Efficient Pricing of Caplets under a Single Factor Affine Interest Rate Process

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1 Introduction

Practitioners have long used Black's option formula for the pricing of caplets, despite the underlying assumptions of Black's model being severely violated in the fixed income market. In particular, Black's formula assumes constant interest rates for discounting and the ability to continuously trade the underlying asset. While the effects of misspecification with respect to both assumptions are mild with equity options, they are crippling in fixed income where the payoff and the discounting factor are highly correlated and the underlying interest rate is impossible to trade. The custom of using Black's model was not mathematically justified until Miltersen, et al (1997) showed that it is possible to represent the payoff of *simple* forward rates in such a way that under a change of numeraire the forward rate is a martingale under the forward measure. In particular, the forward rate could be a log-normal martingale, satisfying the requirements for Black's formula. Unfortunately, much like equity options, the implied volatility of the forward rate is incompatable with the assumption of log-normal forward rates; see, for example, Anderson and Andreasen (2000). Most attempts to compensate for this phenomenon extend the model. Wu and Zhang (2006), for instance, borrow from Heston and model stochastic volatility. In this paper I diverge from these previous attempts and model the short interest rate directly as an affine process. This method is advantageous for a number of reasons: short interest rate models are popular and familiar, the volatility of the forward rate is stochastic, and the numerical procedure for calculation is accurate and quick.

2 Assumptions and Preliminaries

2.1 Dynamics of Interest Rates

Let $(\Omega, \mathcal{F}, \mathbb{P}^R)$ be a probability space with a filtration generated by the one dimensional Brownian motion W_t^R. In this space exists a risk-free asset M_t and a measure $\mathbb{P}$ equivalent to $\mathbb{P}^R$ under which any asset with M_t as the numeraire is a local martingale.

Definition 1.
$$r_t := \lim_{T \to t} \frac{\ln (M_T/M_t)}{T - t} = \frac{\partial \ln(M_t)}{\partial t}$$

r_t is henceforth refered to as the *interest rate*.

Solving this ordinary differential equation (ODE) for M_t yields $M_t = M_0 e^{\int_0^t r_s ds}$.

The interest rate satisfies the following stochastic differential equation (SDE):

$$dr_t = \alpha(r_t, t)dt + \sigma(r_t, t)dW_t$$

Where $\alpha : \mathbb{R}^2 \to \mathbb{R}$ and $\sigma : \mathbb{R}^2 \to \mathbb{R}_+$ with σ satisfying $\mathbb{E}\left[\int_0^t \sigma^2 ds\right] < \infty$.

Remark 1. The condition on σ guarantees that any local martingale with respect to W_t is also a martingale.

Finally, the interest rate is an affine process.

Definition 2. A process is said to be *affine* if $\alpha(r_t, t) = \mu(t) + \gamma(t)r_t$ and $\sigma^2(r_t, t) = \omega(t) + \xi(t)r_t$ for some $\mu, \gamma, \omega, \xi : \mathbb{R}_+ \to \mathbb{R}$.

2.2 Bonds

Definition 3. A *bond* is an asset with payoff function $f(r_T, T, T) = 1$.

Proposition 1. *Let r_t satisfy Definition 2. Then*

$$f(r_0, 0, T) = e^{-A(0,T)r_0 + C(0,T)}$$

is the price of a bond where $A(t, T)$ and $C(t, T)$ are deterministic functions of time.

Proof. Since discounted assets are martingales under $\mathbb{P}$:

$$f(r_0, 0, T) = \mathbb{E}\left[\frac{M_0}{M_T}\right] = \mathbb{E}\left[e^{-\int_0^T r_t dt}\right]$$

By Feynman-Kac, $e^{-\int_0^t r ds} f(r, t, T)$ (with dummy variable r) then satisfies the following partial differential equation (PDE):

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\alpha(r, t) + \frac{\partial^2 f}{\partial r^2}\frac{\sigma^2(r,t)}{2} - rf = 0 \\ \qquad\qquad\qquad f(r, T) = 1 \ \forall r \end{cases}$$

$$= \begin{cases} \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}(\mu + \gamma r) + \frac{\partial^2 f}{\partial r^2}\frac{(\omega + \xi r)}{2} - rf = 0 \\ \qquad\qquad\qquad f(r, T) = 1 \ \forall r \end{cases} \tag{1}$$

