Efficient Pricing of Caplets under a Single Factor Affine Interest Rate Process

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1 Introduction

A common method for pricing caplets uses Black's option formula. While in use by practitioners since the inception of Black's formula, there was no justification for its use in fixed income markets until Miltersen, Sandmann, and Sondermann (1997). They showed that log-normally distributed (simple) forward rates under the forward measure are compatible with the condition for no arbitrage in the fixed income market. Under the assumption of log-normal forward rates, caplets can be priced using Black's option formula. However, the convenience of an analytic solution is frequently unjustified by empirical data, as noted by Andersen and Lund (1997). The formula is still useful since one can generate an implied forward volatility with which to compute other derivative prices, but then caplet prices are essentially state variables instead of derivative securities on some more primitive state variable. A more theoretically appealing model would be one which accounts for the stochastic volatility to begin with, much like the Constant Elasticity of Volatility model attempts to explain the volatility smile found with equity options. Given the widespread use of short interest rate models, this paper attempts to rectify the issue by giving simple and efficient numerical methods to price caplets under any single factor affine interest rate process.

2 Assumptions and Preliminaries

2.1 Dynamics of Interest Rates

Let $(\Omega, \mathcal{F}, \mathbb{P}^R)$ be a probability space with a filtration generated by the one dimensional Brownian motion W_t^R. In this space exists a risk-free asset M_t and a measure $\mathbb{P}$ equivalent to $\mathbb{P}^R$ under which any asset with M_t as the numeraire is a local martingale.

Definition 1.

$$r_t := \lim_{T \to t} \frac{\ln \left(M_T / M_t\right)}{T - t} = \frac{\partial \ln(M_t)}{\partial t}$$

r_t is henceforth refered to as the *interest rate*.

The interest rate process satisfies the following stochastic differential equation (SDE):

$$dr_t = \alpha(r_t, t)dt + \sigma(r_t, t)dW_t$$

Where $\alpha : \mathbb{R}^2 \to \mathbb{R}$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}_+$ with σ satisfying $\mathbb{E}\left[\int_0^t \sigma^2 ds\right] < \infty$.

Remark 1. The condition on σ guarantees that any local martingale with respect to W_t is also a martingale.

Finally, the interest rate process is an affine process.

Definition 2. A process is said to be *affine* if $\alpha(r_t, t) = \mu(t) + \gamma(t)r_t$ and $\sigma^2(r_t, t) = \omega(t) + \xi(t)r_t$ for some μ, γ, ω, $\xi : \mathbb{R}_+ \to \mathbb{R}$.

2.2 Bonds

Definition 3. A *bond* is an asset with payoff function $f(r_T, T, T) = 1$.

Proposition 1. *Let r_t satisfy Definition 2. Then*

$$f(r_0, 0, T) = e^{-A(0,T)r_0 + C(0,T)}$$

is the price of a bond where $A(t, T)$ and $C(t, T)$ are deterministic functions of time.

Proof. Since discounted assets are martingales under $\mathbb{P}$:

$$f(r_0, 0, T) = \mathbb{E}\left[\frac{M_0}{M_T}\right] = \mathbb{E}\left[e^{-\int_0^T r_t dt}\right]$$

By Feynman-Kac, $e^{-\int_0^t r ds} f(r, t, T)$ (with dummy variable r) then satisfies the following partial differential equation (PDE):

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\alpha(r,t) + \frac{\partial^2 f}{\partial r^2}\frac{\sigma^2(r,t)}{2} - rf = 0 \\ f(r,T) = 1 \ \forall r \end{cases}$$

$$= \begin{cases} \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}(\mu + \gamma r) + \frac{\partial^2 f}{\partial r^2}\frac{(\omega+\xi r)}{2} - rf = 0 \\ f(r,T) = 1 \ \forall r \end{cases} \tag{1}$$

Substituting Proposition 1 into this PDE,

$$\left(-\frac{dA}{dt}r + \frac{dC}{dt}\right) f(r,t,T) - Af(r,t,T)(\mu + \gamma r) + A^2 f(r,t,T)\frac{(\omega + \xi r)}{2} - rf(r,t,T) = 0$$

$$\left(-\frac{dA}{dt}r + \frac{dC}{dt}\right) - A(\mu + \gamma r) + A^2\frac{(\omega + \xi r)}{2} - r = 0$$

