Multiplicative Latent Force Models

Using Neumann Series Expansions

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Highlights

We introduce the multiplicative latent force model

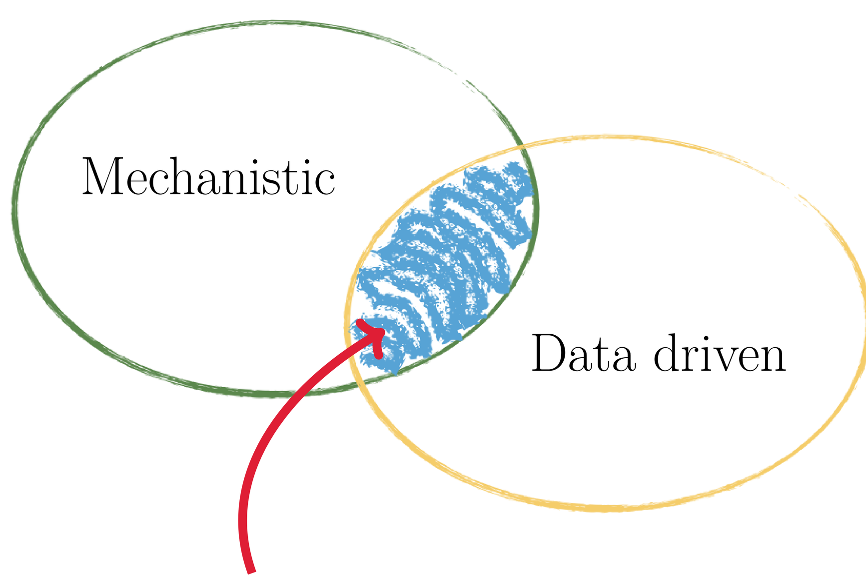
- A compromise between mechanistic and data driven approaches
- Providing controllable model geometry
- At the expense of tractable inference

To solve we introduce an approximation method

- Completing the model using a series expansion of the solution
- Exploiting the resulting conditional independence

Hybrid Modelling

Bayesian modelling of dynamic systems must often attempt to balance physical models with the appeal of the data driven paradigm. This can be problematic when a realistic model is hard to motivate, and yet data is sparse relative to the system complexity.



The linear latent force model [1] exists in this intersection by combining simple linear dynamics with a flexible additive Gaussian process (GP) force

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{S}\mathbf{g}(t)$$

in an attempt to construct a practical class of hybrid mechanistic models of dynamic systems. This system is easily solved and the trajectories are given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{S}\mathbf{g}(\tau)\,\mathrm{d}\tau.$$

We observe that the state is a linear transformation of the latent force and therefore leads to a tractable joint Gaussian distribution - as we consider richer models this feature is lost.

Multiplicative Latent Forces

We extend the latent force model to allow for multiplicative interactions between the latent forces and states

Simple linear flow

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A}_0 + \sum_{r=1}^{R}\mathbf{A}_r g_r(t)\right)\mathbf{x}(t).$$

Multiplicative GP modulation

This combines the flexibility of GP methods with the possibility to embed prior geometric knowledge, but in general it is no longer possible to form a simple expression for the state as a transformation of the latent force.

Neumann Series Method

We introduce an approximation method using the truncated series expansion obtained after M iterates of the map

$$\begin{aligned}\mathbf{x}_n \mapsto \mathbf{x}_{n+1} &= \mathbf{K}[\mathbf{g}]\mathbf{x}_n \\ &\triangleq \mathbf{x}_n(t_0) + \int_{t_0}^{t}\mathbf{A}(\tau)\mathbf{x}_n(\tau)\,\mathrm{d}\tau\end{aligned}$$

where

$$\mathbf{A}(t) = \mathbf{A}_0 + \sum_r \mathbf{A}_r \cdot g_r(t)$$

is a matrix-valued GP.

