Commutative Algebra

Notes on MATH 7830

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January 25, 2023

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Let R be a commutative Noetherian ring, and let M be an R-module. What does it mean for an element $r \in R$ to be a **zero-divisor**? It simply means that for some $m \neq 0$, $r \cdot m = 0$.

$$zdr_R(M) = \{r \in R | r \text{ is a zero divisor on } M\} = \bigcup_{\mathfrak{p} \in \operatorname{Ann}_R M} \mathfrak{p}.$$

We can say $r \in R$ is a non-zero divisor if it is not a zero divisor (abbrev. nzd). Fix a sequence $\mathbf{x} = x_1, \ldots, x_n \in R$.

Definition 1. We say that $\mathbf{x}$ is a **weakly M-regular** sequence on M if x_{i+1} is not a zero divisor on $\frac{M}{(x_1,\ldots,x_i)M}$ for all applicable i. It becomes a **regular sequence** if in addition $\frac{M}{\mathbf{x}M} \neq 0$.

Example. If $R = \Bbbk[x_1, \ldots, x_n]$, and note $\mathbf{x} = x_1, \ldots, x_n$ is a regular sequence on R.

We now introduce Koszul complexes. Given $r \in R$, we can write $K(r, R)$ to be the complex

$$0 \to R \to R \to 0.$$

there $R \to R$ is the homothetic map multiplication by r. The left first copy of R is labeled degree 1. Here, taking the homology functor of the sequence provides 0 on the left R if and only if r is a nzd. We have

$$K(\mathbf{x}, R) = \bigotimes_{i=1}^{n} K(x_i, R).$$

We will get

$$0 \to R \to R^n \to R^{\binom{n}{2}} \to \ldots \to R^{\binom{n}{2}} \to R^n \to R \to 0.$$

(exercise calculate the first and last maps). Given $M \in \mathcal{C}(R)$,

$$K(\mathbf{x}, M) = K(\mathbf{x}, R) \otimes_R M.$$

If M is just an R-module, it is merely replacing copies of R with copies of M. We denote $H_i(\mathbf{x}, M) = H_i(K(\mathbf{x}, M))$. Note

$$H_0(\mathbf{x}, M) = \frac{M}{\mathbf{x}M}.$$

$$H_1(\mathbf{x}, M) = \{m \in M | x_i \cdot M = 0 \forall i\} = (0 :_M (\mathbf{x})).$$

Remark: Note

$$K(\mathbf{x}, M) = K(x_1, R) \otimes K(x_2, R) \otimes \ldots \otimes K(x_n R) \otimes_R M$$
$$K(x_1, R) \otimes K(\mathbf{x}_{\geq 2}, M).$$

So we have

$$K(\mathbf{x}, M) = K(x_1, K(\mathbf{x}_{\geq 2}, M)).$$

In many proofs in this course, being able to decompose the Koszul complex in this way will allow us to do induction.

Remark:
We have $X, Y \in \mathcal{C}(R)$, we get the isomorphism

$$X \otimes_R Y \to Y \otimes_R X.$$

via $x \otimes_R y \mapsto (-1)^{(x)(y)} y \otimes_R x$ For any $\sigma \in S_n$,

$$K(x_1, \ldots, x_n) \cong K(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, R).$$

Also, we have a second perspective on Koszul complexes: that they are the iterated mapping cones. Given a morphism of complexes

$$f : X \to Y.$$

recall the **cone** is defined

$$\text{cone}(f) = \left(Y \oplus \Sigma X, \begin{pmatrix} \partial^Y & f \\ 0 & -\partial^X \end{pmatrix} \right).$$

We get that

$$0 \to Y \to \text{cone}(f) \to \Sigma X \to 0.$$

$y \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}$, $\begin{pmatrix} y \\ \Sigma x \end{pmatrix} \mapsto \Sigma x$. The long exact sequence in homology yields

$$\ldots \to H_i(X) \to H_i(Y) \to H_i(\text{cone}(f)) \to H_i(\Sigma X) \cong H_{i-1}(X) \to \ldots.$$

Where the connecting map $H_i(X) \to H_i(Y)$ is just $H_i(f)$.

Now consider $x \in R$, and the homothetic map $f : R \to R$.