For the above expression to hold, the following ordinary differential equations (ODEs) must hold:

$$\begin{cases} A^2\frac{\xi}{2} - \gamma A - 1 = \frac{dA}{dt} \\ A(T,T) = 0 \end{cases} \tag{2}$$

$$\begin{cases} \frac{dC}{dt} = \mu A - A^2\frac{\omega}{2} \\ C(T,T) = 0 \end{cases} \tag{3}$$

Clearly these ODEs have unique solutions, from which it follows that

$$f(r_0, 0, T) = e^{-A(0,T)r_0 + C(0,T)}$$

□

Remark 2. If the ODEs cannot be solved analytically it is computationally trivial to solve them numerically.

Proposition 2. *Let r_t satisfy Definition 2. Then*

$$df(r_t,t) = r_t f(r_t,t,T)dt - A(t,T)\sigma(r_t,t)f(r_t,t,T)dW_t$$

Proof. By Ito's Lemma,

$$df(r_t,t,T) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial r}\alpha(r_t,t)dt + \frac{\partial^2 f}{\partial r^2}\frac{\sigma^2(r_t,t)}{2}dt + \frac{\partial f}{\partial r}\sigma(r_t,t)dW_t$$

By Feynman-Kac,

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\alpha(r_t,t) + \frac{\partial^2 f}{\partial r^2}\frac{\sigma^2(r_t,t)}{2} = r_t f(r_t,t)$$

Substituting this into the differential of $f(r_t,t,T)$,

$$df(r_t,t,T) = r_t f(r_t,t,T)dt + \frac{\partial f}{\partial r}\sigma(r_t,t)dW_t$$

By Proposition 1,

$$\frac{\partial f}{\partial r} = -A(t,T)f(r_t,t,T)$$

This implies the following differential:

$$df(r_t,t,T) = r_t f(r_t,t,T)dt - A(t,T)f(r_t,t,T)\sigma(r_t,t)dW_t$$

□

2.3 Caplet

Definition 4. A *caplet* is an asset with payoff function

$$(L_{t^*,T} - k)\mathbb{I}_{L_{t^*,T}>k},\ k \in \mathbb{R}_+,\ 0 < t^* < T$$

Where $L_{t^*,T} := -\frac{\ln(f(r_{t^*},t^*,T))}{T-t^*}$. This is paid at time T.

Remark 3. The Black caplet model postulates the dynamics of the forward rate of a specific future date. The forward rate is defined as follows:

$$F(t,t^*,T)=\frac{(T-t)L_{t,T}-(t^*-t)L_{t,t^*}}{T-t^*}$$

The payoff under this formulation is $(F(t^*,t^*,T)-k)\mathbb{I}_{F(t^*,t^*,T)>k}$.
By the definition of forward rate,

$$F(t^*,t^*,T)=\frac{(T-t^*)L_{t^*,T}}{T-t^*}=L_{t^*,T}$$

Therefore the two payoffs are identical.

Proposition 3. *Let r_t satisfy Definition 2 and let the current price of a caplet be $c(r_0,0,t^*,T)$. Then*

$$c(r_0,0,t^*,T)=f(r_0,0,T)\left(\mathbb{E}^F\left[L_{t^*,T}\mathbb{I}_{L_{t^*,T}>k}\right]-k\mathbb{E}^F\left[\mathbb{I}_{L_{t^*,T}>k}\right]\right) \tag{4}$$

Where the expectation is taken under the forward measure and the dynamics of r_t under this measure are

$$dr_t=\left(\alpha(r_t,t)-\sigma^2(r_t,t)A(t,T)\right)dt+\sigma(r_t,t)dW_t^F \tag{5}$$

Proof. Since discounted assets are martingales under $\mathbb{P}$:

$$c(r_0,0,t^*,T)=\mathbb{E}\left[e^{-\int_0^T r_t dt}\left(L_{t^*,T}\mathbb{I}_{L_{t^*,T}>k}-k\mathbb{I}_{L_{t^*,T}>k}\right)\right]$$

Defining a Radon-Nikodym derivative

$$Z_t:=\frac{e^{-\int_0^t r_s ds}f(r_t,t,T)}{f(r_0,0,T)}$$

Substituting into the pricing formula,

$$c(r_0,0,t^*,T)=f(r_0,0,T)\mathbb{E}\left[Z_T\left(L_{t^*,T}\mathbb{I}_{L_{t^*,T}>k}-k\mathbb{I}_{L_{t^*,T}>k}\right)\right]$$

