1 Backward and Forward Kolmogorov Equation

This paper is focused on deriving both the Backward and Forward Kolmogorov equations. These equations answer questions around how the density of a continuous stochastic differential equation evolves in both space and time. This paper assumes an understanding of the theory of Brownian Motion and stochastic calculus (eg, existence and solutions to stochastic differential equations). For this paper, the stochastic differential equation under consideration is one in which a transition density exists and in which the dynamics are:

$$dX_t = \alpha(X_t, t)dt + \sigma(X_t, t)dW_t \tag{1}$$

Where α and σ satisfy the "usual" conditions and dW_t is an increment of Brownian Motion.

2 Evolution of Expectation of Stochastic Different Equations

Definition 1. The *generator* of a diffusion Z_t is defined as

$$\lim_{T\to 0} \frac{\mathbb{E}[f(Z_T)] - f(z)}{T}$$

Theorem 1. *Let X_t be the solution to 1. Then the generator A of X_t is the operator $\alpha(x,t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2}{\partial x^2}$. For the remainder of this paper the generator will be denoted A.*

Proof. By Ito's Lemma,

$$\lim_{T\to 0} \frac{\mathbb{E}[f(X_T)] - f(x)}{T}$$

$$= \lim_{T\to 0} \frac{f(x) + \mathbb{E}\left[\int_0^T \left(\alpha(X_s,s)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(X_s,s)\frac{\partial^2 f}{\partial x^2}\right)ds + \int_0^T \sigma(X_s,s)dW_s\right] - f(x)}{T}$$

$$= \lim_{T\to 0} \frac{\int_0^T \left(\mathbb{E}\left[\alpha(X_s,s)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(X_s,s)\frac{\partial^2 f}{\partial x^2}\right]\right)ds}{T}$$

$$= \mathbb{E}\left[\alpha(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}\right]$$

$$= \alpha(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2}$$

□

Theorem 2. *Let X_t be the solution to 1 and define $\tau = T - s$. Then for any Borel measurable function h, $u(\tau, x) = \mathbb{E}[h(X_T)|\mathcal{F}_s]$ satisfies the partial differential equation $\frac{\partial u}{\partial \tau} = Au$ with terminal condition $u(0,x) = h(x)$ where A is the generator of X_t, and $\mathcal{F}_t$ is the filtration generated by dW_t.*

Proof. By definition,

$$
\begin{aligned}
Au &= \lim_{r\to 0} \frac{\mathbb{E}\left[u(\tau, X_{s+r})\right] - u(\tau, x)}{r} \\
&= \lim_{r\to 0} \frac{\mathbb{E}\left[\mathbb{E}\left[h(X_{T+r})|\mathcal{F}_{s+r}\right]\right] - u(\tau, x)}{r} \\
&= \lim_{r\to 0} \frac{\mathbb{E}\left[h(X_{T+r})|\mathcal{F}_{s}\right] - u(\tau, x)}{r} \\
&= \lim_{r\to 0} \frac{u(\tau + r, x) - u(\tau, x)}{r} \\
&= \frac{\partial u}{\partial \tau}
\end{aligned}
$$

□

3 Backward Kolmogorov Equation

The backward Kolmogorov equation is an application of 4. It describes, for a fixed point at time T, how the conditional density of X_t evolves. I write this conditional density as $p(s, T, x, y)$ where s is the "current" time, x is the "current" value of X_t, T is the "terminal" time, and y is the "dummy" variable for integration. The expectation $\mathbb{E}[g(X_T)|\mathcal{F}_s]$ can thus be written as $\int_{\mathbb{R}} g(y)p(s, T, x, y)dy$. In the special case that X_t is time homogeneous, this can be written as $\int_{\mathbb{R}} g(z)p(\tau, z)dz$ where $z = y - x$ and $\tau = T - s$.

Theorem 3. *Let X_t be defined as in 1. If it exists, and fixing T and y, the transition density $p(s, T, x, y)$ of X_t is a solution to the following partial differential equation:*

$$\frac{\partial p}{\partial s} + Ap = 0$$

With terminal condition $p(T, T, x, y) = \delta(x - y)$. A operates on x.

