## CSSS 510: Lab 6

Multinomial Logit

#### 0. Agenda

- 1. Deriving a likelihood function for the multinomial model
- 2. Fitting a multinomial logit model using optim() and glm()
- 3. Simulating predicted values and confidence intervals
- 4. Evaluating goodness of fit

Recall from lecture the multinomial logit model:

$$Pr(y_i = j | \mathbf{x_i}) = \frac{\exp(\mathbf{x_i}\beta_j)}{\sum_{l=1}^{M} \exp(\mathbf{x_i}\beta_l)}$$

If we make the identifying assumption that  $\beta_1=0$ , then we have

For 
$$j=1$$
 
$$\Pr(y_i=1|\pmb{x_i}) = \frac{1}{1+\sum_{l=2}^{M} \exp(\pmb{x_i}\pmb{\beta_l})}$$
 For  $j>2$  
$$\Pr(y_i=j|\pmb{x_i}) = \frac{\exp(\pmb{x_i}\pmb{\beta_j})}{1+\sum_{l=2}^{M} \exp(\pmb{x_i}\pmb{\beta_l})}$$

$$Pr(y_i = j | \mathbf{x_i}) = \frac{\exp(\mathbf{x_i}\beta_j)}{\sum_{l=1}^{M} \exp(\mathbf{x_i}\beta_l)}$$

In many ways, this is the most tractable model we've seen so far.

1. We assume that  $\Pr(y_i = j | x)$  is a function of the linear combination  $x_i \beta_j$ . One key difference between this and previous models is that  $\beta_j$  differs for *each outcome*. We therefore have a set of  $\beta$  for each outcome category.

We then take two simple steps.

- 2. We take the exponential of  $x_i\beta_j$  to ensure that the probabilities are nonnegative.
- 3. We then divide each  $\exp(\mathbf{x}_i\beta_j)$  and divide it by  $\sum_{l=2}^{M}\exp(\mathbf{x}_l\beta_l)$  to make the probabilities sum to 1.

Our model looks very similar to logit but with more than two categories. Indeed, multinomial logit can be thought of as a way to estimate separate logits for all categories all at once.

Like ordered probit, the likelihood function for multinomial logit finds the  $\beta_j$  that make the observed data most likely. That is, it maximizes the likelihood that  $y_{ij}$  falls into category j. All other categories  $\neq$  j are irrelevant.

$$\mathcal{L}(\beta_2, ..., \beta_m | \mathbf{y}, \mathbf{X}) = \prod_{i=1}^N \prod_{j=1}^M p_{ij}^{y_{ij}}$$

$$\mathcal{L}(\beta_2, ..., \beta_m | \mathbf{y}, \mathbf{X}) = \prod_{i=1}^N \prod_{j=1}^M \left[ \frac{\exp(\mathbf{x}_i \beta_j)}{1 + \sum_{l=2}^M \exp(\mathbf{x}_i \beta_l)} \right]^{y_{ij}}$$

$$\mathcal{L}(\beta_2, ..., \beta_m | \mathbf{y}, \mathbf{X}) = \sum_{i=1}^N \sum_{i=1}^M y_{ij} \log \frac{\exp(\mathbf{x}_i \beta_j)}{1 + \sum_{l=2}^M \exp(\mathbf{x}_i \beta_l)}$$

Our coefficients can be interpreted in a similar way as logit coefficients:  $\beta$  is the change in the log of the odds of one outcome versus another outcome.

Suppose that we have outcome m and outcome n. We will then have a  $\beta$  that corresponds to the change in the log of the odds of outcome m versus outcome n.

$$\log \frac{\Pr(y = m | \mathbf{x}_c, \beta_2, ..., \beta_M)}{\Pr(y = n | \mathbf{x}_c, \beta_2, ..., \beta_M)} = \mathbf{x}_c(\beta_m - \beta_n)$$

If the kth covariate increases by 1, then the log of the odds of category m versus category n increases by the *difference* of their coefficients.

