An exact solution method for
binary equilibrium problems with compensation
and the power market uplift problem

Daniel Huppmann, Sauleh Siddiqui

huppmann@iiasa.ac.at, siddiqui@jhu.edu

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This document summarizes the mathematical formulation for easier reference of the GAMS implementation provided at `http://danielhuppmann.github.io/binary_equilibrium/`. All equation numbers are identical to those in the published version of the manuscript.

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Theoretical formulation (Section 3.4)

$$\min_{\substack{x_i,y_i,\widetilde{y}_i^{(\overline{\mathbf{x}}_i)},\widetilde{\lambda}_i^{(\overline{\mathbf{x}}_i)} \\ \kappa_i^{(\overline{\mathbf{x}}_i)},\zeta_i^{(\overline{\mathbf{x}}_i)}}} \quad F\left((x_i,y_i)_{i\in I}\right) + G\left((\zeta_i^{(\overline{\mathbf{x}}_i)})_{i\in I}\right) \tag{14a}$$

$$\text{s.t.} \quad \nabla_{y_i} f_i\left(\mathbf{1},\widetilde{y}_i^{(\mathbf{1})},y_{-i}\right) + \left(\widetilde{\lambda}_i^{(\mathbf{1})}\right)^T \nabla_{y_i} g_i\left(\mathbf{1},\widetilde{y}_i^{(\mathbf{1})}\right) = 0 \tag{14b}$$

$$0 \leq -g_i\left(\mathbf{1},\widetilde{y}_i^{(\mathbf{1})}\right) \perp \widetilde{\lambda}_i^{(\mathbf{1})} \geq 0 \tag{14c}$$

$$\nabla_{y_i} f_i\left(\mathbf{0},\widetilde{y}_i^{(\mathbf{0})},y_{-i}\right) + \left(\widetilde{\lambda}_i^{(\mathbf{0})}\right)^T \nabla_{y_i} g_i\left(\mathbf{0},\widetilde{y}_i^{(\mathbf{0})}\right) = 0 \tag{14d}$$

$$0 \leq -g_i\left(\mathbf{0},\widetilde{y}_i^{(\mathbf{0})}\right) \perp \widetilde{\lambda}_i^{(\mathbf{0})} \geq 0 \tag{14e}$$

$$f_i\left(\mathbf{1},y_i^{(\mathbf{1})},y_{-i}\right) + \kappa_i^{(\mathbf{1})} - \zeta_i^{(\mathbf{1})} - \kappa_i^{(\mathbf{0})} + \zeta_i^{(\mathbf{0})} = f_i\left(\mathbf{0},y_i^{(\mathbf{0})},y_{-i}\right) \tag{14f}$$

$$\kappa_i^{(\mathbf{1})} + \zeta_i^{(\mathbf{1})} \leq x_i\,\widetilde{K} \tag{14g}$$

$$\kappa_i^{(\mathbf{0})} + \zeta_i^{(\mathbf{0})} \leq \left(1-x_i\right)\widetilde{K} \tag{14h}$$

$$\widetilde{y}_i^{(\mathbf{0})} - x_i\,\widetilde{K} \leq y_i \leq \widetilde{y}_i^{(\mathbf{0})} + x_i\,\widetilde{K} \tag{14i}$$

$$\widetilde{y}_i^{(\mathbf{1})} - \left(1-x_i\right)\widetilde{K} \leq y_i \leq \widetilde{y}_i^{(\mathbf{1})} + \left(1-x_i\right)\widetilde{K} \tag{14j}$$

$$x_i \in \{0,1\},\, \left(y_i,\widetilde{y}_i^{(\overline{\mathbf{x}}_i)}\right) \in \mathbb{R}^{3m},\, \left(\lambda_i^{(\overline{\mathbf{x}}_i)},\kappa_i^{(\overline{\mathbf{x}}_i)},\zeta_i^{(\overline{\mathbf{x}}_i)}\right) \in \mathbb{R}_+^{2k+4}$$

The power market application (Section 4)

The generator's optimization problem

Each generator $i \in I$ seeks to maximize her profits from generating and selling electricity over the time horizon $t \in T$:

$$\min_{x_{ti}, y_{ti}, z_{ti}^{\mathrm{on}}, z_{ti}^{\mathrm{off}}} \quad -p_{tn(i)} y_{ti} + c_i^G y_{ti} + c_i^{\mathrm{on}} z_{ti}^{\mathrm{on}} + c_i^{\mathrm{off}} z_{ti}^{\mathrm{off}} \tag{16a}$$

$$\text{s.t.} \quad x_{ti} g_i^{min} \leq y_{ti} \leq x_{ti} g_i^{max} \quad \left(\alpha_{ti}^{\mathrm{on}}, \beta_{ti}^{\mathrm{on}}\right) \tag{16b}$$

$$x_{ti} - x_{(t-1)i} = z_{ti}^{\mathrm{on}} - z_{ti}^{\mathrm{off}} \tag{16c}$$

$$x_{ti} \in \{0,1\}, \quad y_{ti}, z_{ti}^{\mathrm{on}}, z_{ti}^{\mathrm{off}} \in \mathbb{R}_+$$

