# Commutative Algebra
Notes on MATH 7830

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## Contents

## 1 January 18

Let $R$ be a commutative Noetherian ring, and let $M$ be an $R$-module. What does it mean for an element $r \in R$ to be a **zero-divisor**? It simply means that for some $m \neq 0$, $r \cdot m = 0$.

$$zdr_R(M) = \{r \in R | r \text{ is a zero divisor on } M\} = \bigcup_{\mathfrak{p} \in \operatorname{Ann}_R M} \mathfrak{p}.$$

We can say $r \in R$ is a non-zero divisor if it is not a zero divisor (abbrev. nzd). Fix a sequence $\mathbf{x} = x_1, \ldots, x_n \in R$.

**Definition 1.** We say that $\mathbf{x}$ is a **weakly $M$-regular** sequence on $M$ if $x_{i+1}$ is not a zero divisor on $\frac{M}{(x_1,\ldots,x_i)M}$ for all applicable $i$. It becomes a **regular sequence** if in addition $\frac{M}{\mathbf{x}M} \neq 0$.

**Example.** If $R = \Bbbk[x_1, \ldots, x_n]$, and note $\mathbf{x} = x_1, \ldots, x_n$ is a regular sequence on $R$.

We now introduce Koszul complexes. Given $r \in R$, we can write $K(r, R)$ to be the complex

$$0 \to R \to R \to 0.$$

there $R \to R$ is the homothetic map multiplication by $r$. The left first copy of $R$ is labeled degree 1. Here, taking the homology functor of the sequence provides 0 on the left $R$ if and only if $r$ is a nzd. We have

$$K(\mathbf{x}, R) = \bigotimes_{i=1}^{n} K(x_i, R).$$

We will get

$$0 \to R \to R^n \to R^{\binom{n}{2}} \to \ldots \to R^{\binom{n}{2}} \to R^n \to R \to 0.$$

(exercise calculate the first and last maps). Given $M \in \mathcal{C}(R)$,

$$K(\mathbf{x}, M) = K(\mathbf{x}, R) \otimes_R M.$$

If $M$ is just an $R$-module, it is merely replacing copies of $R$ with copies of $M$. We denote $H_i(\mathbf{x}, M) = H_i(K(\mathbf{x}, M))$. Note

$$H_0(\mathbf{x}, M) = \frac{M}{\mathbf{x}M}.$$

$$H_1(\mathbf{x}, M) = \{m \in M | x_i \cdot M = 0 \forall i\} = (0 :_M (\mathbf{x})).$$

Remark: Note

$$K(\mathbf{x}, M) = K(x_1, R) \otimes K(x_2, R) \otimes \ldots \otimes K(x_n R) \otimes_R M$$
$$K(x_1, R) \otimes K(\mathbf{x}_{\geq 2}, M).$$

So we have

$$K(\mathbf{x}, M) = K(x_1, K(\mathbf{x}_{\geq 2}, M)).$$

In many proofs in this course, being able to decompose the Koszul complex in this way will allow us to do induction.

Remark:
We have $X, Y \in \mathcal{C}(R)$, we get the isomorphism

$$X \otimes_R Y \to Y \otimes_R X.$$

via $x \otimes_R y \mapsto (-1)^{(x)(y)} y \otimes_R x$ For any $\sigma \in S_n$,

$$K(x_1, \ldots, x_n) \cong K(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, R).$$

Also, we have a second perspective on Koszul complexes: that they are the iterated mapping cones. Given a morphism of complexes

$$f : X \to Y.$$

recall the **cone** is defined

$$\text{cone}(f) = \left( Y \oplus \Sigma X, \begin{pmatrix} \partial^Y & f \\ 0 & \partial^X \end{pmatrix} \right).$$

We get that

$$0 \to Y \to \text{cone}(f) \to \Sigma X \to 0.$$

$y \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}$, $\begin{pmatrix} y \\ \Sigma x \end{pmatrix} \mapsto \Sigma x$. The long exact sequence in homology yields

$$\ldots \to H_i(X) \to H_i(Y) \to H_i(\text{cone}(f)) \to H_i(\Sigma X) \cong H_{i-1}(X) \to \ldots.$$

Where the connecting map $H_i(X) \to H_i(Y)$ is just $H_i(f)$.

Now consider $x \in R$, and the homothetic map $f : R \to R$.

**Example.** $\text{cone}(f) = \left( R \oplus \Sigma R, \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right) = K(x, R)$. Ditto for the homothetic map on modules.

$$\text{cone}(M \to M) = K(x, M).$$

Thus, $K(\mathbf{x}, M) = K(x_1, K(\mathbf{x}_{\geq 2}, M)$ is $\text{cone}(K(\mathbf{x}_{\geq 2}, M) \to K(\mathbf{x}_{\geq 2}, M))$. This gives

$$H_i(\mathbf{x}_{\geq 2}, M) \to H_i(\mathbf{x}_{\geq 2}, M) \to H_i(\mathbf{x}, M) \to H_{i-1}(\mathbf{x}_{\geq 2}, M) \to \ldots$$

where the connecting morphism is multiplication by $x_1$ up to sign. By looking at the images/cokernels/kernels of one segment in this sequence, we get induced SES

$$0 \to H_i(\mathbf{x}_{\geq 2}, M)/x_1 H_i(\mathbf{x}_{\geq 2}, M) \to H_i(\mathbf{x}, M) \to (0 :_{H_{i-1}(\mathbf{x}_{\geq 2}, M)} x_1) \to 0.$$

If $M$ is an $R$-module, $\mathbf{x} = x_1, \ldots, x_n \subset R$,

$$K(\mathbf{x}, M) \twoheadrightarrow H_0(\mathbf{x}, M) = \frac{M}{\mathbf{x}M}.$$

So,

$$K(\mathbf{x}, M) \to \frac{M}{\mathbf{x}M}.$$

is a weak equivalence if and only if

$$H_i(\mathbf{x}, M) = 0 \forall i \geq 1.$$

**Lemma 1.** When $\mathbf{x}$ is a weakly $M$-regular,

$$K(\mathbf{x}, M) \twoheadrightarrow \frac{M}{\mathbf{x}M}$$

which is also a weak equivalence.