### 2. Fitting a multinomial logit model

```
rm(list=ls())
# Load data and libraries
library(simcf)
                          # for mlogit simulators
library(tile)
                         # for graphics
## Loading required package: grid
library(RColorBrewer) # for colors
library(MASS)
                         # for murnorm()
library(nnet)
                          # for multinom()
# homemade extractor function for multinom() coef's
coef.multinom <- function(x) {
 nlevel <- length(mlogit.result$lev)</pre>
 ncoef <- length(mlogit.result$coefnames)</pre>
 coef <- x$wts[(ncoef+2):length(x$wts)]</pre>
 coef[-((0:(nlevel-2))*(ncoef+1) + 1)]
# Load data (in simcf library)
data(gator)
gator <- as.data.frame(cbind(food, size, female))</pre>
```

### 2. Fitting a multinomial logit model

```
# Estimate MNL using the nnet library
model <- food ~ size + female
mlogit.result <- multinom(model, Hess=TRUE)
## # weights: 12 (6 variable)
## initial value 64.818125
## iter 10 value 48 292498
## final value 48.291021
## converged
pe <- coef(mlogit.result)
vc <- solve(mlogit.result$Hess)
                               # var-cov matrix
se <- sqrt(diag(vc))
pe
## [1] 4.8969032 -2.5259623 -0.7899437 -1.9471694 0.1337791 0.3817674
Se
## 2:(Intercept)
                  2:size 2:female 3:(Intercept)
                                                              3:size
      1.7063258
                   0.8475594
                               0.7122227 1.5305908 0.5181170
##
      3:female
##
##
   0.9083499
```

### 2. Fitting a multinomial logit model

```
gator$foodnm[gator$food==1]<-"Invertebrates"
gator$foodnm[gator$food==2]<-"Fish"
gator$foodnm[gator$food==3]<-"Other"
model <- foodnm ~ size + female
mlogit.result <- multinom(model, data=gator, Hess=TRUE)</pre>
## # weights: 12 (6 variable)
## initial value 64.818125
## iter 10 value 48.291150
## final value 48,291021
## converged
mlogit.result
## Call:
## multinom(formula = model, data = gator, Hess = TRUE)
##
## Coefficients:
                 (Intercept) size female
##
## Invertebrates -4.896281 2.525668 0.7897795
## Other
         -6.843614 2.659585 1.1714407
##
## Residual Deviance: 96.58204
## ATC: 108.582
```

