Existence of Expectation Representation of Preferences under Uncertainty

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1 Introduction

Expected Utility Theory (EUT) is the dominant economic paradigm for modeling preferences under uncertainty. Classic EUT is justified by the von Nuemann-Morgenstern (1947) theorem, which provides four axioms under which an expected utility representation exists to express preferences under uncertainty. In 1953, Allais (1953) challenged this theorem through a thought experiment (the "Allais Paradox") where most individuals would prefer gambles that violated EUT. Subsequent empirical work (for example, List (2005)) provides additional evidence that classic EUT is not sufficiently rich to capture human behavior under uncertainty. Prospect Theory, introduced by Kahneman and Tversky (1979) has offered an alternative to Expected Utility Theory (EUT) and addressed these empirical discrepancies in classic EUT. Kahneman and Tversky particularly challenged the "independence" or "substitution" axiom of the von Nuemann-Morgenstern proof. Prospect Theory posits that individuals evaluate outcomes relative to a reference point and weigh probabilities non-linearly. The outcomes are represented by a (not-everywhere-differentiable) "value" function instead of a utility function, and the output of the value function in each state is multiplied by a function of the probability of that state occurring, successfully explaining anomalies such as the Allais Paradox.

Kahneman and Tversky's paper sparked additional research and interest in modeling behavior under uncertainty without relying on EUT. Quiggin (1982) introduced rank-dependent weightings on the probabilities; representing utility as the sum of the utility in each potential outcome weighted by a function of the (cumulative) probability of each outcome. Tversky and Kahneman (1992) incorporated Quiggin's approach, extending Prospect Theory into Cumulative Prospect Theory (CPT). Research by Wakker and Tversky (1993) further refined the theoretical underpinnings of CPT and contrasted it with EUT.

While this literature builds a mathematically rigorous alternative to EUT, financial and economic literature continues to be dominated by EUT. Notable exceptions include Barberis and Huang (2008) for financial asset pricing, Schmidt and Zank (2008) for analyzing risk-aversion with Prospect Theory, and Gonzalez and Wu (1999) and Prelec (1998) for statistically estimating the cumulative weighting function.

In this paper, I prove the existence of an expectation representation of preferences without requiring the independence assumption of von Nuemann and Morgenstern. I show that a measure exists such that preferences which violate the independence axiom can still be represented by an expectation. This approach has two major benefits over the CPT. First, the approach is a parsimonious extension of EUT: while a new measure does "re-weight" probabilities as in CPT, it does not require rank-ordering preferences, accumulating probabilities, or using a non-differentiable value function. Second, by allowing an expectation representation, the vast literature built on EUT remains sound despite Kahneman and Tversky's objections to the independence axiom. I give a practical example by showing that preferences consistent with Allais' paradox can be represented by expectations.

2 The von Nuemann-Morgenstern Axioms

Let X represent the consumption set available to economic agents. A bundle $x \in X$ is a set of consumption options within X. A gamble σ is a random variable that chooses (potentially) different consumption bundles within X for various states of the world. In the language of probability theory, $\sigma : \Omega \to X$. I allow gambles to also exist in the consumption set, allowing me to state that $\sigma \in X$. Unless otherwise stated, I denote "deterministic" consumption bundles as $x_i \in X$ while I denote gambles as $\sigma_i \in X$.

The von Nuemann-Morgenstern theorem states that given a set of axioms, there exists a function $u : X \to \mathbf{R}$ such that for all $\sigma_1, \sigma_2 \in X$, $\mathbb{E}[u(\sigma_1)] \geq \mathbb{E}[u(\sigma_2)]$ if and only if $\sigma_1 \succeq \sigma_2$. These axioms are:

1. **Completeness**. $\forall x_1,\, x_2 \in X$, either $x_1 \succeq x_2$, $x_2 \succeq x_1$, or both.
2. **Transitivity**. $\forall x_1,\, x_2,\, x_3 \in X$, if $x_1 \succeq x_2$ and $x_2 \succeq x_3$, then $x_1 \succeq x_3$.
3. **Continuity**. $\forall x_1,\, x_2,\, x_3 \in X$ satisfying $x_1 \succeq x_2 \succeq x_3$, there exists $\alpha \in [0,\, 1]$ such that $\alpha x_1 + (1-\alpha)x_3 \sim x_2$.
4. **Independence**. $\forall x_1,\, x_2,\, x_3 \in X$ and $\alpha \in [0,\, 1]$, $x_1 \succeq x_2 \Leftrightarrow \alpha x_1 + (1-\alpha)x_3 \succeq \alpha x_2 + (1-\alpha)x_3$

Only the first two axioms are required for rationality. The third is required to ensure the existence of a utility function (agents may be rational even if no utility function exists). The last axiom, per von Nuemann-Morgenstern's proof, is required to ensure that preferences in an uncertain world can be represented by taking expectations.