*Proof.* When $n = 1$,

$$0 \to M \to M \to 0$$

has zero homology at degree 1 if and only if $x$ is a nonzero divisor on $M$.

Now say when $n \geq 2$, we know that $K(\mathbf{x}, M) = K(x_n, K(\mathbf{x}_{\leq n-1}, M))$. By our induction hypothesis,

$$K(\mathbf{x}_{\leq n-1}, M) \twoheadrightarrow \frac{M}{(\mathbf{x}_{\leq n-1})M}.$$

We have

$$K(x, R) = (0 \to R \to R \to 0).$$

is semi-free.

$$K(\mathbf{x}, M) = K(x_n, R) \otimes_R K(\mathbf{x}_{\leq n-1}, M) \to K(x_n, \frac{M}{\mathbf{x}_{\leq n-1}M}).$$

**Exercise 1**

Prove this using the Koszul homology long exact sequence.

□

**Definition 2.** $\mathbf{x}$ is **Koszi-regular** on $M$ if

$$K(\mathbf{x}, M) \twoheadrightarrow^{\sim} \frac{M}{\mathbf{x}M}.$$

. Note that $x_1, \ldots, x_n$ is Koszi-regular on $M$ if and only if any permutation

$$x_{\sigma(1)}, \ldots, x_{\sigma(n)}$$

is Koszi-regular on $M$ for any $\sigma \in S_n$.

**Exercise 2**

(Weakly) regular sequences are senitive to permutations.

**Theorem 1.** Say $\mathbf{x} \subset J(R)$ and $M \neq 0$ is finitely generated as an $R$-module. Then the following are equivalent:

1. $\mathbf{x}$ is regular ($\equiv$ weakly regular).
2. $H_i(\mathbf{x}, M) = 0$ for all $i \geq 1$.
3. $H_1(\mathbf{x}, M) = 0$.

Our main application is when $R$ is a local ring and $\mathbf{x} \subset \mathfrak{m}_R$. We use Nakayama's lemma: $J(R) \neq M$, so regularity is equivalent to weak regularity.

*Proof.* We know $1 \Rightarrow 2 \Rightarrow 3$. It remains to show $3 \Rightarrow 1$. We want to examine $H_*(x_1, \ldots, x_{n-1}, x_n, M)$. The module

$$K(\mathbf{x}, M) = K(x_n, K(\mathbf{x}_{\leq n-1}, M))$$

provides long exact sequence containing

$$0 \to H_i(\mathbf{x}_{\leq n-1}, M)/(x_n)H_i(\mathbf{x}_{\leq n-1}, M) \to H_i(\mathbf{x}, M) \to \left(0 :_{H_{i-1}(\mathbf{x}_{\leq n-1}, M)} x_n\right).$$

We have that

$$H_1(\mathbf{x}, M) = 0 \Rightarrow H_1(\mathbf{x}_{\leq n-1}, M) = (x_n)H_1(\mathbf{x}_{\leq n-1}, M).$$

so apply Nakayama's. We are doing the proof of equivalence by induction on $n$ (it is already proven for $n = 1$ ), so we have

$$x_1, \ldots, x_{n-1}.$$

is $M$-regular. This implies further that

$$H_i(\mathbf{x}_{\leq n-1}, M) = 0.$$

for all $i \geq 1$. Moreover, applying this to our exact sequence above, $(0 : x_n) = 0$, so $H_0(\mathbf{x}_{\leq n}, M) = \ker\left(\frac{M}{\mathbf{x}_{\leq n-1}M} \to \frac{M}{\mathbf{x}_{\leq n-1}M}\right)$. □

**Corollary 1.** $\mathbf{x} \subset J(R)$, $M$ finitely generated. The property that $\mathbf{x}$ is $M$-regular does not depend on the ordering of $\mathbf{x}$.

**Lemma 2.** Suppose we have a sequence $x_1, \ldots, x_n \subset R$ (now we drop the assumption regarding the Jacobson radical). Let $M$ be an $R$-module. The following are equivalent:

1. $\mathbf{x}$ is Koszi-regular on $M$.
2. $\{x_1^{a_1}, \ldots, x_n^{a_n}\}$ is Koszi-regular on $M$ for any choice $a_i \geq 1$.
3. $\mathbf{x}^{\mathbf{a}}$ is Koszi-regular on $M$ for some $\mathbf{a} \geq (1, \ldots, 1)$.

*Proof.* It suffices to prove $x_1, \ldots, x_n$ is Koszi-regular on $M$ if and only if $x_1^a, \ldots, x_n$ for some $a \geq 1$. Recall that Koszi-regularity means

$$K(x_1^a, x_2, \ldots, x_n, M) \to^{\sim} K(x_1^a, \frac{M}{(x_{\geq 2})M}).$$

Replacing $M$ with $\frac{M}{(\mathbf{x}_{\geq 2})M}$, we are reduced to proving $x$ is weakly $M$-regular if and only if $x^a$ is weakly $M$-regular for some $a \geq 1$. $x$ is not a zero divisor on $M$ if and only if $x^a$ is not a zero divisor on $M$ for some or all $a \geq 1$. □

**Exercise 3**

(this is also a theorem, called the rigidity of Koszul homology). If we take $\mathbf{x} \subset J(R)$ and $M$ a finitely generated $R$-module, then $H_i(\mathbf{x}, M) = 0$ for some $i \geq 0$ implies that $H_j(\mathbf{x}, M) = 0$ for all $j \geq i$.