Proof. Let $u_\delta(\tau, x) = \mathbb{E}\left[\delta(X_T - y)|\mathcal{F}_s\right]$. By basic properties of delta functions,

$$u_\delta(\tau, x) = \mathbb{E}\left[\delta(X_T - y)|\mathcal{F}_s\right] = \int_{\mathbb{R}} \delta(\hat{y} - y)p(s, T, x, \hat{y})d\hat{y} = p(s, T, x, y)$$

From 4, $\frac{\partial u_\delta}{\partial \tau} = Au_\delta$. Since $\tau = T - s$, $\frac{\partial u_\delta}{\partial s} = \frac{\partial u_\delta}{\partial \tau}\frac{\partial \tau}{\partial s} = -\frac{\partial u_\delta}{\partial \tau}$. Putting it all together,

$$\frac{\partial p}{\partial s} + Ap = 0$$

□

4 Forward Kolmogorov Equation

The forward Kolmogorov equation describes, for a fixed point of time s and x), how the conditional density of X_t evolves. In many applications this is a more useful formulation. For many applications we know s and x, and are attempting to understand the potential outcomes in the future.

Theorem 4. *Let X_t be defined as in 1. If it exists, and fixing s and x, the transition density $p(s,T,x,y)$ of X_t is a solution to the following partial differential equation:*

$$\frac{\partial p}{\partial T} + A^* p = 0$$

With initial condition $p(s,s,x,y) = \delta(x-y)$, where $A^* f$ is defined as:

$$-\frac{\partial\left(\alpha(y,t)f\right)}{\partial y} + \frac{1}{2}\frac{\partial^2\left(\sigma(y,t)f\right)}{\partial y^2}$$

A^* operates on y.

Proof. The adjoint of A is defined as an operator $\hat{A}$ such that

$$\int_{\mathbb{R}} f(z)Ag(z)dz = \int_{\mathbb{R}} g(z)\hat{A}f(z)\ \forall f,g \in \mathcal{D}$$

For our purposes, $\mathcal{D}$ is the set of density functions. Since densities integrate to one, these functions satisfy $\lim_{z\to\infty} D(z) = 0$ and $\lim_{z\to-\infty} D(z) = 0\ \forall D \in \mathcal{D}$.

To derive $\hat{A}$,

$$\int_{\mathbb{R}} f(z)Ag(z)dz = \int_{\mathbb{R}} f(z)\alpha(z,t)\frac{\partial g}{\partial z}dz + \frac{1}{2}\int_{\mathbb{R}} f(z)\sigma^2(z,t)\frac{\partial^2 g}{\partial z^2}dz$$

$$= -\int_{\mathbb{R}} \frac{\partial\left(f(z)\alpha(z,t)\right)}{\partial z}g(z)dz - \frac{1}{2}\int_{\mathbb{R}} \frac{\partial\left(f(z)\sigma^2(z,t)\right)}{\partial z}\frac{\partial g}{\partial z}dz$$

$$= -\int_{\mathbb{R}} \frac{\partial\left(f(z)\alpha(z,t)\right)}{\partial z}g(z)dz + \frac{1}{2}\int_{\mathbb{R}} \frac{\partial^2\left(f(z)\sigma^2(z,t)\right)}{\partial z^2}g(z)dz$$

$$\implies \hat{A} = -\frac{\partial\left(\alpha(y,t)f\right)}{\partial y} + \frac{1}{2}\frac{\partial^2\left(\sigma(y,t)f\right)}{\partial y^2} = A^*$$

Armed with the adjoint, I now proceed to directly compute the dynamics of the density.

