Correlated latent variables in m dimensions

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1 Model Description

The model setup is similar to the setup in Stahl (2015). The main features of this model are independent defaults conditional on latent factors, mutually independent losses given default, and a semi-endogenous risk of a liquidity event. The model in this paper distinguishes itself from Stahl (2015) in three important points. First, the process that drives correlation is now an m dimensional process. These processes represent the exposure of the assets to various factors. For instance, a loan to a business in rural New Mexico or the south east may have more risk than a a loan to a business in Texas or San Francisco. A loan to a tech business in Texas may have different risks than a loan to an energy business in Texas. Each of these risks (geographic, industry, or otherwise) can be captured by a distinct dimension of the process.

The second key difference is that these process are jointly normally distributed: in particular, by a multi-dimensional Ornstein-Uhlenbeck process. This assumption is important for several reasons. Normally distributed processes lend themselves to analytic tractability including the ability to easily implement a correlation structure across the factors. The normality also allows for relatively simple transition from a CAPM and Merton type framework. While there are many benefits to using Gaussian processes, such processes have the unfortunate property of having the positive probability of negative values. As the probabilities of default are affine functions of the processes, this feature of Gaussian models allows for negative probabilities of default. However, the probability of negative values need not be large; and since the integral of the process is used for determining the factors affecting the default probability the probability of having negative probabilities becomes essentially negligible.

The third difference is that the risk of a liquidity crisis is an affine function of the assets' exposures. Modeling liquidity exposure in this manner allows liquidity risk to be charged at the loan level and provides a relatively simple method to quantify and aggregate liquidity risk in terms of common funding practices like FHLB and FRB pledging. However, this flexible modeling comes at the price of losing the linearity of the portfolio in the sense of Tasche (2008). The main drawback is that Euler contributions no longer are the "correct" method for aggregating risk as the sum of the risk contributions no longer equal the entire portfolio risk. While concerning, this drawback is overcome by using some sensible and straightforward criteria in section 4.4. This paper introduces the model and then uses examples to demonstrate how such a model may be used in an actual loan portfolio.

2 General framework

In Stahl (2015), the goal was to find an analytical expression for the characteristic function of the portfolio loss so that numerical inversion could be efficiently applied to recover the density function of the loss distribution. This paper adopts a nearly identical framework: a portfolio $X_t = \sum_j X_t^j$ contains n assets X_t^j with mutually independent exposures l_j. Each asset has a probability of default. In Stahl (2015) the probability of default was driven by a single Brownian motion; in this paper this is extended so that the instantaneous probability of default for asset j is an affine function of an m dimensional mean-reverting Levy process L_t with long run expected value equal to a vector of ones. Letting the filtration generated by the Levy process be denoted $\mathcal{F}_t$ and the filtration of the entire process X_t be denoted $\mathcal{G}_t$, the probability of default for asset X_t^j conditioned on $\mathcal{G}_t$ over the interval $[t, T]$ is $\left(w_j^T \int_t^T L_s ds\right) p_j$ where w_j is a constant normal vector, p_j is a constant, and a superscript T denotes the transpose. Since the instantaneous default probability is $w_j^T L_t p_j dt$ which has long run expected value $p_j dt$, it is convenient to think of p_j as the long run instantaneous probability per unit time. Following the logic in (Stahl 2015), the approximate characteristic function of $X_T - X_t$ conditioned on $\mathcal{G}_t$ is

$$\phi_{X_T}(u) = \mathbb{E}\left[e^{\sum_j \left(w_j^T \int_t^T L_s ds\right) p_j (\phi_j(u)-1)} | \mathcal{G}_t\right] \tag{1}$$

Where $\phi_j(u)$ is the characteristic function of l_j. $\phi_{X_T}(u)$ can be written as

$$\phi_{X_T}(u) = \mathbb{E}\left[e^{\sum_k \int_t^T L_s ds \sum_j w_{j,k} p_j (\phi_j(u)-1)} | \mathcal{G}_t\right] \tag{2}$$

$$= \mathbb{E}\left[e^{z^T \int_t^T L_s ds} | \mathcal{G}_t\right] = \mathbb{E}\left[e^{z^T (Y_T - Y_t)} | \mathcal{G}_t\right] \tag{3}$$

Where $z = \sum_j w_{j,k} p_j (\phi_j(u) - 1),\ k \in [1, ..., m]$ and $Y_t = \int_0^t L_s ds$. But this is just the moment generating function of Y_t. As long as the moment generating function of Y_t has an analytical expression, the characteristic function has a closed form solution and can be efficiently inverted. Examples of Y_t that satisfy this constraint in a diffusion setting can be found in Duffie et. al. (2000) while examples of Y_t that are the more general Levy process is found in Filipovic (2001). As far as the author knows, there is no more general formulation of the basic loss distribution. Within the framework the problem reduces to finding suitable processes Y_t.