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# 2 January 23

Let $R$ be a commutative and Noetherian ring, and $M, N \in \mathcal{C}(R)$. Note

$$\mathrm{RHom}_R(M, N) = \mathrm{Hom}_R(pM, N).$$

where $pM \xrightarrow{\sim} M$ is a $K$-projective resolution. Recall

$$\mathrm{Ext}_R^*(M, N) = H^*(\mathrm{RHom}_R(M, N)).$$

For any $M, N, P \in \mathcal{C}(R)$, there exists

$$\theta : \mathrm{RHom}_R(M, N) \otimes_R^L P \to \mathrm{RHom}_R(M, N \otimes_R^L P).$$

**Lemma 3.** This is a weak equivalence when $P$ is **perfect**. In particular

$$P \xrightarrow{\sim} (0 \to P_b \to \ldots \to P_c \to 0)\,.$$

Where $P_i$ is finitely generated as a projective $R$-module. We get a morphism of complexes

$$\mathrm{Hom}_R(pM, N) \otimes_R p(P) \to \mathrm{Hom}_R(pM, N \otimes_R p(P)).$$

Defined by

$$f \otimes x \mapsto \Big(m \mapsto (-1)^{|x||m|} f(m) \otimes x\Big)\,.$$

In the category of modules over $R$, if we look at

$$\mathrm{Hom}_R(M, N) \otimes_R P \to \mathrm{Hom}_R(M, N \otimes_R P)$$

to prove this when $P$ is a finitely generated projective.

**Lemma 4.** Rees' Lemma. Let $\mathbf{x} \subset R$ be a finite subset. Let $M, N$ be $R$-modules. Let $N$ be an $R$-module such that $\mathbf{x}N = 0$. And let $M$ be an $R$-module such that $\mathbf{x}$ is Koszi-regular on $M$. This means that

$$K(\mathbf{x}, M) \xrightarrow{\sim} \frac{M}{\mathbf{x}M}.$$

**Lemma 5.**

$$\mathrm{RHom}_R(N, \frac{M}{\mathbf{x}M} \xrightarrow{\sim} \mathrm{RHom}_R(N, M) \otimes_R \bigwedge^{*}(\Sigma R^c).$$

In particular,

$$\mathrm{Ext}_R^*(N, \frac{M}{\mathbf{x}M} \cong \mathrm{Ext}_R^*(N, M) \otimes_R \bigwedge^{*}(\Sigma R^c).$$

Where $c$ denotes the rank of the free module.

**Corollary 2.**

$$\inf \mathrm{Ext}_R^*(N, M) = \inf \mathrm{Ext}_R^*(N, \frac{M}{\mathbf{x}M} + c.$$

We also have

$$\mathrm{Ext}_R^*(N, M) \cong \mathrm{Ext}_R^{*+c}(N, \frac{M}{\mathbf{x}M}).$$

Recalling the alternating product complex will have zero differentials.

*Proof.* We want to compute

$$\mathrm{RHom}_R(N, \frac{M}{\mathbf{x}M} \xrightarrow{\sim} \mathrm{RHom}_R(N, K(\mathbf{x}, M)) \xrightarrow{\sim} \mathrm{RHom}_R(N, M \otimes^L_R K(\mathbf{x}, R)).$$

$$\xrightarrow{\sim} \mathrm{RHom}_R(N, M) \otimes^L_R K(\mathbf{x}, R).$$

since $K(\mathbf{x}, R)$ is perfect. Since $\mathbf{x} \cdot N = 0$, $\mathbf{x} \cdot \mathrm{Ext}^*_R(N, M) = 0$ (Exercise, show this is true). Using this and long exact sequence associated to Koszul complexes, one can calculate the isomorphism at the level of Ext.

Alternatively,

$$\mathrm{RHom}_R(N, M) \cong \mathrm{Hom}_R(N, I).$$

where $M \cong I$ is an injective resolution ($\cong$ denotes weak equivalence in $M \cong I$). Now

$$\mathbf{x} \cdot \mathrm{Hom}_R(N, I) = 0.$$

$$\mathrm{RHom}_R(N, M) \otimes_R K(\mathbf{x}, R)$$

$$\cong \mathrm{Hom}_R(N, I) \otimes_R K(\mathbf{x}, R)$$

$$\cong \mathrm{Hom}_R(N, I) \otimes_R K(\mathbf{0}, R).$$

where $\mathbf{0}$ is a zero sequence of length $c$. To get the in particular part of lemma 3, take homology. The details are an exercise. $\square$

If we want to compute $\mathrm{Ext}^n_R(N, \frac{M}{\mathbf{x}M})$ it would be

$$\left(\mathrm{Ext}_R(N, M) \otimes \bigwedge(\Sigma R^c)\right)^n.$$

$$= \bigoplus_i \mathrm{Ext}^i_R(N, M) \otimes_R \left(\bigwedge(\Sigma R^c)\right)^{n-i}.$$

So

$$\mathrm{Ext}^n_R(N, \frac{M}{\mathbf{x}M}) \cong \bigoplus_i \mathrm{Ext}^i_R(N, M) \otimes_R R^{cchoosei-n}.$$

If we had a $\mathbb{Z}$-graded object $V$, we think of it having upper and lower gradings via

$$V^i = V_{-i}.$$

Notation-wise, the supremum of the graded object $V$,

$$\mathrm{sup} V^* = \sup\{i \mid V^i \neq 0\}.$$

$$\inf V^* = \inf\{i \mid V^i \neq 0\}.$$

We brought all of this up to discuss **depth**. Now fix $I \subset R$ an ideal. We can define for any $M \in \mathcal{C}(R)$,

$$\mathrm{depth}_R(I, M) = \mathrm{inf}\mathrm{Ext}^*_R\left(\frac{R}{I}, M\right).$$

This is called the $I$-depth of $M$. We could get a few important properties.

**Remark.** We have the following.

1. Given an exact sequence $0 \to L \to M \to N \to 0$ of complexes, we get a long exact sequence in Ext. If the Ext groups for $L$ and $N$ vanish, then so too must those of $M$. Hence we get

$$\mathrm{depth}_R(I, M) \geq \min\{\mathrm{depth}_R(I, L), \mathrm{depth}_R(I, N)\}.$$

This is all from that exact sequence

$$\mathrm{Ext}^i_R\left(\frac{R}{I}, L\right) \to \mathrm{Ext}^i_R\left(\frac{R}{I}, M\right) \to \mathrm{Ext}^i_R\left(\frac{R}{I}, N\right) \to \mathrm{Ext}^{i+1}_R\left(\frac{R}{I}, L\right) \to \ldots.$$

**Theorem 2.** Let $\mathbf{x} = x_1, \ldots, x_c$ be a generating set for the ideal $I$. Then we can compute

$$\operatorname{depth}_R(I, M) = c - \sup H_*(\mathbf{x}, M).$$

This is true for any $M \in \mathcal{C}(R)$.

