Representation Theory
Notes on MATH 6260

Daniel Koizumi

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Course structure: We will look at representation theory on finite groups first, and then proceed to look at it over compact groups.

Textbook: J.P. Serre: Linear Representations of Finite Groups

Definition 1. Let V be a vector space over $\mathbb{C}$, and G be a group. A representation (π, V) of G on V is a homomorphism

$$\pi : G \to GL(V)$$

$\dim V = \dim \pi$. V is the representation space of (π, V).

Denote by $Rep(G)$ all representations of G. We would like to say that it is a category. Take two representations (π, V) and (ν, U). Look at a linear map $\varphi : V \to U$. If

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\pi(g)"'] & U \arrow[d, "\nu(g)"] \\ V \arrow[r, "\varphi"'] & U \end{tikzcd}$$

commutes, for all $g \in G$, we call φ an intertwining operator. We can also define the sum of intertwining operators, which would be an intertwining operator. $\mathrm{Hom}_{\mathbb{C}}(V, U)$ of linear maps is a complex vector space. The intertwining maps form a linear subspace of the Hom. We denote this space $\mathrm{Hom}_G(V, U)$.

Assume we have a representation (π, V). A linear subspace $U \subset V$ is G-invariant if $\pi(g)(U) \subset U$. We can define $\nu(g)$ as $\pi(g)|_U : U \to U$. (ν, U) is a representation of G. This constructs **subrepresentations of** (π, V). In this case, the inclusion $i : U \to V$ is an intertwining map. We define quotient representations. Look at a quotient space $\frac{V}{U}$. We can define $\rho(g)(v+U) = \pi(g)(v) + U$. This is well defined. If $\varphi : (\pi, V) \to (\nu, U)$ is a morphism of representations, $\ker\varphi = \{v \in V | \varphi(v) = 0\}$. Look at $v \in \ker\varphi$,

$$\varphi(\pi(g)v) = \nu(g)\varphi(v) = 0$$

so $\pi(g)v \in \ker\varphi$. $\ker\varphi$ is a subrepresentation of (π, V). Can look at the subspace $\mathrm{im}\varphi$, which is also an invariant under G. $\mathrm{im}\varphi$ is a subrepresentation of (ν, U).

$$\nu(g)u = \nu(g)\varphi(v) = \varphi(\pi(g)v) \in \mathrm{im}\varphi$$

If we have two representations (π, V), (ν, U). We can define $V \oplus U = W$. For any $g \in G$, we can define $\pi \oplus \nu(g)(v, u) = (\pi(g)(v), \nu(g)(u))$. This is a **direct sum of** (π, V), (ν, U). If we have a morphism $V \xrightarrow{\varphi} U$,

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & U \\
V/ker\varphi \arrow[r, "\sim"] \arrow[ur, dashed] & im\varphi \arrow[u]
\end{tikzcd}$$

The resulting map $\frac{V}{\ker\varphi} \to U$ is a linear isomorphism, but it is also an isomorphism in the category of representations.

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Take (π, V) a representation of G. Take nonzero $v \in V$.Consider $\{\pi(g)v | g \in G\}$, which is a finite set since we are assuming that G is finite.

We can look at $U = \mathrm{span}(\{\pi(g)v | g \in G\})$, which is finite dimensional since it is spanned by the vectors. Note $v \in U$. U is a G-invariant subspace of V.

$$u = \sum c_g \pi(g)v \in \mathbb{C}$$

$$\pi(h)\mu = \sum c_g \pi(hg)v$$

U is a subrepresentation of V.

The representation (π, V) is irreducible if it is not the zero representation and the only G-invariant subspaces of V are V and $\{0\}$. First remark: irreducible representations have to be finite dimensional. If V is irreducible, $v \in V$ nonzero, $U = span\{\pi(g)v | g \in G\}$ contains $v \neq 0$. By irreducibility, $U \neq 0$ implies $U = V$. Every representation of a finite group contains an irreducible subrepresentation. This is because looking at one $span\{\pi(g)v | g \in G\}$, which is finite dimensional, we can continue taking subrepresentations until either the dimension is 1 or we hit an irreducible.

Assume we take (π, V), U invariant subspace. π induces a subrepresentation (ν, U).

Theorem 1. (Mascke) There exists a subrepresentation (ρ, W) such that $\pi = \nu \oplus \rho$. This tells us $V = U \oplus W$ and they are both G-invariant.

Proof. Pick a linear complement W of U. Then $V = U \oplus W$ and we can define a linear projection $p : V \to U$. We say $[G] = \mathrm{Card(G)}$. Write

$$Q = \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1}) p \pi(g).$$

We would like to show that Q is a projection. Note that becuase U is invariant, $\mathrm{im}(Q) \subset U$. We have Qu for $u \in U$,

$$Qu = \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1}) p \pi(g) u = u.$$

Hence $\mathrm{im}(Q) = U$. Note $Q^2 = Q$. $Qv \in U$, so $Q^2 v = Qv$. Hence Q is a projection on $\mathrm{im}(Q)$ along $\ker(Q)$. Let us calculate $Q\pi(h)v = \frac{1}{[G]}\sum_{g\in G}\pi(g^{-1})p\pi(g)\pi(h)v = \frac{1}{[G]}\sum_{g\in G}\pi(g^{-1})p\pi(gh)v$.

$$= \frac{1}{[G]}\sum_{g\in G}\pi(hg^{-1})p\pi(g)v = \pi(h)\frac{1}{[G]}\sum_{g\in G}\pi(g^{-1})p\pi(g).$$

So $Q\pi(h) = \pi(h)Q$, so Q is an intertwining map. $\mathrm{im}(Q)$ is an invariant subspace. We also have $\ker(Q)$ is an invariant subspace. $\ker(Q)$ is the desired W. □

(π, V) is a finite dimensional representation.

Theorem 2. Any finite dimensional representation is a finite direct sum of irreducible subrepresentations.

Suppose we have a morphism $(\pi, V) \xrightarrow{\varphi} (\nu, U)$.

1. (π, V) is irreducible. Two cases occur:
   - $\ker\varphi = \{0\} \Rightarrow \varphi$ injects.
   - $\ker\varphi = V \Rightarrow \varphi = 0$
2. (ν, U) is irreducible representation. Two cases occur again:
   - $\mathrm{im}(\varphi) = \{0\} \Rightarrow \varphi = 0$
   - $\mathrm{im}(\varphi) = U \Rightarrow \varphi$ surjects.

Proposition 1. If (π, V) and (ν, U) are two irreducible representations, then either $\mathrm{Hom}_G(V, U) = \{0\}$ or (π, V) and (ν, U) are isomorphic.

Lemma 1. (Schur's) If (π, V) is an irreducible representation of finite group G, then $\mathrm{Hom}_G(V, V) = \mathbb{C}\cdot I$ (the set of complex multiples of the identity map). Here, we use the fact that we're working over $\mathbb{C}$ since we need eigenvalues.

Given $\varphi \in \mathrm{Hom}_G(V, V)$. Note that φ has an eigenvalue λ with eigenvector v. $\varphi - \lambda I$ has nonzero kernel containing v, which implies the kernel is the entire space V.

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Theorem 3. Any representation of a finite group G is a direct sum of irreducible representations.

Proof. (for infinite dimensional representations V): Denote by $\mathcal{S}$ the set of all irreducible subrepresentations of (π, V). If we have a family (ν_i, U_i), $i \in I$, a family of representations, we can form the direct sum of the representations. We can form

$$U = \bigoplus_{i\in I} U_i$$

which is the set of families of vectors parametrized by I, $(u_i : i \in I)$ where $u_i = 0$ except for finitely many i. We define the sum and scalar products accordingly. We can then define the representation $\nu = \bigoplus_{i\in I}\nu_i$ by

$$\nu(g)(u_i : i \in I) = (\nu_i(g)u_i : i \in I)$$

We can look at $\mathcal{T}$, the set consisting of subsets $I \subset \mathcal{S}$ such that

$$\bigoplus_{i\in I}\nu_i$$

is a subrepresentation of (π, V). Now this set is ordered by inclusion. To apply Zorn's lemma, we need to show that totally ordered subsets are bounded. Suppose we have a totally ordered subset $\mathcal{C}$. It is an exercise to show that the union over elements of $\mathcal{C}$ is a bound in $\mathcal{T}$. Zorn's lemma would then show that there exists $M \in \mathcal{T}$ that is maximal under the partial order. Now the subrepresentation

$$\bigoplus_{i \in M} \nu_i$$

which is either the whole representation π or a strict subrepresentation. Assume the latter holds. Then by a theorem from a previous class (Maschke's) the representation can be written

$$\left(\bigoplus_{i \in M} \nu_i\right) \oplus \rho$$

but note that ρ has an irreducible subrepresentation ω, which we can move over to the left representation. $\square$

