

1 Introduction

The problem of calibrating option prices to market values (the “inverse problem”) is non-trivial especially with complex pricing models with many parameters. A naive approach is to perform optimization by minimizing a distance between the prices provided by the market and the modeled prices by varying the input parameters. However, this can be computationally intensive. The problem is not convex and there may be a plethora of local minima. The parameter surface may have many “flat” areas leading to unstable parameter solutions.

Due to the issues around optimization and given that practitioners require efficient market calibration, we choose our pricing modeling framework in light of optimization instead of choosing the framework which may best fit the market. In a now classic paper, Carr, Madan, Geman, and Yor (2003) demonstrate that asset prices may not have a diffusion component and can be modeled as a pure jump process with infinite activity. However, Huang and Wu (2004) show that when accounting for the leverage effect, the diffusion component does have a significant impact. This is due to the empirical fact that asset returns and asset volatility are correlated. One of the only tractable ways to generate this correlation is by modeling time changes as integrated diffusions.

In our study, we find that it is very difficult to solve the inverse problem for Levy-processes of infinite activity. We therefore eschew the empirical results of CGMY and focus only on finite-activity jump diffusions a la Merton (1976). However, we do allow for a time-changed diffusion component. The models thus incorporate popular models like Heston’s as a special case.

All the code is available at the following Github repo: [FFTOptionPricing](#).

2 The model

Levy processes can be constructed with relatively simple characteristic exponents. In this paper we will focus on processes with characteristic exponents of the following form:

$$\begin{aligned}\psi(u) &= -\frac{\sigma^2 u^2}{2} + \lambda \left(1 - e^{\mu_j + \sigma_j^2/2}\right) + \lambda \left(e^{\mu_j u i - \sigma_j^2 u^2/2} - 1\right) \\ &= -\frac{\sigma^2 u^2}{2} - \psi_j(-i) + \psi_j(u)\end{aligned}$$

Where $\psi_j(u) := \lambda \left(e^{\mu_j u i - \sigma_j^2 u^2/2} - 1\right)$ is the characteristic function of the jump component. $\psi(u)$ is the characteristic exponent of a Merton jump-diffusion model. Following Carr and Wu (2004), we also introduce a random time change so that the characteristic function of the normalized log asset is as follows:

$$\mathbb{E}[e^{u i X_t}] = g(-\psi(u), a, a - \sigma_v \rho u \sigma, \sigma_v, v_0)$$

Where $X_t = \log\left(\frac{S_t}{S_0}\right) - rt$, and g is the moment generating function of an integrated CIR process:

$$g(x, a, \kappa, \sigma_v, v_0) = e^{-b(t)v_0 - c(t)}$$

Where

$$\begin{aligned} b(t) &= 2x(1 - e^{-\delta t}) / (\delta + \kappa + (\delta - \kappa)e^{-\delta t}) \\ c(t) &= \left(\frac{a}{\sigma^2}\right) (2\log(1 + (\kappa - \delta)(1 - e^{-\delta t})/2\delta) + (1 - e^{-\delta t})(\kappa - \delta)) \\ \delta &= \sqrt{\kappa^2 + 2x\sigma_v^2} \end{aligned}$$

3 Calibration

Calibration has traditionally taken the following form:

$$\min_{\theta} \sum_k w_k (C_k - C(k; \theta))^2$$

Where w_k is a weight, θ are the parameters describing the (risk-neutral) asset process, C_k is the observed option prices at log-strike k , and $C(k, \theta)$ is the modeled price.