If we look at $K(\mathbf{x}, R) \to \frac{R}{\mathbf{x}R} = \frac{R}{I}$, we +-+. We prove this theorem when $M$ is a module. Koszul complexes revisited. We started by introducing it as a tensor product as short complexes. Instead, we could start with an exterior algebra, end up with the differential. It is the same as giving a map $f : F \to R$ where $F$ is a finite free $R$-module and with fixed chosen basis of rank $c$. One can choose a Koszul complex attached to $f$. Look at

$$K(f) = \left( \bigwedge^{*} (\Sigma F), \partial \right).$$

The former module is an exterior algebra on $F$. Taking a differential of a typical element, it has form $\partial(e_{i_1} \wedge \ldots \wedge e_{i_n}) = \sum_{j=1}^{n} (-1)^{j-1} f(e_{i_j}) e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_n}$.

For example,

$$e_1 \wedge e_2 \xrightarrow{\partial} f(e_1)e_2 - e_1 f(e_2).$$

**Lemma 6.** Suppose we have $\mathbf{x} = x_1, \ldots, x_c \subset R$. For any $y \in (\mathbf{x})$, then

$$K(\mathbf{x}, y; M) \cong K(\mathbf{x}, 0, M).$$

The above is isomorphism as $R$-complexes. The latter is just

$$K(\mathbf{x}, M) \otimes K(0, R).$$

*Proof.* We stare at the following picture: We get

$$\begin{array}{ccc} R^{n+1} & \xrightarrow{[x_1 x_2 \ldots x_c y]} & R \\ \sim\uparrow & & \| \\ R^{n+1} & \xrightarrow[{[x_1 x_2 \ldots x_c 0]}]{} & R \end{array}$$

In particular,

$$\sup H_*(\mathbf{x}, y; M) = 1 + \sup H_*(\mathbf{x}, M).$$

Thus,

$$c + 1 - \sup H_*(\mathbf{x}, y; M) = c - \sup H_*(\mathbf{x}, M).$$

□

**Corollary 3.** (Check this corollary.) The quantity

$$c - \sup H_*(\mathbf{x}, M).$$

is independent of the choice of generating set for the ideal $I$.

**Theorem 3.** We have

$$\operatorname{depth}_R(I, M) = c - \sup H_*(\mathbf{x}, M)$$

where $\mathbf{x} = x_1, \ldots, x_c$ generates the ideal $I$.

*Proof.* We prove this when $M$ is a module. What does it mean for

$$\operatorname{depth}_R(I, M) = 0?.$$

It precisely means that

$$\operatorname{Hom}_R(\frac{R}{I}, M) \neq 0.$$

This is because the zeroth Ext group is the homology. The depth zero is if and only if

$$I \subset \operatorname{zdr}_R(M).$$

which holds if and only if $H_c(\mathbf{x}, M) \neq 0$. This is also if and only if

$$\sup H_*(\mathbf{x}, M) = c.$$

So we can assume that $\operatorname{depth}_R(I, M) \geq 1$. In particular, there exists $y \in I$ which is nonzero divisor on $M$. Then in particular $y$ is Koszi-regular on $M$. We would like to compute

$$\operatorname{Ext}_R^*\left(\frac{R}{I}, \frac{M}{\mathbf{x}M}\right).$$

What is the supremum of the above complex? Rees's lemma (that corollary afterwards) says

$$\inf \operatorname{Ext}_R^*\left(\frac{R}{I}, \frac{M}{yM}\right) = \inf \operatorname{Ext}_R^*\left(\frac{R}{I}, M\right) - 1.$$

This applies because

$$y \cdot \left(\frac{R}{I}\right) = 0.$$

In terms of depth, it tells us that

$$\operatorname{depth}_R(I, M) = 1 + \operatorname{depth}_R(I, \frac{M}{\mathbf{x}M}) = 1 + c - \sup H_*(.$$

We also have

$$H_*(\mathbf{x}, \frac{M}{yM}) = H_*(\mathbf{x}, K(y, M)) = H_*(\mathbf{x}, y; M).$$

We just saw that this is exactly

$$H_*(\mathbf{x}, 0; M).$$

because $y \in (\mathbf{x})$. If we calculate the supremum, the supremums are the same. In particular,

$$\sup H_*(\mathbf{x}, \frac{M}{yM}) = \sup H_*(\mathbf{x}, 0; M) = 1 + \sup H_*(\mathbf{x}, M).$$

So this justifies the proof of the theorem by completing an induction step. □

One huge takeaway from the story: we have that the depth is the longest Koszi-regular sequence in $I$. Next time, we discuss depth in the context of local.

# 3 January 25

Let $R$ be a commutative ring and $I \subset R$ be an ideal. Let $M$ be an $R$-complex. Recall

$$\begin{aligned} \operatorname{depth}_R(I, M) &= \inf \operatorname{Ext}_R^*\left(\frac{R}{I}, M\right) \\ &= c - \sup H_*(\mathbf{x}, M) \\ &= \text{length of any maximal } M\text{-Koszi regular sequence} \\ &= \text{length of any maximal } M\text{-regular sequence in } I. \end{aligned}$$

For the second to last equality, we need $M$ to be a module. For the last equality, we further assume $M$ is finitely generated and $I \subset J(R)$.

**Example.** Say $\mathbf{x} = x_1, \ldots, x_c$ and $\mathbf{y} = y_1, \ldots, y_\alpha$. We have

$$\sup H_*(\mathbf{x}, \mathbf{y}, M) \leq \sup H_*(\mathbf{x}, M) + d..$$

We have $K(\mathbf{x}, \mathbf{y}, M) = K(\mathbf{y}, K(\mathbf{x}, M))$. This implies that $I \subset J$ implies $\operatorname{depth}_R(I, M) \leq \operatorname{depth}_R(J, M)$.

**Exercise 4**

Show depth does not change across two ideals when their radicals are the same.