Definition 2. Denote by $\mathbb{C}[G] = \{f : G \to \mathbb{C}\}$. Note

$$\dim(\mathbb{C}[G]) = [G] = \mathrm{Card}(G)$$

We call the **right regular representation** $R : G \to \mathrm{Aut}(\mathbb{C}[G])$

$$(R(g)f)(h) = f(hg)$$

The resulting automorphisms are linear, and we have the following calculations:

$$(R(gg')f)(h) = f(hgg') = (R(g')f)(hg) = (R(g)R(g')f)(h)$$

and $R(1) = I$. Similarly, we have the **left regular representation** $L : G \to \mathrm{Aut}(\mathbb{C}[G])$ by

$$(L(g)f)(h) = f(g^{-1}h)$$

We can consider group G and G^{opp}, which is G with the reversed operation $g * h = h \cdot g$. We have a homomorphism $G \to G^{opp}$ by

$$g \mapsto g^{-1}$$

which is an isomorphism. Under this isomorphism, right and left regular representations are isomorphic. Let's examine a basis of $\mathbb{C}[G]$, which consists of functions

$$\delta_g : G \to \mathbb{C}$$

which is equal to 1 at g and 0 elsewhere. Of course $(\delta_g : g \in G)$ is a basis of $\mathbb{C}[G]$. The action

$$(R(g)\delta_1)(k) = \delta_1(kg)$$

which is 0 if $k \neq g^{-1}$ and 1 if $k = g^{-1}$, so $R(g)\delta_1 = \delta_{g^{-1}}$. We have hence proved $R(g)\delta_1 = \delta_{g^{-1}}$. Now assume $g \neq 1$. Then $R(g)\delta_1 = \delta_{g^{-1}} \neq \delta_1$, which shows $R(g)$ is not the identity operator. The kernel of $R : G \to \mathrm{GL}(\mathbb{C}[G])$ is hence trivial.

The regular representation $R = \bigoplus_{i=1}^{k} \rho_i$ and $R(g) \neq I$ if and only if $rho_i(g) \neq I$ for some $i \in \{1, \ldots, k\}$. We have the following theorem

Theorem 4. Let $g \in G$ be a non-identity element. Then there exists an irreducible representation (π, V) of G such that $\pi(g) \neq 1$.

Now we see an application of this theorem to an example. Consider an Abelian finite group G. Let's look at what we can say about an irreducible representation (π, V) of G. $\pi(a)\pi(b) = \pi(b)\pi(a)$ for any $a, b \in G$. Then note $\pi(b) \in \mathrm{Hom}_G(V, V)$ (it is an intertwining map). Then by Schur's lemma,

$$\pi(b) = \lambda(b) \cdot I$$

for some complex number $\lambda(b)$. We have $dim(\pi) = 1$, and all irreducible representations are 1-dimensional. We have

$$b \mapsto \lambda(b) \xrightarrow{|\cdot|} \mathbb{R}_+^*$$

Since b has finite order, we have that some power is equal to 1, implying that some power of $|lambda(b)|$ is 1. But this implies $|lambda(b)| = 1$. We call

$$\varphi : G \to \{z \in \mathbb{C} \mid |z| = 1\}$$

the homomorphism a **character of the group** G**.** We claim that if G is abelian, then all irreducible representations are characters. But we also have the converse. Given $a, b \in G$, we have the group commutator $aba^{-1}b^{-1} \in G$. If φ is a character, then $\varphi(aba^{-1}b^{-1})$ is $\varphi(a)\varphi(b)\varphi(a)^{-1}\varphi(b)^{-1} = 1$, which implies that $aba^{-1}b^{-1}$ is the identity. If it weren't, by a previous result an irreducible representation would detect it.

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We proved last time that G is Abelian if and only if all irreducible representations are 1-dimensional. Any irreducible representation (π, V) can hence be written

$$\pi(g) = \varphi(g) \cdot I,$$

where $\varphi : G \to \mathbb{C}^*$ is a **character**, with $|\varphi(g)| = 1$. Such a map factors through to get map $\varphi : G \to \{z \in \mathbb{C}^* ||z| = 1\}$. If we have two characters $\varphi, \psi \in \hat{G}$,

$$(\varphi \cdot \psi)(g) = \varphi(g) \cdot \psi(g)$$

The resulting product is also a character. The character defined by $\varphi : G \to \{1\}$ also serves as the identity. Likewise, the other properties of a group hold for characters of a group (including inverses). Note that $\hat{G}$ is an abelian group, which we will call the **dual group of** G .

Now assume that G is finite (not necessarily abelian). On the space of functions $G \to \mathbb{C}$, which we call $\mathbb{C}[G]$, we define

$$(f|f') = \frac{1}{[G]} \sum_{g \in G} f(g)\overline{f'(g)}$$

The above is an inner product of f with f'.

We calculate the product of the regular representation with this inner product:

$$(R(g)f|f') = \frac{1}{[G]} \sum_{h \in G} (R(g)f)(h)f'(h)$$

$$= \frac{1}{[G]} \sum_{h \in G} f(hg)f'(h)$$

$$= \frac{1}{[G]} \sum_{h \in G} f(h)\overline{f'(hg^{-1})}$$

$$= \frac{1}{[G]} \sum_{h \in G} f(h)(R(g^{-1})f')(h)$$

$$= (f|R(g^{-1})f')$$

which implies $R(g) = R(g^{-1})^*$ (its adjoint), so R is unitary. In this case we call the representation R unitary. Suppose we have two characters $\varphi, \psi \in \hat{G}$. We calculate

$$\varphi(g)(\varphi|\psi)$$

for a given $g \in G$. We have

$$\varphi(g)(\varphi|\psi) = \varphi(g)\frac{1}{[G]}\sum_{h\in G}\varphi(h)\overline{\psi(h)}$$

We can then move $\varphi(g)$ inside:

$$\frac{1}{[G]}\sum_{h\in G}\varphi(gh)\overline{\psi(h)}$$

We can then swap g and h?

$$\frac{1}{[G]}\sum_{h\in H}\varphi(hg)\overline{\psi(h)} = \frac{1}{[G]}\varphi(h)\overline{\psi(hg^{-1})}$$

$$\psi(hg^{-1}) = \psi(h)\psi(g^{-1}) = \psi(h)\psi(g)^{-1} = \psi(h)\overline{\psi(g)}.$$

We have above

$$= \left(\frac{1}{[G]}\sum_{h\in G}\varphi(h)\overline{\psi(h)}\right)\psi(g) = (\varphi|\psi)\cdot\psi(g)$$

If $\varphi \neq \psi$, then there is some $g \in G$ where $\varphi(g) \neq \psi(g)$. This implies $(\varphi|\psi) = 0$, so $\varphi \perp \psi$. This means $[\hat{G}] \leq [G]$, so the dual group is finite. Note

$$\|\varphi\|^2 = (\varphi|\varphi) = \frac{1}{[G]}\sum_{g\in G}\varphi(g)\overline{\varphi(g)} = 1.$$

We have $\mathbb{C}[G]$, and inner product $(\cdot|\cdot)$, with $R(g)$ unitary operators on V. Since the group is Abelian, the unitary operators $R(g)$ commute.

Exercise 1

There exists a basis for V $(e_i; 1 \leq i \leq [G])$ such that the unitary operators are diagonal matrices. To prove this, do this first for a single operator, then inductively do this for all of the operators. In other words $R(g)e_i = \varphi_i(g)e_i$ where $|\varphi_i(g)| = 1$. We have

$$V_i = \mathbb{C}\cdot e_i$$

is invariant for G. As we have proved from last class, $\varphi_i(g)$ is a character of G.

We have

$$(R(g)e_i)(1) = e_i(g)$$

which is also

$$\varphi_i(g)e_i(1)$$

By rescaling the orthonormal basis, we may assume $e_i(1) = 1$. (what does the notation $e_i(g)$ mean). We have $\{\varphi|\varphi \in \hat{G}\}$ is an orthonormal basis of $\mathbb{C}[G]$. Hence we have shown $[\hat{G}] = [G]$. From the structure theorem of finite abelian groups, $\hat{G} \cong G$ (even though there is not necessarily a natural isomorphism). We can then construct

$$\hat{\hat{G}} = (\hat{G})\hat{}$$

and a map $\alpha : G \to \hat{\hat{G}}$ defined by

$$g \mapsto (\varphi \mapsto \varphi(g))$$

the evaluation morphism. We have

$$(\alpha(gg'))(\varphi) = (\alpha(g)\alpha(g'))(\varphi)$$

so it is a group homomorphism. If $\alpha(g) = 1$, then $\varphi(g) = 1$ for all irreducible representations, so $g = 1$. So α is an inclusion (injection) of groups of the same finite order, implying α is isomorphic.

This is a special case of Pontryagin duality, which states there is a natural isomorphism $\hat{\hat{G}} \cong G$, which we will show holds for all abelian locally compact groups. We define a Fourier transform in this context. We can define characters on $\mathbb{R}$ by $x \mapsto e^{i\lambda x}$. This yields an isomorphism $\mathbb{R} \cong \hat{\mathbb{R}}$. This way we can define a Fourier transform on finite abelian group G in an analogous manner: Given a function $f \in \mathbb{C}[G]$, we can define

$$(\mathcal{F}f)(\varphi) = \frac{1}{[G]}\sum_{g\in G} f(g)\overline{\varphi(g)}$$

Note that this is $(f|\varphi)$. Since φ form an orthonormal basis, we can write

$$f = \sum_{\varphi\in\hat{G}} (f|\varphi)\varphi$$

and

$$f(g) = \sum_{\varphi\in\hat{G}} (\mathcal{F}f)(\varphi)\varphi(g) \text{ this is called the inverse Fourier transform on } \mathcal{F}f$$

We can use Bessel's equality, which says that

$$\begin{aligned}\|f\|^2 &= \sum_{\varphi\in\hat{G}} |(f|\varphi)|^2\\ &= \sum_{\varphi\in\hat{G}} |(\mathcal{F}f)(\varphi)|^2\end{aligned}$$

This is called the Plancherel theorem. We will in the future generalize this to abelian locally compact groups. On Friday, we will see how to generalize these ideas to nonabelian groups. Irreducible representations may not be characters.

