Commutative Algebra

Notes on MATH 7830

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Let R be a commutative Noetherian ring, and let M be an R-module. What does it mean for an element $r \in R$ to be a **zero-divisor**? It simply means that for some $m \neq 0$, $r \cdot m = 0$.

$$zdr_R(M) = \{r \in R | r \text{ is a zero divisor on } M\} = \bigcup_{\mathfrak{p} \in \mathrm{Ann}_R M} \mathfrak{p}.$$

We can say $r \in R$ is a non-zero divisor if it is not a zero divisor (abbrev. nzd). Fix a sequence $\mathbf{x} = x_1, \ldots, x_n \in R$.

Definition 1. We say that $\mathbf{x}$ is a **weakly M-regular** sequence on M if x_{i+1} is not a zero divisor on $\frac{M}{(x_1,\ldots,x_i)M}$ for all applicable i. It becomes a **regular sequence** if in addition $\frac{M}{\mathbf{x}M} \neq 0$.

Example. If $R = \Bbbk[x_1, \ldots, x_n]$, and note $\mathbf{x} = x_1, \ldots, x_n$ is a regular sequence on R.

We now introduce Koszul complexes. Given $r \in R$, we can write $K(r, R)$ to be the complex

$$0 \to R \to R \to 0.$$

there $R \to R$ is the homothetic map multiplication by r. The left first copy of R is labeled degree 1. Here, taking the homology functor of the sequence provides 0 on the left R if and only if r is a nzd. We have

$$K(\mathbf{x}, R) = \bigotimes_{i=1}^{n} K(x_i, R).$$

We will get

$$0 \to R \to R^n \to R^{nchoose2} \to \ldots \to R^{nchoose2} \to R^n \to R \to 0.$$

(exercise calculate the first and last maps). Given $M \in \mathcal{C}(R)$,

$$K(\mathbf{x}, M) = K(\mathbf{x}, R) \otimes_R M.$$

If M is just an R-module, it is merely replacing copies of R with copies of M. We denote $H_i(\mathbf{x}, M) = H_i(K(\mathbf{x}, M))$. Note

$$H_0(\mathbf{x}, M) = \frac{M}{\mathbf{x}M}.$$

$$H_1(\mathbf{x}, M) = \{m \in M | x_i \cdot M = 0 \forall i\} = (0 :_M (\mathbf{x})).$$

Remark: Note

$$K(\mathbf{x}, M) = K(x_1, R) \otimes K(x_2, R) \otimes \ldots \otimes K(x_n R) \otimes_R M$$
$$K(x_1, R) \otimes K(\mathbf{x}_{\geq 2}, M).$$

So we have

$$K(\mathbf{x}, M) = K(x_1, K(\mathbf{x}_{\geq 2}, M)).$$

Remark:
We have $X, Y \in \mathcal{C}(R)$, we get the isomorphism

$$X \otimes_R Y \to Y \otimes_R X.$$

via $x \otimes_R y \mapsto (-1)^{(x)(y)} y \otimes_R x$ For any $\sigma \in S_n$,

$$K(x_1, \ldots, x_n) \cong K(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, R).$$

Also, we have a second perspective on Koszul complexes: that they are the iterated mapping cones. Given a morphism of complexes

$$f : X \to Y.$$

recall the **cone** is defined

$$\operatorname{cone}(f) = (Y \oplus \Sigma X, \begin{pmatrix} \partial^Y & f \\ 0 & -\partial^X \end{pmatrix}).$$

We get that

$$0 \to Y \to \operatorname{cone}(f) \to \Sigma X \to 0.$$

$y \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}$, $\begin{pmatrix} y \\ \Sigma x \end{pmatrix} \mapsto \Sigma X$. The long exact sequence in homology yields

$$\ldots \to H_i(X) \to H_i(Y) \to H_i(\operatorname{cone}(f)) \to H_i(\Sigma X) \cong H_{i-1}(X) \to \ldots.$$

Where the connecting map $H_i(X) \to H_i(Y)$ is just $H_i(f)$.

Now consider $x \in R$, and the homothetic map $f : R \to R$.

Example. $\operatorname{cone}(f) = (R \oplus R, \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}) = K(x, R)$. Ditto for the homothetic map on modules.

$$\operatorname{cone}(M \to M) = K(x, M).$$

Thus, $K(\mathbf{x}, M) = K(x_1, K(\mathbf{x}_{\geq 2}, M)$ is $\operatorname{cone}(K(\mathbf{x}_{\geq 2}, M) \to K(\mathbf{x}_{\geq 2}, M))$. This gives

$$H_i(\mathbf{x}_{\geq 2}, M) \to H_i(\mathbf{x}_{\geq 2}, M) \to H_i(\mathbf{x}, M) \to H_{i-1}(\mathbf{x}_{\geq 2}, M) \to \ldots$$

where the connecting morphism is multiplication by x_1 up to sign. By looking at the images/cokernels/kernels of one segment in this sequence, we get induced SES

$$0 \to H_i(\mathbf{x}_{\geq 2}, M)/x_1 H_i(\mathbf{x}_{\geq 2}, M) \to H_i(\mathbf{x}, M) \to (0 :_{H_{i-1}(\mathbf{x}_{\geq 2}, M)} x_1) \to 0.$$

If M is an R-module, $\mathbf{x} = x_1, \ldots, x_n \subset R$,

$$K(\mathbf{x}, M) \twoheadrightarrow H_0(\mathbf{x}, M) = \frac{M}{\mathbf{x}M}.$$

So,

$$K(\mathbf{x}, M) \to \frac{M}{\mathbf{x}M}.$$

is a weak equivalence if and only if

$$H_i(\mathbf{x}, M) = 0 \forall i \geq 1.$$

Lemma 1. When $\mathbf{x}$ is a weakly M-regular,

$$K(\mathbf{x}, M) \twoheadrightarrow \frac{M}{\mathbf{x}M}$$

which is also a weak equivalence.