Example. $\text{cone}(f) = \left(R \oplus R, \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right) = K(x, R)$. Ditto for the homothetic map on modules.

$$\text{cone}(M \to M) = K(x, M).$$

Thus, $K(\mathbf{x}, M) = K(x_1, K(\mathbf{x}_{\geq 2}, M)$ is $\text{cone}(K(\mathbf{x}_{\geq 2}, M) \to K(\mathbf{x}_{\geq 2}, M))$. This gives

$$H_i(\mathbf{x}_{\geq 2}, M) \to H_i(\mathbf{x}_{\geq 2}, M) \to H_i(\mathbf{x}, M) \to H_{i-1}(\mathbf{x}_{\geq 2}, M) \to \ldots$$

where the connecting morphism is multiplication by x_1 up to sign. By looking at the images/cokernels/kernels of one segment in this sequence, we get induced SES

$$0 \to H_i(\mathbf{x}_{\geq 2}, M)/x_1 H_i(\mathbf{x}_{\geq 2}, M) \to H_i(\mathbf{x}, M) \to (0 :_{H_{i-1}(\mathbf{x}_{\geq 2}, M)} x_1) \to 0.$$

If M is an R-module, $\mathbf{x} = x_1, \ldots, x_n \subset R$,

$$K(\mathbf{x}, M) \twoheadrightarrow H_0(\mathbf{x}, M) = \frac{M}{\mathbf{x}M}.$$

So,

$$K(\mathbf{x}, M) \to \frac{M}{\mathbf{x}M}.$$

is a weak equivalence if and only if

$$H_i(\mathbf{x}, M) = 0 \forall i \geq 1.$$

Lemma 1. When $\mathbf{x}$ is a weakly M-regular,

$$K(\mathbf{x}, M) \twoheadrightarrow \frac{M}{\mathbf{x}M}$$

which is also a weak equivalence.

Proof. When $n = 1$,

$$0 \to M \to M \to 0$$

has zero homology at degree 1 if and only if x is a nonzero divisor on M.

Now say when $n \geq 2$, we know that $K(\mathbf{x}, M) = K(x_n, K(\mathbf{x}_{\leq n-1}, M))$. By our induction hypothesis,

$$K(\mathbf{x}_{\leq n-1}, M) \twoheadrightarrow \frac{M}{(\mathbf{x}_{\leq n-1})M}.$$

We have

$$K(x, R) = (0 \to R \to R \to 0).$$

is semi-free.

$$K(\mathbf{x}, M) = K(x_n, R) \otimes_R K(\mathbf{x}_{\leq n-1}, M) \to K(x_n, \frac{M}{\mathbf{x}_{\leq n-1}M}).$$

Exercise 1

Prove this using the Koszul homology long exact sequence.

□

Definition 2. $\mathbf{x}$ is **Koszi-regular** on M if

$$K(\mathbf{x}, M) \twoheadrightarrow^{\sim} \frac{M}{\mathbf{x}M}.$$

. Note that $x_1, \dots, x_n$ is Koszi-regular on M if and only if any permutation

$$x_{\sigma(1)}, \dots, x_{\sigma(n)}$$

is Koszi-regular on M for any $\sigma \in S_n$.

Exercise 2

(Weakly) regular sequences are senitive to permutations.

Theorem 1. Say $\mathbf{x} \subset J(R)$ and $M \neq 0$ is finitely generated as an R-module. Then the following are equivalent:

1. $\mathbf{x}$ is regular ($\equiv$ weakly regular).
2. $H_i(\mathbf{x}, M) = 0$ for all $i \geq 1$.
3. $H_1(\mathbf{x}, M) = 0$.

Our main application is when R is a local ring and $\mathbf{x} \subset \mathfrak{m}_R$. We use Nakayama's lemma: $J(R) \neq M$, so regularity is equivalent to weak regularity.

Proof. We know $1 \Rightarrow 2 \Rightarrow 3$. It remains to show $3 \Rightarrow 1$. We want to examine $H_*(x_1, \ldots, x_{n-1}, x_n, M)$. The module

$$K(\mathbf{x}, M) = K(x_n, K(\mathbf{x}_{\leq n-1}, M))$$

provides long exact sequence containing

$$0 \to H_i(\mathbf{x}_{\leq n-1}, M)/(x_n)H_i(\mathbf{x}_{\leq n-1}, M) \to H_i(\mathbf{x}, M) \to \left(0 :_{H_{i-1}(\mathbf{x}_{\leq n-1}, M)} x_n\right).$$