5 January 21

Last time, we discussed for a finite abelian group G, dual group $\hat{G}$, the set $\{\varphi : G \to \{z \in \mathbb{C} \big| |z| = 1\}\}$. We proved that $[\hat{G}] = [G]$ and that $R = \bigoplus_{\varphi\in\hat{G}} \varphi$. What can we hope for in general?

Example. Let's look at $G = S_n$, the permutation group on n elements, also known as the symmetric group.

1. $n = 2$. We have $[S_2] = 2$, $\mathbb{Z}_2$. We have two characters: the trivial character sending all elements of the group to 1, and **the sign character** sending even permutations to 1 and odd permutations to -1.
2. Now if we look at S_3, which has order 6, the group is not abelian and it has two characters as above *triv* and *sgn*. We proved last time that if all irreducible representations have dimension 1, then the group is abelian. So there is an irreducible representation σ of S_3 that is not 1 dimensional.

   Claim: there exists a 2-dimensional irreducible σ unique up to isomorphism.

   We are generalizing $R = \bigoplus_{\varphi\in\hat{G}} \varphi$ in that each representation will appear with multiplicity equal to its dimension. Write $\hat{G}$ to be the set of isomorphism classes of irreducible representation of G. $\hat{G}$ is a finite set, but it is difficult to define a group structure on the set (can't compose a 2-dimensional representation with a 1-dimensional). We can, however, introduce additional structure on $\hat{G}$. If we have two irreducible representations, we can take their tensor product. Even though the tensor product may not be irreducible, it can be written as a direct sum of irreducibles. There is a theorem in Alg Geo/Num Theory called Tanaka Duality related to this, which (generalizes Pontryagin duality?).

Remark. S_n has no other characters except *triv* and *sgn*.

If $g \in S_n$ is a transposition, then it satisfies

$$g^2 = 1$$

So ψ is a character of S_n, we have $\psi(g)^2 = 1$ implies $\psi(g) = \pm 1$. Now all transpositions are conjugate. That is, if g, g' are transpositions, then $g' = hgh^{-1}$ for some $h \in S_n$. This implies

$$\psi(g') = \psi(h)\psi(g)\psi(h)^{-1} = \psi(g)$$

since multiplication on $\mathbb{C}$ is commutative. So $\psi(g) = 1$ on all transpositions, or $\psi(g') = -1$ on all transpositions. Since the transpositions generate the group, this determines ψ. So this implies our earlier claim that there is a 2-dimensional representation somehow.

Theorem 5. Let (π, V) be a finite dimensional representation of a finite group G. There exists an inner product $(\cdot|\cdot)$ on V such that (π, V) is a unitary representation. Note we did not say that this inner product is unique, and it need not be.

Proof. We can define an inner product on V by fixing a basis e_i, so we can write any $u \in V$ as

$$u = \sum_{i=1}^{n} u_i e_i$$

and then we can write

$$\langle u|v\rangle = \sum_{i=1}^{n} u_i \overline{v_i}$$

We can then define a new inner product

$$(u|v) = \frac{1}{[G]} \sum_{g \in G} \langle \pi(g)u|\pi(g)v\rangle$$

What we get is linear in the first variable because it is a linear combination of functions which are linear in the first variable. Likewise for the second variable. We also have

$$(u|v) = \overline{(v|u)}$$

We have

$$(u|u) = \frac{1}{[G]} \sum_{g \in G} \langle \pi(g)u|\pi(g)u\rangle \geq 0$$

Now if the inner product is zero, because each of the terms of the sum are nonnegative reals, each term in the sum is also zero, implying $\langle u|u\rangle = 0$.

We can now calculate $(\pi(g)u|\pi(g)v) = \frac{1}{[G]} \sum_{h \in G} \langle \pi(h)\pi(g)u|\pi(h)\pi(g)v\rangle$. Now this is

$$\frac{1}{[G]} \sum_{h \in G} \langle \pi(hg)u, \pi(hg)v\rangle$$

$$= \frac{1}{[G]} \sum_{h \in G} \langle \pi(h)u|\pi(h)v\rangle = (u|v)$$

so that $\pi(g)$ is unitary. So if (π, V) is finite dimensional, (ν, U) is a subrepresentation, we can introduce inner product $(\cdot|\cdot)$ and write $V = U \oplus U^{\perp}$. This gives an alternate proof of Theorem 1 on January 14. $\square$

We now discussion **Orthogonality relations**. Recall

$$\mathbb{C}[G]$$

which has natural product

$$(f|f') = \frac{1}{[G]}\sum_{g\in G} f(g)\overline{f'(g)}$$

We would like to compare representations (π, V), (ν, U). We look at a linear map $A \in \mathrm{Hom}_{\mathbb{C}}(U, V)$, and examine

$$B = \frac{1}{[G]}\sum_{g\in G} \pi(g)A\nu(g)^{-1}$$

which is essentially an average on A after twisting before and afterwards by the two representations. We have $B \in \mathrm{Hom}_{\mathbb{C}}(U, V)$, of course, but we moreover have

$$\pi(g)B = \frac{1}{[G]}\sum_{h\in G} \pi(g)\pi(h)A\nu(h^{-1})$$

$$= \frac{1}{[G]}\sum_{h\in G} \pi(gh)A\nu(h^{-1})$$

$$= \frac{1}{[G]}\sum_{h\in G} \pi(h)A\nu(h^{-1}g) = \frac{1}{[G]}\sum_{h\in G} \pi(h)A\nu(h^{-1})\nu(g)$$

$$= B\nu(g)$$

(by changing variables in sum). We have shown that the average is an intertwining map over the representations π, ν, so $B \in \mathrm{Hom}_G(U, V)$. Let's relate this to Schur's lemma. If (π, V), (ν, U) are irreducible representations, then there are two cases (we examine the second next class):

1. If $(\pi, V), (\nu, U)$ are not isomorphic
2. $(\nu, U) = (\pi, V)$.

We have 1 implies $\mathrm{Hom}_G(U, V) = \{0\}$. Here's why we have the two cases above: Given an intertwining map $U \to V$, the kernel of the map is either 0 or the entire set. In the former case, the image must be nonzero so the entire set V, which implies isomorphism. In the latter, the map is the zero map.

Back to the first case, we have

$$\sum_{g\in G} \pi(g)A\nu(g^{-1}) = 0$$

for all A in case 1. Now A can be represented as a matrix if we fix orthonormal bases $(e_1, \dots, e_n)$ of U, $(f_1, \dots, f_m)$ of V. We plug in A_{pq}, which is zero everywhere except where it is 1 in the p-th row and q-th column. We have

$$\sum_{j,k}\sum_{g\in G} \pi(g)_{ij}(A_{pq})_{jk}\nu(g^{-1})_{kl} = 0$$

We have $A_{jk} = \delta_{jp}\delta_{kq}$, so the sum is

$$\sum_{g\in G} \pi(g)_{ip}\nu(g^{-1})_{ql} = 0$$

If we have a unitary matrix, we have $\nu(g^{-1})_{ql} = \overline{\nu(g)_{lq}}$. We have

$$(\pi_{ip}|\nu_{lq}) = 0$$

6 January 24

Let G be an Abelian group. Schur's lemma says that if (π, V) is an irreducible representation of G, and a linear map $A \in \mathrm{Hom}_G(V, V)$ such that $\pi(g)A = A\pi(g)$, then Schur's lemma implies $A = \lambda I$. For any $g \in G$, Schur's lemma implies that $\pi(g) = \varphi(g)I$. Assume that the dimension of V is greater than 1. We can take a strict nonzero subspace $U \subsetneq V$ (by choice which is dimension 1). It is an invariant subspace, and so we can get a smaller representation, which contradicts the irreducibility of $\pi(g)$.

We stopped last time at where G is a finite group, and took two representations

$$(\pi, V) \ncong (\nu, U)$$

are irreducible. Then we can take a basis $e_1, \dots, e_n$ in V and $f_1, \dots, f_m$ in U which are orthonormal. We can then represent $\pi(g)$ as a matrix $(\pi(g)_{ij})$ and $\nu(g) = (\nu(g)_{k\ell})$ which are $n \times n$ and $m \times m$ matrices, respectively. Then

$$(\pi_{ij} | \nu_{k\ell}) = 0$$

Here the matrix coefficients are characters.

