Let k be an arbitrary field and $k(t)$ be a simple transcendental extension. We hope to understand intermediate extensions $k \subset L \subset k(t)$. To state Lüroth's theorem, we need a definition and a lemma:

Lemma 1. If $f \in k(t)[x]$ is nonzero, f can be uniquely written

$$\frac{P(t,x)}{Q(t)}$$

in such a way that $Q(t)$ and $P(t,x)$ have no common factors in the UFD $k[t,x]$, and Q is monic.

Proof. First, write

$$f = \frac{p_0(t)}{q_0(t)} + \frac{p_1(t)}{q_1(t)}x + \ldots + \frac{p_n(t)}{q_n(t)}x^n.$$

Then, we can note that

$$f = \frac{\sum_{i=0}^{n} q_0(t) \cdot \ldots \cdot \hat{q_i(t)} \cdot \ldots \cdot q_n(t)p_i(t)x^i}{q_0(t) \cdot \ldots \cdot q_n(t)}.$$

Now we have expressed f as a fraction of polynomials in $k[t,x]$. Since $k[t,x]$ is a UFD, we can write the fraction in a reduced form. The reduced form is unique if we further multiply both numerator and denominator by a constant term such that the denominator is monic. $\square$

Definition 1. Let $u \in k(t)[x]$. By the previous lemma, there exists unique P, Q, where Q is monic, such that

$$u = \frac{P(t,x)}{Q(t)}.$$

We define the **height of** u to be the number

$$\max\left(\deg_t(P), \deg_t(Q)\right).$$

We first go through some lemmas before stating Lüroth's theorem.

Lemma 2. If $f \in k(t)[x]$ ($f = \frac{P(t,x)}{Q(t)}$) is monic in x,

$$\mathrm{ht}(f) = \deg_t(P(t,x)).$$

Furthermore, we have that $P(t,x)$ is not divisible by any non-unit of $k[t]$.

Proof. First, write

$$f(x) = \frac{p_n(t)x^n + \ldots + p_0(t)}{Q(t)}.$$

Because f is monic, $p_n(t) = Q(t)$, implying $\deg_t(P) \geq \deg_t(Q)$. A restatement of the second conclusion of the theorem is that $\gcd(p_0, p_1, \ldots, p_n) = 1$. Since the expression above is assumed to be a reduced fraction, no factor of Q also divides all p_i simultaneously. This means

$$\gcd(Q, p_0, p_1, \ldots, p_n) = 1.$$

But $Q = p_n$ implies

$$\gcd(p_0, p_1, \ldots, p_n) = 1.$$

$\square$

Corollary 1. If $f, g \in k(t)[x] - \{0\}$, are monic in x,

$$\text{ht}(f \cdot g) = \text{ht}(f) + \text{ht}(g).$$

Proof. We have $f \cdot g$ is monic in x. If $f = \frac{P}{Q}$ and $g = \frac{P'}{Q'}$,

$$f \cdot g = \frac{P \cdot P'}{Q \cdot Q'}.$$

This fraction is in a reduced form and $Q \cdot Q'$ is monic. Hence

$$\text{ht}(f \cdot g) = \deg_t(P \cdot P') = \deg_t(P) + \deg_t(P') = \text{ht}(f) + \text{ht}(g)$$

by the lemma. $\square$

Lemma 3. Let $u \in k(t) \setminus k$. There exists $u' \in k(t) \setminus k$ such that $k(u') = k(u)$ where $u' = \frac{P'}{Q'}$, $\deg_t(P) > \deg_t(Q)$, and P', Q' are monic.

Proof. First, we know $u = \frac{P}{Q}$ by Lemma . If $\deg_t(Q) < \deg_t(P)$, we can just multiply u by a constant to achieve the desired u'. Otherwise, we can select $\beta \in k$ such that $\deg(P + \beta Q) < \deg(Q)$. We can write

$$u' = \alpha * \frac{Q}{P + \beta Q}$$

where α is chosen so that the resulting fraction has monic numerator and denominator. Note that Q and $P + \beta Q$ do not have common factors. In either case, note $u \in k(u')$ and $u' \in k(u)$, so $k(u) = k(u')$. $\square$

Lemma 4. Given $u = \frac{P(t)}{Q(t)} \in k(t) - k$, verify that t is a root of $P(x) - uQ(x) \in k(u)[x]$. Show further that if $\deg_t(P) > \deg_t(Q)$, and P is monic, then the above polynomial is monic.