Proof. When $n = 1$,

$$0 \to M \to M \to 0$$

has zero homology at degree 1 if and only if x is a nonzero divisor on M.

Now say when $n \geq 2$, we know that $K(\mathbf{x}, M) = K(x_n, K(\mathbf{x}_{\leq n-1}, M))$. By our induction hypothesis,

$$K(\mathbf{x}_{\leq n-1}, M) \twoheadrightarrow \frac{M}{(\mathbf{x}_{\leq n-1})M}.$$

We have

$$K(x, R) = (0 \to R \to R \to 0).$$

is semi-free.

$$K(\mathbf{x}, M) = K(x_n, R) \otimes_R K(\mathbf{x}_{\leq n-1}, M) \to K(x_n, \frac{M}{\mathbf{x}_{\leq n-1}M}).$$

Exercise 1

Prove this using the Koszul homology long exact sequence.

□

Definition 2. $\mathbf{x}$ is **Koszi-regular** on M if

$$K(\mathbf{x}, M) \twoheadrightarrow^{\sim} \frac{M}{\mathbf{x}M}.$$

. Note that $x_1, \ldots, x_n$ is Koszi-regular on M if and only if any permutation

$$x_{\sigma(1)}, \ldots, x_{\sigma(n)}$$

is Koszi-regular on M for any $\sigma \in S_n$.

Exercise 2

(Weakly) regular sequences are senitive to permutations.

Theorem 1. Say $\mathbf{x} \subset J(R)$ and $M \neq 0$ is finitely generated as an R-module. Then the following are equivalent:

1. $\mathbf{x}$ is regular ($\equiv$ weakly regular).
2. $H_i(\mathbf{x}, M) = 0$ for all $i \geq 1$.
3. $H_1(\mathbf{x}, M) = 0$.

Our main application is when R is a local ring and $\mathbf{x} \subset \mathfrak{m}_R$. We use Nakayama's lemma: $J(R) \neq M$, so regularity is equivalent to weak regularity.

Proof. We know $1 \Rightarrow 2 \Rightarrow 3$. It remains to show $3 \Rightarrow 1$. We want to examine $H_*(x_1, \dots, x_{n-1}, x_n, M)$. The module

$$K(\mathbf{x}, M) = K(x_n, K(\mathbf{x}_{\leq n-1}, M))$$

provides long exact sequence containing

$$0 \to H_i(\mathbf{x}_{\leq n-1}, M)/(x_n)H_i(\mathbf{x}_{\leq n-1}, M) \to H_i(\mathbf{x}, M) \to \left(0 :_{H_{i-1}(\mathbf{x}_{\leq n-1}, M)} x_n\right).$$

We have that

$$H_1(\mathbf{x}, M) = 0 \Rightarrow H_1(\mathbf{x}_{\leq n-1}, M) = (x_n)H_1(\mathbf{x}_{\leq n-1}, M).$$

so apply Nakayama's. We are doing the proof of equivalence by induction on n (it is already proven for $n = 1$), so we have

$$x_1, \dots, x_{n-1}.$$

is M-regular. This implies further that

$$H_i(\mathbf{x}_{\leq n-1}, M) = 0.$$

for all $i \geq 1$. Moreover, applying this to our exact sequence above, $(0 : x_n) = 0$, so $H_0(\mathbf{x}_{\leq n}, M) = \ker\left(\frac{M}{\mathbf{x}_{\leq n-1}M} \to \frac{M}{\mathbf{x}_{\leq n-1}M}\right)$. □

Corollary 1. $\mathbf{x} \subset J(R)$, M finitely generated. The property that $\mathbf{x}$ is M-regular does not depend on the ordering of $\mathbf{x}$.

Lemma 2. Suppose we have a sequence $x_1, \dots, x_n \subset R$ (now we drop the assumption regarding the Jacobson radical). Let M be an R-module. The following are equivalent:

1. $\mathbf{x}$ is Koszi-regular on M.
2. $\{x_1^{a_1}, \dots, x_n^{a_n}\}$ is Koszi-regular on M for any choice $a_i \geq 1$.
3. $\mathbf{x}^{\mathbf{a}}$ is Koszi-regular on M for some $\mathbf{a} \geq (1, \dots, 1)$.

Proof. It suffices to prove $x_1, \dots, x_n$ is Koszi-regular on M if and only if $x_1^a, \dots, x_n$ for some $a \geq 1$. Recall that Koszi-regularity means

$$K(x_1^a, x_2, \dots, x_n, M) \to^{\sim} K(x_1^a, \frac{M}{(x_{\geq 2})M}).$$

Replacing M with $\frac{M}{(\mathbf{x}_{\geq 2})M}$, we are reduced to proving x is weakly M-regular if and only if x^a is weakly M-regular for some $a \geq 1$. x is not a zero divisor on M if and only if x^a is not a zero divisor on M for some or all $a \geq 1$. □

Exercise 3

(this is also a theorem, called the rigidity of Koszul homology). If we take $\mathbf{x} \subset J(R)$ and M a finitely generated R-module, then $H_i(\mathbf{x}, M) = 0$ for some $i \geq 0$ implies that $H_j(\mathbf{x}, M) = 0$ for all $j \geq i$.