3 Latent Process

For this paper, the process L_t is assumed to solve the following stochastic differential equation (SDE):

$$dL_t = A(\mathbf{1} - L_t)dt + \Sigma dW_t$$

A is an mxm diagonal matrix with $A_{i,i} > 0,\ i = 1, ..., m$, $\mathbf{1}$ is an $mx1$ vector of ones, Σ is an mxm matrix satisfying $\Sigma\Sigma^T = \Omega$ where Ω is positive semi-definite, and dW_t are independent increments of an $mx1$ Brownian motion. $\Omega_{i,j} = \sigma_i \sigma_j \rho_{i,j},\ i,\ j = 1, ..., m$ where $\sigma_i > 0$, $\rho_{i,j} \in [-1, 1]$, and $\rho_{i,i} = 1$.

3.1 Solution to the SDE

By Ito's Lemma,

$$d\left(e^{At}L_t\right) = Ae^{At}L_t dt + e^{At}A(\mathbf{1} - L_t)dt + e^{At}\Sigma dW_t \tag{4}$$

The matrix exponential is, as usual, defined as $e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!}$. Since A is diagonal, this simplifies to diag $\left(e^{A_{ii}}\right)$.

$$d\left(e^{At}L_t\right) = e^{At}A\mathbf{1}dt + e^{At}\Sigma dW_t \tag{5}$$

Integrating,

$$e^{AT}L_T = L_0 + e^{AT}\mathbf{1} - \mathbf{1} + \int_0^T e^{At}\Sigma dW_t \tag{6}$$

Solving for L_T,

$$L_T = e^{-AT}(L_0 - \mathbf{1}) + \mathbf{1} + e^{-AT}\int_0^T e^{At}\Sigma dW_t \tag{7}$$

As this equation is deterministic in volatility, the distribution of L_t is normal.

3.2 Moments of Y_T

Recall that the probability of default is an affine function of $Y_T = \int_0^T L_t dt$. Since L_t is Gaussian, the integral of L_t is also Gaussian. The moment generating function of a Gaussian random variable with mean μ and variance σ^2 is the following (reference any statistics text book):

$$\phi_{\text{Gaussian}}(u) = e^{u\mu + \frac{u^2\sigma^2}{2}} \tag{8}$$

Thus equation (3) reduces to the computation of the expectation and variance of $z^T Y_T$. The expectation and variance the following:

$$\mathbb{E}[z^T Y_T] = \mu(z) = \sum_{i=1}^{m} z_i T + \frac{z_i}{A_{i,i}} \left(1 - e^{-A_{i,i}T}\right) \left(L_0^i - \mathbf{1}\right) \tag{9}$$

$$\mathbb{V}(z^T Y_T) = \sigma^2(z) = \\ \sum_i \sum_j \rho_{i,j} \frac{z_i z_j \sigma_i \sigma_j}{A_{i,i} A_{j,j}} \left(T - \frac{1 - e^{-A_{i,i}T}}{A_{i,i}} - \frac{1 - e^{-A_{j,j}T}}{A_{j,j}} + \frac{1 - e^{-(A_{i,i} A_{j,j})T}}{A_{i,i} + A_{j,j}} \right) \tag{10}$$

See appendix A (REF) for a full derivation.

4 Liquidity Risk

In Stahl (2015) liquidity risk is modeled by a constant exposure (fixing time) and a probability of a “crisis” that is linear in the dollar credit loss. This allowed the characteristic function of the portfolio loss to retain the same analytical form. In this paper this concept is generalized further so that the exposure in the case of a liquidity “crisis” is an affine function of the individual asset balances at time t. Using this method retains the analytical characteristic function but also allows liquidity capital to be pushed down to individual loans. In the case of a liquidity event (which increases in probability as credit losses mount), loans typically must be liquidated at large discounts to par to meet obligations. The probability of this event occurring (at time T) is qX_T where X_T is the realization of the losses within the portfolio. In Stahl (2015) the exposure was a fixed constant λ.

