Here is a topological proof of a useful fact. The proof is fairly hand-wavy, so please let me know of any potential errors.

Proposition 1. Suppose $p : \tilde{X} \to X$ is a covering space and that X has triangulation by simplicial complex $\mathcal{T}$. In particular, there is a homeomorphism $|\mathcal{T}| \to X$. Then there exists simplicial complex $\mathcal{S}$ with homemorphism $|\mathcal{S}| \to \tilde{X}$. Furthermore, images of faces under this map get mapped homeomorphically to faces of $\mathcal{T}$.

We prove this in the 2-dimensional case. The steps generalize to higher dimensions by a similar argument used to construct the 2-skeleton of $\mathcal{S}$.

Proof. Immediately, we can construct the zero skeleton of $\mathcal{S}$ by taking the preimage of the zero-skeleton of $\mathcal{T}$. Using the path-lifting property, we can also construct the 1-skeleton. Note that $\mathcal{S}^1$ injects into $\tilde{X}$ because of the uniqueness of the path-lifting property. Two distinct paths cannot cross each other, or else we could get two distinct lifts from the point at the intersection. See the figure below:

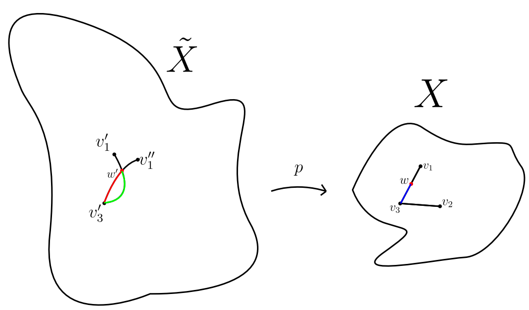

Figure 1: Above, suppose we have two distinct paths that cross each other. Then because of the construction of the skeleton, they had to be lifts of the same path in X. However, this would mean that we get a contradiction to uniqueness of the lift of the blue path on the right starting from w. The red and green paths are both lifts of the blue path.

We thus get a 1-skeleton injecting into $\tilde{X}$. We construct the 2-skeleton as follows. Given a face between vertices v, v', v'' in the 1-skeleton whose images are the vertices of a face in $\mathcal{T}$, we can consider the space Y given by the union of the edge connecting $p(v)$ and $p(v')$ and the edge connecting $p(v'')$ and $p(v')$. Homotope Y to the boundary of the simplex to get a homeomorphism $f : Y \times [0,1] \cong \Delta^2$ (such that $f|_{Y\times\{0\}}$ was the inclusion $Y \subset X$, which lifts) to the 2-simplex sitting in X connecting the images of the points. For a picture of this, see Figure 2.

By the homotopy extension property, there exists a unique lift of the map $f : Y \times [0,1] \to X$ to a map $g : Y \times [0,1] \to \tilde{X}$. The restriction of p to $p^{-1}(\text{Int}(\Delta^2)) \cap g(Y \times [0,1])$ is a degree 1 covering since this is the case near any of the vertices. Hence we can say $p^{-1}(\text{Int}(\Delta^2)) \cap g(Y \times [0,1])$ in $\tilde{X}$ is the interior of the lift of a 2-face in $\tilde{X}$ connecting any three vertices $p(v), p(v'), p(v'')$ in the 0-skeleton of $\mathcal{T}$. Putting the information

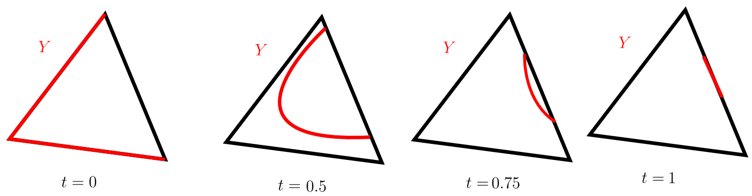

Figure 2: We carefully construct a homotopy that yields a homeomorphism $Y \times [0,1] \cong \Delta^2$

together, we can a 2-skeleton. Again by the path-lifting property, two distinct faces in $\mathcal{S}$ cannot intersect each other. Hence, we get an injection $|\mathcal{S}| \to \tilde{X}$. This is also a surjection since we lifted any 2-face in $\mathcal{T}$. Because continuity and openness are local properties, we can check that $|\mathcal{S}| \to \tilde{X}$ satisfies these properties. Indeed, we have a open and continuous map $X \to |\mathcal{T}|$, and p is also open and continuous. The composition $|\mathcal{S}| \to \tilde{X} \to X \to |\mathcal{T}|$ is open and continuous. $\square$