Today, we look at local rings. Say $(R, m, k)$ is a local ring with $k = \frac{R}{m}$. There is a natural notion of depth by just selecting the maximal ideal.

$$\operatorname{depth}(m, M) = \inf \operatorname{Ext}^*_R(k, M).$$

Can compute with $\mathbf{x}$ a system of parameters for $R$. That is, the radical of the ideal they generate is $m$ and it is of minimal length among such ideals. As $R$ is Noetherian, we note $m = (x_1, \ldots, x_n)$. The above depth quantity is

$$n - \sup H_*(\mathbf{x}, M)..$$

We also note that when $d = \dim(R)$, the system of parameters has length $d$.

We move towards the Ausland Buchsbaum equality.

**Definition 3.** Let $F$ be an $R$-complex, then $F$ has **finite flat dimension**. If $F$ is weakly equivalent to

$$0 \to F_b \to \ldots \to F_a \to 0$$

where $F_i$ is flat, will write $\operatorname{flatdim}_R F < \infty$.

**Example.** Perfect examples satisfy the equality and so do Koszul complexes. Any flat module as well.

If the flat dimension of some $F$ is finite, then

$$\operatorname{Tor}_i^R(\bullet, F) = 0 \ \forall |i| \gg 0$$

on Mod $R$. This is because

$$\operatorname{Tor}_i^R(M, F) = H_i(M \otimes_R (0 \to F_b \to \ldots \to F_a \to 0)),.$$

in particular $\operatorname{Tor}_i^R(M, F) = 0$ for $i \notin [a, b]$. Fact: This property cahracterizes the flat dimension being finite.

**Theorem 4.** (Auslander-Buchsbaum equality). Say $(R, m, k)$ is local. When the flat dimension of $F$ is finite,

$$\operatorname{depth}_R(M \otimes_R^\ell F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^\ell F).$$

For any $R$-complex $M$.

Let us look at the case $M = R$. It says

$$\operatorname{depth}_R(F) = \operatorname{depth}_R(R \otimes_R^\ell F) = \operatorname{depth}(R) - \sup H_*(k \otimes_R^\ell F).$$

Suppose we take a finitely generated module. Let $N$ be a finitely generated $R$-module. We can write down the resolution of $N$. Such an $N$ has a **minimal free resolution**. Take a minimal generating set for $N$, $R^{b_0} \twoheadrightarrow \varepsilon N$. It has kernel which is also minimally generated, say by $R^{b_1}$. We can continue the process to get a complex $G$ of free modules.It is called a semi-free resolution of $N$. Weak equivalence from the first degree to $N$.

$$\partial G \subset mG.$$

**Definition 4.** $G$ is called **the free minimal free resolution of $N$.**

If we look at

$$\operatorname{Tor}_i^R(k,N) = H_i(k \otimes_R G) = (k \otimes_R G)_i.$$

So we have

$$\operatorname{Tor}_i^R(k,N) = 0.$$

if and only if $(G)_i = 0$. So flat dimension is finite if and only if $N$ has a finite free resolution. In particular, precisely when $N$ is perfect.

$$\sup \operatorname{Tor}_*^R(k,N) = \text{length of } G = \text{proj } \dim_R(N).$$

Back to AB-equality, a second special case we could look at is as follows: If $N$ is a finitely generated module with finite projective dimension, then

$$\operatorname{depth}_R(N) = \operatorname{depth}(R) - \operatorname{projdim}_R(N).$$

**Corollary 4.** (of the above special case). If the projective dimension of $N$ is finite,

$$\operatorname{depth}_R(N) \leq \operatorname{depth}_R(R)..$$

Equality holds if and only if $N$ is projective (or precisely when it is free in this case). In general, when the projective dimension is infinite, the inequality is false in general. Mention example to Sri in next lecture. Hint: start with ring of depth 0.

Subtracting variants of AB equalities, we get

$$\operatorname{depth}_R(F) - \operatorname{depth}_R(M \otimes_R^{\ell} F) = \operatorname{depth}(R) - \operatorname{depth}(M)..$$

When the Koszul complex is acyclic, ie $H_*(\mathbf{x}, M) = 0$, then the $\operatorname{depth}_R(\mathbf{x}, M) = \infty$.

*Proof.* (proof of the AB-equality). We would like to compute the depth. So we compute

$$\operatorname{RHom}_R(k, M \otimes_R^{\ell} F).$$

Note we have a map from $\operatorname{RHom}_R(k,M) \otimes_R^{\ell} F$. That map is a quasi-isomorphism because $k$ is a finitely generated module, so $G \xrightarrow{\sim} k$ $G_i$ is finite free, also (using?) flat dimension of $F$ is finite.

$$\operatorname{Hom}_R(N,M) \otimes F \xrightarrow{\sim} \operatorname{Hom}_R(N, M \otimes_R F).$$

$N$ is finitely generated $R$-module and $F$ flat. Verify that the above weak equivalence indeed exists. Key observation: $\operatorname{RHom}_R(k,M)$ is weakly equivalent to the complex of $k$-vector spaces.

$$\operatorname{RHom}_R(k,M) \otimes_R^k \xrightarrow{\sim} \operatorname{RHom}_R(k,M) \otimes_k^{\ell} \left(k \otimes_R^{\ell} F\right).$$

$$\begin{aligned}\operatorname{Ext}_R^*(k, M \otimes_R F) &= H^*(\operatorname{RHom}(k,M) \otimes_k^{\ell} \left(k \otimes_R^{\ell} F\right)) \\ &= \operatorname{Ext}_R^*(k,M) \otimes_k H_*(k \otimes_R^{\ell} F).\end{aligned}$$

If the reader wishes to find a reference, this is Foxby's proof. □

Take any $R$-complex, $M$. Let $s = \sup H_*(M)$. Say $s < \infty$. Then

$$\operatorname{depth}_R(M) \geq -s$$

with equality if and only if $\operatorname{depth}_R H_s(M) = 0$. If $M$ was a module, recall the depth is 0 if and only if

$$\inf \operatorname{Ext}_R^*(k,M) = 0.$$

In particular, $H_R(k,M) \neq 0$ or $k \xhookrightarrow{M}$, or $m$ is an associated prime of $M$. One proof: if $M$ as above,

$$\mathrm{Ext}^s_R(N,M) \cong \mathrm{Hom}_R(N, H_s(M)).$$

where $N$ is any $R$-module. Check this (it is not that difficult, using a projective resolution of $M$ ). Key: $M$ is isomorphic to $M'$ with $M'_i = 0$ for all $i > s$.