Demand for electricity and network constraints

The other side of the market is a player seeking to maximize the welfare (utility) of consumers while guaranteeing feasibility of the transmission system, given locational marginal prices p_{tn}. A set of units $j \in J$ consume electricity (load d_{tj}), each located at a specific node $n(j)$. The sets I_n and J_n are the generators and load units located at node n, respectively. There are a set of power lines $l \in L$ connecting the nodes; the direct-current load flow (DCLF) characteristics are captured using the susceptance matrix B_{nm} (node-to-node) and network transfer matrix H_{nl} (node-to-line mapping). This approach is equivalent to a power transfer distribution factor (PTDF) matrix.

$$\min_{d_{tj}, \delta_{tn}} \quad \sum_{j \in J} p_{tn(j)} \left(d_{tj} + \sum_{m \in N} B_{nm} \delta_{tm} \right) - u_{tj}^D d_{tj} \tag{18a}$$

$$\text{s.t.} \quad d_{tj}^{max} - d_{tj} \geq 0 \quad (\nu_{tj}) \tag{18b}$$

$$f_l^{max} - \sum_{n \in N} H_{ln} \delta_{tn} \geq 0 \quad (\mu_{tl}^+) \tag{18c}$$

$$f_l^{max} + \sum_{n \in N} H_{ln} \delta_{tn} \geq 0 \quad (\mu_{tl}^-) \tag{18d}$$

$$\pi - \delta_{tn} \geq 0 \quad (\xi_{tn}^+) \tag{18e}$$

$$\pi + \delta_{tn} \geq 0 \quad (\xi_{tn}^-) \tag{18f}$$

$$\delta_{t\hat{n}} = 0 \quad (\gamma_t) \tag{18g}$$

Sets & Mappings

$n, m \in N$	... nodes
$t \in T$	... time step, hours
$i \in I$	... generators, power plant units
$j \in J$	... load, demand units
$l \in L$	... power lines
$i \in I_n, j \in J_n$	... generator/load unit mapping to node n
$n(i), n(j)$	... node mapping to generator i/load unit j
$\phi \in \Phi$	... set of dispatch options (schedules) for each generator
$t \in T_\phi$	... hours in which a generator is active in dispatch option ϕ

Primal variables

x_{ti}	... on/off decision for generator i in hour t
$z_{ti}^{\mathrm{on}}, z_{ti}^{\mathrm{off}}$	... inter-temporal start-up/shut-down decision
y_{ti}	... actual generation by generator i in hour t
y_{ti}^{on}	... generation if binary variable is fixed at $\overline{\mathbf{x}}_{\mathbf{i}}$
d_{tj}	... demand by unit j in hour t
δ_{tn}	... voltage angle

Dual variables

$\alpha_{ti}^{\mathrm{on}}, \beta_{ti}^{\mathrm{on}}$	... dual to minimum activity/maximum generation capacity
ν_{tj}	... dual to maximum load constraint
$\mu_{tl}^{+}, \mu_{tl}^{-}$	... dual to voltage angle band constraints
$\xi_{tn}^{+}, \xi_{tn}^{-}$	... dual to thermal line capacity constraints
γ_t	... dual to slack bus constraints

Switch and compensation variables

p_{tn}	... locational marginal price
$\kappa_{ti}^{\mathrm{on}}, \kappa_{ti}^{\mathrm{off}}$	... switch value (defined per time step)
ζ_i	... compensation payment (defined over entire time horizon)

Parameters

c_i^G	... linear generation costs
$c_i^{\mathrm{on}}, c_i^{\mathrm{off}}$	... start-up/shut-down costs
c_ϕ^D	... commitment costs in dispatch option ϕ (start-up, shut-down)
g_i^{min}	... minimum activity level if power plant is online
g_i^{max}	... maximum generation capacity
x_i^{init}	... power plant status at start of model horizon ($t = 0$)
u_{tj}^D	... utility of demand unit j for using electricity
d_{tj}^{max}	... maximum load of unit j
f_l^{max}	... thermal capacity of power line l
B_{nk}, H_{lk}	... line/node susceptance/network transfer matrices