By Proposition 2, the volatility of Z_t is $-A(t,T)Z_t\sigma(r_t,t)$. By Girsonov's theorem, $W_t^F := W_t + \int_0^t A(s,T)\sigma(r_s,s)ds$ is a Brownian motion under the forward measure and the pricing formula can be written as

$$c(r_0,0,t^*,T) = f(r_0,0,T)\mathbb{E}^F\left[L_{t^*,T}\mathbb{I}_{L_{t^*,T}>k} - k\mathbb{I}_{L_{t^*,T}>k}\right]$$

Recalling that $dr_t = \alpha(r_t,t)dt + \sigma(r_t,t)dW_t$, under the forward measure r_t has the following dynamics:

$$\begin{aligned} dr_t &= \alpha(r_t,t)dt + \sigma(r_t,t)(dW_t^F - A(t,T)\sigma(r_t,t)dt) \\ &= \big(\alpha(r_t,t) - \sigma^2(r_t,t)A(t,T)\big)\,dt + \sigma(r_t,t)dW_t^F \end{aligned}$$

□

Remark 4. Let r_t satisfy Definition 2. Using the definition of the forward rate, the forward rate is the following function of the short rate:

$$\frac{(A(t,t^*) - A(t,T))\,r_t + C(t,T) - C(t,t^*)}{T - t^*}$$

Since the randomness of this function is derived from the short rate process, in general the diffusion of the forward rate is non-deterministic, deviating from the assumptions of the Black model.

3 Computation of Caplets

If r_t satisfies Definition 2 such that $\xi(t) \neq 0$, there generally exists no analytic density from which to calculate the (forward) probability of the interest rate terminating in the money; necessitating the use of numerical methods to find a solution. However the usual numerical solutions for option pricing in this scenario are either difficult or inefficient. Monte Carlo methods for solving Equation 4 are subject to discretization error since the analytic distribution is in general unknown, necessitating an Euler-like simulation scheme. In addition, to simulate Equation 5 requires computing or estimating $A(t,T)$ at each time node. In fact, it would be more efficient to simply simulate the risk-neutral process. However, simulating the risk-neutral process compounds the discretization error since the integral must be approximated by a sum. Another common numerical method is to discretize Equation 1 and solve the PDE numerically. Unfortunately this method is not very accurate for the pricing of interest rate derivatives as noted by Büttler (1995). Further, it would be difficult to generalize the boundary conditions to accommodate any single factor affine model.

3.1 Alternate Numerical Solution

Proposition 4. *Let* $p(r,t) := d\mathbb{P}^F(r_t \leq r)$ *be known. Then* $\mathbb{E}^F\left[L_{t^*,T}\mathbb{I}_{L_{t^*,T}>k}\right] - k\mathbb{E}^F\left[\mathbb{I}_{L_{t^*,T}>k}\right]$ *can be priced.*

Proof. This follows from the definition of expectations. □

By Proposition 4, Equation 4 can be approximately solved if the probability density of r_t under the forward measure can be numerically approximated. By the Fokker-Planck equation, $p(r,t)$ satisfies the PDE

$$\begin{cases} \frac{\partial p}{\partial t} = -\frac{\partial}{\partial r}p(r,t)\left(\alpha(r,t) - \sigma^2(r,t)A(t,T)\right) + \frac{\partial^2}{\partial r^2}\frac{1}{2}p(r,t)\sigma^2(r,t) \\ p(r,0) = \delta(r) \end{cases} \tag{6}$$

This equation is far simpler to solve numerically than Equation 1 since the boundary conditions are necessarily zero. However, the initial condition is not easily discretized.

