4 21-01-25

4.1 Question 1

Problem:
Consider the scalar function

$$f(x,y) = x^2 e^{xy} + \sin(xy^2)$$

a) Compute the first-order partial derivatives

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}.$$

b) Compute the second-order partial derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}.$$

c) Show that the mixed partials are equal, i.e.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

4.2 Question 2

Problem:
Let

$$\Phi(x,y,z) = x^2 y + y^2 z + z^2 x$$

a) Compute the gradient

$$\nabla \Phi(x,y,z) := \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial z} \end{pmatrix}$$

b) Evaluate $\nabla \Phi(x,y,z)$ at the point $(1,2,3)$ to show that

$$\nabla \Phi(1,2,3) = \begin{pmatrix} 13 \\ 13 \\ 10 \end{pmatrix}$$

4.3 Question 3

Problem:
You have the scalar function

$$u(x, y, z) = \ln\big(x^2 + y^2 + z^2\big),$$

where x, y, z depend on a parameter t via

$$x(t) = \cos(t), \quad y(t) = \sin(t), \quad z(t) = t.$$

a) Find the partial derivatives
$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial z}$$

b) State the appropriate chain rule for $\frac{\mathrm{d}u}{\mathrm{d}t}$ in this context.

c) Apply your chain rule to calculate $\frac{\mathrm{d}u}{\mathrm{d}t}$. Show that your answer can be simplified to
$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{2t}{1 + t^2}$$

4.4 Question 4

Problem:
Consider the vector field

$$\mathbf{u}(x, y, z) = \begin{pmatrix} xy \\ yz \\ zx \end{pmatrix}$$

a) Compute the divergence
$$\nabla \cdot \mathbf{u}(x, y, z) := \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}.$$

b) Evaluate $\nabla \cdot \mathbf{u}(x, y, z)$ at the point $(0, 1, 0)$ to show
$$\nabla \cdot \mathbf{u}(0, 1, 0) = 1$$

c) In fluid mechanics, a flow is *incompressible* if the velocity field $\mathbf{u}$ satisfies $\nabla \cdot \mathbf{u} = 0$ everywhere. Is the flow defined by $\mathbf{u}$ as above incompressible? If not, can you construct a different flow $\mathbf{v}$ such that the combined flow $\mathbf{u} + \mathbf{v}$ is incompressible?

4.5 Question 5

Problem:
Consider the fluid flow defined by

$$\mathbf{v}(x, y, z) = \begin{pmatrix} e^x \sin y \\ xz \\ y + z^2 \end{pmatrix}.$$

a) Compute the divergence

$$\nabla \cdot \mathbf{v}(x, y, z).$$

b) Evaluate your expression at $(0, 0, 0)$.

c) Find all the points $(x, y, z) \in \mathbb{R}^3$ at which $\nabla \cdot \mathbf{v}(x, y, z) = 0$.

d) Compute the curl

$$\nabla \times \mathbf{v}(x, y, z).$$

Show that the result can be written as

$$\nabla \times \mathbf{v}(x, y, z) = \begin{pmatrix} 1 - x \\ 0 \\ z - e^x \sin y \end{pmatrix}.$$

e) For the vector field $\mathbf{v}$ above, verify explicitly the identity

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$