We have that

$$H_1(\mathbf{x}, M) = 0 \Rightarrow H_1(\mathbf{x}_{\leq n-1}, M) = (x_n)H_1(\mathbf{x}_{\leq n-1}, M).$$

so apply Nakayama's. We are doing the proof of equivalence by induction on n (it is already proven for $n = 1$), so we have

$$x_1, \ldots, x_{n-1}.$$

is M-regular. This implies further that

$$H_i(\mathbf{x}_{\leq n-1}, M) = 0.$$

for all $i \geq 1$. Moreover, applying this to our exact sequence above, $(0 : x_n) = 0$, so $H_0(\mathbf{x}_{\leq n}, M) = \ker\left(\frac{M}{\mathbf{x}_{\leq n-1}M} \to \frac{M}{\mathbf{x}_{\leq n-1}M}\right)$. $\square$

Corollary 1. $\mathbf{x} \subset J(R)$, M finitely generated. The property that $\mathbf{x}$ is M-regular does not depend on the ordering of $\mathbf{x}$.

Lemma 2. Suppose we have a sequence $x_1, \ldots, x_n \subset R$ (now we drop the assumption regarding the Jacobson radical). Let M be an R-module. The following are equivalent:

1. $\mathbf{x}$ is Koszi-regular on M.
2. $\{x_1^{a_1}, \ldots, x_n^{a_n}\}$ is Koszi-regular on M for any choice $a_i \geq 1$.
3. $\mathbf{x}^{\mathbf{a}}$ is Koszi-regular on M for some $\mathbf{a} \geq (1, \ldots, 1)$.

Proof. It suffices to prove $x_1, \ldots, x_n$ is Koszi-regular on M if and only if $x_1^a, \ldots, x_n$ for some $a \geq 1$. Recall that Koszi-regularity means

$$K(x_1^a, x_2, \ldots, x_n, M) \to^{\sim} K(x_1^a, \frac{M}{(x_{\geq 2})M}).$$

Replacing M with $\frac{M}{(\mathbf{x}_{\geq 2})M}$, we are reduced to proving x is weakly M-regular if and only if x^a is weakly M-regular for some $a \geq 1$. x is not a zero divisor on M if and only if x^a is not a zero divisor on M for some or all $a \geq 1$. $\square$

Exercise 3

(this is also a theorem, called the rigidity of Koszul homology). If we take $\mathbf{x} \subset J(R)$ and M a finitely generated R-module, then $H_i(\mathbf{x}, M) = 0$ for some $i \geq 0$ implies that $H_j(\mathbf{x}, M) = 0$ for all $j \geq i$.

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Let R be a commutative and Noetherian ring, and $M, N \in \mathcal{C}(R)$. Note

$$\mathrm{RHom}_R(M, N) = \mathrm{Hom}_R(pM, N).$$

where $pM \xrightarrow{\sim} M$ is a K-projective resolution. Recall

$$\mathrm{Ext}_R^*(M, N) = H^*(\mathrm{RHom}_R(M, N)).$$

For any $M, N, P \in \mathcal{C}(R)$, there exists

$$\theta : \mathrm{RHom}_R(M, N) \otimes_R^L P \to \mathrm{RHom}_R(M, N \otimes_R^L P).$$

Lemma 3. This is a weak equivalence when P is **perfect**. In particular

$$P \xrightarrow{\sim} (0 \to P_b \to \ldots \to P_c \to 0)\,.$$

Where P_i is finitely generated as a projective R-module. We get a morphism of complexes

$$\mathrm{Hom}_R(pM, N) \otimes_R p(P) \to \mathrm{Hom}_R(pM, N \otimes_R p(P)).$$

Defined by

$$f \otimes x \mapsto \left(m \mapsto (-1)^{|x||m|} f(m) \otimes x\right).$$

In the category of modules over R, if we look at

$$\mathrm{Hom}_R(M, N) \otimes_R P \to \mathrm{Hom}_R(M, N \otimes_R P)$$

to prove this when P is a finitely generated projective.

Lemma 4. Rees' Lemma. Let $\mathbf{x} \subset R$ be a finite subset. Let M, N be R-modules. Let N be an R-module such that $\mathbf{x}N = 0$. And let M be an R-module such that $\mathbf{x}$ is Koszi-regular on M. This means that

$$K(\mathbf{x}, M) \xrightarrow{\sim} \frac{M}{\mathbf{x}M}.$$

Lemma 5.

$$\mathrm{RHom}_R(N, \frac{M}{\mathbf{x}M} \xrightarrow{\sim} \mathrm{RHom}_R(N, M) \otimes_R \bigwedge^*(\Sigma R^c).$$

In particular,

$$\mathrm{Ext}_R^*(N, \frac{M}{\mathbf{x}M} \cong \mathrm{Ext}_R^*(N, M) \otimes_R \bigwedge^*(\Sigma R^c).$$

Where c denotes the rank of the free module.