4.1 Characteristic function and Loss distribution Moments

From Stahl (2015), equation (3), and equation (8), the characteristic function including liquidity risk is

$$\phi_{X_T^\lambda} = e^{\mu(z^\lambda) + \frac{\sigma^2(z^\lambda)}{2}} \tag{11}$$

Where $z^\lambda = \sum_j w_{j,k} p_j (\phi_j(u^\lambda) - 1)$ and $u^\lambda = u - iq\left(e^{ui\lambda} - 1\right)$. The moments for the loss distribution are

$$\mathbb{E}[X_T] = \mu(d) \tag{12}$$

Where d is the vector $d = \frac{1}{i}\sum_j p_j \phi_j'(0) w_{j,k}$.

$$\mathbb{V}(X_T) = \mu(d^2) + \sigma^2(d) \tag{13}$$

Where d^2 is the vector $d^2 = -\sum_j p_j \phi_j''(0) w_{j,k}$.

$$\mathbb{E}[X_T^\lambda] = \mathbb{E}[X_T](1 + q\lambda) \tag{14}$$

$$\mathbb{V}(X_T^\lambda) = \mathbb{V}(X_T)(1 + q\lambda)^2 + \mathbb{E}[X_T] q \lambda^2 \tag{15}$$

4.2 Risk Contributions

The departure from Stahl (2015) is that the exposure is decomposed as $\lambda = \lambda_0 + \sum_j r_j b_j$ where b_j is the balance of the asset. The r_j can be thought of as the liquidity equivalent to the "loss given default" from credit risk vernacular. r_j is affected by the market for the asset. If X_t^j is a conforming mortgage it will likely have a low r_j since there tends to be a deep and liquid secondary market for conforming mortgages. Likewise loans pledged to the Federal Home Loan Bank or the Federal Reserve as part of a contingency funding plan will have a lower r_j since they already have a buyer. The random variable describing the portfolio loss is the following:

$$X_T^\lambda = \sum_j X_T^j + q\left(\sum_j X_T^j\right)\left(\lambda_0 + \sum_j b_j r_j\right) \tag{16}$$

For the computation of the entire portfolio distribution there is no difference how λ is decomposed so long as λ stays fixed. However, decomposing λ

makes a tremendous difference when allocating risk to sub-portfolios.

Denote the whole portfolio risk by $\rho(S)$ where $S : \Omega \to \mathbb{R}^n$ is a random variable representing a risky portfolio. ρ is typically a statistic similar to Value at Risk (VaR) or expected shortfall (Artzner 1999). The portfolio ρ can be found for X_T^λ by inverting the characteristic function (11) to recover the density as described in Stahl (2015).

Pricing, optimization, and performance monitoring requires the allocation of risk capital to any subset of the portfolio. There are a number of methods for allocating this risk to the portfolio. A method favored by a number of authors (Tasche 2008, LIST MORE) is the Euler allocation.

Definition 1. The *Euler allocation* for risk to a sub-portfolio, denoted $\rho_B(S)$, is as follows:

$$\rho_B(S) = \left.\frac{d\rho(S+hB)}{dh}\right|_{h=0}$$

Theorem 1 (Euler)**.** If ρ is homogeneous of degree one, then for all $B_i \subset S$ satisfying $B_i \cap B_j = \varnothing$,

$$\sum_i \rho_{B_i}(S) = \rho(S) \tag{17}$$

A particularly simple form for ρ_{B_i} is when $\rho(S) = \mu(S) + c\sigma(S)$. It is typical to let c be the value that sets $\rho(S)$ equal to some portfolio level risk metric such as VaR. The risk contribution for $\mu(S) + c\sigma(S)$ is

$$\rho_{B_i} = \mu(B_i) + c\frac{\mathrm{cov}(B_i,\, S)}{\sigma(S)} \tag{18}$$

4.3 Application to X_T

In the case of X_T, it is useful to introduce the following "partial" variance function

$$\sigma^2(z^1, z^2) = \\ \sum_i \sum_j \rho_{i,j} \frac{z_i^1 z_j^2 \sigma_i \sigma_j}{A_{i,i} A_{j,j}} \left(T - \frac{1-e^{-A_{i,i}T}}{A_{i,i}} - \frac{1-e^{-A_{j,j}T}}{A_{j,j}} + \frac{1-e^{-(A_{i,i}+A_{j,j})T}}{A_{i,i}+A_{j,j}} \right) \tag{19}$$

This function is related to 10 by $\sigma^2(z,z) = \sigma^2(z)$. Using this function, the risk contribution is as follows (see appendix for full derivation):

$$\frac{1}{i} p_j \phi_j'(0) \left(\mu(w_j) + \sigma^2(w_j, d)\right) - p_j \phi''(0) \mu(w_j) \tag{20}$$