We now provide an invariant interpretation. Define

$$M(\pi) = \text{span of } \pi_{ij}, 1 \leq i, j \leq n$$

which we call the matrix coefficient subspace attached to π. Let us take $v \in V$, linear form $\varphi \in V^*$. The matrix coefficient corresponding to G and φ is

$$g \mapsto \varphi\left(\pi(g)v\right)$$

We can then write

$$\pi(g)e_i = \sum_{j=1}^{n} \pi(g)_{ji} e_j$$

And we can write

$$v = \sum_{i=1}^{n} c_i e_i$$

$$\pi(g)v = \sum_{i=1}^{n} c_i \pi(g) e_i$$

$$= \sum_{i,j=1}^{n} c_i \pi(g)_{ji} e_j$$

Applying φ we get

$$\varphi(\pi(g)v) = \sum_{i,j=1}^{n} c_i \pi(g)_{ji} \varphi(e_j)$$

We have just shown that $M(\pi)$ is independent of choice of basis. We can then reformulate $(\pi_{ij}, \nu_{k\ell}) = 0$. It says: if (π, V) is not isomorphic to (ν, U), then $M(\pi) \perp M(\nu)$. Now we can make a definition

$$\hat{G} = \text{the set of equivalence classes of irreducible reps of } G$$

So $M(\pi)$ depends only on the equivalence class of the representation, and we denote the equivalence class of π as $[\pi]$. This makes a well defined map

$$[\pi] \mapsto M(\pi)$$

If we look at the sum

$$\sum_{[\pi] \in \hat{G}} \dim(M(\pi)) \leq \dim(\mathbb{C}[G]) = [G]$$

We have argued $\hat{G}$ is a finite set, which we called the **dual of** G**.** We don't know how large $M(\pi)$ is. We do have a bound $|\hat{G}| \leq |G|$. If the group is Abelian then, as we have shown before, we have equality. Otherwise, the inequality is strict because some $M(\pi)$ must be of dimension greater than 1. We do need to show that if a representation is not one dimensional, then $M(\pi)$ is not of dimension 1.

We are looking at the situation $(\pi, V) = (\nu, U)$. In this case, given a linear map $A \in \mathrm{Hom}_{\mathbb{C}}(V, V)$, we have an average

$$B = \frac{1}{[G]} \sum_{g \in G} \pi(g) A \pi(g^{-1})$$

This implies $B \in \mathrm{Hom}_G(V, V)$. Since π is irreducible, $B = \lambda I$ by Schur's lemma. The variant that we get from our argument is the following: that

$$\frac{1}{[G]} \sum_{g \in G} \pi(g) A \pi(g^{-1}) = \lambda I$$

We now calculate λ. The trick is to take the trace of this expression:

$$tr(B) = \lambda \cdot tr(I) = \lambda \cdot \dim(\pi)$$

But from the left hand side for the average expression, we get

$$tr(B) = \frac{1}{[G]} \sum_{g \in G} tr(\pi(g) A \pi(g^{-1})$$

We can conduct cyclic permutation on the product to move $\pi(g)$ to the back, making

$$tr(B) = \frac{1}{[G]} \sum_{g \in G} tr(A \pi(g^{-1}) \pi(g)) = \frac{1}{[G]} \sum_{g \in G} tr(A) = tr(A)$$

This yields

$$\begin{aligned} \lambda &= \frac{tr(A)}{\dim(\pi)} \cdot \frac{1}{[G]} \sum_{g \in G} \pi(g) A \pi(g^{-1}) \\ &= \frac{tr(A)}{\dim(\pi)} \cdot I \end{aligned}$$

We would like to see what happens if we plug in matrices. That is, looking at coefficients, we get for the second to last expression above,

$$\frac{tr(A)}{\dim(\pi)} \cdot \frac{1}{[G]} \sum_{j,k=1}^{n} \pi(g)_{ij} A_{jk} \pi(g^{-1})_{k\ell}$$

and

$$\frac{tr(A)}{\dim(\pi)} \frac{tr(A)}{\dim(\pi)} \delta_{i\ell}$$

We pick a matrix that is one at index (p, q) and zero elsewhere. We have

$$A_{jk} = (\delta_{jp} \delta_{kq})_{pq}$$

Looking at the above summation, we have

$$\frac{1}{[G]} \sum_{g \in G} \pi(g)_{ip} \overline{\pi(g)_{ql}^{-1}} = (\pi_{ip} | \pi_{\ell q})$$

We have an orthonormal basis that is invariant under the group action. We also have $tr(A) = \delta_{pq}$, and

$$(\pi_{ip} | \pi_{\ell q}) = \frac{1}{\dim \pi} \delta_{i\ell} \delta_{pq}$$

This is the celebrated **Schur orthogonality relation**. The ultimate conclusion is that

$$\dim(M(\pi)) = (\dim\pi)^2$$

which gives us what we wanted to show from earlier in the lecture. Adding to our earlier result, we have

$$\sum_{[\pi]\in\hat{G}} \dim(\pi)^2 = \sum_{[\pi]\in\hat{G}} \dim(M(\pi)) \leq [G]$$

We examine an example:

Example. Look at S_3, the first nonabelian symmetry group. It has two representations $triv$, sgn. It must have at least one representation that is not one dimensional by the nonabelianness of S_3. We call one such representation σ (it could have more). First, let us calculate

$$\sum_{[\pi]\in\hat{S_3}} \dim(\pi)^2 = 1 + 1 + (\dim(\sigma)) \leq 6 = [S_3]$$

But $\dim(\sigma)$ is at least 2, meaning that $\dim(\sigma) = 2$. This shows there is only one two dimensional representation up to equivalence. In the next two classes, we will figure out how to find σ.

On Wednesday, we will figure out how to show the above inequality before the example is actually an equality.

7 January 26

We have a finite group G. For any class $[\pi] \in \hat{G}$, we have $M(\pi) \subset \mathbb{C}[G]$, matrix representations of π. We can consider the direct sum

$$M = \bigoplus M(\pi) \subset \mathbb{C}[G]$$

Our next step is to prove equality. We have

1. $M(\pi)$ are G-invariant for R. Fixing a basis $e_1, \ldots, e_n$ of V. In this case, $\pi(g)e)i = \sum \pi(g)_{ji}e_j$.

   $$(\pi(g)_{ji})\text{span}(M(\pi))$$

   Let's examine $\pi(\bullet)_{ij}$, and apply $R(g)$. We have

   $$(R(g)\pi_{ij})(h) = \pi_{ij}(hg) = \sum_{k=1}^{n} \pi(h)_{ik}\pi(g)_{kj}$$

   $$R(g)\pi_{ij} = \sum_{k=1}^{n} \pi(g)_{kj}\pi_{ik} \in M(\pi)$$

   (we are concluding it is in $M(\pi)$.

2. M is G-invariant for R. Suppose that $M \neq \mathbb{C}[G]$. This implies that $M^\perp \neq 0$. The orthogonal complement of an invariant subspace is invariant, so $M^\perp$ is G-invariant. It hence contains some G-invariant subspace U such that its representation (ν, U) is irreducible. Let's look at (ν, U). We pick a basis $f_1, \ldots, f_m$ of U. We look at

   $$\nu(g)f_i = \sum_{j=1}^{m} \nu(g)_{ji}f_j$$

   We can write

   $$(\nu(g)f_i)(h) = \sum_{j=1}^{m} \nu(g)_{ji}f_j(h)$$

Now this is equal to $f_i(hg)$ because ν is a restriction of R. We now evaluate this at $h = 1$. We have

$$f_i(g) = \sum_{j=1}^{m} \nu(g)_{ji} f_j(1)$$

We can then conclude that f_i is a linear combination of $\nu_{ji} \in M(\nu)$. The conclusion is $f_i \in M(\nu)$. This implies $(f_i|f_i) = 0$ since f_i are in a subspace of $M^\perp$, so $f_i = 0$. This shows $M = \mathbb{C}[G]$.

What is $M(\pi)$? Take the vector space V of π, and its dual V^*. We have a natural map

$$V^* \otimes V \xrightarrow{\alpha} M(\pi)$$

defined by

$$f \otimes v \mapsto f(\pi(\bullet)v) \in \mathbb{C}[G]$$

On V^* we have the dual representation $V \to V^*$ defined by

$$g \mapsto \pi(g)^*$$

which is not a representation because taking adjoints flips order. We do have the representation $\pi^*(g) = \pi(g^{-1})^*$. If we have a representation π we thus have a natural representation π^*. We can thus view $V^* \otimes V$ as being acted on by $G \times G$. Note $M(\pi)$ is invariant for the left regular representation. This yields an isomorphism of irreducible representations. Here we have a decomposition of irreducible representations of M. The dimension of $M(\pi)$ is hence $(\dim \pi)^2$.

Theorem 6.

$$\bigoplus_{[\pi] \in \hat{G}} M(\pi) = \mathbb{C}[G]$$

and

$$[G] = \sum_{[\pi] \in \hat{G}} \dim(\pi)^2$$

Now we introduce another important tool, the notion of the character of a representation.

Definition 3. Let (π, V) be finite dimensional. We define the **character** of the representation to be

$$\mathrm{ch}(\pi) = \mathrm{tr}(\pi(\bullet)) \in \mathbb{C}[G]$$

Take a basis $e_1, \dots, e_n$ of V. We have

$$\pi(g) \to (\pi(g)_{ij})$$

and

$$\mathrm{tr}(\pi(g)) = \sum_{i=1}^{n} \pi(g)_{ii}$$

Let's list some properties of the character.

