Efficient Pricing of Caplets under a Single Factor Affine Yield Interest Rate Process

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1 Introduction

Standard pricing for caplets uses Black's caplet formula. The formula assumes lognormal (simple) forward rates. Under this assumption caplets can be analytically priced; however the convenience of an analytical solution is frequently unjustified by empirical data. The caplets priced in this way are still useful since one can generate an implied forward volatility with which to compute other derivative prices, but then caplet prices are essentially state variables instead of derivative securities on some more primitive state variable. A more theoretically appealing model would be one which correctly prices caplets to begin with, much like the Heston model attempts to explain the volatility smile found with equity options. Given the widespread use of spot interest rate models, this paper attempts to rectify the issue by giving simple and efficient numerical methods to price caplets under any single factor affine yield interest rate process.

2 Assumptions and Prelimaries

2.1 Dynamics of Interest Rates

Assume the existence of a risk-neutral measure under which the spot interest rate process satisfies the following SDE:

$$dr_t = \alpha(r_t, t) + \sigma(r_t, t)dW_t$$

Where $\alpha : \mathbb{R}^2 \to \mathbb{R}$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}_+$ satisfy $\mathbb{E}[(\cdot)^2] < \infty$. In addition, let the interest rate process be affine.

Definition 1. An interest rate process is said to be *affine* if $\alpha(r_t,t) = \mu(t) + \gamma(t)r_t$ and $\sigma^2(r_t,t) = \omega(t) + \xi(t)r_t$ for some μ, γ, ω, $\xi : \mathbb{R}_+ \to \mathbb{R}$

2.2 Bonds

Definition 2. A *bond* is a security with payoff function $f(r_T,T) = 1$.

Proposition 1. *Let r_t satisfy Definition 1. Then*

$$f(r_0,0) = e^{-A(0,T)r_0 + C(0,T)}$$

is the price of a bond.

Proof. Since a risk-neutral measure exists, any derivative security may be priced as the discounted expected value of its payoff under this measure. Therefore

$$f(r_0,0) = \mathbb{E}\left[e^{-\int_0^T r_t dt}\right]$$

By Feynman-Kac, $e^{-\int_0^t r ds} f(r,t)$ (with dummy variable r) then satisfies the following PDE:

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\alpha(r,t) + \frac{\partial^2 f}{\partial r^2}\frac{\sigma^2(r,t)}{2} - rf = 0 \\ f(r,T) = 1 \ \forall r \end{cases}$$

$$= \begin{cases} \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}(\mu + \gamma r) + \frac{\partial^2 f}{\partial r^2}\frac{(\omega + \xi r)}{2} - rf = 0 \\ f(r,T) = 1 \ \forall r \end{cases} \tag{1}$$

Substituting Proposition 1 into this PDE,

$$\left(-\frac{dA}{dt}r + \frac{dC}{dt}\right) f(r,t) - Af(r,t)(\mu + \gamma r) + A^2 f(r,t)\frac{(\omega + \xi r)}{2} - rf(r,t) = 0$$

$$\left(-\frac{dA}{dt}r + \frac{dC}{dt}\right) - A(\mu + \gamma r) + A^2\frac{(\omega + \xi r)}{2} - r = 0$$

For the above expression to hold, the following ODEs must hold:

$$\begin{cases} A^2\frac{\xi}{2} - \gamma A - 1 = \frac{dA}{dt} \\ A(T,T) = 0 \end{cases} \tag{2}$$

$$\begin{cases} \frac{dC}{dt} = \mu A - A^2\frac{\omega}{2} \\ C(T,T) = 0 \end{cases} \tag{3}$$

Clearly these ODEs have unique solututions, from which it follows that

$$f(r_0, 0) = e^{-A(0,T)r_0 + C(0,T)}$$

□

Remark 1. If the ODEs cannot be solved analytically it is computationally trivial to solve them numerically.