Substituting Proposition 1 into this PDE,

$$\left(-\frac{dA}{dt}r + \frac{dC}{dt}\right) f(r, t, T) - Af(r, t, T)(\mu + \gamma r) + A^2 f(r, t, T)\frac{(\omega + \xi r)}{2} - rf(r, t, T) = 0$$

$$\left(-\frac{dA}{dt}r + \frac{dC}{dt}\right) - A(\mu + \gamma r) + A^2\frac{(\omega + \xi r)}{2} - r = 0$$

For the above expression to hold, the following ODEs must hold:

$$\begin{cases} A^2\frac{\xi}{2} - \gamma A - 1 = \frac{dA}{dt} \\ \qquad A(T, T) = 0 \end{cases} \tag{2}$$

$$\begin{cases} \frac{dC}{dt} = \mu A - A^2\frac{\omega}{2} \\ \qquad C(T, T) = 0 \end{cases} \tag{3}$$

Clearly these ODEs have unique solutions, from which it follows that

$$f(r_0, 0, T) = e^{-A(0,T)r_0 + C(0,T)}$$

$\square$

Proposition 2. *Let r_t satisfy Definition 2. Then*

$$df(r_t, t, T) = r_t f(r_t, t, T)dt - A(t, T)\sigma(r_t, t)f(r_t, t, T)dW_t$$

Proof. By Ito's Lemma,

$$df(r_t, t, T) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial r}\alpha(r_t, t)dt + \frac{\partial^2 f}{\partial r^2}\frac{\sigma^2(r_t, t)}{2}dt + \frac{\partial f}{\partial r}\sigma(r_t, t)dW_t$$

By Feynman-Kac,

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\alpha(r_t, t) + \frac{\partial^2 f}{\partial r^2}\frac{\sigma^2(r_t, t)}{2} = r_t f(r_t, t)$$

Substituting this into the differential of $f(r_t, t, T)$,

$$df(r_t, t, T) = r_t f(r_t, t, T)dt + \frac{\partial f}{\partial r}\sigma(r_t, t)dW_t$$

By Proposition 1,

$$\frac{\partial f}{\partial r} = -A(t, T)f(r_t, t, T)$$

This implies the following differential:

$$df(r_t, t, T) = r_t f(r_t, t, T)dt - A(t, T)f(r_t, t, T)\sigma(r_t, t)dW_t$$

$\square$

2.3 Caplet

Definition 4. A *caplet* is an asset with payoff function

$$(L_{t^*, T} - k)\mathbb{I}_{L_{t^*, T} > k}, \ \ k \in \mathbb{R}_+, \ 0 < t^* < T$$

Where $L_{t^*, T} := -\frac{\ln(f(r_{t^*}, t^*, T))}{T - t^*}$. This is paid at time T.

Remark 2. The Black caplet model models the dynamics of the forward rate of a specific future

date. The forward rate (with continuous compounding) is defined as follows:

$$F(t, t^*, T) = \frac{(T-t)L_{t,T} - (t^*-t)L_{t,t^*}}{T-t^*}$$

The payoff under this formulation is $(F(t^*, t^*, T) - k)\mathbb{I}_{F(t^*,t^*,T)>k}$.

By the definition of forward rate,

$$F(t^*, t^*, T) = \frac{(T-t^*)L_{t^*,T}}{T-t^*} = L_{t^*,T}$$

Therefore the two payoffs are identical.

Proposition 3. *Let r_t satisfy Definition 2 and let the current price of a caplet be $c(r_0, 0, t^*, T)$.*
Then

$$c(r_0, 0, t^*, T) = f(r_0, 0, T) \left(\mathbb{E}^F \left[L_{t^*,T} \mathbb{I}_{L_{t^*,T}>k} \right] - k \mathbb{E}^F \left[\mathbb{I}_{L_{t^*,T}>k} \right] \right) \tag{4}$$

Where the expectation is taken under the forward measure and the dynamics of r_t under this mea-
sure are

$$dr_t = \left(\alpha(r_t, t) - \sigma^2(r_t, t) A(t, T) \right) dt + \sigma(r_t, t) dW_t^F \tag{5}$$