1. $\mathrm{ch}(\pi)(1) = \mathrm{tr}(id) = \dim(\pi)$
2. $g, h \in G$, can calculate hgh^{-1}. So $\mathrm{ch}(\pi)(hgh^{-1}) = \mathrm{tr}(\pi(hgh^{-1})) = \mathrm{tr}(\pi(h)\pi(g)\pi(h)^{-1})$ and

$$\mathrm{tr}(\pi(g)\pi(h^{-1})\pi(h)) = \mathrm{tr}(\pi(g))$$

by the fact that you can do cyclic permutations in products without changing the trace. That is, $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$. We proved the fact that characters are constant on conjugacy classes.

3. If we have a representation $\pi = \nu \oplus \rho$, then
$$\mathrm{ch}(\pi) = \mathrm{ch}(\nu) + \mathrm{ch}(\rho)$$
Let's look at the matrix for π:
$$\begin{bmatrix} \nu & 0 \\ 0 & \rho \end{bmatrix}$$
by choosing a basis for V. The trace is accordingly the sum of traces. By Maschke's theorem, the characters of irreducible representations determine those for all representations.

Now let's take two irreducible representations π, ν. Let's assume that $[\pi] \neq [\nu]$. Then we can look at
$$(\mathrm{ch}(\pi)|\mathrm{ch}(\nu))$$
The former character is in $M(\pi)$, the latter is in $M(\nu)$ and $M(\pi) \perp M(\nu)$, so the inner product is 0. Characters of nonisomorphic irreducible representations are orthogonal to each other. The second question is what happens with the norm of a character? What is
$$(\mathrm{ch}(\pi)|\mathrm{ch}(\pi)) = \sum_{i=1}^{n}\sum_{j=1}^{n}(\pi_{ii}|\pi_{jj})$$
Now note that $(\pi_{ii}|\pi_{jj}) = 0$ unless $i = j$ by the orthogonality relation. We get a single term
$$\sum_{i=1}^{n}(\pi_{ii}|\pi_{ii}) = \sum_{i=1}^{n}\frac{1}{\dim(\pi)} = 1$$
Hence
$$(\mathrm{ch}(\pi) : [\pi] \in \hat{G})$$
is an orthonormal family in $\mathbb{C}[G]$.

Exercise 2

Look at S_3. You can two irreducible characters triv, sgn. There is a third irreducible character. From there, you can find an orthonormal basis for $\mathbb{C}[S_3]$. From there, you can find the third irreducible character.

We will see later how to do this in other ways.

8 January 28

Reminder of last time: Let G be a finite group, $\hat{G}$ be its dual, which is the isomorphism classes of irreducible representations. Each element $[\pi]$ is attached to its character $\mathrm{ch}(\pi) \in \mathbb{C}[G]$. We proved last time the Schur orthogonality relations, which say that
$$(\mathrm{ch}(\pi)|\mathrm{ch}(\nu)) = \delta_{\pi\nu} \text{ for } [\pi], [\nu] \in \hat{G}$$
And $\{\mathrm{ch}(\pi)|[\pi] \in \hat{G}\}$ is an orthogonal family. So let's describe the first reformulation of the Schur Orthogonality relations.

Assume that π, ν are finite dimensional representations on spaces V, U respectively. We can look at the space of morphisms
$$\mathrm{Hom}_G(\pi, \nu)$$
which are linear maps $V \to U$ that commute with the action. Since the space is finite dimensional, Hom is a finite dimensional vector space. We have the following theorem:

Theorem 7. $\dim_{\mathbb{C}}\mathrm{Hom}(\pi,\nu) = (\mathrm{ch}(\pi)|\mathrm{ch}(\nu))$. The former gives homological information about the category of representations.

Proof. Let's check what happens when π, ν are irreducible. We know from Schur's lemma there are two options:

1. $[\pi] \neq [\nu]$, in which case $\mathrm{Hom}_G(\pi,\nu) = \{0\}$.
2. $[\pi] = [\nu]$, in which case $\mathrm{Hom}_G(\pi,\pi) = \mathbb{C}I$.

In case 1, the dimension is 0, and in the latter case, the dimension is 1. These also match with $(\mathrm{ch}(\pi)|\mathrm{ch}(\nu))$. The rest is based on Maschke's theorem. Suppose we have $\pi = \pi_1 \oplus \pi_2$. This means

$$\mathrm{Hom}_G(\pi,\nu) = \mathrm{Hom}_G(\pi_1 \oplus \pi_2, \nu)$$

$$= \mathrm{Hom}_G(\pi_1,\nu) \oplus \mathrm{Hom}_G(\pi_2,\nu)$$

(exercise: prove this equality). If we prove this,

$$\dim(\mathrm{Hom}_G(\pi_1 \oplus \pi_2,\nu)) = \dim(\mathrm{Hom}_G(\pi_1,\nu)) + \dim(\mathrm{Hom}_G(\pi_2,\nu))$$

We have

$$\dim(\mathrm{Hom}_G(\pi_1 \oplus \pi_2 \oplus \ldots \oplus \pi_k, \nu)$$

$$= \sum_{i=1}^{n} \dim\left(\mathrm{Hom}_G(\pi_i,\nu)\right)$$

We can do the second argument on the second variable. An analogous formula:

$$\dim\left(\mathrm{Hom}_G(\pi, \nu_1 \oplus \ldots \oplus \nu_k)\right)$$

$$= \sum_{j=1}^{k} \dim\left(\mathrm{Hom}_G(\pi,\nu_j)\right.$$

We know that the character of a direct sum is a direct sum of characters, and additivity applies to the right side $(\mathrm{ch}(\pi)|\mathrm{ch}(\nu))$. □

The characters describe the structure of the category. How big is $\hat{G}$? If π is a finite dimensional representation of G, then $\mathrm{ch}(\pi)$ is constant on conjugacy classes. We can look at the vector subspace of all functions on $\mathbb{C}[G]$ constant on conjugacy classes. These are called the **central functions**.

1. The dimension of the space of central functions is equal to the number of conjugacy classes.
2. $\{\mathrm{ch}(\pi)|\pi \in \hat{G}\}$ implies $\mathrm{Card}(\hat{G}) \leq$ the number of conjugacy classes. We would like to show that this is an equality. If G is abelian, then note the number of conjugacy classes is $[G]$, but this is actually $[\hat{G}]$. This result is a generalization of that result.

Theorem 8. If we take $f \in M(\pi)$, $[\pi] \in \hat{G}$, we can form

$$\frac{1}{[G]} \sum_{h \in G} f(hgh^{-1})$$

$$= \frac{f(1)}{\dim(\pi)} \cdot \mathrm{ch}(\pi)(g)$$

We have to assume that we can take an orthonormal basis for $M(\pi)$.

Proof. Take $f = \pi_{ij}$ (matrix coefficients in some orthonormal basis $(e_1, \ldots, e_n)$ with respect to a G-invariant inner product). It's enough to prove the formula for all such f 's by linearity. Plugging this in,

$$\frac{1}{[G]} \sum_{h \in G} \pi(hgh^{-1})_{ij} = \frac{1}{[G]} \sum_{h \in G} (\pi(h)\pi(g)\pi(h)^{-1})_{ij}$$

$$\frac{1}{[G]} \sum_{h \in G} \sum_{\ell,k=1}^{n} \pi(h)_{ik}\pi(g)_{k\ell}\pi(h)^{-1}_{\ell j}$$

Since $\pi(h)$ is unitary, $\pi(h)^{-1}_{\ell j} = \overline{\pi(h)_{\ell j}}$. Now we have

$$\sum_{k,\ell=1}^{n} \pi(g)_{k\ell} \frac{1}{[G]} \sum_{h \in G} \pi(h)_{ik}\overline{\pi(h)_{j\ell}}.$$

$$= \sum_{k,\ell=1}^{n} \pi(g)_{k\ell} \cdot \frac{1}{\dim(\pi)} \delta_{ij}\delta_{k\ell}.$$

$$= \frac{\delta_{ij}}{\dim(\pi)} (\mathrm{ch}(\pi))(g)$$

and note

$$\delta_{ij} = I_{ij} = \pi(1)_{ij}$$

giving us the theorem result. □

We want to show that characters span the space of central functions. Let's take φ a central function on conjugacy classes. Take $f \in M(\pi)$. Assume that $\varphi \perp \mathrm{ch}(\pi), \pi \in \hat{G}$. We show

$$(\varphi | f) = 0$$

which gives that $\varphi \perp M(\pi)$ for any $\pi \in \hat{G}$. This implies $\varphi \perp \mathbb{C}[G]$, which gives $\varphi = 0$. In other words, $\{\mathrm{ch}(\pi) | \pi \in \hat{G}\}$ is an orthonormal basis of the space of central functions. We calculat e

$$(\varphi | f) = \frac{1}{[G]} \sum_{g \in G} \varphi(g)\overline{f(g)}$$

The first step is to split the sum over conjugacy classes, and φ is constant over each mini-sum. We can then apply the earlier theorem.