Proof. The first assertion is just

$$P(t) - \frac{P(t)}{Q(t)} Q(t) = 0.$$

The second claim is just because the leading coefficient of $P(x) - uQ(x)$ is equal to that of $P(x)$ if $\deg_t(P) > \deg_t(Q)$. $\square$

We would like to prove the following theorem:

Theorem 1. (Lüroth's Theorem) Let k be an arbitrary field. If $k(t)$ is a simple transcendental extension of k, and

$$k \subset L \subset k(t).$$

is an arbitrary intermediate field extension $L \neq k$, then L is also a simple transcendental extension over k, generated by an element $u \in k(t)$ of minimal possible height. We further have $[L : k] = \text{ht}(u)$.

The proof requires some steps. Given u an element of minimal height in $L - k$, we write

$$u = \frac{P(t)}{Q(t)}$$

using Lemma , and denote the height of u by $n \in \mathbb{Z}$. By Lemma , we can assume that $\deg(P) > \deg(Q)$ and P is monic.

Lemma 5. For any $f \in L[x]$, $\mathrm{ht}(f)$ is either 0 or is at least n. $P(x) - uQ(x)$ is either irreducible in $L[x]$ or is divisible by a non-unit element of $k[x]$.

Proof. We also show that $P(x) - u \cdot Q(x)$ has height n. By Lemma , we can construct $f' = \frac{P'}{Q'}$ such that both numerator and denominator are monic with

$$\deg_t(P') > \deg_t(Q').$$

By construction of u, $\mathrm{ht}(f') \geq \mathrm{ht}(u) = n$ or $\mathrm{ht}(f') = 0$. The conclusion is given by $\mathrm{ht}(f) = \mathrm{ht}(f')$. We calculate

$$\mathrm{ht}\left(P(x) - \frac{P(t)}{Q(t)}Q(x)\right) = \mathrm{ht}\left(\frac{P(x)Q(t) - P(t)Q(x)}{Q(t)}\right).$$

Note that $Q(t)$ does not share a factor with $P(x)Q(t) - P(t)Q(x)$, for then it would share a factor with $P(t)Q(x)$. Because $k[t, x]$ is a UFD, this would mean $Q(t)$ shares a factor with $P(t)$. In particular, the expression in the height above is the fraction expression used in the definition of height. The height is thus

$$\max(\deg_t(P(x)Q(t) - P(t)Q(x)), \deg_t(Q(t)) = \max(\deg_t(P(t), \deg_t(Q(t))))$$

because $\deg(P) > \deg(Q)$. The second conclusion is from the fact that if $P(x) - uQ(x) = s(t, x)w(t, x)$ for $s, w \in L[x]$, we can assume s, w are monic in x since $P(x) - uQ(x)$ is. Corollary 1 says that $\mathrm{ht}(s) + \mathrm{ht}(w) = \mathrm{ht}(P(x) - uQ(x)) = n$. Hence one of $\mathrm{ht}(s), \mathrm{ht}(w)$ is 0 and one of them is n. $\square$

Lemma 6. If $P(x) - uQ(x)$ is divisible in $L[x]$ by an element in $k[x]$, then this element must divide both $P(x)$ and $Q(x)$. Deduce that this element must be a unit.

Proof. The second claim immediately follows from the first because we assume $\frac{P}{Q}$ is a reduced fraction. To prove the first claim, if $P - uQ$ is divisible by an element $s(x) \in k[x]$ in $L[x]$, we have there exists $w(x, t) \in L[x]$ such that

$$s(x)w(x, t) = P(x) - uQ(x).$$

In L, we can extend $\{1, u\}$ to a basis for L as a k-vector space using Zorn's lemma. Say

$$L = k \oplus k \cdot u \oplus W.$$

for complementary k-vector subspace $W \subset L$. Denote the L coefficients for P, Q, w, s by P_i, Q_i, w_i, s_i respectively (in the first two cases they are coefficients in k) and

$$w_i(t) = a_i + b_i u + c_i(t)$$

for $w_i(t) \in L$, $a_i, b_i \in k$, and $c_i(t) \in W$. Then for all i, $s(x) \cdot w(x, t) = P(x) - uQ(x)$ gives

$$\sum_{j+k=i} s_j(a_k + b_k u + c_k(t)) = P_i - uQ_i.$$

By linear independence,

$$s(x) \cdot (a_0 + a_1 x + \ldots + a_{\deg_x(w)}) = P$$

and

$$s(x) \cdot (b_0 + b_1 x + \ldots + b_{\deg_x(w)} x^{\deg_x(w)}) = Q.$$

$\square$

Finally, we can prove Lüroth's theorem as follows. $P(x) - uQ(x)$ is an irreducible polynomial with t as a root. This implies that $P(x) - uQ(x)$ is a minimal polynomial of t over L and $k(u)$. In particular,

$$[L : k] = [k(u) : k] = \deg(P(x)).$$

Hence, we have $[L : k(u)] = 1$ and $L = k(u)$.