This theorem is powerful: it provides theoretical justification for using the mathematical language of expected utility as a toolkit for explaining human preferences and behavior. However, the proof relies heavily on all four axioms. If one of the axioms is shown to be false, the proof fails.

The proof is constructive, demonstrating how a utility function can be created. The construction relies on all four axioms. However, utility functions may still exist if the third and fourth axioms are relaxed. They simply cannot be constructed in the same way as in the theorem. In other words, if the independence axiom is not valid, expected utility under the real-world measure *may* still represent preferences, but it is not *guaranteed* to do so.

3 The Allais Paradox

In 1953, Allais proposed the following two sets of choices of gambles:

3.0.1 Choice 1

$$\begin{cases} 1\text{M} & \text{with probability } 1 \end{cases}$$

OR

$$\begin{cases} 0 & \text{with probability } 0.01, \\ 1\text{M} & \text{with probability } 0.89, \\ 5\text{M} & \text{with probability } 0.1. \end{cases}$$

3.0.2 Choice 2

$$\begin{cases} \\ 0\text{M} & \text{with probability } 0.89, \\ 1\text{M} & \text{with probability } 0.11. \end{cases}$$

OR

$$\begin{cases} 0 & \text{with probability } 0.9, \\ 5\text{M} & \text{with probability } 0.1. \end{cases}$$

If the von Nuemann-Morgenstern theorem holds (using both the existence of a utility function and the independence axiom), the second choice can be decomposed as

$$\begin{aligned} 0.89u(0) + 0.11u(1) &\,|\, 0.9u(0) + 0.1u(5) \\ \Leftrightarrow u(1) - 0.89u(1) &\,|\, 0.01u(0) + 0.1u(5) \\ \Leftrightarrow u(1) - 0.89u(1) &\,|\, 0.01u(0) + 0.1u(5) \\ \Leftrightarrow u(1) &\,|\, 0.01u(0) + 0.1u(5) + 0.89u(1) \end{aligned}$$

This final statement is simply the utility of the first choice. Hence if an agent chooses option one in choice one, he or she should choose option one in choice two. Likewise, if the agent prefers option two in choice one, he or she should choose option two in choice two.

In reality, many people would prefer the first option for choice one and the second option for choice two. Even the remote possibility of missing out on 1 million dollars is too big a gamble. For the remainder of this paper the independence axiom is dropped. I show that even without this axiom an expectation representation exists for preferences under uncertainty.

4 Existence of Utility Function and Probability Measure

Theorem 1. *Assume the first three von Nuemann-Morgenstern axioms, which implies the existence of an (ordinal) utility function g representing preferences such that $g(x_1) \geq g(x_2) \Leftrightarrow x_1 \succeq x_2$. Assume a set of independent gambles. Then there exists a function $u : X \to \mathbb{R}$ and a probability measure $\mathbb{Q}$ equivalent to the "physical" measure $\mathbb{P}$ such that $\mathbb{E}_Q[u(\sigma_1)] \geq \mathbb{E}_Q[u(\sigma_2)]$ if and only if $\sigma_1 \succeq \sigma_2$.*

Proof. The proof that a measure exists is nearly trivial and is by construction. Debreu's Representation Theorem axioms 1, 2, and 3 are sufficient and necessary for the existence of the ordinal utility function g. By the Transitivity axiom, the utility of a gamble must be bounded by the utility in the best and worst outcomes. Represent each gamble by a vector v_i of outcomes from gamble σ_i. Since the actual utility of gamble σ_i (denoted $g(x_i^*)$ where x_i^* is the "equivalent" deterministic bundle) lies in the relative interior of v_i, there exists at least one (and typically many) strictly positive vectors q_i such that $g(x_i)^T q_i = g(x_i^*)$. The full measure $\mathbb{Q}$ is then recovered by taking the Cartesian product of each q_i and multiplying each row. The resulting vector has length $\prod_i \|v_i\|$. In many cases the utility function u can simply be chosen to be g. If for a given application utility needs to be positive, u can be defined as $u = a + bg$ where a and b are chosen such that u is positive.

