# Modern Algebra II Notes on MATH 6320

Daniel Koizumi

January 16, 2023

### Contents

## 1 January 11

As always, let R be a ring. We noted

$$D(R) \cong \mathrm{KProj}(R)$$
.

To recap, a complex  $P \in \mathcal{C}(\operatorname{Proj}(R))$  is K projective we have the existing map in the following diagram:

For example,

**Example.**  $P \in \mathcal{C}(\operatorname{Proj}(R))$  with  $P_i = 0$  for all  $i \ll 0$ . To sketch a proof of this claim, construct liftings one step at a time.

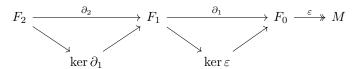
$$(\qquad \dots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \dots \longrightarrow P_{a+1} \xrightarrow{\tilde{\alpha}} P_a \xrightarrow{\tilde{\alpha}} 0 \longrightarrow \dots ) \xrightarrow{\alpha} Y$$

Here,  $\tilde{\alpha}(S)$  should satisfy

- $\varepsilon \tilde{\alpha} = \alpha(S)$
- $\partial \tilde{\alpha}(S) = \tilde{\alpha}(\partial S)$ .
- $\varepsilon$  is surjective implies that  $\varepsilon$  is surjective on boundaries.
- $H(\varepsilon)$  is isomorphic (H is the homology functor) implies that  $\varepsilon: Z(X) \twoheadrightarrow Z(Y)$
- $\ker(\varepsilon)$  is acyclic.

Once you construct these maps, the claim holds.

Note that every module has a K-projective resolution. Every module M admits a surjection  $F_0 woheadrightarrow M$  from a free module  $F_0$ , call this map  $\varepsilon$ . The kernel of  $\varepsilon$  then admits a surjection  $F_1 woheadrightarrow \ker \varepsilon$  from a free module  $F_1$ . Continuing the process one more step yields the following diagram:



If we continue this process, we get the K-projective resolution. To make the choice of resolution canonical, we choose the free modules to be freely generated by the elements of the kernels that they surject onto. This yields a functor from modules to K-projective resolutions.

**Definition 1.** A K-projective resolution of  $M \in \mathcal{C}(R)$  is a morphism  $\varepsilon : P \to M$  such that

- $\varepsilon$  is a quasi-isomorphism, and
- P is K-projective.

We can make this functorial by taking  $R \times M \rightarrow M$ ,

$$\varepsilon(r,m) = rm.$$

We then set

$$\bigoplus_{m \in M} R \times m = R^{\oplus M} \twoheadrightarrow M.$$

The last surjection above is the  $\varepsilon$  map.

**Theorem 1.** For all  $M \in \mathcal{C}(R)$ , there exists a surjective K-projective resolution

$$P \twoheadrightarrow M$$

which is a weak equivalence.

**Definition 2.** An R-complex F is semi-free if it has a filtration

$$(0) \subset F_0 \subset F_1 \subset \ldots \subset \bigcup_{n \ge 0} F_n = F$$

such that

- 1.  $F(n) \subset F$  is a subcomplex,
- 2. F(n+1)/F(n) is a graded free module with trivial boundary maps  $(\partial=0)$ . In particular,  $\partial(F_{n+1})\subset F_n$ .

### Exercise 1

Show that semi-free implies K-projective. Also show that unions are K-projective.

**Example.** Take a complex of free modules

$$\ldots \to F_{n+1} \to F_n \to 0.$$

Let F(n) be the truncation consisting of degrees n and lower. Then  $\frac{F(n+1)}{F(n)}$  is precisely  $\Sigma^{n+1}F_{n+1}$ .

**Theorem 2.** Each  $M \in \mathcal{C}(R)$  has a semi-free resolution

$$F \rightarrow M$$
.

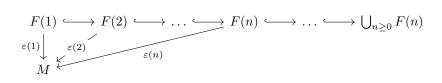
(The map above is a weak equivalence.)

#### Corollary 1. Every K-projective is a retract of a semi-free.

*Proof.* (Corollary proof): Let P be K-projective. The following diagram completes the proof:



*Proof.* (Theorem proof): We use the small object argument. We want to construct



such that  $\frac{F(n+1)}{F(n)}$  is a graded complex of free modules with zero differential which is surjective on cycles. We need that  $\varepsilon(1)$  is surjective on cycles. We have

$$F(1) = H(F(1)) \longrightarrow H(F(n)) \xrightarrow{\phi} H(F(n+1))$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \qquad$$

for all n where the kernel of  $\phi$  contains the kernel of  $H(\varepsilon(n))$ , so that  $H(\varepsilon(n))$  is surjective for all n.

Next, we note  $\ker(H(\varepsilon(n))) \subset H(F(n))$  goes to 0 under H(i(n)). We have  $H(F) \xrightarrow{\sim} H(M)$ .

**Remark.** Given  $\varepsilon: X \to Y$  where  $Z(\varepsilon)$  and  $H(\varepsilon)$  are surjective, we have that  $\varepsilon$  is surjective. See the four lemmas applied to the following diagrams:

Say we have calculated  $\varepsilon(n): F(n) \to M$ . Choose cycles  $\{z_{\lambda}\}$  in F(n) that map to the generating set of  $\ker(H(\Sigma(n)))$ . Pick  $w_{\lambda} \in M$  such that  $\partial w_{\lambda} = \varepsilon(n)(z_{\lambda})$ . Set

$$F(n+1) = F(n) \oplus_{\lambda} R \cdot e_{\lambda}.$$

Above, we have  $|e_{\lambda}|=|z_{\lambda}|+1$  and  $\partial|_{F(n)}=\partial^{F(n)}$ . (Attached are some iPad notes, where the proof continues.)