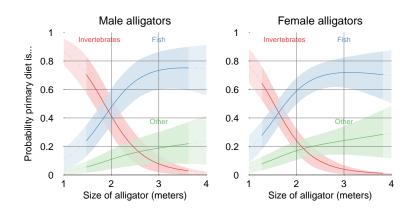
```
# Simulate parameters from predictive distributions
sims <- 10000
simbetas <- mvrnorm(sims,pe,vc)</pre>
                                    # draw parameters, using MASS::murnorm
simB <- array(NA, dim = c(sims,3,2)) # re-arrange simulates to array format
simB[,,1] \leftarrow simbetas[,1:3]
                                    # for MNL simulation
simB[,,2] \leftarrow simbetas[,4:6]
# Create full factorial set of counterfactuals
sizerange <- seq(1,4,by=0.1) # range of counterfactual sizes
femalerange \leftarrow c(0,1)
                              # range of counterfactual sexes
xhyp1 <- cfFactorial(size = sizerange, female = femalerange)</pre>
# Simulate expected probabilities with 68% CI
mlogit.ev1 <- mlogitsimev(xhyp1, simB, ci=0.68)</pre>
```

```
# Plot expected values for all combinations of size and sex in 2 plots
# Get 3 colors
cols <- brewer.pal(3, "Set1")</pre>
# Create one trace for each predicted category of the response, and each sex
trace1a <- lineplot(x=xhyp1$x$size[xhyp1$x$female==0],
                   v=mlogit.ev1$pe[xhyp1$x$female==0,1],
                   lower=mlogit.ev1$lower[xhvp1$x$female==0.1.].
                   upper=mlogit.ev1$upper[xhyp1$x$female==0,1,],
                   ci=list(mark="shaded"),
                   extrapolate=list(data=cbind(size.female).
                                   cfact=xhvp1$x\xhvp1$x$female==0.].
                                   omit.extrapolated=FALSE),
                   col=cols[1].
                   plot=1
trace2a <- lineplot(x=xhvp1$x$size[xhvp1$x$female==0].
                   v=mlogit.ev1$pe[xhyp1$x$female==0,2],
                   lower=mlogit.ev1$lower[xhyp1$x$female==0,2,],
                   upper=mlogit.ev1$upper[xhvp1$x$female==0.2.].
                   ci=list(mark="shaded").
                   extrapolate=list(data=cbind(size,female),
                                   cfact=xhvp1$x[xhvp1$x$female==0.].
                                   omit.extrapolated=FALSE).
                   col=cols[3],
                   plot=1
```

```
trace3a <- lineplot(x=xhyp1$x$size[xhyp1$x$female==0],
                    v=mlogit.ev1$pe[xhvp1$x$female==0.3].
                    lower=mlogit.ev1$lower[xhyp1$x$female==0,3,],
                    upper=mlogit.ev1$upper[xhyp1$x$female==0,3,],
                    ci=list(mark="shaded"),
                    extrapolate=list(data=cbind(size,female),
                                     cfact=xhyp1$x[xhyp1$x$female==0,],
                                     omit.extrapolated=FALSE).
                    col=cols[2].
                    plot=1
trace4a <- lineplot(x=xhyp1$x$size[xhyp1$x$female==1],
                    v=mlogit.ev1$pe[xhyp1$x$female==1,1],
                    lower=mlogit.ev1$lower[xhvp1$x$female==1.1.].
                    upper=mlogit.ev1$upper[xhyp1$x$female==1,1,],
                    ci=list(mark="shaded"),
                    extrapolate=list(data=cbind(size.female).
                                     cfact=xhyp1$x[xhyp1$x$female==1,],
                                     omit.extrapolated=FALSE),
                    col=cols[1],
                    plot=2
```

```
trace5a <- lineplot(x=xhyp1$x$size[xhyp1$x$female==1],
                    v=mlogit.ev1$pe[xhvp1$x$female==1,2].
                    lower=mlogit.ev1$lower[xhyp1$x$female==1,2,],
                    upper=mlogit.ev1$upper[xhyp1$x$female==1,2,],
                    ci=list(mark="shaded"),
                    extrapolate=list(data=cbind(size,female),
                                      cfact=xhyp1$x[xhyp1$x$female==1,],
                                      omit.extrapolated=FALSE).
                    col=cols[3].
                    plot=2
trace6a <- lineplot(x=xhyp1$x$size[xhyp1$x$female==1],</pre>
                    v=mlogit.ev1$pe[xhyp1$x$female==1,3],
                    lower=mlogit.ev1$lower[xhvp1$x$female==1.3.].
                    upper=mlogit.ev1$upper[xhyp1$x$female==1,3,],
                    ci=list(mark="shaded"),
                    extrapolate=list(data=cbind(size.female).
                                      cfact=xhyp1$x[xhyp1$x$female==1,],
                                      omit.extrapolated=FALSE),
                    col=cols[2],
                    plot=2
```

```
linelabels <- textTile(labels=c("Invertebrates".
                                "Fish".
                                "Other"),
                       x = c(1.75. 3.
                                                   3).
                       v = c(0.95, 0.95,
                                                  0.375).
                       col=c(cols[1], cols[2], cols[3]),
                       cex = 0.75,
                       plot=c(1,2)
at.x <-c(1,2,3,4)
at.v \leftarrow c(0,0.2,0.4,0.6,0.8,1)
# Plot traces using tile
file <- "gatorsEV"
tile(trace1a.
     trace2a.
    trace3a.
    trace4a.
    trace5a.
    trace6a,
    linelabels.
    RxC = c(1.2).
    limits = c(1,4,0,1),
     output = list(outfile=file, width=7),
     xaxis = list(at=at.x).
     yaxis = list(at=at.y, major=FALSE),
     xaxistitle = list(labels="Size of alligator (meters)"),
     vaxistitle = list(type="first", labels="Probability primary diet is...", x=0.1).
     plottitle = list(labels=c("Male alligators", "Female alligators"), v=1),
     gridlines = list(type="xy")
```