Proposition 2. *Let r_t satisfy Definition 1. Then*

$$df(r_t, t) = f(r_t, t)r_t dt - A(t,T)\sigma(r_t, t)f(r_t, t)dW_t$$

Proof. By Ito's Lemma,

$$df(r_t, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\alpha(r_t, t) + \frac{\partial^2 f}{\partial r^2}\frac{\sigma^2(r_t, t)}{2} + \frac{\partial f}{\partial r}\sigma(r_t, t)dW_t$$

By Feynman-Kac,

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\alpha(r_t, t) + \frac{\partial^2 f}{\partial r^2}\frac{\sigma^2(r_t, t)}{2} = r_t f(r_t, t)$$

Substituting this into the differential of $f(r_t, t)$,

$$df(r_t, t) = r_t f(r_t, t) + \frac{\partial f}{\partial r}\sigma(r_t, t)dW_t$$

By Proposition 1,

$$\frac{\partial f}{\partial r} = -A(t,T)f(r_t,t)$$

This implies the following differential:

$$df(r_t,t) = r_t f(r_t,t)dt - A(t,T)f(r_t,t)\sigma(r_t,t)dW_t$$

$\square$

2.3 Caplet

Definition 3. A *caplet* is a derivative security with payoff function

$$f(L_{t*,T},t*) = (L_{t*,T} - k)\mathbb{I}_{L_{t*,T}>k},\ k \in \mathbb{R}_+,\ 0 < t* < T$$

Where $L_{t*,T} := -\frac{\ln(f(r_{t*},t*))}{(T-t*)}$. This is paid at time T.

Remark 2. The Black caplet model determines the dynamics of the forward rate of a specific future date and treats the final possible value as the term in the payoff function. The forward rate is defined as follows:

$$F(0,t*,T) = \frac{TL_{0,T} - (t*)L_{0,t*}}{T-t*}$$

As $t \to t*$,

$$F(t*,t*,T) = \frac{(T-t*)L_{t*,T}}{T-t*} = L_{t*,T}$$

Therefore the two payoffs are identical.

Proposition 3. *Let r_t satisfy Definition 1 and let the current price of a caplet be $c(r_0,0)$. Then*

$$c(r_0,0) = f(r_0,0)\left(\mathbb{E}^F\left[L_{t*,T}\mathbb{I}_{L_{t*,T}>k}\right] - k\mathbb{E}^F\left[\mathbb{I}_{L_{t*,T}>k}\right]\right) \tag{4}$$

Where the probability is taken under the forward measure and the dynamics of r_t under this measure are

$$dr_t = \left(\alpha(r_t,t) - \sigma^2(r_t,t)A(t,T)\right)dt + \sigma(r_t,t)dW_t^F \tag{5}$$

Proof. Since a risk-neutral measure exists,

$$c(r_0, 0) = \mathbb{E}\left[e^{-\int_0^T r_t dt}\left(L_{t*,T}\mathbb{I}_{L_{t*,T}>k} - k\mathbb{I}_{L_{t*,T}>k}\right)\right]$$

Defining a Radon-Nikodym derivative

$$Z_t := \frac{e^{-\int_0^t r_s ds} f(r_t, t)}{f(r_0, 0)}$$

Substituting into the pricing formula,

$$c(r_0, 0) = f(r_0, 0)\mathbb{E}\left[Z_{t*}\left(L_{t*,T}\mathbb{I}_{L_{t*,T}>k} - k\mathbb{I}_{L_{t*,T}>k}\right)\right]$$

By Proposition 2, the volatility of Z_t is $-A(t,T)Z_t\sigma(r_t,t)$. By Girsonov's theorem, $W_t^F := W_t + \int_0^t A(s,T)\sigma(r_s,s)ds$ is a Brownian Motion under the forward measure and the pricing formula can be written as

$$c(r_0, 0) = f(r_0, 0)\mathbb{E}^F\left[L_{t*,T}\mathbb{I}_{L_{t*,T}>k} - k\mathbb{I}_{L_{t*,T}>k}\right]$$

Recalling that $dr_t = \alpha(r_t,t)dt + \sigma(r_t,t)dW_t$, under the forward measure r_t has the following dynamics:

$$dr_t = \alpha(r_t,t)dt + \sigma(r_t,t)(dW_t^F - A(t,T)\sigma(r_t,t)dt)$$

$$= \left(\alpha(r_t,t) - \sigma^2(r_t,t)A(t,T)\right)dt + \sigma(r_t,t)dW_t^F$$

By Ito's Lemma, $L_{t,T}$ has the following dynamics under the forward measure:

$$dL_t = \frac{dA}{dt}r_t dt - \frac{dC}{dt}dt + A(t,T)\left(\left(\alpha(r_t,t) - \sigma^2(r_t,t)A(t,T)\right)dt + \sigma(r_t,t)dW_t^F\right)$$
$$= \left(\frac{dA}{dt}r_t - \frac{dC}{dt} + A(t,T)\left(\alpha(r_t,t) - \sigma^2(r_t,t)A(t,T)\right)\right)dt + \sigma(r_t,t)A(t,T)dW_t^F$$

□

Remark 3. Recall Remark 2. Using the definition of the forward rate, the diffusion of the forward rate is the following:

$$\frac{1}{T-t*}\left(TA(t,T)-(t*)A(t,t*)\right)\sqrt{\omega+\xi r_t}dW_t^F$$

In general the volatility term in the diffusion is non-deterministic, an important deviation from the assumptions of the Black model.