Corollary 2.

$$\inf \mathrm{Ext}_R^*(N, M) = \inf \mathrm{Ext}_R^*(N, \frac{M}{\mathbf{x}M} + c.$$

We also have

$$\mathrm{Ext}_R^*(N, M) \cong \mathrm{Ext}_R^{*+c}(N, \frac{M}{\mathbf{x}M}).$$

Recalling the alternating product complex will have zero differentials.

Proof. We want to compute

$$\mathrm{RHom}_R(N, \frac{M}{\mathbf{x}M} \xrightarrow{\sim} \mathrm{RHom}_R(N, K(\mathbf{x}, M)) \xrightarrow{\sim} \mathrm{RHom}_R(N, M \otimes^L_R K(\mathbf{x}, R)).$$

$$\xrightarrow{\sim} \mathrm{RHom}_R(N, M) \otimes^L_R K(\mathbf{x}, R).$$

since $K(\mathbf{x}, R)$ is perfect. Since $\mathbf{x} \cdot N = 0$, $\mathbf{x} \cdot \mathrm{Ext}^*_R(N, M) = 0$ (Exercise, show this is true). Using this and long exact sequence associated to Koszul complexes, one can calculate the isomorphism at the level of Ext.

Alternatively,

$$\mathrm{RHom}_R(N, M) \cong \mathrm{Hom}_R(N, I).$$

where $M \cong I$ is an injective resolution ($\cong$ denotes weak equivalence in $M \cong I$). Now

$$\mathbf{x} \cdot \mathrm{Hom}_R(N, I) = 0.$$

$$\mathrm{RHom}_R(N, M) \otimes_R K(\mathbf{x}, R)$$
$$\cong \mathrm{Hom}_R(N, I) \otimes_R K(\mathbf{x}, R)$$
$$\cong \mathrm{Hom}_R(N, I) \otimes_R K(\mathbf{0}, R).$$

where $\mathbf{0}$ is a zero sequence of length c. To get the in particular part of lemma 3, take homology. The details are an exercise. □

If we want to compute $\mathrm{Ext}^n_R(N, \frac{M}{\mathbf{x}M})$ it would be

$$\left(\mathrm{Ext}_R(N, M) \otimes \bigwedge (\Sigma R^c)\right)^n.$$

$$= \bigoplus_i \mathrm{Ext}^i_R(N, M) \otimes_R \left(\bigwedge (\Sigma R^c)\right)^{n-i}.$$

So

$$\mathrm{Ext}^n_R(N, \frac{M}{\mathbf{x}M}) \cong \bigoplus_i \mathrm{Ext}^i_R(N, M) \otimes_R R^{cchoosei-n}.$$

If we had a $\mathbb{Z}$-graded object V, we think of it having upper and lower gradings via

$$V^i = V_{-i}.$$

Notation-wise, the supremum of the graded object V,

$$\mathrm{sup} V^* = \sup\{i \mid V^i \neq 0\}.$$

$$\inf V^* = \inf\{i \mid V^i \neq 0\}.$$

We brought all of this up to discuss **depth**. Now fix $I \subset R$ an ideal. We can define for any $M \in \mathcal{C}(R)$,

$$\mathrm{depth}_R(I, M) = \mathrm{inf}\mathrm{Ext}^*_R\left(\frac{R}{I}, M\right).$$

This is called the I-depth of M. We could get a few important properties.

Remark. We have the following.

1. Given an exact sequence $0 \to L \to M \to N \to 0$ of complexes, we get a long exact sequence in Ext. If the Ext groups for L and N vanish, then so too must those of M. Hence we get

$$\mathrm{depth}_R(I, M) \geq \min\{\mathrm{depth}_R(I, L), \mathrm{depth}_R(I, N)\}.$$

This is all from that exact sequence

$$\mathrm{Ext}^i_R\left(\frac{R}{I}, L\right) \to \mathrm{Ext}^i_R\left(\frac{R}{I}, M\right) \to \mathrm{Ext}^i_R\left(\frac{R}{I}, N\right) \to \mathrm{Ext}^{i+1}_R\left(\frac{R}{I}, L\right) \to \dots.$$

Theorem 2. Let $\mathbf{x} = x_1, \ldots, x_c$ be a generating set for the ideal I. Then we can compute

$$\operatorname{depth}_R(I, M) = c - \operatorname{sup} H_*(\mathbf{x}, M).$$

This is true for any $M \in \mathcal{C}(R)$.