Proposition 5. *Let* $p(r,t)$ *exist. Then* $F(r,t) := \mathbb{P}^F(r_t < r)$ *satisfies*

$$\begin{cases} \frac{\partial F}{\partial t} = -\left(\alpha(r,t) - \sigma^2(r,t)A(t,T) - \frac{1}{2}\xi\right)\frac{\partial F}{\partial r} + \frac{1}{2}\sigma^2(r,t)\frac{\partial^2 F}{\partial r^2} \\ F(r,0) = \mathbb{I}_{r>r_0} \end{cases} \tag{7}$$

Proof.

$$\frac{\partial}{\partial r}F(r,t) = p(r,t)$$

$$\implies \begin{cases} \frac{\partial^2 F}{\partial t \partial r} = -\frac{\partial}{\partial r}\left(\frac{\partial F}{\partial r}\left(\alpha(r,t) - \sigma^2(r,t)A(t,T)\right)\right) + \frac{\partial^2}{\partial r^2}\left(\frac{1}{2}\frac{\partial F}{\partial r}\sigma^2(r,t)\right) \\ \frac{\partial F}{\partial r} = \delta(r) \end{cases}$$

Integrating with respect to r,

$$\begin{cases} \frac{\partial F}{\partial t} = -\frac{\partial F}{\partial r}\left(\alpha(r,t) - \sigma^2(r,t)A(t,T)\right) + \frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\partial F}{\partial r}\sigma^2(r,t)\right) + c(t) \\ \hfill F(r,0) = \mathbb{I}_{r>r_0} \end{cases}$$

Since at the boundaries $\lim_{r\to\Omega_+} F(r,t) = 1$ and $\lim_{r\to\Omega_-} F(r,t) = 0$, $\frac{\partial F}{\partial t}$ is equal to zero at the boundaries, which implies that $c(t)$ is also zero.

$$\begin{cases} \frac{\partial F}{\partial t} = -\frac{\partial F}{\partial r}\left(\alpha(r,t) - \sigma^2(r,t)A(t,T)\right) + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}\sigma^2(r,t) + \frac{1}{2}\xi\frac{\partial F}{\partial r} \\ \hfill F(r,0) = \mathbb{I}_{r>r_0} \end{cases}$$

$$\begin{cases} \frac{\partial F}{\partial t} = -\left(\alpha(r,t) - \sigma^2(r,t)A(t,T) - \frac{1}{2}\xi\right)\frac{\partial F}{\partial r} + \frac{1}{2}\sigma^2(r,t)\frac{\partial^2 F}{\partial r^2} \\ \hfill F(r,0) = \mathbb{I}_{r>r_0} \end{cases}$$

□

3.2 Numerical Results

To approximately solve Equation 7, I use an implicit finite difference scheme. The initial condition is discretized by $F(r,0) = 0\ \forall r < r_0$, $F(r,0) = 1\ \forall r > r_0$, and $F(r,0) = .5,\ r = r_0$. The functions $A(t,T)$ and $C(t,T)$ are approximated via an Euler discretization scheme in which the time step is the same size as the one used for the numerical solution to Equation 7. Discretizing Equation 7 results in a vector of approximate values of $F(r,t^*)$ denoted $\tilde{F}(r,t^*)$ corresponding to each discrete node of r. The approximation of the forward expectation therefore is computed as follows:

$$\sum_{i=g(k)}^{m}\left(\frac{A(t^*,T)r_i - C(t^*,T) + \frac{A(t^*,T)(r_{i+1}-r_i)}{2}}{T-t^*} - k\right)\left(\tilde{F}(r_{i+1},t^*) - \tilde{F}(r_i,t^*)\right)$$

Where $g(k)$ is a function mapping the value of k to the smallest value of i such that $\frac{A(t^*,T)r_i - C(t^*,T)}{T-t^*} > k$. For optimal accuracy r_0 and t^* should be discrete values on their respective domains. The complete algorithm for computing the price of the caplet is given in the appendix.

3.2.1 Vasicek

Proposition 6. *In the case that the interest rate process follows* Vasicek's *model:*

$$dr_t = \alpha(b - r_t)dt + \sigma dW_t,\ \ \alpha,\ \ b,\ \ \sigma\ \ \in \mathbb{R}_+ \tag{8}$$

The price of the caplet is

$$c(r_0, 0, t^*, T) = f(r_0, 0, T)\left(\sigma_L \phi(z) + (\mu_L - k)(1 - \Phi(z))\right)$$