3 SDE governing $L_{t,T}$

$$dL_{t,T}=-\left(\frac{r_t-A^2(t,T)\frac{\omega+\xi r_t}{2}}{T-t}-\frac{A(t,T)\sqrt{\omega+\xi r_t}dW_t}{T-t}+\frac{\ln(f(r_t,t))}{(T-t)^2}\right)$$

$$=\frac{A^2(t,T)\frac{\omega+\xi r_t}{2}-r_t}{T-t}+\frac{A(t,T)\sqrt{\omega+\xi r_t}dW_t}{T-t}-\frac{\ln(f(r_t,t))}{(T-t)^2}$$

$$=\frac{A^2(t,T)\frac{\omega+\xi\left(\frac{C(t,T)+L_{t,T}(T-t)}{A(t,T)}\right)}{2}-\frac{C(t,T)+L_{t,T}(T-t)}{A(t,T)}}{T-t}+\frac{A(t,T)\sqrt{\omega+\xi\left(\frac{C(t,T)+L_{t,T}(T-t)}{A(t,T)}\right)}dW_t}{T-t}+\frac{L_{t,T}}{T-t}dt$$

$$=\frac{A^2(t,T)\frac{\omega+\xi\left(\frac{C(t,T)+L_{t,T}(T-t)}{A(t,T)}\right)}{2}-\frac{C(t,T)+L_{t,T}(T-t)}{A(t,T)}}{T-t}+\frac{A(t,T)\sqrt{\omega+\xi\left(\frac{C(t,T)+L_{t,T}(T-t)}{A(t,T)}\right)}dW_t}{T-t}+\frac{L_{t,T}}{T-t}dt$$

$$=\frac{-A^2(t,T)\frac{\omega+\xi\left(\frac{C(t,T)+L_{t,T}(T-t)}{A(t,T)}\right)}{2}-\frac{C(t,T)+L_{t,T}(T-t)}{A(t,T)}}{T-t}+\frac{A(t,T)\sqrt{\omega+\xi\left(\frac{C(t,T)+L_{t,T}(T-t)}{A(t,T)}\right)}dW_t^F}{T-t}+\frac{L_{t,T}}{T-t}dt$$

4 Backset Interest Rate

Definition 4. *Backset Interest Rate* is an asset with payoff function $L_{t*,T}$ at T.

Proposition 4. *The price of the backset interest rate is*

$$s_{0,T} = f(0, r_0)\mathbb{E}^F[-\ln(f(t*, T))]$$

Proof.

$$s_{0,T} = \mathbb{E}\left[-e^{-\int_0^T r_t dt}\ln(f(t*, r_{t*}))\right]$$

$$= \mathbb{E}\left[-e^{-\int_0^{t*} r_t dt}e^{-\int_{t*}^T r_t dt}\ln(f(t*, r_{t*}))\right]$$

$$= \mathbb{E}\left[-e^{-\int_0^{t*} r_t dt}\mathbb{E}\left[e^{-\int_{t*}^T r_t dt}|\mathcal{F}_{t*}\right]\ln(f(t*, r_{t*}))\right]$$

$$= \mathbb{E}\left[-e^{-\int_0^{t*} r_t dt}f(t*, r_{t*})\ln(f(t*, r_{t*}))\right]$$

$$= f(0, r_0)\mathbb{E}\left[-\frac{e^{-\int_0^{t*} r_t dt}f(t*, r_{t*})}{f(0, r_0)}\ln(f(t*, r_{t*}))\right]$$

$$= f(0, r_0)\mathbb{E}\left[-Z_{t*}\ln(f(t*, r_{t*}))\right]$$

$$= f(0, r_0)\mathbb{E}^F\left[-\ln(f(t*, r_{t*}))\right]$$

□

Let $\xi \neq 0$. Then $s_{0,T} > 0$. PROVE THIS.

Since $s_{0,T} > 0$, the dynamics can be represented as follows:

$$d\frac{s_{t,T}}{f(t, r_t)} = \frac{s_{t,T}}{f(t, r_t)}\sigma_s dW_t^F$$

Where σ_s is an adapted process.