4.4 Application to X_T^λ

While the Euler allocation principal is useful for X_T, it does not apply for X_T^λ. X_T^λ is not "linear" with respect to the individual assets in the following sense:

$$\nexists f : \mathbb{R} \to \mathbb{R} \mid hX_T^\lambda = hf\left(\sum_j X_T^j\right)$$

In fact,

$$hX_T^\lambda = h\sum_j X_T^j + qh\left(\sum_j X_T^j\right)\left(\lambda_0 + h\sum_j b_j r_j\right)$$

This equation is non-linear in h and so the homogeneity constraint on ρ will not hold. However, it is still possible to allocate risk to each asset within the portfolio using simple and intuitive criteria. To develop these criteria, it is important to note that the homogeneity constraint does still hold if $r_j = 0\,\forall j$. Indeed, the risk contribution in this case is the following:

$$\begin{aligned}\rho_j(X_T^\lambda) = {} & \frac{1}{i} p_j \phi_j'(0) \mu(w_j)\left(1 + q\lambda_0 + q\lambda_0^2\right) \\ & + \left(\frac{1}{i} p_j \phi_j'(0) \sigma^2(w_j, d) - p_j \phi''(0)\mu(w_j)\right)(1 + q\lambda_0)^2 \end{aligned} \tag{21}$$

The availability of the homogeneous risk measure for the case where $r_j = 0$ helps inform the criteria for risk contributions in the case without a homogeneous risk function.

1. If $r_j = 0$, then X_j must not increase the impact of a liquidity event
2. If $r_j = 0\,\forall j$, risk contributions must equal equation (21)
3. $\sum_j \rho_i = \rho$

Using these criteria, the covariance risk contributions to the portfolio variance is as follows:

$$
\begin{aligned}
\rho_j(X_T^\lambda) = &\frac{1}{i} p_j \phi_j'(0)\mu(w_j)\left(1 + q\lambda_0 + q\lambda_0^2\right) + r_j b_j q \mathbb{E}[X_T] \\
&+ \left(\frac{1}{i} p_j \phi_j'(0)\sigma^2(w_j, d) - p_j \phi''(0)\mu(w_j)\right)(1 + q\lambda_0)^2 \\
&+ 2r_j b_j q \mathbb{V}(X_T) + r_j b_j q^2 \mathbb{V}(X_T)(\lambda + \lambda_0) + r_j b_j(\lambda_0 + \lambda) q \mathbb{E}[X_T] \quad (22)
\end{aligned}
$$

5 Worked Example

5.1 Overview

The goal in this section is to apply the framework of section 4 to a simple portfolio. The framework can be summarized as follows:

1. Find the portfolio risk measure $\rho(X_T^\lambda)$ (eg, VaR)
2. Find c such that $\mathbb{E}\left[X_T^\lambda\right] + c\sqrt{\mathbb{V}(X_T^\lambda)} = \rho(X_T^\lambda)$
3. Allocate risk to each asset via equation (22)

5.2 Example Portfolio

Consider 5 loans with the following features:

	X_T^1	X_T^2	X_T^3	X_T^4	X_T^5
p	.005	.01	.015	.02	.025
l	987	2104	1264	576	377
r	.13	.15	.18	.14	.78

For simplicity in this example $b_j = l_j \, \forall j$ and l_j are constants. Each loan has exposure to 3 risk drivers Y which is parameterized as follows:

$$
A = \begin{bmatrix} .3 & 0 & 0 \\ 0 & .2 & 0 \\ 0 & 0 & .1 \end{bmatrix} \quad (23)
$$

$$\Sigma = \begin{bmatrix} .2 & 0 & 0 \\ .02 & .09797959 & 0 \\ -.09 & .07960842 & .2748863 \end{bmatrix} \tag{24}$$

$$L_0 = \begin{bmatrix} 1.1 \\ .9 \\ .7 \end{bmatrix} \tag{25}$$

Where the matrix Σ is constructed from the correlation matrix

$$\rho = \begin{bmatrix} 1 & .2 & -.3 \\ .2 & 1 & .1 \\ -.3 & .1 & 1 \end{bmatrix} \tag{26}$$

and the matrix of volatilities

$$\sigma = \begin{bmatrix} .2 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & .3 \end{bmatrix} \tag{27}$$

such that

$$\Omega = \sigma\rho\sigma$$

Figures 1 and 2 shows sample paths of the process L and Y.