Where

$$\mu_L = \frac{A(t^*, T)\mu_r - C(t^*, T)}{T - t^*}$$

$$\sigma_L = \frac{A(t^*, T)\sigma_r}{T - t^*}$$

$$\mu_r = \left(e^{-\alpha t^*} r_0 - \sigma^2 \left(\frac{2 - 2e^{-\alpha t^*} - e^{-\alpha(T - t^*)} + e^{-\alpha T}}{2}\right) + b\left(1 - e^{-\alpha t^*}\right)\right)$$

$$\sigma_r = \sqrt{\left(1 - e^{-2\alpha t^*}\right)\frac{\sigma^2}{2\alpha}}$$

$$z = \frac{k - \mu_L}{\sigma_L}$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$A(t, T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha}$$

$$C(t, T) = \left(\frac{\sigma^2}{2\alpha^2} - b\right)(T - t) + \frac{b}{\alpha}\left(1 - e^{-\alpha(T-t)}\right) + \frac{\sigma^2}{2\alpha^2}\left(\frac{2(e^{-\alpha(T-t)} - 1)}{\alpha} - \frac{e^{-2\alpha(T-t)} - 1}{2\alpha}\right)$$

Proof. The proof is given in the appendix. □

This analytic solution facilitates the numerical analysis of the algorithm. Using parameter values $\alpha = 1$, $b = .1$, $\sigma = .03$, $r_0 = .10$, $k = .10$, $T = 1$, and $t^* = .5$, the analytic solution is $.00471525$.

Space is discretized with m nodes on $[-.1,\ .5]$ and time with n nodes.

	$n = 60$, $m = 72$	$n = 150$, $m = 180$	$n = 300$, $m = 360$	$n = 600$, $m = 720$
Value	.00472895	.0047172	.00471583	.00471545
Time (s)	0	0	.016	.047
Relative error	.29048 %	.04124 %	.01225 %	.00420 %
Decrease in error		7.0438	3.3667	2.9178

3.2.2 Cox Ingersoll Ross

When $\mu(t)$, $\gamma(t)$, $\xi(t)$ are constants and $\omega = 0$ then r_t follows the *Cox Ingersoll Ross* model:

$$dr_t = \alpha(b - r_t)dt + \sigma\sqrt{r_t}dW_t,\ \ \alpha,\ \ b,\ \ \sigma \in \mathbb{R}_+$$

This model is given parameter values $\alpha = 1$, $b = .1$, $\sigma = .12$, $r_0 = .10$, $k = .10$, $t^* = .5$, and $T = 1$. Space is discretized on $[0,\ .5]$.

	$n = 60$, $m = 60$	$n = 150$, $m = 150$	$n = 300$, $m = 300$	$n = 600$, $m = 600$	$n = 60000$ $m = 6000$
Value	.00592512	.00591677	.00591565	.0059153	.00591509
Time (s)	0	0	.016	.031	38.625
Relative error	.16957 %	.02840 %	.00947%	.00355 %	–
Decrease in error		5.9708	2.9989	2.6676	–

Here the 6000 by 60000 mesh is considered the "exact" solution.

4 Extensions

The method used to compute caplets can be extended to any path independent payoff of assets that follow singe factor SDEs. For instance, an option on a bond can be computed under a single factor HJM framework. Perhaps the optimal use for this method of computation is a path dependent payoff that is a linear function of the underlying: for example, a derivative security with payoff function $\int_0^T S_t dt$. In a manner similar to Monte Carlo simulation, Greeks can also be calculated by taking either the derivative of the payoff function or by further differencing the terminal distribution.

5 Conclusion

The algorithm presented quickly and accurately prices caplets under the assumption that the short interest rate follows a single factor affine yield process. A downside is that as either time or the short rate changes the entire computation must be redone. This is similar to a Monte Carlo solution in that only a single price can be computed at a time. Standard PDE methods shine in this area since the time and space mesh are discretized for any time and space value that the underlying process may take. A further compromise (shared by standard PDE methods but not by Monte Carlo) is that generalizing the result to a three or four factor model would be computationally prohibitive. Still, this technique gives greater flexibility to the pricing of caplets than analytic formulas based on the assumption of log-normal forward rates; generalizing the rather restrictive assumptions of Black's model and doing so with computational accuracy.

