1 Simple derivation of Black Scholes

The goal of this paper is to provide a rigorous, simple derivation of the Black Scholes formula. The only pre-requisite mathematical and economic theory is basic (continuous) stochastic differential equations and the economic notion of efficiency. Enumerating these pre-requisites:

1. I use stochastic calculus and regard the existence of solutions to integrals such as $dS_t = \alpha(S,t)dt + \sigma(S,t)dW_t$ to be established and rigorous (as it is).

2. I use the economic concept that if two payoffs are equivalent, then economic agents will be indifferent between them and the existence of a market will force the two payoffs to have the same time 0 price.

Note I do not use Feynman-Kac, Girsonov, or the First Fundamental Theorem of Asset Pricing. However, I do need Girsonov for the derivation, so in the next section I prove it.

2 Girsonov Proof

Theorem 1. *Let $Z_t = e^{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds}$ where dW_s is an increment of (standard) Brownian Motion and θ is measurable by the filtration generated by W_t and satisfies the "usual" conditions. Then $d\tilde{\mathbb{P}}_t = Z_t d\mathbb{P}_t$ is a probability measure. Further, $d\tilde{W}_t = dW_t + \theta_t dt$ is a standard Brownian Motion under $\tilde{\mathbb{P}}$.*

Proof. The proof requires two parts. Proof that $d\tilde{\mathbb{P}}$ is a probability measure:

A probability measure requires that $\int_\Omega d\tilde{\mathbb{P}} = 1$. Since $\mathbb{E}[Z_T] = 1$, this is self-evident. Since $d\mathbb{P}$ is a probability measure (by assumption), then $d\tilde{\mathbb{P}}$ is a probability measure if it is an equivalent measure to $d\mathbb{P}$. Note that equivalence is a stronger statement, but by proving it I prove the weaker statement that $d\tilde{\mathbb{P}}$ is a probability measure. Equivalence is also immediately evident when noting that Z_t is almost-surely positive.

It remains to prove that $d\tilde{W}_t = dW_t + \theta_t dt$ is a standard Brownian Motion under $\tilde{\mathbb{P}}$.

$$\tilde{\mathbb{E}}\left[e^{ui\tilde{W}_T}\right] = \mathbb{E}\left[Z_T e^{ui\left(\int_0^T dW_t + \int_0^T \theta_t dt\right)}\right]$$
$$= \mathbb{E}\left[e^{\int_0^T (ui-\theta_t)dW_t - \frac{1}{2}\int_0^T \theta_t^2 dt + \int_0^T ui\theta_t dt}\right]$$

Completing the square,

$$= \mathbb{E}\left[e^{\int_0^T (ui-\theta_t)dW_t - \frac{1}{2}\int_0^T (ui-\theta_t)^2 dt + \frac{1}{2}(ui)^2 T}\right]$$

$$= \mathbb{E}\left[e^{-\frac{1}{2}u^2 T}\right]$$

Recognizing this as the characteristic function of a standard Brownian Motion, the proof is complete. $\square$

This proof implies that for any measurable function g, $\mathbb{E}[Z_T g(W_T)]$ can be written as $\int_\Omega g(\omega) d\tilde{\mathbb{P}}(\omega) = \tilde{\mathbb{E}}[g(W_T)]$.

3 Derivation of Black Scholes

Let there be a market in which two assets exist. Canonically these two assets are a stock and a bond. Let these markets be infinitely liquid so that there are buyers and sellers at any price and transaction costs are zero. The goal is to construct a portfolio from these two assets that replicates the payoff of an option. Mathematically, an option is a function h of one or more assets in the market. For a European call option, $h(x) = \mathbb{I}_{x>K}(x - K)$ where $\mathbb{I}$ is the indicator function and K is the "strike" price.

I now assume that the assets have the following dynamics:

$$\begin{aligned} dS_t =& \alpha(S_t, t)dt + \sigma(S_t, t)dW_t \\ dM_t =& r_t M_t dt \end{aligned} \tag{1}$$

Besides the "usual" constraints on α and σ, I also require that these functions are chosen such that S_t is positive almost surely. Additionally, r_t may be stochastic though it is measurable with respect to the Brownian Motion W_t.

Theorem 2. *Let 1 hold. Then a contingent claim on S_t has the following price:*

$$v(S_t, t) = M_t \mathbb{E}^{\mathbb{M}} \left[\frac{h(S_T)}{M_T} | \mathcal{F}_t \right] = S_t \mathbb{E}^{\mathbb{S}} \left[\frac{M_T h(S_T)}{S_T} | \mathcal{F}_t \right]$$