$$\begin{tikzcd}
\dots \arrow[r] & M_{s+1} \arrow[r, "\partial"] & M_s \arrow[r] \arrow[d] & M_{s-1} \arrow[r] \arrow[d, equal] & \dots \arrow[d, equal] \\
& 0 \arrow[r] & M_s/\partial(M_{s+1}) & M_{s-1} \arrow[r] & \dots \\
& H_s(M) \arrow[r, equal] & Z_s(M)/\partial(M_{s-1}) \arrow[u, hook] & &
\end{tikzcd}$$

In particular,

$$0 \to \Sigma^s H_s(M) \xhookrightarrow{M}\to M'' \to 0.$$

where the second map is isomorphism in homology in degrees $\leq s-1$ and $H_i(M'') = 0$ for $i \geq s$. Let $\mathbf{x} = x_1, \dots, x_n$ be a generating set for $m$. We get

$$H_{i+1}(\mathbf{x}, M'') \to H_i(\mathbf{x}, \Sigma^s H_s(M)) \to H_i(\mathbf{x}, M) \to H_i(\mathbf{x}, M'') \to .$$

We get

$$H_j(M'') = 0.$$

for all $j \geq s-1$. So $M'' \xrightarrow{\sim} M'''$ with $M'''_j$ for $j \geq s$. If we look at $K(\mathbf{x}, M''')$, how far does the complex go? The complex is zero for degrees $j \geq s+n+1$. Thus $H_j(\mathbf{x}, M''') = 0$ for $j \geq s+n+1$. Now we know

$$H_i(\mathbf{x}, \Sigma^s H_s(M)).$$

Thus

$$H_j(\mathbf{x}, M) = 0.$$

for $j \geq n+s+1$ implies that

$$\mathrm{depth}_R(M) \geq -s.$$

And

$$\begin{aligned} H_{n+s}(\mathbf{x}, M) &\cong H_{n+s}(\mathbf{x}, \Sigma^s H_s(M)) \\ &\cong H_n(\mathbf{x}, H_s(M)). \end{aligned}$$

Where these are isomorphisms. We get

$$H_n(\mathbf{x}, H_s(M)) \neq 0$$

implying $\mathrm{depth} H_s(M) = 0$ . This completes the proof of the fact we wanted to prove above.

1. Say $\mathrm{flatdim}_R(F) < \infty$. Then $\forall\ M$,

$$\mathrm{depth}_R(M \otimes^\ell_R F) = \mathrm{depth}_R(M) - \sup H_*(k \otimes^\ell_R F)..$$

2. $s = \sup H_*(M)$ is finite.

$$\mathit{depth}_R(M) \geq -s.$$

with equality if and only if $\mathrm{depth}_R(H_s(M)) = 0$.

APplication. Say that $F$ is a finite free complex,

$$0 \to F_d \to \ldots \to F_0 \to 0$$

where the length of the homology modules are all finite and nonzero.

$$0 < \mathrm{length}(H_*(F)) < \infty.$$

The claim is that for any $M$,

$$\mathrm{depth}_R(M) = d - \sup H_*(F \otimes_R M).$$

Normally, we would apply this in the case when we are looking at a system of parameters and the Koszul complex associated to it. This states a generalization of that fact.

When $mM \neq M$, one can check that $H_*(F \otimes_R M) \neq 0$ (one of the modules is nonzero). Then one gets that $d \geq \mathrm{depth}_R(M)$. Then one gets that

$$d \geq \mathrm{depth}_R(M).$$

Over any local ring $R$, there exists $M$ such that $M \neq mM$ and $\mathrm{depth}_R(M) = \dim(R)$. So $d \geq \dim(R)$. So $d \geq \dim(R)$. This is called the new intersection theorem. Hochster in the 70s, Andre in 2016, Bhatt in 2021. In the last five minutes, we sketh the proof.

*Proof.* Say $s = \sup H_*(F \otimes_R M) < \infty$. The infinite and zero cases are an exercise in homological algebra. If we look at prime $p \neq m$, look at

$$H_i(F \otimes_R M)_p = H_i(F_p \otimes_{R_p} M_p).$$

We have

$$H(F_p) = 0$$

because the length is finite. In particular,

$$H_*(F_p \otimes_{R_p} M_p) = 0.$$

Thus

$$H_i(F \otimes_R M).$$

is $m$-* * * torsion. ie, each $a \in H_i(F \otimes_R M)$ is such that $m^n \cdot a = 0$ for some $n$. IN particular,

$$\mathrm{depth} H_s(F \otimes_R M) = 0$$

so $s = -\mathrm{depth}(F \otimes_R M) = -[\mathrm{depth}_R(M) - \sup H(k \otimes_R^\ell F)]$. Say $\partial F \subset mF$. Solving for the depth, we get

$$\mathrm{depth}_R(M) = \sup H(k \otimes_R^\ell F) - s.$$

The supremum is $d$. This concludes the proof. □

# 4 January 30

Recap: Let $I \subset R$ be an ideal in a commutative Noetherian ring, and let $M$ be and $R$-complex. We recall depth of $I$ in $M$ is given by choosing $\mathbf{x} = x_1, \ldots, x_c$ a generating set, and then

$$\mathrm{depth}_R(I, M) = c - \sup H_*(\mathbf{x}, M).$$

For reference, the definition was that depth was

$$\inf \mathrm{Ext}_R^*(R/I, M).$$

Let $(R, \mathfrak{m}, k)$ be local.

$$\mathrm{depth}_R(M) = \mathrm{depth}(\mathfrak{m}, M).$$

We discussed last time that

$$\mathrm{depth}_R(M) \geq -\sup H_*(M).$$

With equality if and only if $\mathfrak{m} \in \mathrm{Ass} H_s(M)$ where $s = \sup H_*(M)$.