Proof. Since discounted assets are martingales under $\mathbb{P}$:

$$c(r_0, 0, t^*, T) = \mathbb{E} \left[e^{-\int_0^T r_t dt} \left(L_{t^*,T} \mathbb{I}_{L_{t^*,T}>k} - k \mathbb{I}_{L_{t^*,T}>k} \right) \right]$$

Define the Radon-Nikodym derivative

$$Z_t := \frac{e^{-\int_0^t r_s ds} f(r_t, t, T)}{f(r_0, 0, T)}$$

Substituting into the pricing formula,

$$c(r_0, 0, t^*, T) = f(r_0, 0, T) \mathbb{E} \left[Z_T \left(L_{t^*,T} \mathbb{I}_{L_{t^*,T}>k} - k \mathbb{I}_{L_{t^*,T}>k} \right) \right]$$

By Proposition 2, the volatility of Z_t is $-A(t, T) Z_t \sigma(r_t, t)$. By Girsonov's theorem, $W_t^F :=$

$W_t + \int_0^t A(s,T)\sigma(r_s,s)ds$ is a Brownian motion under the forward measure and the pricing formula can be written as

$$c(r_0, 0, t^*, T) = f(r_0, 0, T)\mathbb{E}^F\left[L_{t^*,T}\mathbb{I}_{L_{t^*,T}>k} - k\mathbb{I}_{L_{t^*,T}>k}\right]$$

Recalling that $dr_t = \alpha(r_t,t)dt + \sigma(r_t,t)dW_t$, under the forward measure r_t has the following dynamics:

$$dr_t = \alpha(r_t,t)dt + \sigma(r_t,t)(dW_t^F - A(t,T)\sigma(r_t,t)dt)$$

$$= \left(\alpha(r_t,t) - \sigma^2(r_t,t)A(t,T)\right)dt + \sigma(r_t,t)dW_t^F$$

□

Remark 3. Let r_t satisfy Definition 2. Using the definition of the forward rate, the forward rate is the following function of the short rate:

$$\frac{\left(A(t,t^*) - A(t,T)\right)r_t + C(t,T) - C(t,t^*)}{T - t^*}$$

Since the randomness of this function is derived from the short rate process, in general the diffusion of the forward rate is non-deterministic, deviating from the assumptions of the Black model.

3   Computation of Caplets

If r_t satisfies Definition 2 such that $\xi(t) \neq 0$, there generally exists no analytic density from which to calculate the (forward) probability of the interest rate terminating in the money; necessitating the use of numerical methods to find a solution. However the usual numerical solutions for option pricing in this scenario are either difficult or inefficient. Monte Carlo methods for solving Equation 4 are subject to discretization error since the analytic distribution is in general unknown, necessitating an Euler-like simulation scheme. In addition, to simulate Equation 5 requires computing or estimating $A(t,T)$ at each time node. In fact, it would be more efficient to simply simulate the risk-neutral process. However, simulating the risk-neutral process compounds the discretization error since the integral must be approximated by a sum. Another common numerical method is to discretize Equation 1 and solve the PDE numerically. Unfortunately this method is not very

accurate for the pricing of interest rate derivatives as noted by Büttler (1995). Further, it would be difficult to generalize the boundary conditions to accommodate any single factor affine model.

3.1 Alternate Numerical Solution

Proposition 4. *Let* $p(r,t) := d\mathbb{P}^F(r_t \leq r)$ *be known. Then* $\mathbb{E}^F\left[L_{t^*,T}\mathbb{I}_{L_{t^*,T}>k}\right] - k\mathbb{E}^F\left[\mathbb{I}_{L_{t^*,T}>k}\right]$ *can be priced.*

Proof. This follows from the definition of expectations. $\qquad\square$

By Proposition 4, Equation 4 can be approximately solved if the probability density of r_t under the forward measure can be numerically approximated. By the Fokker-Planck equation, $p(r,t)$ satisfies the PDE

$$\begin{cases} \frac{\partial p}{\partial t} = -\frac{\partial}{\partial r}p(r,t)\left(\alpha(r,t) - \sigma^2(r,t)A(t,T)\right) + \frac{\partial^2}{\partial r^2}\frac{1}{2}p(r,t)\sigma^2(r,t) \\ \\ \qquad\qquad\qquad\qquad\qquad\qquad p(r,0) = \delta(r) \end{cases} \tag{6}$$

This equation is far simpler to solve numerically than Equation 1 since the boundary conditions are necessarily zero. However, the initial condition is not easily discretized.

