Here is a topological proof of a useful fact. The proof is fairly hand-wavy, so please let me know of any potential errors.

**Proposition 1.** Suppose $p : \tilde{X} \to X$ is a covering space and that $X$ has triangulation by simplicial complex $\mathcal{T}$. In particular, there is a homeomorphism $|\mathcal{T}| \to X$. Then there exists simplicial complex $\mathcal{S}$ with homemorphism $|\mathcal{S}| \to \tilde{X}$. Furthermore, images of faces under this map get mapped homeomorphically to faces of $\mathcal{T}$.

We prove this in the 2-dimensional case. The steps generalize to higher dimensions by a similar argument used to construct the 2-skeleton of $\mathcal{S}$.

*Proof.* Immediately, we can construct the zero skeleton of $\mathcal{S}$ by taking preimages of points. Using the path-lifting property, we can also construct the 1-skeleton. Note that $\mathcal{S}^1$ injects into $\tilde{X}$ because of the uniqueness of the path-lifting property. Two distinct paths cannot cross each other, or else we could get two distinct lifts from the point at the intersection. See the figure below:

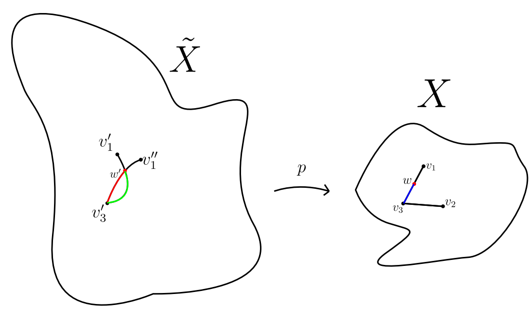


Figure 1: Above, suppose we have two distinct paths that cross each other. Then because of the construction of the skeleton, they had to be lifts of the same path in $X$. However, this would mean that we get a contradiction to uniqueness of the lift of the blue path on the right starting from $w$. The red and green paths are both lifts of the blue path.

We thus get a 1-skeleton injecting into $\tilde{X}$. We construct the 2-skeleton as follows. Given a face between vertices $v, v', v''$ in the 1-skeleton whose images are the vertices of a face in $\mathcal{T}$, we can consider the space $Y$ given by the union of the edge connecting $p(v)$ and $p(v')$ and the edge connecting $p(v'')$ and $p(v')$. Homotope $Y$ to the boundary of the simplex to get a homeomorphism $f : Y \times [0,1] \cong \Delta^2$ (such that $f|_{Y\times\{0\}}$ was the inclusion $Y \subset X$, which lifts) to the 2-simplex sitting in $X$ connecting the images of the points. For a picture of this, see Figure 2.

By the homotopy extension property, there exists a unique lift of the map $f : Y \times [0,1] \to X$ to a map $g : Y \times [0,1] \to \tilde{X}$. The restriction of $p$ to $p^{-1}(\mathrm{Int}(\Delta^2)) \cap g(Y \times [0,1])$ is a degree 1 covering since this is the case near any of the vertices. Hence we can say $p^{-1}(\mathrm{Int}(\Delta^2)) \cap g(Y \times [0,1])$ in $\tilde{X}$ is the interior of the lift of a 2-face in $\tilde{X}$ connecting any three vertices $p(v), p(v'), p(v'')$ in the 0-skeleton of $\mathcal{T}$. Putting the information

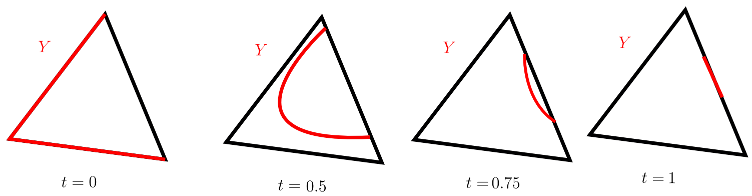


Figure 2: We carefully construct a homotopy that yields a homeomorphism $Y \times [0, 1] \cong \Delta^2$

together, we can a 2-skeleton. Again by the path-lifting property, two distinct faces in $\mathcal{S}$ cannot intersect each other. Hence, we get an injection $|\mathcal{S}| \to \tilde{X}$. This is also a surjection since we lifted any 2-face in $\mathcal{T}$. Because continuity and openness are local properties, we can check that $|\mathcal{S}| \to \tilde{X}$ satisfies these properties. Indeed, we have a open and continuous map $X \to |\mathcal{T}|$, and $p$ is also open and continuous. The composition $|\mathcal{S}| \to \tilde{X} \to X \to |\mathcal{T}|$ is open and continuous. $\square$