**Exercise 5**

Say $M$ i bounded,

$$0 \to M_b \to \ldots \to M_a \to 0.$$

$$\operatorname{depth}_R(M) \geq \inf\{\operatorname{depth}(M_i) - i | a \leq i \leq b\}$$

with an analogous statement for homology:

$$\operatorname{depth}_R(M) \geq \inf\{\operatorname{depth}(H_i(M)) - i | \inf H_*(M) \leq i \leq \sup H_*(M)\}.$$

The latter statement is a strengthening of the inequality above with minus sup because $\operatorname{depth}(H_i(M)) \geq 0$.

$R$ not necessarily local

**Remark.** Let us assume that $H_*(M)$ is bounded. Take $\mathbf{x} = x_1, \ldots, x_c$. We have

$$\sup H_*(M) \leq \sup H_*(\mathbf{x}, M) \leq \sup H_*(M) + c.$$

Call the former inequality (1) and the latter (2).

**Lemma 7.** (a) Inequality (2) always holds and equality holds if and only if $\operatorname{depth}_R(\mathbf{x}, H_s(M)) = 0$. (b) Inequality (1) holds when $(\mathbf{x}) \subset J(R)$ and $H_i(M)$ is finitely generated $\forall i$. Equality holds when $\mathbf{x}$ is $H_s(M)$-regular where $s = \sup H_*(M)$.

*Proof.* We can reduce the proof to when $c = 1$ and apply induction. Recall

$$H(\mathbf{x}, M) = H(x_1, K[x_2, \ldots, x_c; M]).$$

When $c = 1$, the exact sequence looks like

$$H_i(M) \xrightarrow{x} H_i(M) \to H_i(x, M) \to H_{i-1}(M).$$

The exactness gives us inequality 2 immediately. More precisely, for $i \geq s+1$, one gets $H_i(x, M) = 0$. Moreover

$$H_{s+1}(x, M) \neq 0 \Rightarrow x \text{ is a zero ——— }.$$

This settles (a). We have

$$H_i(M) \neq 0 \Rightarrow H_i(x, M) \neq 0.$$

by Nakayama's lemma. Moreover, this implies

$$\sup H_*(x, M) \geq \sup H_*(M).$$

We have

$$0 \to H_{s+1}(x, M) \to H_s(M) \xrightarrow{x} H_s(M).$$

□

**Corollary 5.** (2) implies that

$$\operatorname{depth}_R(\mathbf{x}, M) \geq -\sup H_*(M).$$

with equality if and only if $(\mathbf{x}) \subset \mathfrak{p} \in \operatorname{Ass} H_s(M)$. Or $\operatorname{depth}(\mathbf{x}, H_s(M)) = 0$ (check the former?).

**Proposition 1.** Let $R$ be a local ring, and $M$ be any complex with $H_*(M)$ bounded. Then for any $I \subset R$,

$$\operatorname{depth}_R(M) \leq \operatorname{depth}_R(I, M) + \dim(\frac{R}{I}).$$

In particular, if $M$ is any finitely generated module,

$$\operatorname{depth}_R(M) \leq \inf\{\dim\left(\frac{R}{\mathfrak{p}}\right) \mid \mathfrak{p} \in \operatorname{Ass}(M)\}$$

$$\leq \dim_R(M).$$

As we will get into later, Cohen-Macaulay modules satisfy equality.

*Proof.* Let $I = (y_1, \ldots, y_c)$. Let $x_1, \ldots, x_d$ be such that they generate an ideal whose radical is the maximal ideal $\mathfrak{m}/I = \mathfrak{m}_{R/I}$. They are a system of parameters. Thus $\sqrt{(\mathbf{y}, \mathbf{x})} = \mathfrak{m}_R$. So we can apply the lemma (b) to $K(\mathbf{y}, M)$. We get that

$$\sup H_*(\mathbf{y}, M) \leq \sup H_*(\mathbf{x}, K(\mathbf{y}, M)).$$

The latter is $\sup H_*(\mathbf{x}, \mathbf{y}; M)$. Thus

$$d + c - \sup H_*(\mathbf{y}, M) \geq d + c - \sup H_*(\mathbf{x}, \mathbf{y}, M).$$

The former is $d + \operatorname{depth}(I, M) \geq \operatorname{depth}_R(M)$. □

Let $R$ be a local ring with residue field $k = \frac{R}{\mathfrak{m}_R}$. The Auslander-Buchsbaum equality states that if $F$ is an $R$-complex with $\operatorname{flatdim}_R(F) < \infty$, then for any $R$-complex $M$,

$$\operatorname{depth}(M \otimes_R^{\ell} F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^{\ell} F).$$

**Proposition 2.** If $I \subset R$ ideal,

$$\operatorname{depth}(I, M) = \inf\{\operatorname{depth} M_{\mathfrak{p}} \mid I \subset \mathfrak{p}\}.$$

*Proof.* $I \subset \mathfrak{p}$ implies

$$\operatorname{depth}(I, M) \leq \operatorname{depth}(\mathfrak{p}, M) \leq \operatorname{depth}_{R_{\mathfrak{p}}}(\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}}).$$

Check the last inequality using Koszul homology. Say $I = (x_1, \ldots, x_c)$, $s = \sup H_*(\mathbf{x}, M), \mathfrak{p} \in \operatorname{Ass} H_s(\mathbf{x}, M)$, then the depth of $H_s(\mathbf{x}, M)_{\mathfrak{p}}$ is zero in the ring $R_{\mathfrak{p}}$.

Consider $K(\mathbf{x}, M)_{\mathfrak{p}} = K(\mathbf{x}, M_{\mathfrak{p}})$. We have

$$\sup H_*(K(\mathbf{x}, M_{\mathfrak{p}})) = \sup H_*(\mathbf{x}, M) = s.$$

$$\operatorname{depth}_{R_{\mathfrak{p}}} K(\mathbf{x}, M_{\mathfrak{p}}) = -s.$$

$$\operatorname{depth}_{R_{\mathfrak{p}}}(K(\mathbf{x}, R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}).$$

$$= \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \sup H_*\left(k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} K(\mathbf{x}, R_{\mathfrak{p}})\right).$$

implying

$$-s \geq \operatorname{depth} M_{\mathfrak{p}} - c.$$

$$\operatorname{depth}_R(I, M) \geq \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}.$$

Using Auslander-Buchsbaum. □

**Remark.** The proof shows

$$\operatorname{depth}_{\mathfrak{p}}(I, M) = \operatorname{depth}_{\mathfrak{p}} M_{\mathfrak{p}}.$$

$\forall \mathfrak{p} \in \operatorname{Ass} H_s(\mathbf{x}, M)$.