Proposition 5. *Let* $p(r,t)$ *exist. Then* $F(r,t) := \mathbb{P}^F(r_t < r)$ *satisfies*

$$\begin{cases} \frac{\partial F}{\partial t} = -\left(\alpha(r,t) - \sigma^2(r,t)A(t,T) - \frac{1}{2}\xi\right)\frac{\partial F}{\partial r} + \frac{1}{2}\sigma^2(r,t)\frac{\partial^2 F}{\partial r^2} \\ \\ \qquad\qquad\qquad\qquad\qquad\qquad F(r,0) = \mathbb{I}_{r>r_0} \end{cases} \tag{7}$$

Proof.

$$\frac{\partial}{\partial r}F(r,t) = p(r,t)$$

$$\Longrightarrow \begin{cases} \frac{\partial^2 F}{\partial t \partial r} = -\frac{\partial}{\partial r}\left(\frac{\partial F}{\partial r}\left(\alpha(r,t) - \sigma^2(r,t)A(t,T)\right)\right) + \frac{\partial^2}{\partial r^2}\left(\frac{1}{2}\frac{\partial F}{\partial r}\sigma^2(r,t)\right) \\ \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{\partial F}{\partial r} = \delta(r) \end{cases}$$

Integrating with respect to r,

$$\begin{cases} \frac{\partial F}{\partial t} = -\frac{\partial F}{\partial r}\left(\alpha(r,t) - \sigma^2(r,t)A(t,T)\right) + \frac{\partial}{\partial r}\left(\frac{1}{2}\frac{\partial F}{\partial r}\sigma^2(r,t)\right) + c(t) \\ \\ \qquad\qquad\qquad\qquad\qquad\qquad F(r,0) = \mathbb{I}_{r>r_0} \end{cases}$$

Since at the boundaries $\lim_{r \to \Omega_+} F(r, t) = 1$ and $\lim_{r \to \Omega_-} F(r, t) = 0$, $\frac{\partial F}{\partial t}$ is equal to zero at the boundaries, which implies that $c(t)$ is also zero.

$$\begin{cases} \frac{\partial F}{\partial t} = -\frac{\partial F}{\partial r} \left(\alpha(r, t) - \sigma^2(r, t) A(t, T) \right) + \frac{1}{2} \frac{\partial^2 F}{\partial r^2} \sigma^2(r, t) + \frac{1}{2} \xi \frac{\partial F}{\partial r} \\ \\ \qquad\qquad\qquad\qquad\qquad\qquad F(r, 0) = \mathbb{I}_{r > r_0} \end{cases}$$

$$\begin{cases} \frac{\partial F}{\partial t} = - \left(\alpha(r, t) - \sigma^2(r, t) A(t, T) - \frac{1}{2} \xi \right) \frac{\partial F}{\partial r} + \frac{1}{2} \sigma^2(r, t) \frac{\partial^2 F}{\partial r^2} \\ \\ \qquad\qquad\qquad\qquad\qquad\qquad F(r, 0) = \mathbb{I}_{r > r_0} \end{cases}$$

$\square$

3.2   Numerical Results

To approximately solve Equation 7, I use an implicit finite difference scheme. The initial condition is discretized by $F(r, 0) = 0 \ \forall r < r_0$, $F(r, 0) = 1 \ \forall r > r_0$, and $F(r, 0) = .5$, $r = r_0$. The functions $A(t, T)$ and $C(t, T)$ are approximated via an Euler discretization scheme in which the time step is the same size as the one used for the numerical solution to Equation 7. Discretizing Equation 7 results in a vector of approximate values of $F(r, t^*)$ denoted $\tilde{F}(r, t^*)$ corresponding to each discrete node of r. The approximation of the forward expectation therefore is computed as follows:

$$\sum_{i=g(k)}^{m} \left(\frac{A(t^*, T) r_i - C(t^*, T) + \frac{A(t^*, T)(r_{i+1} - r_i)}{2}}{T - t^*} - k \right) \left(\tilde{F}(r_{i+1}, t^*) - \tilde{F}(r_i, t^*) \right)$$

Where $g(k)$ is a function mapping the value of k to the smallest value of i such that $\frac{A(t^*, T) r_i - C(t^*, T)}{T - t^*} > k$. For optimal accuracy r_0 and t^* should be discrete values on their respective domains. The complete algorithm for computing the price of the caplet is given in the appendix.