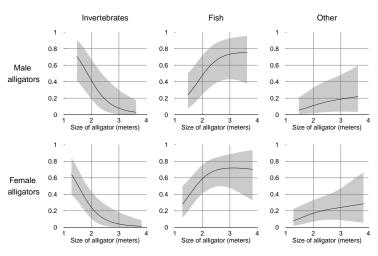
```
## Alternate version with 95% CIs and one plot per trace (6 plots)
# Simulate expected probabilities with 95% CI
mlogit.ev1 <- mlogitsimev(xhvp1, simB, ci=0.95)
# Create one trace for each predicted category of the response, and each sex
trace1a <- lineplot(x=xhvp1$x$size[xhvp1$x$female==0].
                    v=mlogit.ev1$pe[xhvp1$x$female==0.1].
                    lower=mlogit.ev1$lower[xhyp1$x$female==0,1,],
                    upper=mlogit.ev1$upper[xhyp1$x$female==0,1,],
                    ci=list(mark=c("shaded", "dashed")).
                    extrapolate=list(data=cbind(size,female),
                                     cfact=xhyp1$x[xhyp1$x$female==0,],
                                     omit.extrapolated=TRUE).
                    plot=1
trace2a <- lineplot(x=xhyp1$x$size[xhyp1$x$female==0],</pre>
                    v=mlogit.ev1$pe[xhyp1$x$female==0,2],
                    lower=mlogit.ev1$lower[xhyp1$x$female==0,2,],
                    upper=mlogit.ev1$upper[xhyp1$x$female==0,2,],
                    ci=list(mark="shaded", "dashed"),
                    extrapolate=list(data=cbind(size,female),
                                     cfact=xhyp1$x[xhyp1$x$female==0,],
                                     omit.extrapolated=TRUE).
                    plot=3
```

```
trace3a <- lineplot(x=xhyp1$x$size[xhyp1$x$female==0],</pre>
                    v=mlogit.ev1$pe[xhyp1$x$female==0,3],
                    lower=mlogit.ev1$lower[xhyp1$x$female==0,3,],
                    upper=mlogit.ev1$upper[xhvp1$x$female==0.3.].
                    ci=list(mark="shaded", "dashed"),
                    extrapolate=list(data=cbind(size,female),
                                      cfact=xhvp1$x\xhvp1$x$female==0.].
                                      omit.extrapolated=TRUE).
                    plot=2
trace4a <- lineplot(x=xhyp1$x$size[xhyp1$x$female==1],
                    v=mlogit.ev1$pe[xhyp1$x$female==1,1],
                    lower=mlogit.ev1$lower[xhyp1$x$female==1,1,],
                    upper=mlogit.ev1$upper[xhyp1$x$female==1,1,],
                    ci=list(mark="shaded", "dashed"),
                    extrapolate=list(data=cbind(size,female),
                                      cfact=xhyp1$x[xhyp1$x$female==1,],
                                      omit.extrapolated=TRUE),
                    plot=4
```

```
trace5a <- lineplot(x=xhyp1$x$size[xhyp1$x$female==1],</pre>
                    v=mlogit.ev1$pe[xhyp1$x$female==1,2],
                    lower=mlogit.ev1$lower[xhyp1$x$female==1,2,],
                    upper=mlogit.ev1$upper[xhvp1$x$female==1.2.].
                    ci=list(mark="shaded", "dashed"),
                    extrapolate=list(data=cbind(size,female),
                                      cfact=xhvp1$x\xhvp1$x$female==1.].
                                      omit.extrapolated=TRUE).
                    plot=6
trace6a <- lineplot(x=xhyp1$x$size[xhyp1$x$female==1],
                    v=mlogit.ev1$pe[xhyp1$x$female==1,3],
                    lower=mlogit.ev1$lower[xhyp1$x$female==1,3,],
                    upper=mlogit.ev1$upper[xhyp1$x$female==1,3,],
                    ci=list(mark="shaded", "dashed"),
                    extrapolate=list(data=cbind(size,female),
                                      cfact=xhyp1$x[xhyp1$x$female==1,],
                                      omit.extrapolated=TRUE),
                    plot=5
```

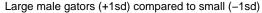
```
# Plot traces using tile
file <- "gatorsEVsep95"
tile(trace1a.
    trace2a,
    trace3a.
    trace4a.
    trace5a.
    trace6a.
    RxC = c(2.3).
    limits = c(1,4,0,1),
     output = list(file=file, width=10),
     xaxis = list(at=at.x).
     yaxis = list(at=at.y, major=FALSE),
     xaxistitle = list(labels="Size of alligator (meters)"),
     maintitle = list(labels="Probability primary diet is...").
     rowtitle = list(labels=c("Male\n alligators", "Female\n alligators"), cex=1.25),
     columntitle = list(labels=c("Invertebrates", "Fish", "Other"), cex=1.25),
     height=list(columntitle=5).
     width=list(rowtitle=1.5),
     gridlines = list(type="xy")
```