9 January 31

We would like to complete a proof from last class. We wanted to show that the space of central functions on G is spanned by characters of irreducible representations $\mathrm{ch}(\pi)$ for $\pi \in \hat{G}$. To prove this, we know that characters

$$\{\mathrm{ch}(\pi) | \pi \in \hat{G}\}$$

is an orthonormal set in $\mathbb{C}[G]$. It is enough to show that if φ is a central function on G such that $(\varphi | \mathrm{ch}(\pi)) = 0$, then $\varphi = 0$. The proof will be based on a formula we proved last time. If we take $f \in M(\pi)$ and average it over conjugacy classes:

$$\frac{1}{[G]} \sum_{h \in \hat{G}} f(hgh^{-1}) = \frac{f(1)}{\dim(\pi)} \cdot \mathrm{ch}(\pi)(g)$$

We want to calculate $(\varphi | f) = \frac{1}{[G]} \sum_{g \in G} \varphi(g)\overline{f(g)}$. This is also

$$\frac{1}{[G]^2} = \sum_{g,h \in G} \varphi(hgh^{-1})\overline{f(hgh^{-1})}$$

Our function f is central, so conjugation by h is irrelevant:

$$\frac{1}{[G]^2}\sum_{g\in G}\varphi(g)\sum_{h\in G}\overline{f(hgh^{-1})}$$

$$=\frac{1}{[G]}\sum_{g\in G}\varphi(g)\frac{1}{[G]}\sum_{h\in G}\overline{f(hgh^{-1})}$$

$$\frac{1}{[G]}\sum_{g\in G}\varphi(g)\frac{\overline{f(1)}}{\dim(\pi)}\overline{\mathrm{ch}(\pi)(g)}$$

$$=\frac{\overline{f(1)}}{\dim(\pi)}\frac{1}{[G]}\sum_{g\in G}\varphi(g)\overline{\mathrm{ch}(\pi)(g)}$$

$$=\frac{\overline{f(1)}}{\dim(\pi)}(\varphi|\mathrm{ch}(\pi))=0$$

We have $\varphi\perp M(\pi)$, $\pi\in\hat{G}$, so $\varphi\perp\bigoplus_{[\pi]\in\hat{G}}M(\pi)=\mathbb{C}[G]$ (this equality is from a previous theorem (Bernsie's theorem?)). So $\varphi=0$.

We have ultimately shown that the dimension of the space of central functions is equal to the number of conjugacy classes which is equal to $\mathrm{Card}\left(\hat{G}\right)$.

Example. We look at $G=S_3$. We have remarked that there are three irreducible representations, so $\mathrm{Card}(\hat{G})=3$. We have the trivial representation, the sgn representation, and then an unknown representation σ which is two dimensional (see last class). We have $[G]=6$. We have $\{1\}$ which is one conjugacy class, the conjugacy class of transpositions,

$$\{(1\ 2),(2\ 3),(3\ 1)\}\ \text{(in cyclic notation)}$$

and the conjugacy class of everything else:

$$\{(3\ 1\ 2),(3\ 2\ 1)\}$$

The third and first ones are even, while the second is odd. The set of even permutations is a normal subgroup, which is kernel of sgn. Note that the character of triv is equal to 1 everywhere. We have

$$\mathrm{ch}(\mathrm{sgn})=\begin{cases}1 & \text{even}\\ -1 & \text{odd}\end{cases}$$

If we look at

$$\mathrm{ch}(\mathrm{triv})-\mathrm{ch}(\mathrm{sgn})=\begin{cases}0 & \text{even}\\ 2 & \text{odd}\end{cases}$$

We note that this difference is orthogonal to $\mathrm{ch}(\sigma)$. So

$$(\mathrm{ch}(\mathrm{triv})-\mathrm{ch}(\mathrm{sgn})|\mathrm{ch}(\sigma))$$

$$=\frac{1}{6}\sum_{g\in S_3}\dots$$

$$=\frac{1}{3}\left(3\mathrm{ch}(\sigma)((1\ 2))\right)=0$$

so $\mathrm{ch}(\sigma)=0$ on transpositions (it is constant on transpositions). We also know $\mathrm{ch}(\sigma)(1)=2$.

$$\mathrm{ch}(\sigma)=\begin{cases}2 & 1\\ 0 & \text{transposition}\\ x & \text{3-cycles}\end{cases}$$

Now note

$$(\mathrm{ch}(\sigma)|\mathrm{ch}(\sigma)) = \frac{1}{6}(2\cdot 1 + 0\cdot 3 + 2x) = \frac{1}{3}(1+x) = 0$$

so $x=-1$. We have σ is an induced representation.

The next step: one technique of constructing representations is to look at group G and its subgroups $H<G$. We first define a functor

$$\mathrm{Rep}(G)$$

and given object (π, V), it satisfies $\pi : G \to \mathrm{GL}(V)$, so if we have a subgroup, we can use the inclusion $\iota : H \hookrightarrow G$ and the induced $\mathrm{Res}_H^G(\pi) : H \to \mathrm{GL}(V)$.

$$(\pi, V) \to \mathrm{Res}_H^G(\pi)$$

We get a functor Res_H^G. If we have two representations $(\pi, V), (\pi', V')$, and the map is an induced map of representations, so $\pi'(g)\circ\varphi = \varphi\circ\pi(g)$ for all $g\in G$.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\pi(g)"'] & V \arrow[d, "\pi'(g)"] \\ V \arrow[r, "\varphi"'] & V \end{tikzcd}$$

so $\varphi \in \mathrm{Hom}_H(V, V')$. We have an adjoint functor which goes in the opposite direction. We would like to construct a functor

$$\mathrm{Rep}(H) \to \mathrm{Rep}(G)$$

which we call induction. Let us recall regular representations, defined on $\mathbb{C}[G]$. Define $(R(g)f)(g') = f(g'g)$. Let's take a representation (ν, U) on $H<G$ (we are not assuming anything about their size, so we will use this when we work on compact groups). Define $\mathrm{Ind}(U) = \{f : G \to U | f(hg) = \nu(h)(f(g)) \forall h \in H, g \in G\}$. We have

$$\mathrm{Ind}(U) \subset \mathbb{C}[G, U]$$

We point out that if we take triv $=\{1\}$, the only representation is $\mathbb{C}$ with the trivial action. In this case, $\mathrm{Ind}(U) = \mathbb{C}[G]$. We define a map $\rho(g) : \mathrm{Ind}(U) \to \mathrm{Ind}(U)$ by

$$(\rho(g)f)(g') = f(g'g)$$

Note that

$$(\rho(g)f)(hg') = f(hg'g) = \nu(h)f(g'g) = \nu(h)(\rho(g)f)(g')$$

We also have

$$\rho(g)f \in \mathrm{Ind}(U)$$

We have that $\rho(g) : \mathrm{Ind}(U) \to \mathrm{Ind}(U)$ where $\rho(1) = \mathrm{id}$. We have $\rho(gg') = \rho(g)\rho(g')$. Therefore $(\rho, \mathrm{Ind}(U))$ is a representation of G defined by $\mathrm{Ind}_H^G(\nu)$, which is called the **induced representation.** The regular representation R is nothing else than $\mathrm{Ind}_{\{1\}}^G(\mathrm{triv})$.

We point out that Ind is a functor, so we if we have (ν, U) and (ν', U') representations of H, let's assume we have $\varphi : U \to U'$ an intertwining map. This means $\nu'(h)\circ\varphi(h) = \varphi\circ\nu(h)$. We have for $f \in \mathrm{Ind}(U)$

$$\Phi(f)(g) = \varphi(f(g))$$

Exercise 3

φ is the intertwining map of $\varphi \mapsto \Phi : \mathrm{Ind}_H^G \to \mathrm{Ind}_H^G$. Check $\Phi = \mathrm{Ind}_H^G(\varphi)$.

Next time, we will see Ind_H^G is a right adjoint of the restriction functor.

10 February 2

Suppose we have a group G and $H < G$. We can first consider categories

$$\text{Rep}(G)$$

and

$$\text{Rep}(H)$$

We of course have the restriction functor

$$\text{Rep}(G) \to \text{Rep}(H)$$

$\text{Res}_H^G(\pi)$ is a restriction of $\pi \in (\text{Rep}(G))$. We also constructed an induction functor, which takes $\nu \in \text{Rep}(H)$, and defines $\text{Ind}_H^G(\nu) \in \text{Rep}(G)$.

Theorem 9. We claim that Ind_H^G is a right adjoint of the functor Res_H^G. This means that if we look at $\text{Hom}_G(\pi, \text{Ind}_H^G(\nu))$ for $\pi \in \text{Rep}(G), \nu \in \text{Rep}(H)$,

$$\text{Hom}_G(\pi, \text{Ind}_H^G(\nu)) = \text{Hom}_H(\text{Res}_H^G(\pi), \nu)$$

via natural isomorphism.