3.2.1   Vasicek

Proposition 6. *In the case that the interest rate process follows* Vasicek's *model:*

$$dr_t = \alpha(b - r_t)dt + \sigma dW_t, \ \alpha, \ b, \ \sigma \ \in \mathbb{R}_+ \tag{8}$$

The price of the caplet is

$$c(r_0, 0, t^*, T) = f(r_0, 0, T) \left(\sigma_L \phi(z) + (\mu_L - k)(1 - \Phi(z)) \right)$$

Where

$$\mu_L = \frac{A(t^*, T)\mu_r - C(t^*, T)}{T - t^*}$$

$$\sigma_L = \frac{A(t^*, T)\sigma_r}{T - t^*}$$

$$\mu_r = \left(e^{-\alpha t^*} r_0 - \sigma^2 \left(\frac{2 - 2e^{-\alpha t^*} - e^{-\alpha(T - t^*)} + e^{-\alpha T}}{2} \right) + b \left(1 - e^{-\alpha t^*} \right) \right)$$

$$\sigma_r = \sqrt{(1 - e^{-2\alpha t^*}) \frac{\sigma^2}{2\alpha}}$$

$$z = \frac{k - \mu_L}{\sigma_L}$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$A(t, T) = \frac{1 - e^{-\alpha(T - t)}}{\alpha}$$

$$C(t, T) = \left(\frac{\sigma^2}{2\alpha^2} - b \right)(T - t) + \frac{b}{\alpha} \left(1 - e^{-\alpha(T - t)} \right) + \frac{\sigma^2}{2\alpha^2} \left(\frac{2(e^{-\alpha(T - t)} - 1)}{\alpha} - \frac{e^{-2\alpha(T - t)} - 1}{2\alpha} \right)$$

Proof. The proof is given in the appendix. □

This analytic solution facilitates the numerical analysis of the algorithm. Using parameter values $\alpha = 1$, $b = .1$, $\sigma = .03$, $r_0 = .10$, $k = .10$, $T = 1$, and $t^* = .5$, the analytic solution is .00471525. Space is discretized with m nodes on $[-.1, \ .5]$ and time with n nodes.

	$n = 60,$ $m = 72$	$n = 150,$ $m = 180$	$n = 300,$ $m = 360$	$n = 600,$ $m = 720$
Value	.00472895	.0047172	.00471583	.00471545
Time (s)	0	0	.016	.047
Relative error	.29048 %	.04124 %	.01225 %	.00420 %
Decrease in error		7.0438	3.3667	2.9178

3.2.2 Cox Ingersoll Ross

When $\mu(t)$, $\gamma(t)$, $\xi(t)$ are constants and $\omega = 0$ then r_t follows the *Cox Ingersoll Ross* model:

$$dr_t = \alpha(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \ \alpha, \ b, \ \sigma \in \mathbb{R}_+$$

This model is given parameter values $\alpha = 1$, $b = .1$, $\sigma = .12$, $r_0 = .10$, $k = .10$, $t^* = .5$, and $T = 1$. Space is discretized on $[0, \ .5]$.

	$n = 60,$ $m = 60$	$n = 150,$ $m = 150$	$n = 300,$ $m = 300$	$n = 600,$ $m = 600$	$n = 60000$ $m = 6000$
Value	.00592512	.00591677	.00591565	.0059153	.00591509
Time (s)	0	0	.016	.031	38.625
Relative error	.16957 %	.02840 %	.00947%	.00355 %	–
Decrease in error		5.9708	2.9989	2.6676	–

Here the 6000 by 60000 mesh is considered the "exact" solution.