As mentioned in the introduction, this problem is not trivial. See Tankov (2004) for details. Since we are dealing with Levy processes, we instead consider minimizing the following:

$$\min_{\theta} ||\psi(u_l; \theta) - \hat{\psi}(u_l; k)||$$

We can borrow from Carr and Madan (1999) and Belomestny and Reiß (2006) to create the estimate $\hat{\psi}$ from the observed option data:

$$\log\left(1 + iu(iu + 1) \int_{-\infty}^{\infty} e^{uix} O(x) dx\right) = \psi(u - i, t)$$

Where $O(x) = C_{x+\log(S_0)+rt}/S_0 - (1 - e^x/S_0)^+$ and $x = k - \log(S_0) - rt$. Since we do not observe a continuum of option prices in the market, we use a monotonic spline to interpolate the option prices. To preserve accuracy, we use a two part spline fit. The first fit uses the realized $O_k = C_k/S_0 - (1 - e^{k-rT})^+$ from the minimum log strike until the last strike that satisfies $e^k/S_0 < 1$. The second fit is the log of normalized option prices $\log(C_k/S_0)$ and is fit from the last strike that satisfies $e^k/S_0 < 1$ until the maximum log strike.

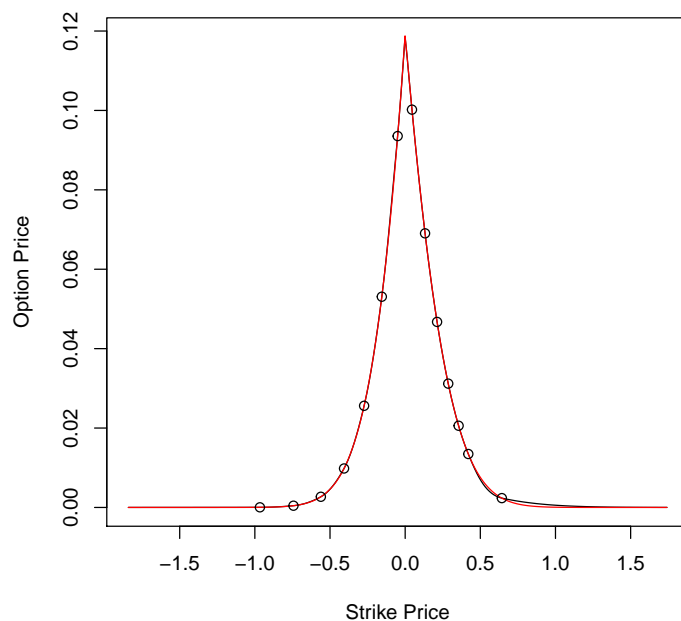
There are three sources of error in the estimate:

1. The observation error in the options themselves
2. The error in the spline approximation
3. The error in numerically integrating the observed options prices

3.1 Spline Error

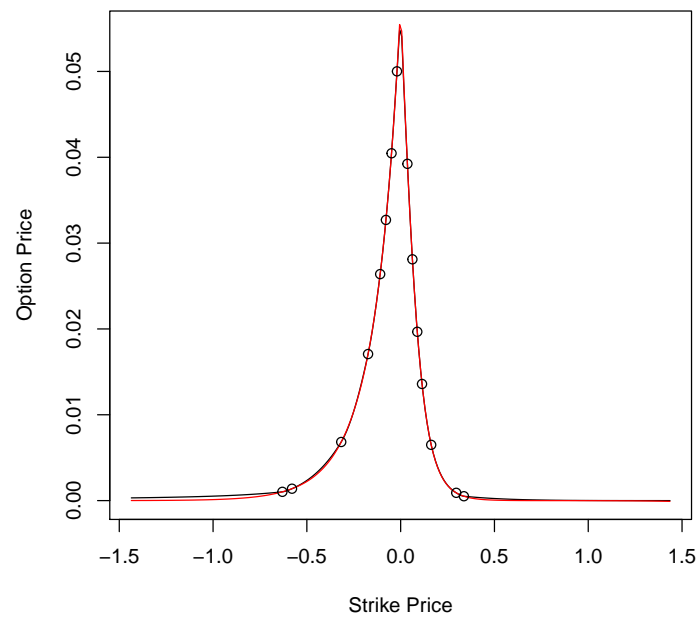
3.1.1 Black Scholes

We first test the spline error on a Black-Scholes model. The parameters chosen are $S = 10$, $r = 0$, $t = 1$, $\sigma = .3$, and the strike array 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16. This model shows very good alignment when choosing a spline on the log of the option price against the strike as can be seen in the following charts. Note that the black line is estimated, while the red line is actual. The red line nearly completely covers the black line.