5 Computation of Caplets

Proposition 5. *In the case that*

$$dr_t = \alpha(b - r_t)dt + \sigma dW_t,\ \alpha,\ b,\ \sigma\ \in \mathbb{R}_+ \tag{6}$$

The price of the caplet is

$$c(r_0,0)=f(r_0,0)\left(\sigma_L\phi(z)+(\mu_L-k)(1-\Phi(z))\right)$$

Where

$$\mu_L=\frac{A(t*,T)\mu_r-C(t*,T)}{T-t*}$$

$$\sigma_L=\frac{A(t*,T)\sigma_r}{T-t*}$$

$$\mu_r=\left(e^{-\alpha t*}r_0-\sigma^2\left(\frac{2-2e^{-\alpha t*}-e^{-\alpha(T-t*)}+e^{-\alpha(T+t*)}}{2\alpha^2}\right)+b\left(1-e^{-\alpha t*}\right)\right)$$

$$\sigma_r=\sqrt{\left(1-e^{-2\alpha t*}\right)\frac{\sigma^2}{2\alpha}}$$

$$z=\frac{k-\mu_L}{\sigma_L}$$

$$\Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy$$

$$\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

Proof. The proof is given in the appendix. □

Remark 4. When r_t satisfies Equation 6, r_t is refered to as the Vasicek model.

If r_t satisfies Definition 1 such that $\xi(t)\neq 0$, there generally exists no analytical density from which to calculate the (forward) probability of the interest rate terminating in the money; necessitating the use of numerical methods to find a solution. However the usual numerical solutions for option pricing in this scenario are either difficult or inefficient.

Remark 5. When $\mu(t)$, $\gamma(t)$, $\xi(t)$ are constants and $\omega=0$ the resulting interest rate process is called the Cox Ingersoll Ross model.

5.1 Monte Carlo

Monte Carlo methods for solving Equation 4 are subject to discretization error since the analytical distribution is unkown, necessitating an Euler-like simulation scheme. In addition, to simulate Equation 5 requires computing $A(t,T)$ at each time node. In fact, it would be more efficient to simply simulate the risk-neutral process. However, simulating the risk-neutral process compounds the discretization error since the integral must be approximated by a sum.

5.2 PDE Methods

Another common numerical method is to discretize Equation 1 and solve the PDE numerically. Unfortunately the boundary conditions are difficult to formulate for square root diffusions and a large mesh would be required to gain acceptable accuracy (especially for interest rates near zero).

5.3 Alternate PDE solution

Proposition 6. *Let* $p(r,t) := d\mathbb{P}^F(r_t \leq r)$ *be known. Then* $c(0,r_0)$ *can be priced.*

Proof.

$$f(r_0,0)\mathbb{E}[(r_T-k)\mathbb{I}_{r_T>k}] = f(r_0,0)\left(\int_k^\infty rp(r,t)dr - \int_k^\infty kp(r,t)dr\right)$$

□

By Proposition 6, Equation 4 can be approximately solved if the probability density of r_t under the forward measure can be numerically approximated. By the Fokker-Planck equation, $p(r,t)$ satisfies the PDE

$$\begin{cases} \frac{\partial p}{\partial t} = -\frac{\partial}{\partial r}p(r,t)\left(\alpha(r,t) - \sigma^2(r,t)A(t,T)\right) + \frac{\partial^2}{\partial r^2}\frac{1}{2}p(r,t)\sigma^2(r,t) \\ \hfill p(r,0) = \delta(r) \end{cases} \tag{7}$$

This equation is far simpler to solve numerically than Equation 1 since the boundary conditions are necessarily zero. However, the initial condition is not easily discretized.

Proposition 7. *Let $p(r,t)$ exist. Then $F(r,t) := \mathbb{P}^F(r_t < r)$ satisfies*

$$\begin{cases} \frac{\partial F}{\partial t} = -\left(\alpha(r,t) - \sigma^2(r,t)A(t,T) - \frac{1}{2}\xi\right)\frac{\partial F}{\partial r} + \frac{1}{2}\sigma^2(r,t)\frac{\partial^2 F}{\partial r^2} \\ F(r,0) = \mathbb{I}_{r>r_0} \end{cases} \tag{8}$$

Proof.