The exposure that each asset has to each of the three risk drivers is represented by the following matrix:

$$w = \begin{bmatrix} .17 & .44 & .39 \\ .04 & .47 & .49 \\ .02 & .87 & .11 \\ .41 & .19 & .40 \\ .52 & .31 & .17 \end{bmatrix} \tag{28}$$

The instantaneous default probability is thus obtained by the multiplication $p^T wL_t dt$. The plots of a sample path for each asset's probability of default is given in 3 and 4.

Sample paths of L and $Y = \int Ldt$

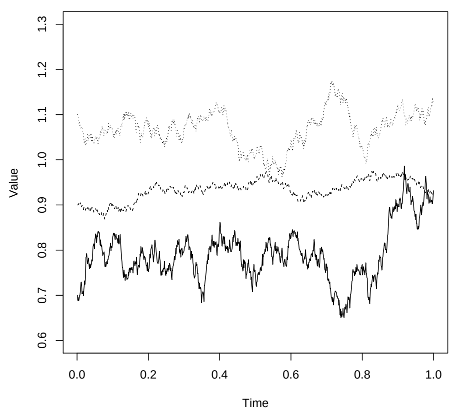

Figure 1

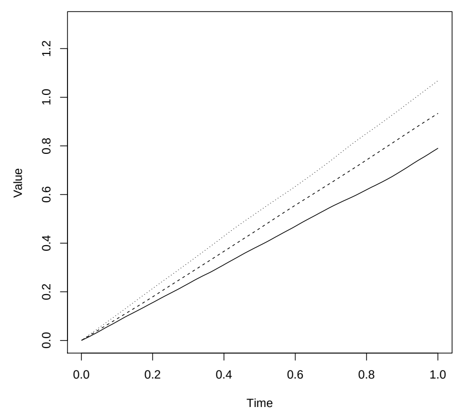

Figure 2

5.3 Total and Marginal Risk

For the sake of this example and without loss of generality, it will be assumed that the risk of the portfolio is adequately represented by $\mathbb{E}[X_T^\lambda] + \sqrt{\mathbb{V}(X_T^\lambda)}$. The following table summarizes the results.

	X_T^1	X_T^2	X_T^3	X_T^4	X_T^5	Total
p	.005	.01	.015	.02	.025	
l	987	2104	1264	576	377	
r	.13	.15	.18	.14	.78	
$\rho_j\left(X_T^\lambda\right)$	27.62	159.80	103.83	36.08	42.38	369.70
$\rho_j\left(X_T^{\lambda^*}\right)$	23.03	170.37	110.02	39.19	27.09	369.70
$\rho_j(X_T)$	20.01	153.34	96.89	32.88	22.01	325.13

$X_T^{\lambda^*}$ represents the risk contributions without charging liquidity risk at the loan level: that is, equation (21) is used (with $\lambda_0 = \lambda$) instead of equation (22). X_T^λ is superior at charging risk to the loans that truly add more to the riskiness of the portfolio. The risk of X_T^5 is sharply increased over $\rho_j(X_T)$ due to its relatively high r, while the risk of X_T^2 barely increases.

Sample paths of the probability of default

Figure 3: Instantaneous probability of default

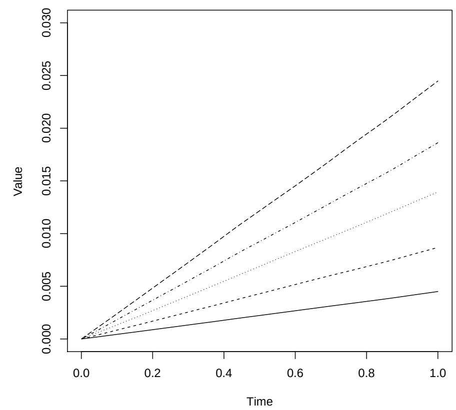

Figure 4: Cumulative probability of default

6 Appendix

6.1 Expectation of $z^T Y_T$

From equation (7) and using the fact that Ito integrals are martingales,

$$\mathbb{E}[z^T L_T] = z^T \left(\mathbf{1} + e^{-AT}(L_0 - \mathbf{1})\right) \tag{29}$$

Integrating and using the fact that e^A is applied component-wise when A is diagonal,

$$\mathbb{E}[z^T Y_T] = \sum_{i=1}^{m} z_i T + \frac{z_i}{A_{i,i}} \left(1 - e^{-A_{i,i}T}\right) \left(L_0^i - \mathbf{1}\right) \tag{30}$$