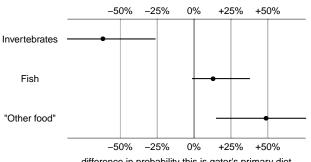




```
# Create a specific comparison for first diffs and relative risks
# Load data (in simcf library)
data(gator)
gator <- as.data.frame(cbind(food, size, female))</pre>
model <- food ~ size + female
xhyp2 <- cfMake(model, gator, nscen=1)
# Scenario 1: Large vs small (male mu + 1sd vs male mu - 1sd), holding male fixed
xhyp2 <- cfName(xhyp2, "Large vs Small Males", scen=1)</pre>
xhyp2 <- cfChange(xhyp2, "size",</pre>
                  x=mean(gator$size[gator$female==0]) + sd(gator$size[gator$female==0]),
                  xore=mean(gator$size[gator$female==0]) - sd(gator$size[gator$female==0]).
                  scen=1)
xhyp2 <- cfChange(xhyp2, "female", x=0, xpre=0, scen=1)</pre>
# Simulate first differences with 95\% CI
mlogit.fd1 <- mlogitsimfd(xhyp2, simB, ci=0.95)
# Simulate relative risks with 95\% CT
mlogit.rr1 <- mlogitsimrr(xhyp2, simB, ci=0.95)
```

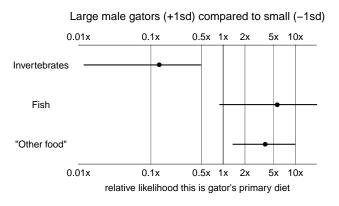
```
# Make ropeladder plot of First Differences
# Make trace of FDs, large vs small males, all categories
traceFD <- ropeladder(x=mlogit.fd1$pe,
                     lower=mlogit.fd1$lower,
                     upper=mlogit.fd1$upper,
                     labels=c("Invertebrates",
                              "Fish",
                              "\"Other food\"").
                     size=0.65.
                     lex=1.75,
                     lineend="square".
                     plot=1
# Make reference line trace for first diffs (at 0)
vertmarkFD <- linesTile(x=c(0,0), y=c(0,1), plot=1)</pre>
# Set tick marks for x axis
xat \leftarrow c(-0.5, -0.25, 0, 0.25, 0.5)
xlab <- c("-50\%", "-25\%", "0\%", "+25\%", "+50\%")
# Make plot with tile
file <- "gatorsFD"
tile(traceFD, vertmarkFD,
    xaxis=list(at=xat, labels=xlab).
    topaxis=list(add=TRUE, at=xat, labels=xlab),
    plottitle=list(labels="Large male gators (+1sd) compared to small (-1sd)"),
    xaxistitle=list(labels="difference in probability this is gator's primary diet").
    width=list(null=4),
    height=list(xaxistitle=3, plottitle=4),
    gridlines=list(type="xt").
    output=list(file=file, width=7)
```