If we look at $K(\mathbf{x}, R) \to \frac{R}{\mathbf{x}R} = \frac{R}{I}$, we +-+. We prove this theorem when M is a module. Koszul complexes revisited. We started by introducing it as a tensor product as short complexes. Instead, we could start with an exterior algebra, end up with the differential. It is the same as giving a map $f : F \to R$ where F is a finite free R-module and with fixed chosen basis of rank c. One can choose a Koszul complex attached to f. Look at

$$K(f) = \left(\bigwedge^{*} (\Sigma F), \partial \right).$$

The former module is an exterior algebra on F. Taking a differential of a typical element, it has form $\partial(e_{i_1} \wedge \ldots \wedge e_{i_n}) = \sum_{j=1}^{n} (-1)^{j-1} f(e_{i_j}) e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_n}$.

For example,

$$e_1 \wedge e_2 \xrightarrow{\partial} f(e_1)e_2 - e_1 f(e_2).$$

Lemma 6. Suppose we have $\mathbf{x} = x_1, \ldots, x_c \subset R$. For any $y \in (\mathbf{x})$, then

$$K(\mathbf{x}, y; M) \cong K(\mathbf{x}, 0, M).$$

The above is isomorphism as R-complexes. The latter is just

$$K(\mathbf{x}, M) \otimes K(0, R).$$

Proof. We stare at the following picture: We get

$$\begin{array}{ccc} R^{n+1} & \xrightarrow{[x_1 x_2 \ldots x_c y]} & R \\ \sim\uparrow & & \| \\ R^{n+1} & \xrightarrow[{[x_1 x_2 \ldots x_c 0]}]{} & R \end{array}$$

In particular,

$$\operatorname{sup} H_*(\mathbf{x}, y; M) = 1 + \sup H_*(\mathbf{x}, M).$$

Thus,

$$c + 1 - \sup H_*(\mathbf{x}, y; M) = c - \sup H_*(\mathbf{x}, M).$$

$\square$

Corollary 3. (Check this corollary.) The quantity

$$c - \sup H_*(\mathbf{x}, M).$$

is independent of the choice of generating set for the ideal I.

Theorem 3. We have

$$\operatorname{depth}_R(I, M) = c - \sup H_*(\mathbf{x}, M)$$

where $\mathbf{x} = x_1, \ldots, x_c$ generates the ideal I.

Proof. We prove this when M is a module. What does it mean for

$$\operatorname{depth}_R(I, M) = 0?.$$

It precisely means that

$$\operatorname{Hom}_R(\frac{R}{I}, M) \neq 0.$$

This is because the zeroth Ext group is the homology. The depth zero is if and only if

$$I \subset \operatorname{zdr}_R(M).$$

which holds if and only if $H_c(\mathbf{x}, M) \neq 0$. This is also if and only if

$$\sup H_*(\mathbf{x}, M) = c.$$

So we can assume that $\operatorname{depth}_R(I, M) \geq 1$. In particular, there exists $y \in I$ which is nonzero divisor on M. Then in particular y is Koszi-regular on M. We would like to compute

$$\operatorname{Ext}_R^*\left(\frac{R}{I}, \frac{M}{\mathbf{x}M}\right).$$

What is the supremum of the above complex? Rees's lemma (that corollary afterwards) says

$$\inf \operatorname{Ext}_R^*\left(\frac{R}{I}, \frac{M}{yM}\right) = \inf \operatorname{Ext}_R^*\left(\frac{R}{I}, M\right) - 1.$$

This applies because

$$y \cdot \left(\frac{R}{I}\right) = 0.$$

In terms of depth, it tells us that

$$\operatorname{depth}_R(I, M) = 1 + \operatorname{depth}_R(I, \frac{M}{\mathbf{x}M}) = 1 + c - \sup H_*(.$$

We also have

$$H_*(\mathbf{x}, \frac{M}{yM}) = H_*(\mathbf{x}, K(y, M)) = H_*(\mathbf{x}, y; M).$$

We just saw that this is exactly

$$H_*(\mathbf{x}, 0; M).$$

because $y \in (\mathbf{x})$. If we calculate the supremum, the supremums are the same. In particular,

$$\sup H_*(\mathbf{x}, \frac{M}{yM}) = \sup H_*(\mathbf{x}, 0; M) = 1 + \sup H_*(\mathbf{x}, M).$$

So this justifies the proof of the theorem by completing an induction step. $\square$

One huge takeaway from the story: we have that the depth is the longest Koszi-regular sequence in I. Next time, we discuss depth in the context of local.