6.2 Variance of $z^T Y_T$

To find the variance of $z^T Y_T$ I introduce the following lemma:

Lemma 1. Let I_t be a bounded Ito integral and C_t be a Riemann integrable function. Then

$$\int_0^T C_t I_t dt = \int_0^T \left(\int_0^T C_t dt - \left(\int_0^t C_s ds \right) \right) dI_t \tag{31}$$

Proof. By Ito's Lemma,

$$d \left(\int_0^T C_t dt \right) I_T = C_t I_t dt + \left(\int_0^t C_s ds \right) dI_t \tag{32}$$

Rearranging and integrating yields equation (31). □

The only part of equation (7) that contributes the variance is the Ito integral. Hence the problem of finding the variance reduces to

$$\mathbb{V}(z^T Y_T) = \mathbb{V} \left(z^T \int_0^T e^{-At} \int_0^t e^{As} \Sigma dW_s dt \right) \tag{33}$$

Letting $C_t = e^{-At}$ and $I_t = \int_0^t e^{As} \Sigma dW_s$ and using lemma 1,

$$= \mathbb{V} \left(z^T \int_0^T \frac{1}{A} \left(e^{-At} - e^{-AT} \right) e^{At} \Sigma dW_t \right) \tag{34}$$

Where $\frac{1}{A}$ is defined as component-wise division and $\mathbf{I}$ is the identity matrix.

$$= \mathbb{V} \left(z^T \int_0^T \frac{1}{A} \left(\mathbf{I} - e^{-A(T-t)} \right) \Sigma dW_t \right) \tag{35}$$

$$= \int_0^T z^T M_t \Sigma \Sigma^T M_t^T z dt \tag{36}$$

Where $M_t = \frac{1}{A} \left(\mathbf{I} - e^{-A(T-t)} \right)$.

$$= \int_0^T \sum_{i=1}^m \sum_{j=1}^m z_i z_j \sigma_i \sigma_j \rho_{i,j} \frac{1}{A_{i,i}} \left(1 - e^{-A_{i,i}(T-t)} \right) \frac{1}{A_{j,j}} \left(1 - e^{-A_{j,j}(T-t)} \right) dt \tag{37}$$

Integrating yields equation (10).

6.3 Expectation of X_T

$$\mathbb{E}[X_T] = \frac{\partial \phi_{X_T}}{\partial u}\Big|_{u=0} \tag{38}$$

$$= \frac{\partial e^{\mu(z)+\frac{\sigma^2(z)}{2}}}{\partial u}\Bigg|_{u=0} \tag{39}$$

$$= e^{\mu(z)+\frac{\sigma^2(z)}{2}} \left(\mu(z') + \frac{1}{2}\sigma^2(z', \mathbf{1})\right)\Bigg|_{u=0} \tag{40}$$

$$= \mu(d) \tag{41}$$

Where $d = z' = \frac{1}{i}\sum_j p_j \phi_j'(0) w_{j,k}$

6.4 Variance of X_T

$$\mathbb{V}(X_T) = \frac{\partial^2 \phi_{X_T}}{\partial u^2}\Big|_{u=0} \tag{42}$$

$$= \frac{\partial^2 e^{\mu(z)+\frac{\sigma^2(z)}{2}}}{\partial u^2}\Bigg|_{u=0} \tag{43}$$

$$= e^{\mu(z)+\frac{\sigma^2(z)}{2}} \big(\mu(z'') + \sigma^2(z')\big)\Bigg|_{u=0} \tag{44}$$

$$= \mu(d^2) + \sigma^2(d) \tag{45}$$

Where $d^2 = z'' = -\sum_j p_j \phi_j''(0) w_{j,k}$

6.5 Expectation of X_T^λ

$$\mathbb{E}[X_T^\lambda] = \frac{\partial \phi_{X_T^\lambda}(u^\lambda)}{\partial u}\Big|_{u=0} \tag{46}$$

$$\mathbb{E}[X_T^\lambda] = \mathbb{E}[X_T](1 + q\lambda) \tag{47}$$

6.6 Variance of X_T^λ

$$\mathbb{V}(X_T^\lambda) = \frac{\partial^2 \phi_{X_T^\lambda}(u^\lambda)}{\partial u^2}\Big|_{u=0} \tag{48}$$

$$\mathbb{V}(X_T^\lambda) = \mathbb{E}[X_T](1 + q\lambda) \tag{49}$$

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