By Dynkan's formula,

$$\mathbb{E}[h(X_T)|\mathcal{F}_s] = h(x) + \mathbb{E}\left[\int_s^T Ahds\right]$$

Substituting for the density,

$$\int_{\mathbb{R}} h(y)p(s,T,x,y)dy = h(x) + \int_s^T \int_{\mathbb{R}} p(s,v,x,y)Ah(y)dydv$$

Taking the derivative with respect to T of both sides,

$$\int_{\mathbb{R}} h(y)\frac{\partial p(s,T,x,y)}{\partial T}dy = \int_{\mathbb{R}} p(s,T,x,y)Ah(y)dy$$

Using the adjoint,

$$\int_{\mathbb{R}} h(y)\frac{\partial p(s,T,x,y)}{\partial T}dy = \int_{\mathbb{R}} h(y)A^*p(s,T,x,y)dy$$

Where the adjoint operates on the y variable. The only way this equation holds for all $h(y)$ is if

$$\frac{\partial p(s,T,x,y)}{\partial T} = A^*p(s,T,x,y)$$

□

5 Examples

5.1 Brownian Motion

5.1.1 Backward equation

When $dX_t = dW_t$, the Backward equation becomes

$$\frac{\partial p}{\partial s} + \frac{1}{2}\frac{\partial^2 p}{\partial x^2} = 0$$

The conditional density of a Brownian Motion is

$$p_{bm}(s,T,x,y) = \frac{1}{\sqrt{2\pi}\sqrt{T-s}}e^{-\frac{(y-x)^2}{2(T-s)}}$$

Taking the first derivative with respect to s:

$$\frac{\partial p_{bm}(s,T,x,y)}{\partial s} = \frac{1}{2}p_{bm}(s,T,x,y)\left(\frac{1}{T-s} - \frac{(y-x)^2}{(T-s)^2}\right)$$

Taking the second derivative with respect to x:

$$\frac{\partial^2 p_{bm}(s,T,x,y)}{\partial x^2} = p_{bm}(s,T,x,y)\left(\frac{(y-x)^2}{(T-s)^2} - \frac{1}{T-s}\right)$$

Combining,

$$\frac{\partial p}{\partial s} + \frac{1}{2}\frac{\partial^2 p}{\partial x^2} = 0$$

5.1.2 Forward equation

The Forward equation becomes

$$\frac{\partial p}{\partial T} - \frac{1}{2}\frac{\partial^2 p}{\partial y^2} = 0$$

Taking the first derivative with respect to T:

$$\frac{\partial p_{bm}(s,T,x,y)}{\partial T} = \frac{1}{2} p_{bm}(s,T,x,y)\left(\frac{(y-x)^2}{(T-s)^2} - \frac{1}{T-s}\right)$$

Taking the second derivative with respect to y:

$$\frac{\partial^2 p_{bm}(s,T,x,y)}{\partial y^2} = p_{bm}(s,T,x,y)\left(\frac{(y-x)^2}{(T-s)^2} - \frac{1}{T-s}\right)$$

Combining,

$$\frac{\partial p}{\partial T} - \frac{1}{2}\frac{\partial^2 p}{\partial y^2} = 0$$

5.2 Geometric Brownian Motion

5.2.1 Backward equation

When $dX_t = \alpha X_t dt + \sigma X_t dW_t$, the Backward equation becomes

$$\frac{\partial p}{\partial s} + \alpha x \frac{\partial p}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 p}{\partial x^2} = 0$$

The conditional density of a Geometric Brownian Motion is

$$p_{gbm}(s,T,x,y) = \frac{1}{\sigma y \sqrt{2\pi}\sqrt{T-s}} e^{-\frac{\left(\log\left(\frac{y}{x}\right) - \left(\alpha - \frac{\sigma^2}{2}\right)(T-s)\right)^2}{2\sigma^2 (T-s)}}$$

To simplify notation, from here on v denotes $\frac{\log\left(\frac{y}{x}\right) - \left(\alpha - \frac{\sigma^2}{2}\right)(T-s)}{T-s}$.