This operator is

- Linear in the state, conditional on the latent forces
- Linear in the latent forces, conditional on the state

Starting then from an initial GP 'guess', conditional on the latent force, we form successive Gaussian additive error updates

$$p(\mathbf{x}_i \mid \mathbf{x}_{i-1}, \mathbf{g}) = \mathcal{N}(\mathbf{x}_i \mid \mathbf{K}[\mathbf{g}]\mathbf{x}_{i-1}, \beta^{-1}I).$$

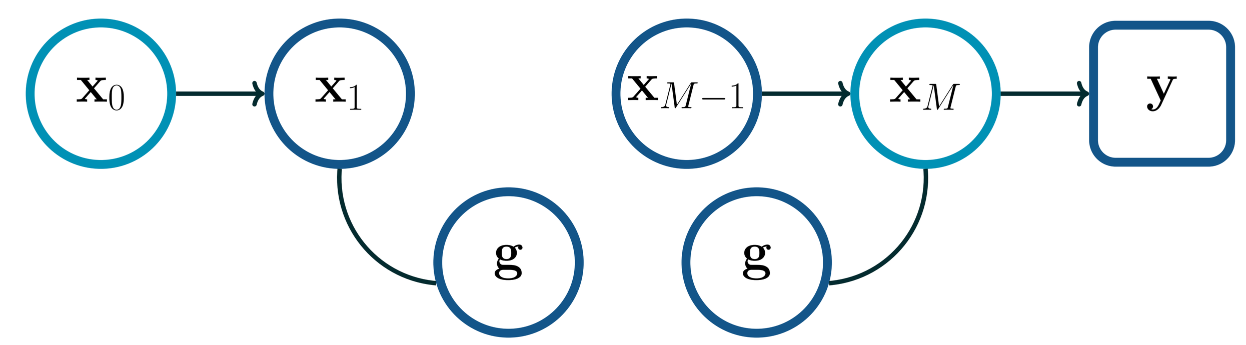
It is possible to marginalise out the states up to the truncation order, but the resulting covariance matrix is a degree $2M$ polynomial in the latent forces making carrying out inference for the latent forces challenging.

Conditional Inference

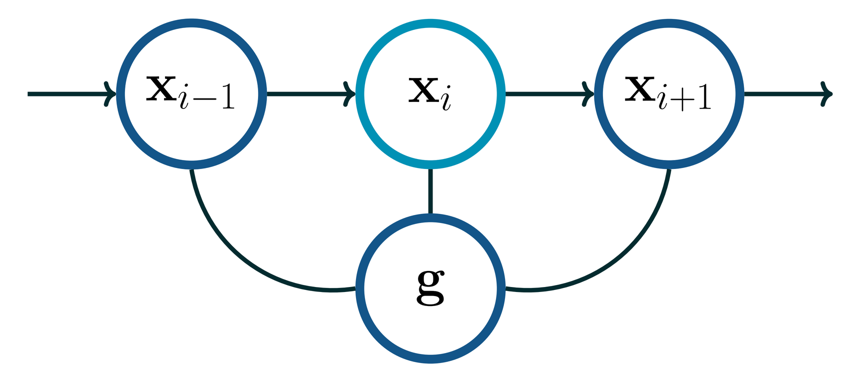
Instead of marginalising out the successive approximations we can retain them to form the (conditional) complete data likelihood

$$\begin{aligned}&\mathbf{p}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_M, \mathbf{y} \mid \mathbf{g}) \\ &\quad = p(\mathbf{x}_0)\left(\prod_{i=1}^{M} p(\mathbf{x}_i \mid \mathbf{x}_{i-1}, \mathbf{g})\right) p(\mathbf{y} \mid \mathbf{x}_M).\end{aligned}$$

This model is equivalent to a linear Gaussian dynamic system leading to tractable inference using Kalman filter methods. Furthermore, the latent forces have a Gaussian distribution after conditioning on the complete set of states and data so that the completed model is well suited to Gibbs and variational methods.