Table 2: Notation for the nodal power market problem

The bi-level multi-objective program

$$\min \sum_{t \in T} \left[\sum_{i \in I} c_i^G y_{ti} + c_i^{\mathrm{on}} z_{ti}^{\mathrm{on}} + c_i^{\mathrm{off}} z_{ti}^{\mathrm{off}} - \sum_{j \in J} u_{tj}^D d_{tj} \right] + \sum_{i \in I} \zeta_i \tag{20a}$$

$$\text{s.t.} \quad \sum_{j \in J_n} d_{tj} - \sum_{i \in I_n} y_{ti} + \sum_{m \in N} B_{nm} \delta_{tm} = 0 \tag{20b}$$

$$0 \leq -u_{tj}^D + p_{tn(j)} + \nu_{tj} \quad \perp \quad d_{tj} \geq 0 \tag{19a}$$

$$0 = \sum_{m \in N} B_{mn} p_{tm} + \sum_{l \in L} H_{ln}\big(\mu_{tl}^+ - \mu_{tl}^-\big)$$
$$+\xi_{tn}^+ - \xi_{tn}^- - \begin{cases} \gamma_t & \text{if } n = \hat{n} \\ 0 & \text{else} \end{cases} \quad , \quad \delta_{tn} \text{ (free)} \tag{19b}$$

$$0 \leq d_{tj}^{max} - d_{tj} \quad \perp \quad \nu_{tj} \geq 0 \tag{19c}$$

$$0 \leq f_l^{max} - \sum_{n \in N} H_{ln} \delta_{tn} \quad \perp \quad \mu_{tl}^+ \geq 0 \tag{19d}$$

$$0 \leq f_l^{max} + \sum_{n \in N} H_{ln} \delta_{tn} \quad \perp \quad \mu_{tl}^- \geq 0 \tag{19e}$$

$$0 \leq \pi - \delta_{tn} \quad \perp \quad \xi_{tn}^+ \geq 0 \tag{19f}$$

$$0 \leq \pi + \delta_{tn} \quad \perp \quad \xi_{tn}^- \geq 0 \tag{19g}$$

$$0 = \delta_{t\hat{n}} \quad , \quad \gamma_t \text{ (free)} \tag{19h}$$

$$0 = c_i^G - p_{tn(i)} + \beta_{ti}^{\mathrm{on}} - \alpha_{ti}^{\mathrm{on}} \quad , \quad y_{ti}^{\mathrm{on}} \text{ (free)} \tag{17a}$$

$$0 \leq -g_{ti}^{min} + y_{ti}^{\mathrm{on}} \quad \perp \quad \alpha_{ti}^{\mathrm{on}} \geq 0 \tag{17b}$$

$$0 \leq g_{ti}^{max} - y_{ti}^{\mathrm{on}} \quad \perp \quad \beta_{ti}^{\mathrm{on}} \geq 0 \tag{17c}$$

$$x_{(t-1)i} + z_{ti}^{\mathrm{on}} - z_{ti}^{\mathrm{off}} = x_{ti} \tag{21a}$$

$$\beta_{ti}^{\mathrm{on}} g_{ti}^{max} - \alpha_{ti}^{\mathrm{on}} g_{ti}^{min} - \kappa_{ti}^{\mathrm{on}} + \kappa_{ti}^{\mathrm{off}} = 0 \tag{21b}$$

$$|\kappa_{ti}^{\mathrm{on}}| \leq x_{ti} \, \widetilde{K} \tag{21c}$$

$$|\kappa_{ti}^{\mathrm{off}}| \leq (1 - x_{ti}) \, \widetilde{K} \tag{21d}$$

$$\sum_{t \in T} \left[\kappa_{ti}^{\mathrm{on}} - c_i^{\mathrm{on}} z_{ti}^{\mathrm{on}} - c_i^{\mathrm{off}} z_{ti}^{\mathrm{off}} \right] + \zeta_i \geq$$
$$\sum_{t \in T_\phi} \left[\beta_{ti}^{\mathrm{on}} g_{ti}^{max} - \alpha_{ti}^{\mathrm{on}} g_{ti}^{min} \right] - c_{\phi i}^D \quad \forall \ \phi \in \Phi \tag{21e}$$

$$0 \leq y_{ti} \leq x_{ti} \, g_{ti}^{max} \tag{21f}$$

$$y_{ti}^{\mathrm{on}} - (1 - x_{ti}) \, g_{ti}^{max} \leq y_{ti} \leq y_{ti}^{\mathrm{on}} + (1 - x_{ti}) \, g_{ti}^{max} \tag{21g}$$