Taking the first derivative with respect to s:

$$\begin{aligned}\frac{\partial p_{gbm}(s,T,x,y)}{\partial s} &= \\ & p_{gbm}(s,T,x,y)\frac{1}{2(T-s)} + p_{gbm}(s,T,x,y)\frac{v}{2} - \\ & \qquad p_{gbm}(s,T,x,y)\alpha\frac{v}{\sigma^2} - p_{gbm}(s,T,x,y)\frac{v^2}{2\sigma^2}\end{aligned}$$

Taking the first derivative with respect to x:

$$\frac{\partial p_{gbm}(s,T,x,y)}{\partial x} = p_{gbm}(s,T,x,y)\left(\frac{v}{\sigma^2 x}\right)$$

Taking the second derivative with respect to x:

$$\frac{\partial^2 p_{gbm}(s,T,x,y)}{\partial x^2} = p_{gbm}(s,T,x,y) * \left(\left(\frac{v}{\sigma^2 x}\right)^2 - \frac{1}{\sigma^2 x^2 (T-s)} - \frac{v}{\sigma^2 x^2}\right)$$

Applying the coefficients,

$$\alpha x \frac{\partial p}{\partial x} = p\alpha\frac{v}{\sigma^2}$$

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 p}{\partial x^2} = p\left(\frac{v^2}{2\sigma^2} - \frac{1}{2(T-s)} - \frac{v}{2}\right)$$

Putting it all together,

$$\begin{aligned}&\frac{\partial p}{\partial s} + \alpha x\frac{\partial p}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 p}{\partial x^2} = \\ &\quad p\frac{1}{2(T-s)} + p\frac{v}{2} - p\alpha\frac{v}{\sigma^2} - p\frac{v^2}{2\sigma^2} + p\alpha\frac{v}{\sigma^2} + p\frac{v^2}{2\sigma^2} - p\frac{1}{2(T-s)} - p\frac{v}{2} = 0\end{aligned}$$

5.2.2 Forward equation

The Forward equation becomes

$$\frac{\partial p}{\partial T} + \frac{\partial\left(\alpha y p\right)}{\partial y} - \frac{1}{2}\frac{\partial^2\left(\sigma^2 y^2 p\right)}{\partial y^2} = 0$$

Taking the first derivative with respect to T:

$$\frac{\partial p_{gbm}(s,T,x,y)}{\partial T} = -\frac{\partial p_{gbm}(s,T,x,y)}{\partial s} = p\alpha\frac{v}{\sigma^2} + p\frac{v^2}{2\sigma^2} - p\frac{1}{2(T-s)} - p\frac{v}{2}$$

Where again $v = \frac{\log\left(\frac{y}{x}\right) - \left(\alpha - \frac{\sigma^2}{2}\right)(T-s)}{T-s}$.

Taking the first derivative with respect to y:

$$\begin{aligned}\frac{\partial\left(\alpha y p\right)}{\partial y} &= \frac{\partial\left(\frac{\alpha}{\sigma\sqrt{2\pi}\sqrt{T-s}}e^{-\frac{(T-s)v^2}{2\sigma^2}}\right)}{\partial y} \\ &= -p\alpha\frac{v}{\sigma^2}\end{aligned}$$

Taking the second derivative with respect to y:

$$\frac{\partial^2\left(\sigma^2 y^2 p\right)}{\partial y^2} = \frac{\partial^2\left(\frac{\sigma y}{\sqrt{2\pi}\sqrt{T-s}} e^{-\frac{(T-s)v^2}{2\sigma^2}}\right)}{\partial y^2}$$

$$= p\frac{v^2}{\sigma^2} - pv - p\frac{1}{T-s}$$

Combining,

$$\frac{\partial p}{\partial T} + \frac{\partial\left(\alpha y p\right)}{\partial y} - \frac{1}{2}\frac{\partial^2\left(\sigma^2 y^2 p\right)}{\partial y^2} =$$

$$p\alpha\frac{v}{\sigma^2} + p\frac{v^2}{2\sigma^2} - p\frac{1}{2(T-s)} - p\frac{v}{2} - p\alpha\frac{v}{\sigma^2} + p\frac{1}{2(T-s)} + p\frac{v}{2} - p\frac{v^2}{2\sigma^2} = 0$$