3.1.2 Heston

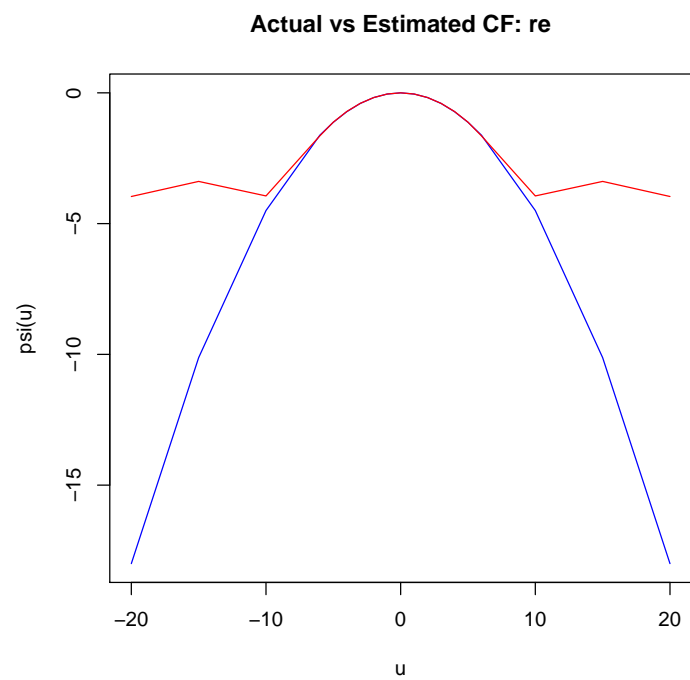
We then fit the spline for a Heston model. The parameters chosen are the strike array (95, 100, 130, 150, 160, 165, 170, 175, 185, 190, 195, 200, 210, 240, 250).

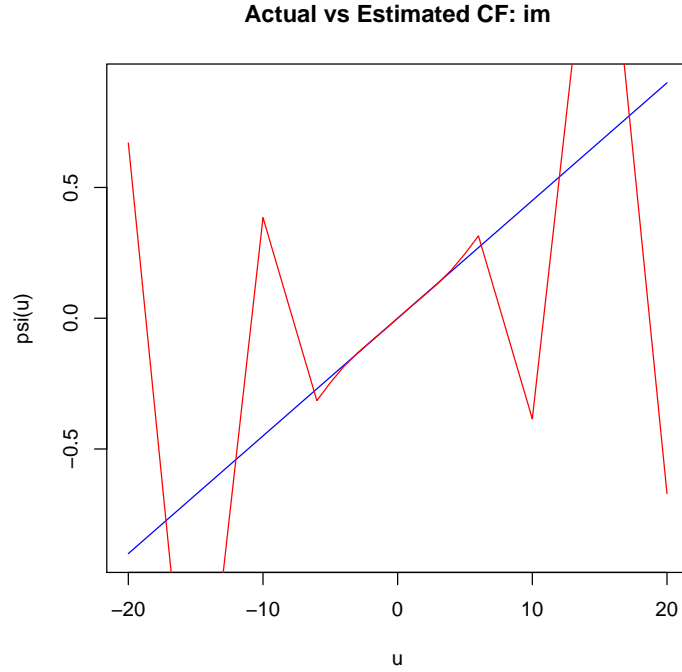


3.2 Numerical Integration Error

3.2.1 Black Scholes

The numerical integration error can be seen as follows:



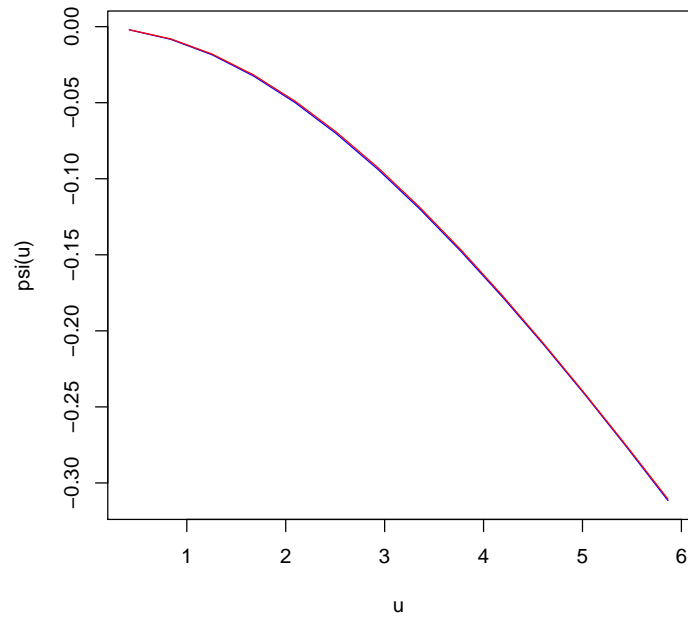


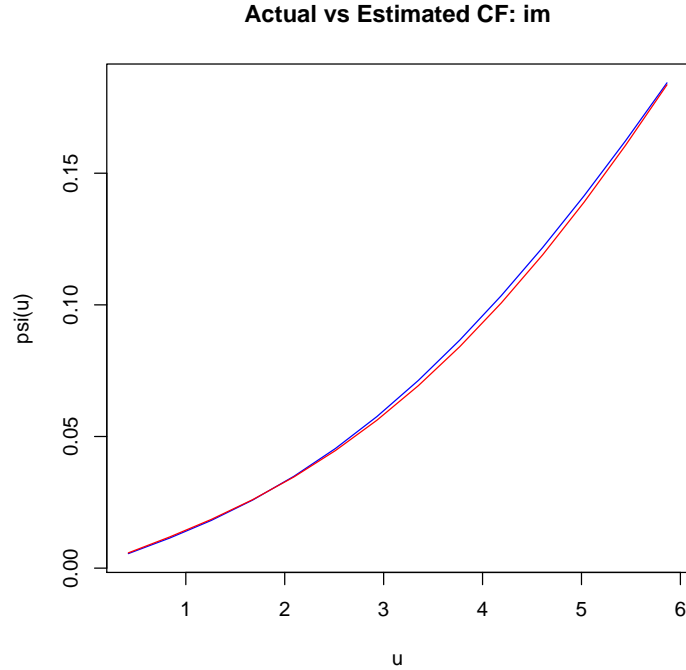
While in theory the integration should be valid over all real numbers, it appears that it is only accurate from $(-2\pi, 2\pi)$. However, since the function is even, we don't benefit from using all the domain and truncate it from $(0, 2\pi)$.

3.2.2 Heston

The following plot shows the integration comparison for a Heston model.

Actual vs Estimated CF: re





The model shows very good alignment.

3.3 Estimates

3.3.1 Black Scholes

Estimating the characteristic function at points u between negative and positive 2π , we use a Cuckoo algorithm to minimize the sum of the norms of the differences between the estimated and modeled characteristic functions. For a simple model like Black Scholes we can also use gradient descent, however for more complicated models the output has many local minima.

```

actual    optimal parameter
1    0.3 0.3007071    sigma

```

3.3.2 Heston

For more complicated models with many local minima, we use the cuckoo search. The search for a Heston model shows the following results:

```

          actual    optimal parameter
1 0.1994994 0.2154307    sigma
2 1.5768000 0.4021933    speed
3 2.8827158 1.7322743    ada_v

```


4	-0.5711000	-0.5626718	rho
5	0.4396985	0.5190957	v0_hat