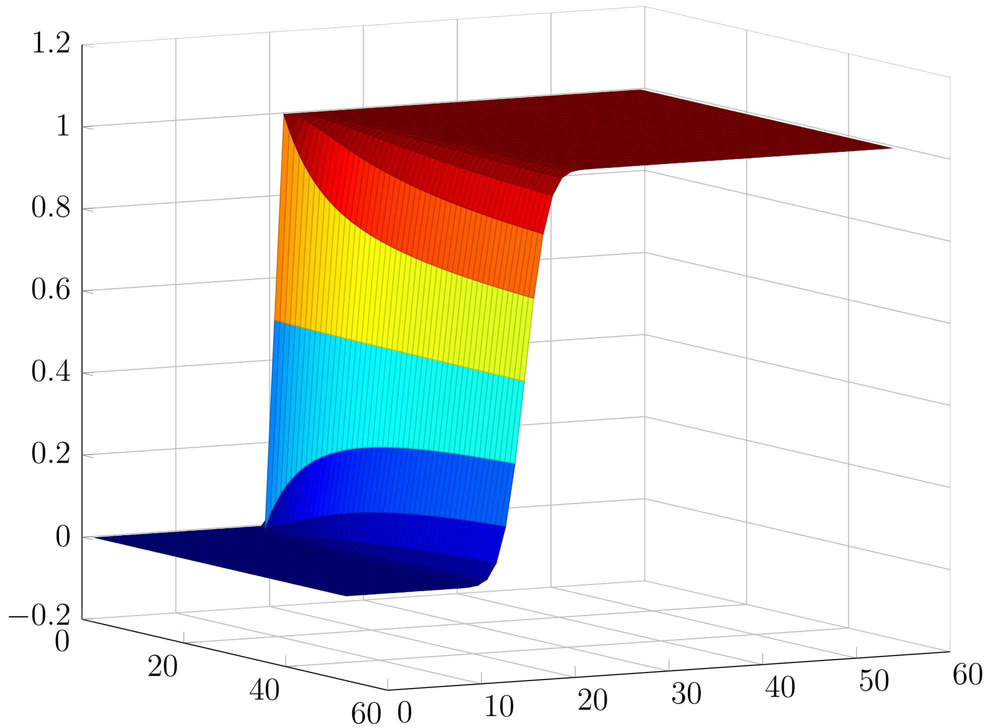
Works Cited

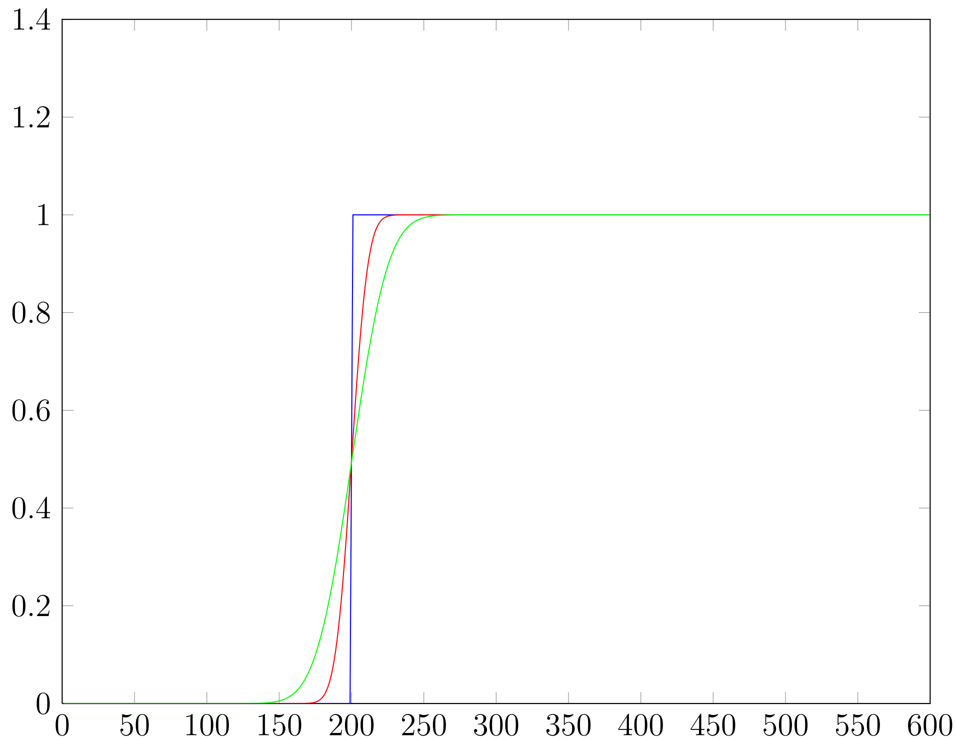
Miltersen, K., Sandmann, K., and Sondermann, D. 1997. "Closed Form Solutions for Term Structure Derivatives with Log-normal Interest Rates." Journal of Finance, 52(1), 409-430.