$$\frac{\partial}{\partial r}F(r,t) = p(r,t)$$

$$\implies \begin{cases} \frac{\partial^2 F}{\partial t \partial r} = -\frac{\partial}{\partial r}\left(\frac{\partial F}{\partial r}\left(\alpha(r,t) - \sigma^2(r,t)A(t,T)\right)\right) + \frac{\partial^2}{\partial r^2}\left(\frac{1}{2}\frac{\partial F}{\partial r}\sigma^2(r,t)\right) \\ \frac{\partial F}{\partial r} = \delta(r) \end{cases}$$

Integrating with respect to r,

$$\begin{cases} \frac{\partial F}{\partial t} = -\frac{\partial F}{\partial r}\left(\alpha(r,t) - \sigma^2(r,t)A(t,T)\right) + \frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\partial F}{\partial r}\sigma^2(r,t)\right) + c(t) \\ F(r,0) = \mathbb{I}_{r>r_0} \end{cases}$$

Since at the boundaries $\lim_{r\to\infty} F(r,t) = 1$ and $\lim_{r\to\Omega_-} F(r,t) = 0$, $\frac{\partial F}{\partial t}$ is equal to zero at the boundaries, which implies that $c(t)$ is also zero.

$$\begin{cases} \frac{\partial F}{\partial t} = -\frac{\partial F}{\partial r}\left(\alpha(r,t) - \sigma^2(r,t)A(t,T)\right) + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2(r,t) + \frac{1}{2}\xi\frac{\partial F}{\partial r} \\ F(r,0) = \mathbb{I}_{r>r_0} \end{cases}$$

$$\begin{cases} \frac{\partial F}{\partial t} = -\left(\alpha(r,t) - \sigma^2(r,t)A(t,T) - \frac{1}{2}\xi\right)\frac{\partial F}{\partial r} + \frac{1}{2}\sigma^2(r,t)\frac{\partial^2 F}{\partial r^2} \\ F(r,0) = \mathbb{I}_{r>r_0} \end{cases}$$

□

6 Numerical Results

To approximately solve Equation 8, I use an implicit finite difference scheme. The initial condition is discretized by $F(r,0) = 0 \; \forall r < r_0$, $F(r,0) = 1 \; \forall r > r_0$, and $F(r,0) = .5, \; r = r_0$. The functions $A(t,T)$ and $C(t,T)$ are approximated via an Euler discretization scheme in which the time step is the same size as the one used for the numerical solution to Equation 8. Discretizing

Equation 8 results in a vector of approximate values of $F(r,T)$ denoted $\tilde{F}(r,T)$ corresponding to each discrete node of r. The approximation of the forward expectation therefore is computed as follows:

$$\sum_{i=g(k)}^{m} \left(r_i + \frac{r_{i+1} - r_i}{2} - k \right) \left(\tilde{F}(r_{i+1}, T) - \tilde{F}(r_i, T) \right)$$

Where $g(k)$ is a function mapping the value of k to the smallest value of i such that $r_i > k$. The complete algorithm for computing the price of the caplet is given in the appendix.

6.1 Vasicek

Since an analytical solution exists for the price of a caplet when r_t satisfies Equation 6, that is the interest rate process that is initially tested. Using parameter values $\alpha = 1$, $b = .1$, $\sigma = .03$, $r_0 = .10$, $k = .10$, and $T = 1$, the analytical solution is $.00703998$. Space is discretized with m nodes on $[-.1,\ .5]$ and time with n nodes.

	$n = 60$, $m = 72$	$n = 150$, $m = 180$	$n = 300$, $m = 360$	$n = 600$, $m = 720$
Value	.0069681	.00702166	.00703268	.00703681
Time (s)	0	0	.031	.109
Relative error	1.02108 %	.26032 %	.10373 %	.04514 %
Decrease in error		3.9224	2.5096	2.2980

6.2 Cox Ingersoll Ross

Subsequently, a Cox Ingersoll Ross process

$$dr_t = \alpha(b - r_t)dt + \sigma\sqrt{r_t}dW_t,\ \ \alpha,\ \ b,\ \ \sigma \in \mathbb{R}_+$$

is tested with parameter values $\alpha = 1$, $b = .1$, $\sigma = .12$, $r_0 = .10$, $k = .10$, $t = 0$, and $T = 1$. Space is discretized on $[0,\ .5]$.

