Modern Algebra II
Notes on MATH 6320

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As always, let R be a ring. We noted

$$D(R) \cong \mathrm{KProj}(R).$$

To recap, a complex $P \in \mathcal{C}(\mathrm{Proj}(R))$ is K projective we have the existing map in the following diagram:

$$\begin{tikzcd} & X \arrow[d, two heads, "\wr"'] \\ P \arrow[ur, dashed, "\exists"] \arrow[r] & Y \end{tikzcd}$$

For example,

Example. $P \in \mathcal{C}(\mathrm{Proj}(R))$ with $P_i = 0$ for all $i \ll 0$. To sketch a proof of this claim, construct liftings one step at a time.

$$\begin{tikzcd} & & & & & & & & X \arrow[d, "\varepsilon", "\wr"'] \\ (\quad \dots \arrow[r] & P_{n+1} \arrow[r] & P_n \arrow[r] & \dots \arrow[r] & P_{a+1} \arrow[r] \arrow[urrrr, dashed, "\tilde{\alpha}"] & P_a \arrow[r] \arrow[urrr] & 0 \arrow[r] & \dots \quad) \arrow[r, "\alpha"] & Y \end{tikzcd}$$

Here, $\tilde{\alpha}(S)$ should satisfy

- $\varepsilon\tilde{\alpha} = \alpha(S)$
- $\partial\tilde{\alpha}(S) = \tilde{\alpha}(\partial S)$.
- ε is surjective implies that ε is surjective on boundaries.
- $H(\varepsilon)$ is isomorphic (H is the homology functor) implies that $\varepsilon : Z(X) \twoheadrightarrow Z(Y)$
- $\ker(\varepsilon)$ is acyclic.

Once you construct these maps, the claim holds.

Note that every module has a K-projective resolution. Every module M admits a surjection $F_0 \twoheadrightarrow M$ from a free module F_0, call this map ε. The kernel of ε then admits a surjection $F_1 \to \ker\varepsilon$ from a free module F_1. Continuing the process one more step yields the following diagram:

$$\begin{tikzcd} F_2 \arrow[rr, "\partial_2"] \arrow[dr] & & F_1 \arrow[rr, "\partial_1"] \arrow[dr] & & F_0 \arrow[r, two heads, "\varepsilon"] & M \\ & \ker\partial_1 \arrow[ur] & & \ker\varepsilon \arrow[ur] & & \end{tikzcd}$$

If we continue this process, we get the K-projective resolution. To make the choice of resolution canonical, we choose the free modules to be freely generated by the elements of the kernels that they surject onto. This yields a functor from modules to K-projective resolutions.

Definition 1. A K-projective resolution of $M \in \mathcal{C}(R)$ is a morphism $\varepsilon : P \to M$ such that

- ε is a quasi-isomorphism, and
- P is K-projective.

We can make this functorial by taking $R \times M \twoheadrightarrow M$,

$$\varepsilon(r, m) = rm.$$

We then set

$$\bigoplus_{m \in M} R \times m = R^{\oplus M} \twoheadrightarrow M.$$

The last surjection above is the ε map.

Theorem 1. For all $M \in \mathcal{C}(R)$, there exists a surjective K-projective resolution

$$P \twoheadrightarrow M$$

which is a weak equivalence.

Definition 2. An R-complex F is semi-free if it has a filtration

$$(0) \subset F_0 \subset F_1 \subset \ldots \subset \bigcup_{n \geq 0} F_n = F$$

such that

1. $F(n) \subset F$ is a subcomplex,
2. $F(n+1)/F(n)$ is a graded free module with trivial boundary maps ($\partial = 0$). In particular, $\partial(F_{n+1}) \subset F_n$.

Exercise 1

Show that semi-free implies K-projective. Also show that unions are K-projective.

Example. Take a complex of free modules

$$\ldots \to F_{n+1} \to F_n \to 0.$$

Let $F(n)$ be the truncation consisting of degrees n and lower. Then $\frac{F(n+1)}{F(n)}$ is precisely $\Sigma^{n+1} F_{n+1}$.

Theorem 2. Each $M \in \mathcal{C}(R)$ has a semi-free resolution

$$F \twoheadrightarrow M.$$

(The map above is a weak equivalence.)

Corollary 1. Every K-projective is a retract of a semi-free.

Proof. (Corollary proof): Let P be K-projective. The following diagram completes the proof:

$$\begin{tikzcd}
 & F \arrow[d, two heads, "\wr"'] \\
P \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, equal] & P
\end{tikzcd}$$

□

Proof. (Theorem proof): We use the small object argument. We want to construct

$$\begin{tikzcd}
F(1) \arrow[r, hook] \arrow[d, "\varepsilon(1)"'] & F(2) \arrow[r, hook] \arrow[dl, "\varepsilon(2)"'] & \dots \arrow[r, hook] & F(n) \arrow[r, hook] \arrow[dlll, "\varepsilon(n)"] & \dots \arrow[r, hook] & \bigcup_{n\geq 0} F(n) \\
M
\end{tikzcd}$$

such that $\frac{F(n+1)}{F(n)}$ is a graded complex of free modules with zero differential which is surjective on cycles. We need that $\varepsilon(1)$ is surjective on cycles. We have

$$\begin{tikzcd}
F(1) \arrow[r, equal] & H(F(1)) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\Longrightarrow"] & H(F(n)) \arrow[r, "\phi"] \arrow[dl, two heads, "H(\varepsilon(n))"] & H(F(n+1)) \\
 & H(M)
\end{tikzcd}$$

for all n where the kernel of ϕ contains the kernel of $H(\varepsilon(n))$, so that $H(\varepsilon(n))$ is surjective for all n.

Next, we note $\ker(H(\varepsilon(n))) \subset H(F(n))$ goes to 0 under $H(i(n))$. We have $H(F) \xrightarrow{\sim} H(M)$.

Remark. Given $\varepsilon : X \to Y$ where $Z(\varepsilon)$ and $H(\varepsilon)$ are surjective, we have that ε is surjective. See the four lemmas applied to the following diagrams:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & B(X) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & H(X) \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
0 \arrow[r] & B(Y) \arrow[r] & Z(Y) \arrow[r] & H(Y) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & Z(X) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \varepsilon(B(X)) \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
0 \arrow[r] & Z(Y) \arrow[r] & Y \arrow[r] & \varepsilon(B(Y)) \arrow[r] & 0
\end{tikzcd}$$

Say we have calculated $\varepsilon(n) : F(n) \to M$. Choose cycles $\{z_\lambda\}$ in $F(n)$ that map to the generating set of $\ker(H(\Sigma(n)))$. Pick $w_\lambda \in M$ such that $\partial w_\lambda = \varepsilon(n)(z_\lambda)$. Set

$$F(n+1) = F(n) \oplus_\lambda R \cdot e_\lambda.$$

Above, we have $|e_\lambda| = |z_\lambda| + 1$ and $\partial|_{F(n)} = \partial^{F(n)}$. (Attached are some iPad notes, where the proof continues.) □