Andersen, T. G., and J. Lund. 1997. "Estimating Continuous-Time Stochastic Volatility Models of the Short-Term Interest Rate." Journal of Econometrics 77, 343-377.

Büttler, H.J. 1995. "Evaluation of callable bonds: Finite difference methods, stability and accuracy". Economic Journal 105, 374-384.

A Plots


1.2
1
0.8
0.6
0.4
0.2
0
−0.2
0
20
40
60
0
10
20
30
40
50
60


1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
250
300
350
400
450
500
550
600

B Proof of Proposition 6

Proof. If $y \sim \mathcal{N}(\mu, \sigma^2)$,

$$\mathbb{E}\left[(y-k)\mathbb{I}_{y>k}\right] = \int_k^\infty y\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy - k\int_k^\infty \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy$$

$$= \int_{\frac{k-\mu}{\sigma}}^\infty (\sigma x+\mu)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx - k\int_{\frac{k-\mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$= \sigma\int_{\frac{k-\mu}{\sigma}}^\infty -d\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right) + \mu\left(1-\Phi\left(\frac{k-\mu}{\sigma}\right)\right) - k\left(1-\Phi\left(\frac{k-\mu}{\sigma}\right)\right)$$

$$= \sigma\phi\left(\frac{k-\mu}{\sigma}\right) + (\mu-k)\left(1-\Phi\left(\frac{k-\mu}{\sigma}\right)\right)$$

By Ito's Lemma,

$$d\left(e^{\alpha t}r_t\right) = \alpha e^{\alpha t}r_t dt + e^{\alpha t}\left(\alpha b - \sigma^2 A(t,T) - \alpha r_t\right)dt + e^{\alpha t}\sigma dW_t^F$$

$$= e^{\alpha t}\left(\alpha b - \sigma^2 A(t,T)\right)dt + e^{\alpha t}\sigma dW_t^F$$

$$e^{\alpha t^*}r_{t^*} = r_0 + \frac{1}{\alpha}\left(e^{\alpha t^*}-1\right)\alpha b - \sigma^2\int_0^{t^*} e^{\alpha t}A(t,T)dt + \int_0^{t^*} e^{\alpha t}\sigma dW_t^F$$

$$r_{t^*} = e^{-\alpha t^*}r_0 + b\left(1-e^{-\alpha t^*}\right) - \sigma^2\int_0^{t^*} e^{-\alpha(t^*-t)}A(t,T)dt + e^{-\alpha t^*}\int_0^{t^*} e^{\alpha t}\sigma dW_t^F$$

Let

$$X_t := e^{-\alpha t}r_0 - \sigma^2\int_0^t e^{-\alpha(t-s)}A(s,T)ds + b\left(1-e^{-\alpha t}\right) + u\left(1-e^{-2\alpha t}\right)\frac{\sigma^2}{4\alpha}$$

For some $u \in \mathbb{R}_+$ and

$$V_t := e^{ur_t - uX_t} = e^{ue^{-\alpha t}\int_0^t e^{\alpha s}\sigma dW_s^F - u^2\left(1-e^{-2\alpha t}\right)\frac{\sigma^2}{4\alpha}}$$

By Ito's Lemma,

$$dV_t = V_t u e^{-\alpha t}e^{\alpha t}\sigma dW_t^F + V_t\frac{1}{2}u^2 e^{-2\alpha t}e^{2\alpha t}\sigma^2 dt - V_t u^2\frac{\sigma^2}{2}dt$$

$$= V_t u \sigma dW_t^F$$

Since $\mathbb{E}\left[\int_0^t u^2\sigma^2 V_t^2 dt\right] < \infty$, V_t is a martingale under the forward measure and satisfies $V_t(r_0, 0) = 1$. Therefore

$$\mathbb{E}^F[e^{ur_t - uX_t}] = 1$$

$$\implies \mathbb{E}^F[e^{ur_t}] = e^{uX_t}$$

Comparing e^{uX_t} to the moment generating function of a normal random variable, the following is clear:

$$r_{t^*} \sim \mathcal{N}\left(e^{-\alpha t^*} r_0 - \sigma^2 \int_0^{t^*} e^{-\alpha(t^*-t)} A(t,T)dt + b\left(1 - e^{-\alpha t^*}\right), \left(1 - e^{-2\alpha t^*}\right)\frac{\sigma^2}{2\alpha}\right)$$