	$n = 60$, $m = 60$	$n = 150$, $m = 150$	$n = 300$, $m = 300$	$n = 600$, $m = 600$	$n = 120000$ $m = 120000$
Value	.00875845	.00880975	.00882109	.00882562	.00882935
Time (s)	0	0	.016	.078	3759.36
Relative error	.80300 %	.22199 %	.09355%	.04225 %	–
Decrease in error		3.6173	2.3730	2.2142	–

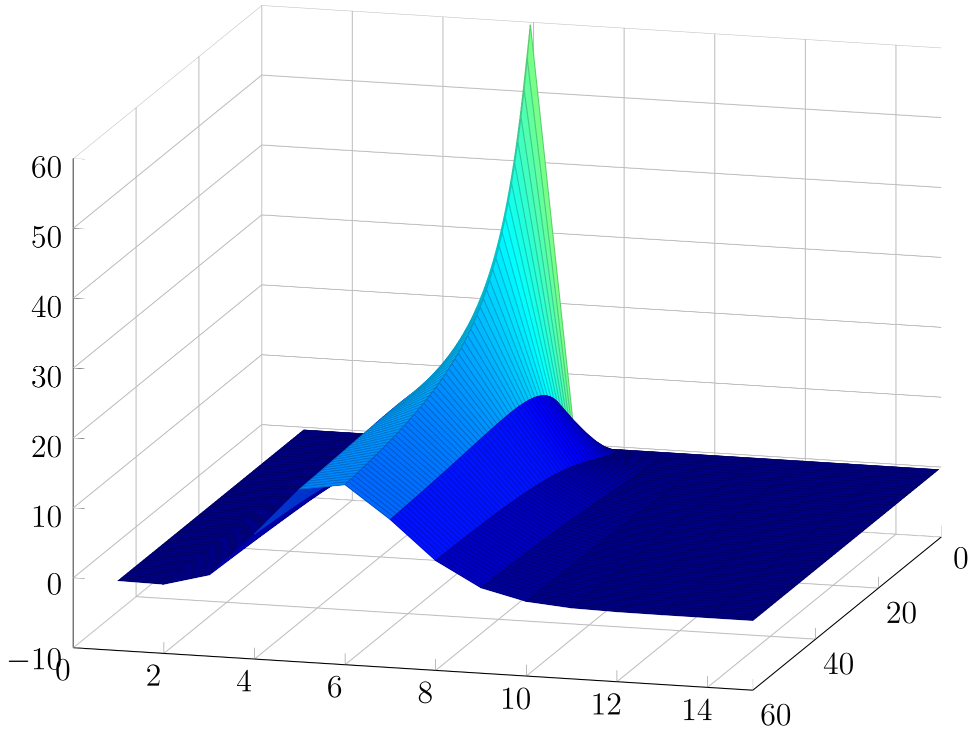
Here the 120000 by 120000 mesh is considered the “exact” solution.