Let's assume we want to construct an irreducible representation of G. If we have such a representation, any homomorphisms in the preceding Hom set must be injective if not 0. In other words, π is isomorphic to a subrepresentation in $\text{Ind}_H^G(\nu)$. (Frobenius reciprocity is mentioned, the functorial form)

Exercise 4

If a representation (ν, V) is a direct sum of two representations $\nu_1 \oplus \nu_2$, then in $\text{Ind}_H^G(\nu) = \text{Ind}_H^G(\nu_1) \oplus \text{Ind}_H^G(\nu_2)$. This implies that the functor Ind_H^G is exact. If G, H are finite, then we look at

$$0 \to \nu' \to \nu \to \nu'' \to 0$$

Because we are working with a semi-simple category, $\nu \cong \nu' \oplus \nu''$. We hence get

$$0 \to \text{Ind}_H^G(\nu') \to \text{Ind}_H^G(\nu') \oplus \text{Ind}_H^G(\nu'') \to \text{Ind}_H^G(\nu'') \to 0$$

First, let us consider $\text{Ind}_H^G(\nu)$, in particular its space $\text{Ind}_H^G(U) = \{f : G \to U | f(hg) = \nu(h)f(g), h \in H, g \in G\}$. So we can take a function $f \in \text{Ind}_H^G(U)$, and take it to U by evaluating it at $1 \in G$. So $e(f) = f(1)$. What can we say about the map? Taking the induced representation ρ on $\text{Ind}_H^G(U)$,

$$e(\rho(h)f) = (\rho(h)f)(1) = f(1 \cdot h) = f(h) = \nu(h)f(1)$$

$$= \nu(h)e(f)$$

We see that $e \circ \rho = \nu(h) \circ e$. Hence $e \in \text{Hom}_H(\text{Res}_H^G(\text{Ind}_H^G(\nu)), \nu)$. Take $\psi \in \text{Hom}_G(\pi, \text{Ind}_H^G(\nu))$, $\text{Res}_H^G(\psi) \in \text{Hom}_H(\text{Res}_H^G(\pi), \text{Ind}_H^G(\nu))$. $e \circ \psi \in \text{Hom}_H(\text{Res}_H^G(\pi), \nu)$. We would like to show that this map A is an isomorphism. Let's define this map $A(\psi) = e \circ \psi$. We construct the inverse map to A. Suppose we have an element $\text{Hom}_H(\text{Res}_H^G(\pi), \nu)$. We would like to create an element $\text{Hom}_G(\pi, \text{Ind}_H^G(\nu))$. We have $\varphi : V \to U$, $\varphi \in \text{Hom}_H(\text{Res}_H^G(\pi), \nu)$.. Take $v \in V$. Let's consider $F_v : G \to U$ given by

$$F_v(g) = \varphi(\pi(g)v).$$

Let's check $F_v(hg) = \varphi(\pi(h)\pi(g)v) = \nu(h)\varphi(\pi(g)v)$, hence F_v is in the induced representation space $\text{Ind}_H^G(U)$. Hence this gives us

$$v \mapsto F_v$$

a map $V \to \mathrm{Ind}(U)$. In fact, this is a linear map. Now what happens with the G action?

$$\pi(g)v \mapsto F_{\pi(g)v}$$

but

$$F_{\pi(g)v}(g') = \varphi(\pi(g')\pi(g)v) = \varphi(\pi(g'g)v) = F_v(g'g) = \rho(g')F_v(g).$$

Hence the bigger map

$$\varphi \mapsto (v \mapsto F_v)$$

$Hom_H(\mathrm{Res}_H^G(\pi), \nu) \to \mathrm{Ind}_H^G(U)$ is in

$$\mathrm{Hom}_G\left(\mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu), \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu)\right)$$

We have $\Phi \in \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$. This gives $B(\varphi) = \Phi$. Claim: A and B are inverses to each other.

$$(A \circ B)\varphi(v) = A(\Phi)(v) = e \circ \Phi(v) = e(F_v) = F_v(1) = \varphi(v)$$

Now

$$(B \circ A)(\psi)(v)(g) = (B\,(A(\psi))\,(v))(g) = A(\psi)(\pi(g)v) = e \circ \psi(\pi(g)v) = (\psi(\pi(g)v))\,(1) = (\rho(g)\psi(v))(1) = \psi(v)(g)$$

so $B \circ A(\psi) = \psi$. This shows the natural isomorphism. This is the proof of Frobenius reciprocity. This is a very on-the-nose proof, but it will show up constantly throughout the course.

11 February 4

Fact: $\mathrm{Ind}_K^G(\nu) \cong \mathrm{Ind}_H^G\left(\mathrm{Ind}_K^H(\nu)\right)$. Frobenius Reciprocity:

$$\mathrm{Hom}_G\left(\pi, \mathrm{Ind}_K^G(\nu)\right) = \mathrm{Hom}_K(\mathrm{Res}_K^G(\pi), \nu)$$

$$= \mathrm{Hom}_K(\mathrm{Res}_K^H \circ \mathrm{Res}_H^G(\pi), \nu)$$
$$= \mathrm{Hom}_H\left(\mathrm{Res}_H^G(\pi), \mathrm{Ind}_K^H(\nu)\right)$$
$$= \mathrm{Hom}_G\left(\pi, \mathrm{Ind}_H^G \circ \mathrm{Ind}_K^H(\nu)\right)$$

Thus this gives us a natural isomorphism

$$\mathrm{Ind}_K^G(\nu) \cong \mathrm{Ind}_H^G\left(\mathrm{Ind}_K^H(\nu)\right)$$

The general theorem: Two adjoint functors are isomorphic. Yoneda's lemma

Induction on stages is useful for calculations. The dimension of induced representations? Assume G is finite. By induction from identity to G of

$$\mathrm{Ind}_{\{1\}}^G(\mathbb{C}) = R(G) = \mathbb{C}[G]$$

If G is finite,

$$\mathrm{Ind}_H^G(U) = \mathbb{C}[G, U] = \mathbb{C}[G] \otimes_{\mathbb{C}} U(\text{space of functions from } G \text{ to } U\ )$$

We have

$$\dim\, \mathrm{Ind}_H^G(U) \le [G] \cdot \dim U$$

Remember that

$$F(hg) = \nu(h)F(g)$$

These functions are determined on each (right) coset Hg by its value at g. Given group G, we have cosets Hence $\mathrm{Ind}(U)$ is the direct sum of functions supported on one coset only. Pick coset C to be an H right-coset.

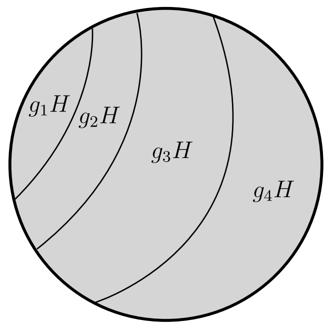

Figure 1: Split G into cosets

Pick

$$Hg_1 = C$$

for some $g_1 \in G$. Pick $(e_i; i \in I)$ a basis of U.

$$e_{c,i}(g) = \begin{cases} 0 & g \in C \\ \nu(gg_C^{-1})e_i & g \in C = Hg_C \end{cases}$$

We see that $\left(e_{c,i} : c \in \frac{G}{H}, i \in I\right)$.

$$\dim \mathrm{Ind}(U) = [\frac{G}{H}] \cdot \dim(U)$$

This gives us the precise formula.

Example. We finish the example of $G = S_3$. $\hat{G}$ consists of $\mathrm{triv}, \mathrm{sgn}, \sigma$. σ is 2-dimensional, and we know its character. We have $A_3 \subset S_3$, the set of even permutations. Denote $a = (2\ 3\ 1)$. Since A_3 is cyclic group (and hence abelian), all of its irreducible representations are one dimensional. In particular, note $\hat{A}_3 = \{1, \varphi, \varphi^2\}$. $\varphi \in \hat{A}_3$ which is nontrivial, $\varphi(a) = e^{2\pi i/3}$ (note that φ is a scalar multiple of the identity transformation, and it has order 3). The other nontrivial element is φ^2. Let's calculate $\mathrm{Res}_{A_3}^{S_3}(\sigma)$. We know

$$\mathrm{ch}\left(\mathrm{Res}_{A_3}^{S_3}(\sigma)\right) = \mathrm{ch}(\sigma)|_{A_3}$$

$$= \begin{cases} 2 & \text{at } 1 \\ -1 & \text{at } A_3 \setminus \{1\} \end{cases}$$

Now this character must be a sum of characters (by Maschke's and the fact that all irreducibles are characters). We claim

$$\mathrm{ch}(\mathrm{Res}_{A_3}^{S_3}(\sigma)) = \mathrm{ch}(\varphi) + \mathrm{ch}(\varphi^{-1})$$

Evaluated at a, the latter is 2 times the real part of $e^{2\pi i/3}$, $\cos(2\pi i/3) = -\frac{1}{2}$. Evaluated at a^2, the latter is that of $\cos(4\pi i/3) = -\frac{1}{2}$. At 1, the latter is 2. We can now calculate $\dim \mathrm{Hom}_{A_3}(\mathrm{Res}_{A_3}^{S_3}(\sigma), \varphi) = 1$. By Frobenius reciprocity, this is

$$\dim \mathrm{Hom}_{S_3}(\sigma, \mathrm{Ind}_{A_3}^{S_3}(\varphi)) = 1$$

The conclusion is $\sigma \cong \mathrm{Ind}_{A_3}^{S_3}(\varphi)$. This gives us the last remaining representation of S_3. But you may ask about φ^{-1}. There is no distinction. By symmetry, $\sigma \cong \mathrm{Ind}_{A_3}^{S_3}(\varphi^{-1})$.