4 Conclusion

The algorithm presented quickly and accurately prices caplets under the assumption that the short interest rate follows a single factor affine yield process. A downside is that as either time or the short rate changes the entire computation must be redone. This is similar to a Monte Carlo solution in that only a single price can be computed at a time. Standard PDE methods shine in this area since the time and space mesh are discretized for any time and space value that the underlying process may take. A further compromise (shared by standard PDE methods but not by Monte Carlo) is that

generalizing the result to a three or four factor model would be computationally prohibitive. Still, this technique gives greater flexibility to the pricing of caplets than analytic formulas based on the assumption of log-normal forward rates; generalizing the rather restrictive assumptions of Black's model and doing so with computational accuracy.

Works Cited

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A   Proof of Proposition 6

Proof. If $y \sim \mathcal{N}\left(\mu, \sigma^2\right)$,

$$\mathbb{E}\left[(y-k)\mathbb{I}_{y>k}\right] = \int_k^\infty y \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy - k \int_k^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

$$= \int_{\frac{k-\mu}{\sigma}}^\infty (\sigma x + \mu) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - k \int_{\frac{k-\mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \sigma \int_{\frac{k-\mu}{\sigma}}^\infty -d\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right) + \mu\left(1 - \Phi\left(\frac{k-\mu}{\sigma}\right)\right) - k\left(1 - \Phi\left(\frac{k-\mu}{\sigma}\right)\right)$$

$$= \sigma\phi\left(\frac{k-\mu}{\sigma}\right) + (\mu - k)\left(1 - \Phi\left(\frac{k-\mu}{\sigma}\right)\right)$$

By Ito's Lemma,

$$d\left(e^{\alpha t} r_t\right) = \alpha e^{\alpha t} r_t dt + e^{\alpha t}\left(\alpha b - \sigma^2 A(t,T) - \alpha r_t\right) dt + e^{\alpha t} \sigma dW_t^F$$

$$= e^{\alpha t}\left(\alpha b - \sigma^2 A(t,T)\right) dt + e^{\alpha t} \sigma dW_t^F$$

$$e^{\alpha t^*} r_{t^*} = r_0 + \frac{1}{\alpha}\left(e^{\alpha t^*} - 1\right) \alpha b - \sigma^2 \int_0^{t^*} e^{\alpha t} A(t,T) dt + \int_0^{t^*} e^{\alpha t} \sigma dW_t^F$$

$$r_{t^*} = e^{-\alpha t^*} r_0 + b\left(1 - e^{-\alpha t^*}\right) - \sigma^2 \int_0^{t^*} e^{-\alpha(t^*-t)} A(t,T) dt + e^{-\alpha t^*} \int_0^{t^*} e^{\alpha t} \sigma dW_t^F$$

Let

$$X_t := e^{-\alpha t} r_0 - \sigma^2 \int_0^t e^{-\alpha(t-s)} A(s,T) ds + b\left(1 - e^{-\alpha t}\right) + u\left(1 - e^{-2\alpha t}\right) \frac{\sigma^2}{4\alpha}$$

For some $u \in \mathbb{R}_+$ and

$$V_t := e^{u r_t - u X_t} = e^{u e^{-\alpha t} \int_0^t e^{\alpha s} \sigma dW_s^F - u^2\left(1 - e^{-2\alpha t}\right)\frac{\sigma^2}{4\alpha}}$$

By Ito's Lemma,

$$dV_t = V_t u e^{-\alpha t} e^{\alpha t} \sigma dW_t^F + V_t \frac{1}{2} u^2 e^{-2\alpha t} e^{2\alpha t} \sigma^2 dt - V_t u^2 \frac{\sigma^2}{2} dt$$

$$= V_t u \sigma dW_t^F$$

Since $\mathbb{E}\left[\int_0^t u^2 \sigma^2 V_t^2 dt\right] < \infty$, V_t is a martingale under the forward measure and satisfies $V_t(r_0, 0) = 1$. Therefore

$$\mathbb{E}^F[e^{ur_t - uX_t}] = 1$$

$$\implies \mathbb{E}^F[e^{ur_t}] = e^{uX_t}$$

Comparing e^{uX_t} to the moment generating function of a normal random variable, the following is clear:

$$r_{t^*} \sim \mathcal{N}\left(e^{-\alpha t^*}r_0 - \sigma^2 \int_0^{t^*} e^{-\alpha(t^*-t)}A(t,T)dt + b\left(1 - e^{-\alpha t^*}\right), \left(1 - e^{-2\alpha t^*}\right)\frac{\sigma^2}{2\alpha}\right)$$