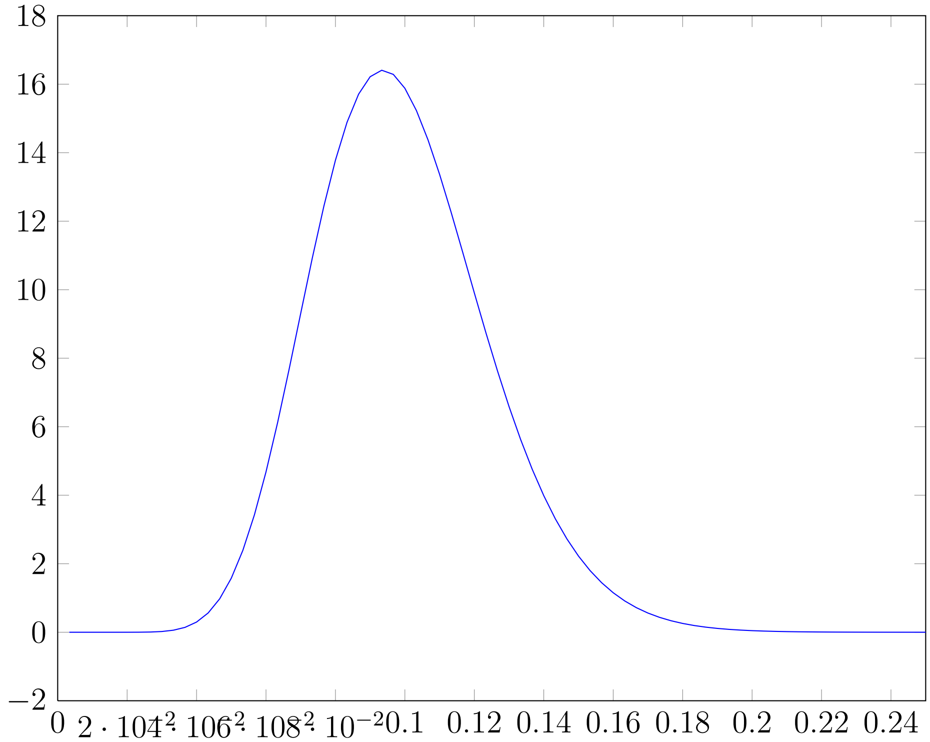
7 Conclusion

The algorithm presented quickly and accurately prices caplets under the assumption that the spot interest rate follows a single factor affine yield process. This technique gives greater flexibility to the pricing of caplets than analytical formulas based on the assumption of lognormal forward rates.

A Plots

The following plots should provide intuition into the numerical results. The first is a surface plot of the solution to the transition density of a Cox Ingersoll Ross process, the second a cross section of the surface plot.


60
50
40
30
20
10
0
−10
0
2
4
6
8
10
12
14
0
20
40
60


18
16
14
12
10
8
6
4
2
0
−2
0
2 · 10−2
4 · 10−2
6 · 10−2
8 · 10−2
0.1
0.12
0.14
0.16
0.18
0.2
0.22
0.24

B Proof of Proposition 5

Proof. If $y \sim \mathcal{N}(\mu, \sigma^2)$,

$$\mathbb{E}\left[(y-k)\mathbb{I}_{y>k}\right] = \int_k^\infty y\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy - k\int_k^\infty \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy$$

$$= \int_{\frac{k-\mu}{\sigma}}^\infty (\sigma x+\mu)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx - k\int_{\frac{k-\mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$= \sigma\int_{\frac{k-\mu}{\sigma}}^\infty -d\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right) + \mu\left(1-\Phi\left(\frac{k-\mu}{\sigma}\right)\right) - k\left(1-\Phi\left(\frac{k-\mu}{\sigma}\right)\right)$$

$$= \sigma\phi\left(\frac{k-\mu}{\sigma}\right) + (\mu-k)\left(1-\Phi\left(\frac{k-\mu}{\sigma}\right)\right)$$

By Ito's Lemma,

$$d\left(e^{\alpha t}r_t\right) = \alpha e^{\alpha t}r_t dt + e^{\alpha t}\left(\alpha b - \sigma^2 A(t,T) - \alpha r_t\right)dt + e^{\alpha t}\sigma dW_t^F$$

$$= e^{\alpha t}\left(\alpha b - \sigma^2 A(t,T)\right)dt + e^{\alpha t}\sigma dW_t^F$$

$$e^{\alpha t*}r_{t*} = r_0 + \frac{1}{\alpha}\left(e^{\alpha t*}-1\right)\alpha b - \sigma^2\int_0^{t*} e^{\alpha t}A(t,T)dt + \int_0^{t*} e^{\alpha t}\sigma dW_t^F$$

$$r_{t*} = e^{-\alpha t*}r_0 + b\left(1-e^{-\alpha t*}\right) - \sigma^2\int_0^{t*} e^{-\alpha(t*-t)}A(t,T)dt + e^{-\alpha t*}\int_0^{t*} e^{\alpha t}\sigma dW_t^F$$

Let

$$X_t := e^{-\alpha t}r_0 - \sigma^2\int_0^t e^{-\alpha(t-s)}A(s,T)ds + b\left(1-e^{-\alpha t}\right) + u\left(1-e^{-2\alpha t}\right)\frac{\sigma^2}{4\alpha}$$

For some $u \in \mathbb{R}_+$ and

$$V_t := e^{ur_t - uX_t} = e^{ue^{-\alpha t}\int_0^t e^{\alpha s}\sigma dW_s^F - u^2\left(1-e^{-2\alpha t}\right)\frac{\sigma^2}{4\alpha}}$$

By Ito's Lemma,

$$dV_t = V_t u e^{-\alpha t}e^{\alpha t}\sigma dW_t^F + V_t\frac{1}{2}u^2 e^{-2\alpha t}e^{2\alpha t}\sigma^2 dt - V_t u^2\frac{\sigma^2}{2}dt$$

$$= V_t u \sigma dW_t^F$$

Since $\mathbb{E}\left[\int_0^t u^2\sigma^2 V_t^2 dt\right] < \infty$, V_t is a martingale under the forward measure and satisfies $V_t(r_0, 0) = 1$. Therefore

$$\mathbb{E}^F[V_t] = 1$$

$$\mathbb{E}^F[e^{ur_t - uX_t}] = 1$$

$$\mathbb{E}^F[e^{ur_t}] = e^{uX_t}$$

Comparing e^{uX_t} to the moment generating function of a normal random variable, it is clear

$$r_t* \sim \mathcal{N}\left(e^{-\alpha t*}r_0 - \sigma^2\int_0^{t*} e^{-\alpha(t*-t)}A(t,T)dt + b\left(1-e^{-\alpha t*}\right), \left(1-e^{-2\alpha t*}\right)\frac{\sigma^2}{2\alpha}\right)$$

