

# 1 Assumptions

We combine the work by Carr and Wu (2004) and the work of Duffie, Pan, and Singleton (2000) to create a very general framework for jump diffusions and stochastic volatility. Due to specification and calibration issues, we restrict our attention to jump-diffusions and do not consider processes with jumps of infinite activity.

## 1.1 Specification of stochastic volatility

Following Carr and Wu, we specify the stochastic time change rather than directly specifying the volatility. The time change is assumed to take the following form:

$$\tau = \int_0^t v_s ds$$

$$v_t = v_0 + \int_0^t a(1 - kv_s)ds + \int_0^t \eta\sqrt{v_s}dW_s^2 + \sum_{j=1}^{N_t} z_j$$

Where  $N_t$  follows a Poisson process and the  $z_j$  are independent draws from an almost surely positive distribution. In this paper, we assume  $z_j \sim \exp(q)$  and that it is independent of every other source of randomness. Following Carr and Wu, we set the parameters such that the long run expectation of  $v_t$  is 1. Since the time change impacts the frequency of jumps linearly, the frequency of jumps can be modeled by  $\lambda v_s$ . Hence to adjust the drift to make the long run expectation of  $v_t$  be 1, we adjust the drift as follows:  $a \left( 1 - \left( \frac{\lambda \mathbb{E}[z_j]}{a} + 1 \right) v_s \right)$  where for simplicity we let  $k = \frac{\lambda \mathbb{E}[z_j]}{a} + 1$ .

## 1.2 Specification of the log asset price

The log asset price is assumed to follow the following form:

$$x_t = \log \left( \frac{S_t}{S_0} \right) = \left( \alpha - \frac{\sigma^2}{2} \right) t + \sigma W_t^1 + \sum_{j=1}^{N_t} y_j$$

Here  $dW_t^1 dW_t^2 = \rho dt$  and  $N_t$  is the same jump process as the one in the stochastic time change dynamics.  $y_j$  is assumed to be Gaussian and independent of every other source of randomness.

## 1.3 Risk Neutral log asset price

Following Carr and Wu, the risk neutral log price can be modeled as follows (note that the market is incomplete):

$$x_t = \log \left( \frac{S_t}{S_0} \right) = rt - \left( \frac{\sigma^2}{2} - \lambda \left( e^{\mu_y + \frac{\sigma_y^2}{2}} - 1 \right) \right) t + \sigma \tilde{W}_t^1 + \sum_{j=1}^{N_t} y_j$$

## 1.4 Analytical Characteristic Function

Following Carr and Wu, the full time changed  $x_\tau$  has the following characteristic function:

$$\phi_x(u) = \hat{\mathbb{E}} \left[ e^{uirt} e^{\tau\psi(u)} \right]$$

Where

$$\psi(u) = \lambda \left( e^{iu\mu_y - \frac{u^2\sigma_y^2}{2}} - 1 \right) - \frac{\sigma^2}{2}u^2 - \left( \frac{\sigma^2}{2} - \lambda \left( e^{\mu_y + \frac{\sigma_y^2}{2}} \right) \right) ui$$

Under  $\hat{\mathbb{P}}$ ,  $v_s$  has the following dynamics:

$$v_t = v_0 + \int_0^t a \left( 1 - \left( k - \frac{iu\rho\sigma\eta}{a} \right) v_s \right) ds + \int_0^t \eta \sqrt{v_s} d\hat{W}_s^2 + \sum_{j=1}^{\hat{N}_t} z_j$$

Where  $\hat{N}_t$  has jump frequency  $v_s \lambda e^{iu y}$ . By Duffie, Pan, and Singleton (2000), such a characteristic function has a semi-analytical solution.

## 1.5 ODE for characteristic function

### 1.5.1 General case

Consider the following functions:

$$\mu(x) = K_0 + K_1 x, \sigma^2(x) = H_0 + H_1 x, \lambda(x) = l_0 + l_1 x, R(x) = \rho_0 + \rho_1 x$$

By Duffie, Pan, and Singleton (2000), for processes  $X_t$  defined as

$$X_t = X_0 + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dW_s + \sum_j^{N_t} Y_j$$

with jump frequency  $\lambda(X_s)$ , the following holds:

$$g(u, x, t, T) := \mathbb{E} \left[ e^{-\int_t^T R(X_s) ds} e^{cX_t} \right]$$

has solution

$$e^{\alpha(t) + \beta(t)x}$$

where

$$\begin{aligned} \beta'(t) &= \rho_1 - K_1 \beta(t) - \frac{\beta^2(t) H_1}{2} - \int_{\Omega} l_1 \left( e^{\beta(t)z} - 1 \right) \mu(dz) \\ \alpha'(t) &= \rho_0 - K_0 \beta(t) - \frac{\beta^2(t) H_0}{2} - \int_{\Omega} l_0 \left( e^{\beta(t)z} - 1 \right) \mu(dz) \end{aligned}$$

with  $\beta(T) = c, \alpha(T) = 0$ .

### 1.5.2 Application to this paper

The process  $v_t$  under  $\hat{\mathbb{P}}$  has this same structure with the following parameters:

$$K_0 = a, K_1 = -a \left( k - \frac{iu\rho\sigma\eta}{a} \right)$$

$$H_0 = 0, H_1 = \eta$$

$$l_0 = 0, l_1 = \lambda e^{iu z}$$

$$\rho_0 = 0, \rho_1 = -\psi(u)$$

$$c = 0$$

Substituting and simplifying yields the following ODEs:

$$\beta'(t) = -\psi(u) + (a + \lambda \mathbb{E}[z_j] - iu\rho\sigma\eta) \beta(t) - \frac{\beta^2(t)\eta}{2} - \int_{\Omega} \left( e^{(\beta(t)+iu)z} - e^{iu z} \right) \lambda \mu(dz)$$

$$\alpha'(t) = -a\beta(t)$$

with  $\beta(T) = 0, \alpha(T) = 0$ .