\alpha\rho}} n_{ijt}^{\frac{\rho(1-\alpha)}{1-\alpha\rho}} \right)^{\frac{1-\alpha\rho}{\rho}}$$

which together with the previous result, it allows me to find an expression for utility that depends only on the number of commuting and non-commuting trips

$$U_{ij} = \left(T_{ij}^{\alpha} \left(\sum_t a_t^{\frac{1-\rho}{1-\alpha\rho}} n_{ijt}^{\frac{\rho(1-\alpha)}{1-\alpha\rho}} \right)^{\frac{1-\alpha\rho}{\rho}} \right)^{1-\beta} \left(r_i^{\gamma-1} M_{ij} \right)^{\beta}$$

2.2.1 Optimal number of commuting trips

The previous expression of utility depends on the number of commuting trips only through the terms T_{ij} and M_{ij}. The intuition of this result is simple, the more times you travel to work, the more income you earn increasing M_{ij}. However, the more times you travel to work, the less time you have to enjoy non-commuting trips T_{ij}. This creates a tension for an interior solution to exist.

Taking the natural logarithm to the expression of utility, the first order condition with respect to x_{ij} is

$$-\alpha(1-\beta)\frac{W + t_{ij}}{T_{ij}} + \beta \frac{w_j W - c_{ij}}{M_{ij}} = 0$$

which can be solved to yield an expression for the optimal number of commuting trips

$$x_{ij} = \frac{\beta}{\alpha(1-\beta)+\beta} \frac{T - \sum_k n_{ijk} t_{ik}}{W + t_{ij}} + \frac{\alpha(1-\beta)}{\alpha(1-\beta)+\beta} \frac{\sum_k n_{ijk} c_{ik}}{w_j W - c_{ij}}$$

Using this result, I can find expressions for the available time T_{ij} and monetary budget M_{ij} exclusively as a function of the non-commuting trips:

$$T_{ij} = \frac{\alpha(1-\beta)}{\alpha(1-\beta)+\beta} \frac{v_{ij} T - \sum_k n_{ijk} \tau_{ijk}}{v_{ij}}$$

$$M_{ij} = \frac{\beta}{\alpha(1-\beta)+\beta}\left(v_{ij}T - \sum_k n_{ijk}\tau_{ijk}\right)$$

where $v_{ij} \equiv \frac{w_j W - c_{ij}}{W + t_{ij}}$ is the value of time of the i to j commuter and $\tau_{ijk} \equiv v_{ij}t_{ik} + c_{ik}$ is the generalized cost of the non-commuting trip.

Using these results, utility can be written as

$$U_{ij} \propto v_{ij}^{-\alpha(1-\beta)} r_i^{(\gamma-1)\beta}\left(v_{ij}T - \sum_k n_{ijk}\tau_{ijk}\right)^{\alpha(1-\beta)+\beta}\left(\sum_t a_t^{\frac{1-\rho}{1-\alpha\rho}} n_{ijt}^{\frac{\rho(1-\alpha)}{1-\alpha\rho}}\right)^{\frac{(1-\alpha\rho)(1-\beta)}{\rho}}$$

This expression encapsulates the key idea of adding non-commuting trips in a QSM. While more trips to any destination increases utility, they also consume time that could be use for working to earn an income to enjoy consumption.

2.2.2 Optimal distribution of non-commuting trips

Taking the natural logarithm to the equation for utility, the first order conditions with respect to n_{ijk} is

$$-(\alpha(1-\beta)+\beta)\frac{\tau_{ijk}}{v_{ij}T - \sum_k n_{ijk}\tau_{ijk}} + (1-\beta)(1-\alpha)\frac{a_t^{\frac{1-\rho}{1-\alpha\rho}} n_{ijt}^{\frac{\rho(1-\alpha)}{1-\alpha\rho}-1}}{\sum_t a_t^{\frac{1-\rho}{1-\alpha\rho}} n_{ijt}^{\frac{\rho(1-\alpha)}{1-\alpha\rho}}} = 0$$

Multiplying by n_{ijt} and summing across destinations, it yields

$$(1-\alpha)(1-\beta) = (\alpha(1-\beta)+\beta)\frac{\sum_k n_{ijk}\tau_{ijk}}{v_{ij}T - \sum_k n_{ijk}\tau_{ijk}}$$

which can be rearranged to obtain

$$\sum_k n_{ijk}\tau_{ijk} = (1-\alpha)(1-\beta)v_{ij}T$$

This equation says that the generalized cost of non-commuting trips is proportional to the value of time of an i to j commuter.