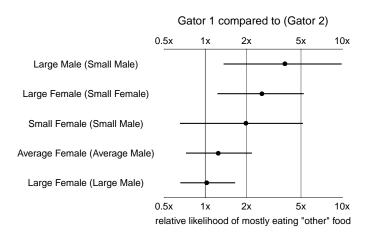
difference in probability this is gator's primary diet

```
*******************************
# Make ropeladder plot of Relative Risks
# Make trace of RRs. large vs small males, all categories
traceRR <- ropeladder(x=mlogit.rr1$pe,</pre>
                      lower=mlogit.rr1$lower,
                      upper=mlogit.rr1$upper,
                      labels=c("Invertebrates".
                               "Fish",
                               "\"Other food\""),
                      size=0.65.
                      lex=1.75.
                      lineend="square",
                      plot=1
# Make reference line trace for relative risks (at 1)
vertmarkRR <- linesTile(x=c(1,1), y=c(0,1), plot=1)</pre>
# Set tick marks for x axis
xat <- c(0.01, 0.1, 0.5, 1, 2, 5, 10)
# Make plot with tile
file <- "gatorsRR"
tile(traceRR, vertmarkRR,
     xaxis=list(log=TRUE, at=xat, labels=paste0(xat,"x")),
     topaxis=list(add=TRUE, log=TRUE, at=xat, labels=paste0(xat, "x")),
     plottitle=list(labels="Large male gators (+1sd) compared to small (-1sd)").
     xaxistitle=list(labels="relative likelihood this is gator's primary diet"),
     width=list(null=4),
     height=list(xaxistitle=3, plottitle=4),
     gridlines=list(type="xt"),
     output=list(file=file, width=7)
```



```
# Create several comparisons for first diffs and relative risks
xhyp3 <- cfMake(model, gator, nscen=5)
# Scenario 1: Large vs small (male mu + 1sd vs male mu - 1sd), holding male fixed
xhyp3 <- cfName(xhyp3, "Large Male (Small Male)", scen=1)
xhyp3 <- cfChange(xhyp3, "size",
                 x=mean(gator$size[gator$female==0]) + sd(gator$size[gator$female==0]).
                 xpre=mean(gator$size[gator$female==0]) - sd(gator$size[gator$female==0]),
                 scen=1)
xhvp3 <- cfChange(xhvp3, "female", x=0, xpre=0, scen=1)</pre>
# Scenario 2: Large vs small (female mu + 1sd vs female mu - 1sd), holding female fixed
xhvp3 <- cfName(xhvp3, "Large Female (Small Female)", scen=2)
xhvp3 <- cfChange(xhvp3, "size",
                 x=mean(gator$size[gator$female==1]) + sd(gator$size[gator$female==1]),
                 xpre=mean(gator$size[gator$female==1]) - sd(gator$size[gator$female==1]).
                 scen=2)
xhyp3 <- cfChange(xhyp3, "female", x=1, xpre=1, scen=2)</pre>
# Scenario 3: Female vs male holding size fixed at mean of all gators
xhyp3 <- cfName(xhyp3, "Average Female (Average Male)", scen=3)
xhyp3 <- cfChange(xhyp3, "size",
                 x=mean(gator$size).
                 xpre=mean(gator$size),
                 scen=3)
xhvp3 <- cfChange(xhvp3, "female", x=1, xpre=0, scen=3)</pre>
```

```
# Scenario 4: Female vs male holding size fixed at high of all gators
xhyp3 <- cfName(xhyp3, "Large Female (Large Male)", scen=4)
xhvp3 <- cfChange(xhvp3, "size",
                  x=mean(gator$size) + sd(gator$size).
                  xpre=mean(gator$size) + sd(gator$size),
                  scen=4)
xhvp3 <- cfChange(xhvp3, "female", x=1, xpre=0, scen=4)
# Scenario 5: Female vs male holding size fixed at low of all gators
xhyp3 <- cfName(xhyp3, "Small Female (Small Male)", scen=5)
xhyp3 <- cfChange(xhyp3, "size",
                  x=mean(gator$size) - sd(gator$size),
                  xpre=mean(gator$size) - sd(gator$size).
                  scen=5)
xhyp3 <- cfChange(xhyp3, "female", x=1, xpre=0, scen=5)</pre>
# Simulate first differences with 95\% CI
mlogit.fd2 <- mlogitsimfd(xhyp3, simB, ci=0.95)
# Simulate relative risks with 95\% CI
mlogit.rr2 <- mlogitsimrr(xhyp3, simB, ci=0.95)
sorted <- rev(order(mlogit.rr2$pe[.3]))
# Make trace of RRs, large vs small males, all categories
traceRR2 <- ropeladder(x=mlogit.rr2$pe[sorted.3].
                       lower=mlogit.rr2$lower[sorted,3,1],
                       upper=mlogit.rr2$upper[sorted,3,1],
                       labels=rownames(xhvp3$x)[sorted].
                       size=0.65.
                       lex=1.75,
                       lineend="square".
                       plot=1
```



#### 2. Evaluating Goodness of Fit

```
## A cross-validated goodness of fit test
## A simple leave-one-out cross-validation function for multinom; returns predicted probs
loocv <- function (obj, model, data) {
  ncat <- length(obj$lev)</pre>
  m <- nrow(data)
 form <- model
 loo <- matrix(NA, nrow=m, ncol=ncat)
 for (i in 1:m) {
    i.mlogit <- multinom(model, data=data[-i,])
    loo[i,] <- predict(i.mlogit, newdata = data[i,], type="probs")</pre>
  }
  100
predIS <- predict(mlogit.result, type="probs")</pre>
predCV <- loocv(mlogit.result, model, gator)</pre>
ncat <- 3
predIScat <- apply(predIS, 1, function(x, ncat) order(x)[ncat], ncat=ncat)</pre>
predCVcat <- apply(predCV, 1, function(x, ncat) order(x)[ncat], ncat=ncat)</pre>
pcpIS <- mean(predIScat==gator$food)
pcpCV <- mean(predCVcat==gator$food)
```

## 2. Evaluating Goodness of Fit

pcpIS

## [1] 0.2372881

pcpCV

## [1] 0.559322