

Here is a fairly standard topological proof of a useful fact in topology:

Proposition 1. Suppose $p : \tilde{X} \rightarrow X$ is a covering space and that X has triangulation by 2-dimensional simplicial complex $|\mathcal{T}|$. In particular, there is a homeomorphism $|\mathcal{T}| \rightarrow X$. Then there exists 2-dimensional simplicial complex \mathcal{S} with homomorphism $|\mathcal{S}| \rightarrow \tilde{X}$.

Proof. Immediately, we can construct the zero skeleton of \mathcal{S} by taking preimages of points. Using the path-lifting property, we can also construct the 1-skeleton. Note that \mathcal{S}^1 injects into \tilde{X} because of the uniqueness of the path-lifting property. Two distinct paths cannot cross each other, or else we could get two distinct lifts from the point at the intersection. See the figure.

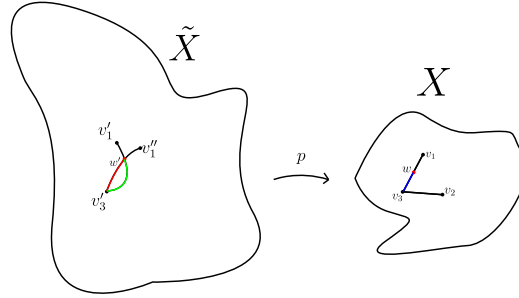


Figure 1: Above, suppose we have two distinct paths that cross each other. Then because of the construction of the skeleton, they had to be lifts of the same path in X . However, this would mean that we get a contradiction to uniqueness of the lift of the blue path on the right starting from w . The red and green paths are both lifts of the blue path.

We thus get a 1-skeleton injecting into \tilde{X} . We construct the 2-skeleton as follows. Given a face between vertices v, v', v'' in the 1-skeleton whose images are the vertices of a face in \mathcal{T} , we can consider the space Y given by the union of the edge connecting $p(v)$ and $p(v')$ and the edge connecting $p(v'')$ and $p(v')$. Homotope Y to the boundary of the simplex to get a homeomorphism $f : Y \times [0, 1] \cong \Delta^2$ (such that $f|_{Y \times \{0\}}$ was the inclusion $Y \subset X$, which lifts) to the 2-simplex sitting in X connecting the images of the points. For a picture of this, see Figure 2.

By the homotopy extension property, there exists a unique lift of the map $f : Y \times [0, 1] \rightarrow X$ to a map $g : Y \times [0, 1] \rightarrow \tilde{X}$. The restriction of p to $p^{-1}(\text{Int}(\Delta^2)) \cap g(Y \times [0, 1])$ is a degree 1 covering since this is the case near any of the vertices. Hence we can say $p^{-1}(\text{Int}(\Delta^2)) \cap g(Y \times [0, 1])$ in \tilde{X} is the interior of the lift of a 2-face in \tilde{X} connecting any three vertices $p(v), p(v'), p(v'')$ in the 0-skeleton of \mathcal{T} . Putting the information together, we can a 2-skeleton. Again by the path-lifting property, two distinct faces in \mathcal{S} cannot intersect each other. Hence, we get an injection $|\mathcal{S}| \rightarrow \tilde{X}$. This is also a surjection since we lifted any 2-face in \mathcal{T} . Because continuity and openness are local properties, it is left to the reader to check that indeed the map $|\mathcal{S}| \rightarrow \tilde{X}$ satisfies these properties. Indeed, we have a open and continuous map $X \rightarrow |\mathcal{T}|$, and p is also open and continuous. What can be said about the composition $|\mathcal{S}| \rightarrow \tilde{X} \rightarrow X \rightarrow |\mathcal{T}|$? \square

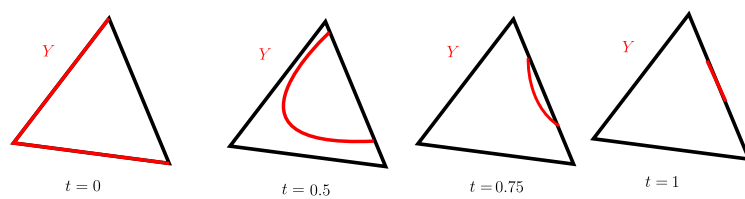


Figure 2: We carefully construct a homotopy that yields a homeomorphism $Y \times [0, 1] \cong \Delta^2$