□

5 Selecting a Probability Measure

The previous section shows that a measure could be constructed. While in principal any q_i that satisfies $u(v_i)^T q_i = u(x_i^*)$ can be chosen, parsimony suggests finding a vector that is "similar" to the real-world probabilities. This motivates a series of constrained minimization problems (one for each gamble σ_i). A natural choice for objective function is the Kullback-Leibler (KL) divergence:

$$
\begin{aligned}
\min_q \quad & \sum_j q_{i,j} \ln\left(p_{i,j}/q_{i,j}\right) \\
\text{s.t.} \quad & \sum_j u(v_{i,j}) q_{i,j} = u(x_i^*) \\
& \sum_j q_{i,j} = 1
\end{aligned}
\tag{1}
$$

Where $u(x_i^*)$ is the "actual" utility of the gamble σ_i and whose ordering of the outcomes under $\mathbb{Q}$ across gambles aligns with the ordering of preferences. This problem is solvable via Lagrange multipliers, with final multiplier solvable via efficient numerical methods.

5.1 Examples

5.1.1 Allias Paradox choice 1

Consider the first choice of the Allias Paradox. The outcomes can be written in matrix form as follows (numbers in millions):

$$X^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 5 \end{bmatrix}$$

Assuming a starting wealth of \$100,000 and a log utility function, the (ordinal) utilities associated with these outcomes are:

$$M^T = \begin{bmatrix} 13.911 & 11.513 \\ 13.911 & 13.911 \\ 13.911 & 15.445 \end{bmatrix}$$

The function u can be set to the ordinal utility function g. The expected utility from these two outcomes under the real-world measure is

	Expected Utility under $\mathbb{P}$
Choice 1 Option 1	13.911
Choice 1 Option 2	14.040

For the sake of the example, plausible actual utility values could be

	Actual Utility
Choice 1 Option 1	13.911
Choice 1 Option 2	13.900

Solving the Lagrange, a q is chosen as $q(\omega) = [0.0345, 0.918, 0.047]$.
Under this measure, the expected utility for each outcome corresponds to the actual utility:

	Expected Utility under $\mathbb{Q}$
Choice 1 Option 1	13.911
Choice 1 Option 2	13.900

5.1.2 Allias Paradox all choices

Extending the previous example to all choices generates the following q vectors:

$$q_1 = [1.000]$$

$$q_2 = [0.0345, 0.918, 0.047]$$

$$q_3 = [0.831, 0.169]$$

$$q_4 = [0.880, 0.120]$$

Taking the Cartesian product of the four vectors yields the full measure q:

$$\begin{aligned} q(\omega) = [&0.025, 0.003, 0.005, 0.001, \\ &0.671, 0.091, 0.137, 0.019, \\ &0.035, 0.005, 0.007, 0.001] \end{aligned}$$

Under this measure, the expected utility for each outcome is

	Expected Utility under $\mathbb{Q}$
Choice 1 Option 1	13.911
Choice 1 Option 2	13.900
Choice 2 Option 1	11.918
Choice 2 Option 2	11.983

This resolves the paradox.

6 Conclusion

This paper establishes that preferences can be represented by expectations even without relying on the independence axiom of von Neumann and Morgenstern. A measure exists such that preferences violating the independence axiom can still conform to an expectation representation. The proof is exemplified through preferences consistent with Allais' paradox, demonstrating that economic insights from EUT remain robust despite challenges posed by Kahneman and Tversky. This result suggests that the foundational principles of EUT can still apply to decision-making scenarios that deviate from classical assumptions.

Bibliography

1. Allais, M. (1953). Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école américaine. *Econometrica*, 21(4), 503-546.

2. Barberis, N., and Huang, M. (2008). Stocks as lotteries: The implications of probability weighting for security prices. *American Economic Review*, 98(5), 2066-2100.

3. Gonzalez, R., and Wu, G. (1999). On the shape of the probability weighting function. *Cognitive Psychology*, 38(1), 129-166.

4. Kahneman, D. and Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, 47(2), 263-291.

5. List, J. and Haigh, M. (2005). A simple test of expected utility theory using professional traders. List JA, Haigh MS. (2005). A simple test of expected utility theory using professional traders. *Proc Natl Acad Sci USA*, 102(3), 945–948.

6. Neumann, John von and Morgenstern, Oskar (1947). Theory of Games and Economic Behavior. Princeton University Press.

7. Prelec, D. (1998). The probability weighting function. *Econometrica*, 66(3), 497-527.

8. Quiggin, J. (1982). A theory of anticipated utility. *Journal of Economic Behavior and Organization*, 3(4), 323-343.

9. Schmidt, U., and Zank, H. (2008). Risk aversion in cumulative prospect theory. *Management Science*, 54(1), 208-216.

10. Tversky, A., and Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5(4), 297-323.

11. Wakker, P., and Tversky, A. (1993). An axiomatization of cumulative prospect theory. *Journal of Risk and Uncertainty*, 7(2), 147-175.