In this model the function $A(t,T)$ satisfies Equation 2 with $\xi = 0$, $\gamma = -\alpha$, which has the solution

$$\frac{1-e^{-\alpha(T-t)}}{\alpha}$$

$$\implies \sigma^2\int_0^{t*} e^{-\alpha(t*-t)}A(t,T)dt = \sigma^2\int_0^{t*}\frac{e^{-\alpha(t*-t)}}{\alpha} - \frac{e^{-\alpha(t*+T)+2\alpha t}}{\alpha}dt$$

$$= \sigma^2\left(\frac{2-2e^{-\alpha t*} - e^{-\alpha(T-t*)} + e^{-\alpha(T+t*)}}{2\alpha^2}\right)$$

Therefore

$$c(r_0, 0) = f(r_0, 0)\left(\sigma_L\phi(z) + (\mu_L - k)(1-\Phi(z))\right)$$

Where

$$\mu_L = \frac{A(t*,T)\mu_r - C(t*,T)}{T-t*}$$

$$\sigma_L = \frac{A(t*,T)\sigma_r}{T-t*}$$

$$\mu_r = \left(e^{-\alpha t*}r_0 - \sigma^2\left(\frac{2-2e^{-\alpha t*} - e^{-\alpha(T-t*)} + e^{-\alpha(T+t*)}}{2\alpha^2}\right) + b\left(1-e^{-\alpha t*}\right)\right)$$

$$\sigma_r = \sqrt{\left(1-e^{-2\alpha t*}\right)\frac{\sigma^2}{2\alpha}}$$

$$z = \frac{k - \mu_L}{\sigma_L}$$

$\square$

C Algorithm for Pricing Caplets

Data: r_0, μ, γ, ω, ξ, t, n, m, rmax, rmin
Result: The numerical price of a caplet
Define: $\Delta t = t/n$, $\Delta r = (\text{rmax} - \text{rmin})/m$
Adjust rmax, rmin such that $r_0 \Delta r = p$ is an integer
Set a vector v such that $v[1 : p-1] = 0$, $v[p] = .5$, $v[p+1 : n] = 1$
for *i = 1:n-1* **do**
    $A[i+1] = A[i] - \frac{\xi \Delta t}{2} A[i]^2 + \gamma A[i] \Delta t + \Delta t$
    $C[i+1] = C[i] - \left(A[i] \mu \Delta t - A[i]^2 \frac{\omega \Delta t}{2}\right)$
end
for *i = 1:n-1* **do**
    for *j=1:m* **do**
        $a[j] = (((\mu + \gamma(\text{rmin} + \Delta r j)) - \frac{\xi}{2} - A[n-i](\omega + \xi(\text{rmin} + \Delta r j)))/(2\Delta r) - (\omega + \xi(\text{rmin} + \Delta r j))/(2 \Delta x^2))\Delta t$
        $b[j] = 1 + \Delta t \frac{\omega + \xi(rmin - \Delta r j)}{\Delta r^2}$
        $c[j] = (-((\mu + \gamma(\text{rmin} + \Delta r(j+1))) - \frac{\xi}{2} - A[n-i](\omega + \xi(\text{rmin} + \Delta r(j + 1))))/(2\Delta r) - (\omega + \xi(\text{rmin} + \Delta r(j+1)))/(2\Delta x^2))\Delta t$
    end
    $v[m] = v[m] - a[m]$
    $v = tridiagsolve(c[1 : m-1], b, a[1 : m-1], v)$
end
cap=0
for *i=1: n-1* **do**
    if $rmin + \Delta r i + \frac{\Delta r}{2} > 0$ **then**
        cap $= (\text{rmin} + \Delta r i + \frac{\Delta r}{2} - k)(v[i+1] - v[i]) +$ cap
    end
end
cap $=$ cap $\left(e^{-A[m] r_0 + C[m]}\right)$