In this model the function $A(t,T)$ satisfies Equation 2 with $\xi = 0$, $\gamma = -\alpha$, which has the solution

$$\frac{1 - e^{-\alpha(T-t)}}{\alpha}$$

$$\implies \sigma^2 \int_0^{t^*} e^{-\alpha(T-t)}A(t,T)dt = \sigma^2 \int_0^{t^*} \frac{e^{-\alpha(t^*-t)}}{\alpha} - \frac{e^{-\alpha(t^*+T)+2\alpha t}}{\alpha}dt$$

$$= \sigma^2 \left(\frac{2 - 2e^{-\alpha t^*} - e^{-\alpha(T-t^*)} + e^{-\alpha T}}{2}\right)$$

Integrating $\alpha b A(t,T) - \frac{A(t,T)^2 \sigma^2}{2}$ yields $C(t,T)$.

Therefore

$$c(r_0, 0, t^*, T) = f(r_0, 0, T)\left(\sigma_L \phi(z) + (\mu_L - k)(1 - \Phi(z))\right)$$

Where

$$\mu_L = \frac{A(t^*, T)\mu_r - C(t^*, T)}{T - t^*}$$

$$\sigma_L = \frac{A(t^*, T)\sigma_r}{T - t^*}$$

$$z = \frac{k - \mu_L}{\sigma_L}$$

$\square$

B   Algorithm for Pricing Caplets

Data: r_0, k, μ, γ, ω, ξ, t^*, T, n, m, rmax, rmin
Result: The numerical price of a caplet
Define: $\Delta t = t/n$, $\Delta r = (\text{rmax} - \text{rmin})/m$
Adjust rmax, rmin such that $\frac{r_0}{\Delta r} := p$ is an integer
Adjust n such that $tn := (n * t^*)/T$ is an integer
Set a vector v such that $v[1:p-1] = 0$, $v[p] = .5$, $v[p+1:n] = 1$
A[1]=0
C[1]=0
for $i = 1{:}n$ **do**
$\quad \Big|\quad A[i+1] = A[i] - \frac{\xi\Delta t}{2}A[i]^2 + \gamma A[i]\Delta t + \Delta t$
$\quad \Big|\quad C[i+1] = C[i] - \left(A[i]\mu - \frac{\omega}{2}A[i]^2\right)\Delta t$
end
for $i = 1{:}tn$ **do**
$\quad \Big|\quad$ **for** $j=1{:}m$ **do**
$\quad\quad \Big|\quad a[j] = (((\mu + \gamma(\text{rmin} + \Delta r j)) - \frac{\xi}{2} - A[n-i](\omega + \xi(\text{rmin} + \Delta r j)))/(2\Delta r) - (\omega + \xi(\text{rmin} + \Delta r j))/(2\Delta x^2))\Delta t$
$\quad\quad \Big|\quad b[j] = 1 + \Delta t\frac{\omega + \xi(\text{rmin} - \Delta r j)}{\Delta r^2}$
$\quad\quad \Big|\quad c[j] = (-((\mu + \gamma(\text{rmin} + \Delta r(j+1))) - \frac{\xi}{2} - A[n-i](\omega + \xi(\text{rmin} + \Delta r(j+1))))/(2\Delta r) - (\omega + \xi(\text{rmin} + \Delta r(j+1)))/(2\Delta x^2))\Delta t$
$\quad \Big|\quad$ **end**
$\quad \Big|\quad v[m] = v[m] - a[m]$
$\quad \Big|\quad v = tridiagsolve(c[1:m-1], b, a[1:m-1], v)$
end
cap=0
for $i=1{:}\ n\text{-}1$ **do**
$\quad \Big|\quad$ **if** $\left(\left(rmin + \Delta r i + \frac{\Delta r}{2}\right) A[tn] - C[tn]\right)/(T - t^*) - k > 0$ **then**
$\quad\quad \Big|\quad \text{cap} = \left(\left(\left(rmin + \Delta r i + \frac{\Delta r}{2}\right) A[tn] - C[tn]\right)/(T - t^*) - k\right)(v[i+1] - v[i]) + \text{cap}$
$\quad \Big|\quad$ **end**
end
$\text{cap} = \text{cap} * e^{-A[tn]r_0 + C[tn]}$