A Model Moments

For ease of notation, the moments of X are computed at time 0. Extension to times $t < T$ is trivial.

A.1 Expected Value of Z_T

Z_t has dynamics $dZ = \alpha(1-Z)dt + \sigma\sqrt{Z}dW_t$. Integrating and taking expectations,

$$\mathbb{E}[Z_T] = \int_0^T \alpha(1 - \mathbb{E}[Z_s])ds \tag{26}$$

Letting $g(t) = \mathbb{E}[Z_t]$, $g(t)$ solves the following ODE:

$$\frac{dg}{dt} = \alpha - \alpha g,\ g(0) = Z_0 \tag{27}$$

This ODE has the following solution:

$$g(T) = g(0)e^{-\alpha T} + 1 - e^{-\alpha T} = (Z_0 - 1)e^{-\alpha T} + 1 \tag{28}$$

As $T \to \infty$, $g(T) \to 1$.

A.2 Variance of Y_T

As given by Dufresne (2001), the variance of the integral of Z_T is the following:

$$\begin{aligned}\mathbb{V}[Y_T] = \frac{Z_0^2}{\alpha^2} - \frac{2Z_0}{\alpha^2} + \frac{1}{\alpha^2} + \frac{Z_0\sigma^2}{\alpha^3} - \frac{5\sigma^2}{2\alpha^3} + t\left(\frac{2Z_0}{\alpha} - \frac{2}{\alpha} + \frac{\sigma^2}{\alpha^2}\right) + t^2 \\ + e^{-\alpha t}\left(-\frac{2Z_0^2}{\alpha^2} + \frac{4Z_0}{\alpha^2} - \frac{2}{\alpha^2} + \frac{2\sigma^2}{\alpha^3} + t\left(-\frac{2Z_0}{\alpha} + \frac{2}{\alpha} - \frac{2Z_0\sigma^2}{\alpha^2} + \frac{2\sigma^2}{\alpha^2}\right)\right) \\ + e^{-2\alpha t}\left(\frac{Z_0^2}{\alpha^2} - \frac{2Z_0}{\alpha^2} + \frac{1}{\alpha^2} - \frac{Z_0\sigma^2}{\alpha^3} + \frac{\sigma^2}{2\alpha^3}\right) - \mathbb{E}[Y_T]^2\end{aligned} \tag{29}$$

A.3 Expected value without liquidity risk

The expected value of X_T is the following:

$$\mathbb{E}[X_T] = \mathbb{E}\left[\mathbb{E}[X_T|\mathcal{F}_T]\right] \tag{30}$$

$$= \mathbb{E}\left[\sum_j p_j l_j Y_T\right] = \mathbb{E}[Y_T]\sum_j p_j \mathbb{E}[l_j] \tag{31}$$

$$= \mathbb{E}\left[\int_0^T Z_s ds\right]\sum_j p_j \mu_{l_j} = \left(\int_0^T \mathbb{E}\left[Z_s\right] ds\right)\sum_j p_j \mu_{l_j} \tag{32}$$

where $\mathbb{E}[l_j] = \mu_{l_j}$. The expected value of Z_t is $Z_0 e^{-\alpha t} + 1 - e^{-\alpha t}$. Integrating this expected value yields

$$\mathbb{E}[Y_T] = \left(\frac{Z_0 - 1}{\alpha}\right)\left(1 - e^{-\alpha t}\right) + t \tag{33}$$

Substituting this expression into $\mathbb{E}[X_T]$,

$$\mathbb{E}[X_T] = \left(\left(\frac{Z_0 - 1}{\alpha}\right)\left(1 - e^{-\alpha T}\right) + T\right)\sum_j p_j \mu_{l_j} \tag{34}$$

$$= \mu_Y \sum_j p_j \mu_{l_j} \tag{35}$$

Where $\mu_Y = \left(\frac{Z_0 - 1}{\alpha}\right)\left(1 - e^{-\alpha T}\right) + T$.