**Theorem 5.** Suppose we have $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ a local homomorphism of local rings. Let $M$ be an $R$-complex, and $N$ be an $S$-module such that $N$ is flat as an $R$-module. Then

$$\operatorname{depth}_S(N \otimes_R M) = \operatorname{depth}_R(M) + \operatorname{depth}_{(S/\mathfrak{m}_R S}(N/\mathfrak{m}_R N).$$

We apply this when $M = R$ and $N = S$. In this case,

**Corollary 6.**
$$\operatorname{depth}_S(S) = \operatorname{depth}_R(R) + \operatorname{depth}_{S/\mathfrak{m}_R S}(S/\mathfrak{m}_R S).$$

We had an exact

$$R \to S \to \frac{S}{\mathfrak{m}_R S}.$$

Compare: Under the same hypotheses,

$$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}_R S).$$

Later on, Cohen-Macaulayness behaves nicely with exact sequences.

*Proof.* Let $\mathbf{x} = x_1, \ldots, x_c$ be a generating set for $\mathfrak{m}_R$. Pick $\mathbf{y} = y_1, \ldots, y_d$ in $\mathfrak{m}_S$ such that

$$\mathbf{y}\left(\frac{S}{\mathfrak{m}_R S}\right).$$

is the maximal ideal of $\frac{S}{\mathfrak{m}_R S}$. Then $\mathbf{x}S, \mathbf{y}$ generates $\mathfrak{m}_S$. We want to compute depth, so let us consider

$$K(\mathbf{x}, \mathbf{y}, N \otimes_R M) \cong K(\mathbf{y}, N) \otimes_R K(\mathbf{x}, M).$$

(Commutativity of tensor products). $N$ flat over $R$ implies $K(\mathbf{y}, N)$ has finite flat dimension over $R$. We can apply Auslander-Buchsbaum:

$$\operatorname{depth}_R(K(\mathbf{x}, \mathbf{y}, N \otimes_R M)) = \operatorname{depth}_R(K(\mathbf{x}, M)) - \sup(k \otimes_R K(\mathbf{y}, N)).$$

Note

$$(\mathbf{x}, \mathbf{y}) \cdot H(\mathbf{x}, \mathbf{y}, N \otimes_R M) = 0.$$

So

$$\mathfrak{m}_S \cdot H(\mathbf{x}, \mathbf{y}, N \otimes_R M) = 0.$$
$$\mathfrak{m}_R H(\mathbf{x}, \mathbf{y}, N \otimes_R M) = 0.$$

Similarly,

$$\mathfrak{m}_R H(\mathbf{x}, M) = 0.$$

Above

$$-\sup(k \otimes_R K(\mathbf{y}, N)) = -\sup(K\left(\mathbf{y}, \frac{N}{\mathfrak{m}_R N}\right).$$

This implies that

$$\operatorname{depth}_S(N \otimes_R M) = -\sup H_*(\mathbf{x}, \mathbf{y}, N \otimes_R M) = -\sup H_*(\mathbf{x}, M) - \sup H_*(\mathbf{y}, \frac{N}{\mathfrak{m}_R N}).$$

$\square$

**Exercise 6**

Let $M$ be a finitely generated $R$-module with $R$ local. We have a sequence of inequalities

$$\operatorname{depth}(R) - \operatorname{depth}_R(M) \le \operatorname{grade}_R(M) \le \operatorname{codim}_R(M) \le \dim(R) - \dim(M) \le \operatorname{pdim}_R(M)$$

where $\operatorname{grade}_R(M) = \operatorname{depth}_R(\operatorname{ann}_R(M), R)$. Codimension is $\operatorname{ht}(\operatorname{ann}_R(M)) = \inf\{\dim R_{\mathfrak{p}} \mid \mathfrak{p} \supset \operatorname{ann}_R(M)\}$. The last inequality is the most nontrivial inequality, following from the intersection theorem, recalled below:

**Theorem 6.** If we have a finite free complex

$$0 \to F_d \to \ldots \to F_0 \to 0.$$

where $0 < \operatorname{length}(H_*(F)) < \infty$. Then $d \ge \dim(R)$.

This is a consequence of Auslander Buchsbaum and the existence of finitely generated Cohen Macaulay modules.

**Corollary 7.** Let $R$ be a local ring, $M$ be a nontrivial finitely generated $R$-module with $\operatorname{pdim}_R(M) < \infty$. Then for any finitely generated $R$-module $N$,

$$\dim_R(N) - \dim_R(M \otimes_R N) \le \operatorname{pdim}_R(M).$$

Deducing this corollary from the theorem above is an exercise.

Inspired by a result by Serre: Let $R$ be regular local ring. If we take any finite $R$-modules $M, N$, then

$$\dim_R(N) - \dim_R(M \otimes_R N) \le \dim(R) - \dim(M).$$

We discuss Cohen-Macaulay rings on Wednesday.

# 5 February 6

Let $R$ be a local ring $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring. Recall that

$$\operatorname{depth}(R) \le \dim R \le \operatorname{edim}(R).$$

where the last number is **embedding dimension,** defined as

$$\operatorname{edim}(R) = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2).$$

So we have two inequalities we label 1 and 2. Recall when equality holds for 1, $R$ was said to be **Cohen-Macaulay.** 2 holds by Krull's height theorem. We are interested in Cohen Macaulay rings where equality holds throughout. The embedding codepth of $R$ is

$$\operatorname{codepth}(R) = \operatorname{edim}(R) - \operatorname{depth}(R).$$

**Definition 5.** We say $R$ is regular if $\operatorname{codepth}(R) = 0$.