In this model the function $A(t,T)$ satisfies Equation 2 with $\xi = 0$, $\gamma = -\alpha$, which has the solution

$$\frac{1 - e^{-\alpha(T-t)}}{\alpha}$$

$$\implies \sigma^2 \int_0^{t^*} e^{-\alpha(T-t)} A(t,T)dt = \sigma^2 \int_0^{t^*} \frac{e^{-\alpha(t^*-t)}}{\alpha} - \frac{e^{-\alpha(t^*+T)+2\alpha t}}{\alpha} dt$$

$$= \sigma^2 \left(\frac{2 - 2e^{-\alpha t^*} - e^{-\alpha(T-t^*)} + e^{-\alpha T}}{2}\right)$$

Integrating $\alpha b A(t,T) - \frac{A(t,T)^2\sigma^2}{2}$ yields $C(t,T)$.

Therefore

$$c(r_0, 0, t^*, T) = f(r_0, 0, T)\left(\sigma_L \phi(z) + (\mu_L - k)(1 - \Phi(z))\right)$$

Where

$$\mu_L = \frac{A(t^*,T)\mu_r - C(t^*,T)}{T - t^*}$$

$$\sigma_L = \frac{A(t^*,T)\sigma_r}{T - t^*}$$

$$z = \frac{k - \mu_L}{\sigma_L}$$

$\square$

C Algorithm for Pricing Caplets

Data: r_0, k, μ, γ, ω, ξ, t^*, T, n, m, rmax, rmin
Result: The numerical price of a caplet
Define: $\Delta t = t/n$, $\Delta r = (\text{rmax} - \text{rmin})/m$
Adjust rmax, rmin such that $\frac{r_0}{\Delta r} := p$ is an integer
Adjust n such that $tn := (n * t^*)/T$ is an integer
Set a vector v such that $v[1 : p - 1] = 0$, $v[p] = .5$, $v[p + 1 : n] = 1$
A[1]=0
C[1]=0
for *i = 1:n* **do**
  $A[i + 1] = A[i] - \frac{\xi \Delta t}{2} A[i]^2 + \gamma A[i]\Delta t + \Delta t$
  $C[i + 1] = C[i] - \left(A[i]\mu - \frac{\omega}{2} A[i]^2\right) \Delta t$
end
for *i = 1:tn* **do**
  for *j=1:m* **do**
    $a[j] = (((\mu + \gamma(\text{rmin} + \Delta rj)) - \frac{\xi}{2} - A[n - i](\omega + \xi(\text{rmin} + \Delta rj)))/(2\Delta r) - (\omega + \xi(\text{rmin} + \Delta rj))/(2\Delta x^2))\Delta t$
    $b[j] = 1 + \Delta t \frac{\omega + \xi(rmin - \Delta rj)}{\Delta r^2}$
    $c[j] = (-((\mu + \gamma(\text{rmin} + \Delta r(j + 1))) - \frac{\xi}{2} - A[n - i](\omega + \xi(\text{rmin} + \Delta r(j + 1))))/(2\Delta r) - (\omega + \xi(\text{rmin} + \Delta r(j + 1)))/(2\Delta x^2))\Delta t$
  end
  $v[m] = v[m] - a[m]$
  $v = tridiagsolve(c[1 : m - 1], b, a[1 : m - 1], v)$
end
cap=0
for *i=1: n-1* **do**
  if $\left(\left(rmin + \Delta ri + \frac{\Delta r}{2}\right) A[tn] - C[tn]\right) /(T - t^*) - k > 0$ **then**
    cap $= \left(\left(\left(rmin + \Delta ri + \frac{\Delta r}{2}\right) A[tn] - C[tn]\right) /(T - t^*) - k\right) (v[i + 1] - v[i]) +$ cap
  end
end
cap $=$ cap $* e^{-A[tn]r_0 + C[tn]}$