A.4 Expected value with liquidity risk

Denote by X_L the random credit loss associated with liquidity risk. Then

$$\mathbb{E}\left[X_L\right] = \left.\frac{\partial \phi\left(u - iq(e^{u\lambda i} - 1)\right)}{i\partial u}\right|_{u=0} \tag{36}$$

$$= \frac{1}{i}\phi'\left(u - iq(e^{u\lambda i} - 1)\right)\left(1 + q\lambda e^{u\lambda i}\right)\big|_{u=0} \tag{37}$$

$$= \frac{\phi'(0)}{i}(1 + q\lambda) = \mathbb{E}[X_T](1 + q\lambda) \tag{38}$$

A.5 Variance without liquidity risk

The variance of X can be decomposed as follows:

$$\mathbb{V}[X_T] = \mathbb{E}\left[\mathbb{V}[X_T|\mathcal{F}_T]\right] + \mathbb{V}\left[\mathbb{E}[X_T|\mathcal{F}_T]\right] \tag{39}$$

$$\mathbb{V}[X_T|\mathcal{F}_T] = -\left.\frac{\partial^2}{\partial u^2} e^{\sum_j Y_T p_j (e^{uil_j} - 1)}\right|_{u=0} - \mathbb{E}[X_T|\mathcal{F}_T]^2 = -Y_T \sum_j p_j \phi''_{l_j}(0) \tag{40}$$

Note that the variance of the sum of independent Bernoulli random variables is $-\sum_j p_j(1-p_j)\phi''_{l_j}(0)$, but due to the Poisson approximation, the conditional variance of X_T is $-\sum_j p_j\phi''_{l_j}(0)$. Substituting the conditional variance of X_T,

$$\mathbb{V}[X_T] = \mathbb{V}\left[Y_T \sum_j \mu_{l_j} p_j\right] - \mathbb{E}\left[Y_T \sum_j \phi''_{l_j}(0) p_j\right] \tag{41}$$

$$= \mathbb{V}[Y_T]\left(\sum_j \mu_{l_j} p_j\right)^2 - \mu_Y \sum_j \phi''_{l_j}(0) p_j \tag{42}$$

The variance of Y_T can be found in Dufresne (2001) and in the first part of this appendix. Letting σ_Y^2 denote this variance,

$$\mathbb{V}[X_T] = \sigma_Y^2 \left(\sum_j l_j p_j\right)^2 - \mu_Y \sum_j \phi''_{l_j}(0) p_j \tag{43}$$

A.6 Variance with liquidity risk

$$\mathbb{E}\left[X_L^2\right] = -\frac{\partial^2}{\partial u^2}\phi\left(u - iq(e^{u\lambda i}-1)\right)\Big|_{u=0} \tag{44}$$

$$= -\phi''\left(u - iq(e^{u\lambda i}-1)\right)\left(1 + q\lambda e^{u\lambda i}\right)^2\Big|_{u=0}$$

$$+ \frac{1}{i^2}\phi'\left(u + \frac{q(e^{u\lambda i}-1)}{i}\right) qi\lambda^2 e^{u\lambda i}\Big|_{u=0} \tag{45}$$

$$= -\phi''(0)(1+q\lambda)^2 + i\phi'(0)q\lambda^2 \tag{46}$$

$$= \left(\mathbb{V}[X_T] + \mathbb{E}[X_T]^2\right)(1+q\lambda)^2 + \mathbb{E}[X_T]q\lambda^2 \tag{47}$$

$$\mathbb{V}[X_L] = \left(\mathbb{V}[X_T] + \mathbb{E}[X_T]^2\right)(1+q\lambda)^2 + \mathbb{E}[X_T]q\lambda^2 - \mathbb{E}[X_T]^2(1+q\lambda)^2 \tag{48}$$

$$= \mathbb{V}[X_T](1+q\lambda)^2 + \mathbb{E}[X_T]q\lambda^2 \tag{49}$$

A.7 Covariance without liquidity risk

The piecewise covariance of assets and the covariance of each asset with the portfolio are important for the computation of risk contributions to the portfolio. For a comprehensive overview of risk contributions, see Tasche and Acerbi (2007).

A.7.1 Piecewise covariance

The covariance of X_T^j, X_T^k is the following:

$$\begin{aligned}
\mathrm{Cov}(X_T^j, X_T^k) &= \mathbb{E}[X_T^j X_T^k] - \mathbb{E}[X_T^j]\mathbb{E}[X_T^k] && (50)\\
&= \mathbb{E}\left[\mathbb{E}[X_T^j X_T^k | \mathcal{F}_T]\right] - p_j p_k \mu_{l_j}\mu_{l_k}\mathbb{E}[Y_T]^2 && (51)\\
&= p_j p_k \mu_{l,j}\mu_{l_k}\mathbb{E}[Y_T^2] - p_j p_k \mu_{l_j}\mu_{l_k}\mathbb{E}[Y_T]^2 && (52)\\
&= p_j p_k \mu_{l_j}\mu_{l_k}\sigma_Y^2 && (53)
\end{aligned}$$