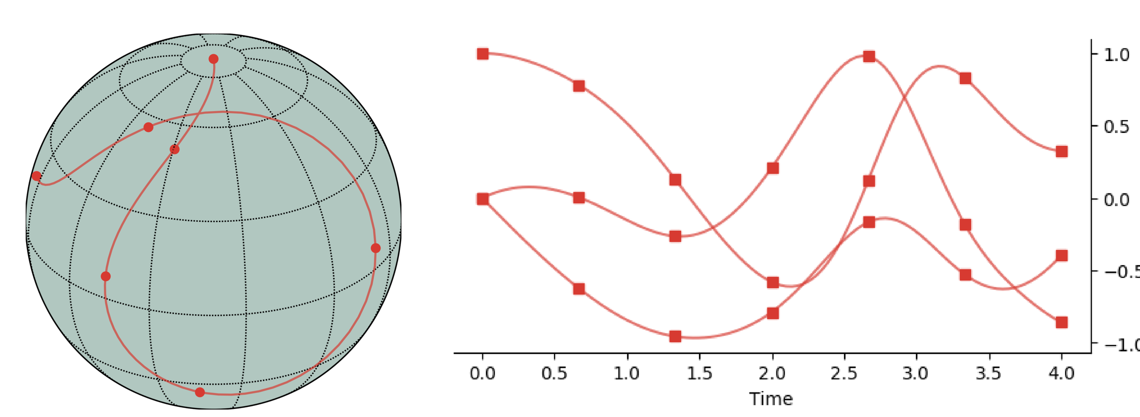
(a) Initial state update **(b)** Final state update



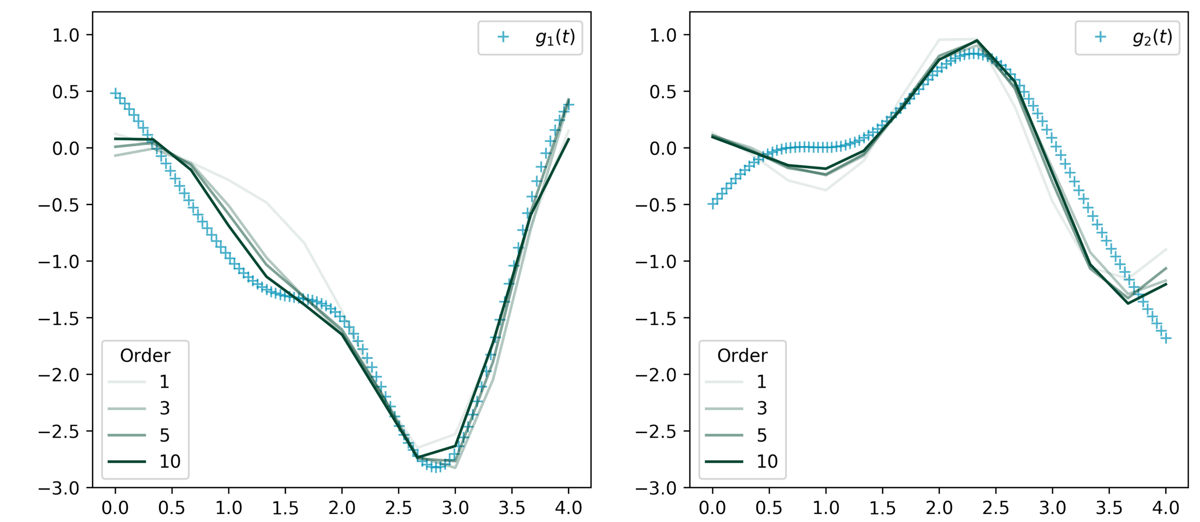
(c) Interior state update

Simulated Dynamic System on S^2

Choice of $\mathbf{A}_r$ allows strong topological constraints. If we chose elements of the Lie algebra $\mathfrak{so}(3)$ then we can simulate dynamic systems on the sphere $S^2 \in \mathbb{R}^3$.



EM Estimation



MAP estimates of the latent forces using data simulated from the model

$$\dot{\mathbf{x}}(t) = (\mathbf{A}_x + \mathbf{A}_y \cdot g_1(t) + \mathbf{A}_z \cdot g_2(t))\,\mathbf{x}(t)$$

where $\mathbf{A}_i$ is the infinitesimal rotation matrix around the i-coordinate axis. Estimation was carried out using the EM algorithm and the estimates converge quickly with respect to the expansion order.

Discussion

We have proposed an extention to the latent force model framework that uses multiplicative interactions to combine flexible modelling of dynamic systems with prior geometric constraints.

By using a series expansion approximation we are able to motivate a complete data model that allows for tractable conditional inference.

In future work we consider extension to the case where we attempt to learn the underlying manifold using a foliation of the latent space.

References

[1] Mauricio Alvarez, David Luengo, and Neil Lawrence. Latent force models. *Proceedings of Machine Learning Research*, pages 9–16, 2009.

[2] Arieh Iserles. On the method of Neumann series for highly oscillatory equations. *BIT Numerical Mathematics*, pages 473–488, 2004.