The ratio of two FOCs is

$$\frac{a_k^{\frac{1-\rho}{1-\alpha\rho}} n_{ijk}^{-\frac{1-\rho}{1-\alpha\rho}}}{a_t^{\frac{1-\rho}{1-\alpha\rho}} n_{ijt}^{-\frac{1-\rho}{1-\alpha\rho}}} = \frac{\tau_{ijk}}{\tau_{ijt}}$$

Rearranging and multiplying by τ_{ijt}

$$a_k^{-1}\tau_{ijk}^{-\frac{\alpha\rho-1}{1-\rho}} n_{ik} a_t \tau_{ijt}^{\frac{\rho(\alpha-1)}{1-\rho}} = n_{ijt}\tau_{ijt}$$

Summing across destinations

$$a_k^{-1}\tau_{ijk}^{-\frac{\alpha\rho-1}{1-\rho}} n_{ik} \sum_t a_t \tau_{ijk}^{\frac{-\rho(1-\alpha)}{1-\rho}} = \sum_t n_{ijt}\tau_{ijt}$$

and solving for n_{ijk}

$$n_{ijk} = \frac{a_k \tau_{ijk}^{\frac{1-\alpha\rho}{1-\rho}}}{\sum_t a_t \tau_{ijk}^{\frac{-\rho(1-\alpha)}{1-\rho}}} \sum_t n_{ijt}\tau_{ijt}$$

which can be further simplied using the previous result

$$n_{ijk} = \frac{a_k \tau_{ijk}^{-\frac{1-\alpha\rho}{1-\rho}}}{\sum_t a_t \tau_{ijt}^{-\frac{\rho(1-\alpha)}{1-\rho}}} (1-\beta)(1-\alpha) v_{ij} T$$

This equation finds the optimal number of non-commuting trips as a function of exogenous objects: the relative attractiveness of locations and the generalized cost to reach them.

Finally, utility for a resident who commutes i to j is

$$U_{ij} \propto v_{ij}^{(1-\alpha)(1-\beta)+\beta} r_i^{(\gamma-1)\beta} \left(\sum_t a_t \tau_{ijk}^{-\frac{\rho(1-\alpha)}{1-\rho}} \right)^{\frac{(1-\rho)(1-\beta)}{\rho}} \equiv A_{ij}^{\frac{(1-\rho)(1-\beta)}{\rho}} v_{ij}^{(1-\alpha)(1-\beta)+\beta} r_i^{(\gamma-1)\beta}$$

where $A_{ij} = \left(\sum_t a_t \tau_{ijk}^{-\frac{\rho(1-\alpha)}{1-\rho}} \right)$.

This is almost equivalent to a basic specification in a QSM. This means that adding non-commuting trips using the above formulation boils down to a way of endogenizing local amenities.

3 Estimation

The number of non-commuting trips for a resident commuting i to j is:

$$n_{ijk} = \frac{a_k \tau_{ijk}^{-\frac{1-\alpha\rho}{1-\rho}}}{\sum_t a_t \tau_{ijt}^{-\frac{\rho(1-\alpha)}{1-\rho}}} (1-\beta)(1-\alpha) v_{ij} T$$

Select a destination k' as a reference point. The proportion of non-commuting to this destination with with respect to any other destination is:

$$\frac{n_{ijk}}{n_{ijk'}} = \frac{a_k \tau_{ijk}^{-\delta}}{a_{k'} \tau_{ijk}^{-\delta}}$$

Taking logs to the previous expression and expanding it to account for measurement errors:

$$\ln\left(\frac{n_{ijk}}{n_{ijk'}}\right) = \ln\left(\frac{a_k}{a_k'}\right) - \delta\left(\frac{\tau_{ijk}}{\tau_{ijk'}}\right) + \epsilon_{ijk}$$

This equation allows to estimate δ.

4 Inversion

The total number of non-commuting trips to location k is

$$N_k = \sum_i \sum_j \frac{a_k^{\frac{\rho}{1-\rho}} \tau_{ijk}^{-\frac{1-\alpha\rho}{1-\rho}}}{\sum_t a_t^{\frac{\rho}{1-\rho}} \tau_{ijt}^{-\frac{\rho(1-\alpha)}{1-\rho}}} \delta v_{ij} R_{ij} T$$

Assume that we observe τ_{ijk}, v_{ij}, R_{ij} and N_k. Assume that we now both the value for α and ρ (or at least the value of $1 - \alpha\rho/1 - \rho$). We need to be able to prove that there exists a (unique) solution for a_k.

The above system of equation is both continuous and homogenous of degree zero in a_k. Brouwer's fixed point theorem tell us that there is at least one solution. A possible way to prove that there is a unique solution is to prove that the above system of equation exhibits gross substitution.

5 Equilibrium