A.7.2 Covariance with the portfolio

The covariance of X_T, X_T^k is the following:

$$\begin{aligned}
\mathrm{Cov}(X_T, X_T^k) &= \mathbb{E}[X_T X_T^k] - \mathbb{E}[X_T]\mathbb{E}[X_T^k] && (54)\\
&= \mathbb{E}\left[\mathbb{E}[X_T X_T^k | \mathcal{F}_T]\right] - p_k \mu_{l_k} \sum_j p_j \mu_{l_j} \mathbb{E}[Y_T]^2 && (55)\\
&= \mathbb{E}\left[\mathbb{E}\left[{X_T^k}^2 + X_T^k \sum_{j \setminus k} X_T^j | \mathcal{F}_T\right]\right] - p_k \mu_{l_k} \sum_j p_j \mu_{l,j} \mathbb{E}[Y_T]^2 && (56)\\
&= -\mathbb{E}[Y_T] p_k \phi''_{l_k}(0) - \mathbb{E}[Y_T^2] p_k^2 \phi''_{l_k}(0)\\
&+ \mathbb{E}[Y_T^2] p_k \mu_{l_k} \sum_{j \setminus k} p_j \mu_{l_j} - p_k \mu_{l_k} \sum_j p_j \mu_{l,j} \mathbb{E}[Y_T]^2 && (57)\\
&= \sigma_Y^2 p_k \mu_{l_k} \sum_j p_j \mu_{l_j} - \mu_Y p_k \phi''_{l_k}(0) && (58)
\end{aligned}$$

Note that $\sum_k \mathrm{Cov}(X_T, X_T^k) = \mathbb{V}[X_T]$, as it should.

A.8 Covariance with liquidity risk

A.8.1 Piecewise covariance

$$\begin{aligned}
\mathrm{Cov}(X_T^j, X_T^k) &= \mathbb{E}[X_T^j X_T^k] - \mathbb{E}[X_T^j]\mathbb{E}[X_T^k] && (59)\\
&= \mathbb{E}\left[\mathbb{E}[X_T^j X_T^k | \mathcal{F}_T]\right] - p_j p_k \mu_{l,j}\mu_{l_k}(1+q\lambda)^2\mathbb{E}[Y_T]^2 && (60)\\
&= p_j p_k \mu_{l_j}\mu_{l_k}(1+q\lambda)^2\mathbb{E}[Y_T^2] - p_j p_k \mu_{l_j}\mu_{l_k}(1+q\lambda)^2\mathbb{E}[Y_T]^2 && (61)\\
&= p_j p_k \mu_{l,j}\mu_{l_k}(1+q\lambda)^2\sigma_Y^2 && (62)
\end{aligned}$$

A.8.2 Covariance with the portfolio

Recalling that $\sum_k \mathrm{Cov}(X_L, X_T^k) = \mathbb{V}[X_T](1+q\lambda)^2 + \mathbb{E}[X_T]q\lambda^2$, the covariance can be found by taking the "difference" of the expression:

$$\mathrm{Cov}(X_L, X_T^k) = \left(\sigma_Y^2 p_k \mu_{l_k} \sum_j p_j \mu_{l_j} - \mu_Y p_k \phi''_{l_k}(0)\right)(1+q\lambda)^2 + p_k \mu_{l_k} q\lambda^2 \quad (63)$$

B Pseudo-Code

Given a characteristic function **phi(u)** that returns the real value and the variables **m** (nodes in u-space), **h** (nodes in x space), **xmax** (truncation of x), the following is the COS algorithm pseudo-code:

Input: m, h, $xmax$
Ouput: Approximate expected value of the distribution
Requirements: Function that returns real value of characteristic function.

```
du ← Pi/xmax
dx ← xmax/(k − 1)
cp ← 2/xmax
Array f[m]                    ▷ f is the characteristic function for each node
Array y[h]                    ▷ y is the approximate density for each node
j = 0
for j < m do
    f[j] ← phi(du * j) * cp   ▷ If phi returns a complex number, use Re(phi)
end for
f[0] = .5 * f[0]
exloss = 0
j, i = 0
for i < h do
    y[i] ← 0
    for j < m do
        y[i] = y[i] + f[j] * cos(du * j * dx * i)
    end for
    exloss = exloss + y[i] * i * dx   ▷ Trapezoidal rule: Integrate y over x
end for
exloss = exloss − xmax * .5 * y[h − 1]   ▷ Trapezoidal rule
return exloss * dx
```