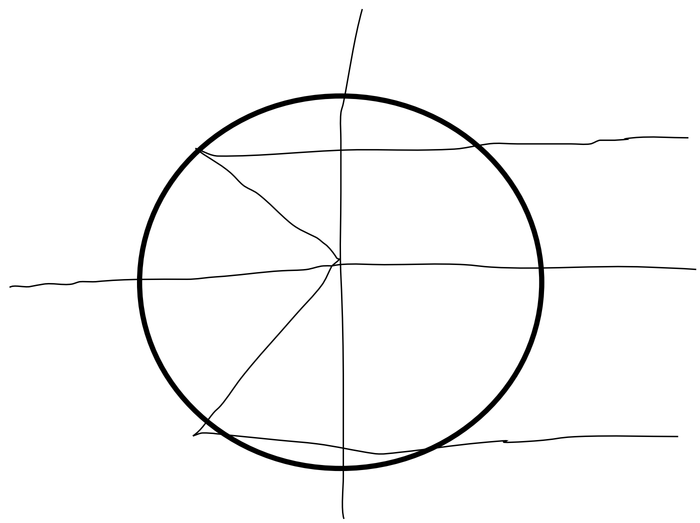

Figure 2: testcirc

12 February 7

Let's assume G is finite. Then if we take $H < G$, and let (ν, U) be an irreducible representation of H. Then it's finite dimensional, and from what we proved last time, $\mathrm{Ind}_H^G(\nu)$ is finite dimensional. By Maschke's theorem,

$$\mathrm{Ind}_H^G(\nu) = \bigoplus_{i=1}^{m} \pi_i$$

where π_i's are irreducible representations of G. Therefore if (π, V) is an irreducible representation of G. As we have discussed,

$$\dim(\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))) = \dim(\mathrm{Hom}_G(\pi, \bigoplus_{i=1}^{m} \pi_i)$$

$$= \sum_{i=1}^{n} \dim\left(\mathrm{Hom}_G(\pi, \pi_i)\right)$$

We know from a previous lecture that each term is zero if $\pi \ncong \pi_i$ and 1 if $\pi \cong \pi_i$.

$$= \text{multiplicity of } \pi \text{ in } \mathrm{Ind}_H^G(\nu)$$

By Frobenius reciprocity,

$$\dim(\mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu)) = \text{mult. of } \nu \text{ in } \mathrm{Res}_H^G(\pi)$$

and $\mathrm{Res}_H^G(\pi) \cong \bigoplus_{j=1}^{k} \nu_j$ where ν_j are irreducible representations of H. Then the multiplicity of π in Ind_H^G =the multiplicity of ν in $\mathrm{Res}_H^G(\pi)$. We would like to calculate the character of induced representations. $\mathrm{Ind}_H^G(\nu)$. For a character to exist, its representation must be finite dimensional. We must assume H, G are finite, and that ν is finite dimensional. As noted before, we have found a basis for $\mathrm{Ind}_H^G(\nu)$. We look at the right coset space $H \setminus G \ni C$. And the space

$$\mathrm{Ind}_H^G(U) = \bigoplus_{C \in H\setminus G} \left\{\text{space of functions in } \mathrm{Ind}_H^G(U) \text{ supported on } C\right\}$$

$f(hg) = \nu(h)f(g)$. $\{e_i, 1 \le i \le n\}$ is a basis of U.

$$e_{c,i}(g) = \begin{cases} 0 & g \notin C \\ \nu(gg_c^{-1})e_i & \end{cases}$$

where g_c is a representative of C. We have

$$\{e_{c,i} | 1 \le i \le n, c \in H \setminus G\}$$

is a basis for $\mathrm{Ind}_H^G(U)$. In particular, $\mathrm{Ind}_H^G(\nu)$ is finite dimensional. Let's take $g \in G$. We are going to calculate how g acts on the basis. We have denoted by ρ the action of the induced representation.

$$(\rho(g)e_{c,i})(g') = e_{c,i}(g'g)$$

$$g'g \notin C \Rightarrow \rho(g)e_{c,i} = 0$$

$g' \notin C_g^{-1} = D$, a coset. We conclude that $\rho(g)$ is a linear combination of such cosets. We have

$$\sum \rho(g)(D,j)(c,i)\rho_{D,j}(g')$$

$(\rho(g)e_{c,i})(g_D) = \nu(g_D g g_C^{-1})e_i$.

$$\sum_{j=1}^n \nu(g_D g g_C^{-1})_{ji} e_j$$

$$= \sum_{j=1}^n \nu(g_D g g_C^{-1})_{ji} e_{D,j}(g_D)$$

$C \cdot g^{-1} = D$ is a relation we are using. $D = C$ (there is a critical error nearby that you should check written notes for). $Hg_C \cdot g^{-1} = Hg_C$, which means $g_C g^{-1} g_C^{-1} \in H$.

$$\mathrm{ch}(\mathrm{Ind}_H^G(\nu))(g) = \sum_{C \cdot g = c} \mathrm{ch}(\nu)(g_C g g_C^{-1})$$

So the above equation is the first observation. Now $g_C g g_C^{-1} \in H$, so we have

$$\sum_{g_C g g_C^{-1} \in H} \mathrm{ch}(\nu)(g_C g g_C^{-1}).$$

Since $\mathrm{ch}(\nu)$ is constant on conjugacy classes, we can conjugate the elemnt in its parameters by some $h \in H$, which we can then tak the average with:

$$\frac{1}{[H]} \sum_{h \in H} \sum_{g_C g g_C^{-1} \in H} \mathrm{ch}(\nu)\left(h g_C g g_C^{-1} h^{-1}\right)$$

Now we have

$$\frac{1}{[H]} \sum_{g' \in G \ni g' g g'^{-1} \in H} \mathrm{ch}(\nu)(g' g g'^{-1})$$

(the difference between this sum and the last sum is that the former only sums over coset representatives g_C, one for each coset C, while here we are summing over all $g' \in G$). We have $\mathrm{ch}(\nu) : H \to \mathbb{C}$. We can extend $\mathrm{ch}(\nu)$ to a function χ_ν on G. It is

$$\chi_\nu(g') = 0, g' \notin H$$

$$\chi_\nu(g') = \mathrm{ch}(\nu)(g'), g' \in H$$

$\mathrm{ch}(\mathrm{Ind}_H^G(\nu))(g) = \frac{1}{[H]}\sum_{g'\in G}\chi_\nu(g'gg'^{-1})$ (this is the formula that will imply Frobenius reciprocity directly, as we will discuss on Wednesday). First observe that the character of Ind_H^G vanishes on all conjugacy classes that don't intersect H.

Example. Note the regular representation $R = \mathrm{Ind}_{\{1\}}^G(\mathrm{triv})$. The character of the trivial representation is 1 at the origin and zero everywhere else.

$$\mathrm{ch}(R(g)) = \begin{cases} [G]\cdot 1 & \text{at } 1 \\ 0 & \text{at } g \neq 1 \end{cases}$$

13 February 9

This marks the last class in which we review representation theory of finite groups. Let G be a finite group and H be a subgroup. We have (ν, U) a finite dimensional representation, we can then talk about $\mathrm{ind}_H^G(U)$. We can then obtain

$$\mathrm{ch}\left(\mathrm{Ind}_H^G(U)\right)$$

We have the visualization We take the average

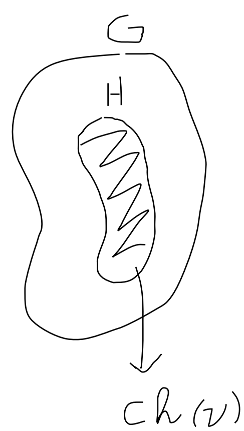

Figure 3: We can extend $\mathrm{ch}(\nu)$ to $\chi_\nu : G \to \mathbb{C}$ with $\chi_\nu|_H = \mathrm{ch}(\nu)$.

$$\mathrm{ch}\left(\mathrm{Ind}_H^G(\nu)\right)(g) = \frac{1}{[G]}\sum_{g'\in G}\chi_\nu\left(g'gg'^{-1}\right)$$

We have $\mathrm{ch}(\mathrm{Ind}_H^G(\nu))(1) = \dim\left(\mathrm{Ind}_H^G(\nu)\right) = [H\setminus G] - \dim(U)$. The more interesting case arises when $H \lhd G$. In this case, the support of $\mathrm{ch}\left(\mathrm{Ind}_H^G(\nu)\right)$ is in H.

$$\mathrm{ch}\left(\mathrm{Ind}_H^G(\nu)\right)(g) = \begin{cases} 0 & g \notin H \\ \frac{1}{[H]}\sum_{g'\in G}\mathrm{ch}(\nu)(g'gg'^{-1}) & g \in H \end{cases}$$

Exercise 5

If you look at the representation $h \mapsto \nu(h)$, $g \in G$, and then if you look at

$$h \mapsto \nu(ghg^{-1})$$

since $ghg^{-1} \in H$, we can see that this is a representation of the group H. Call it $\nu^g(h) = \nu(ghg^{-1})$, the representation **twisted by** g. If ν is irreducible, ν^g stay irreducible. You can have the action of G on the dual space of H, $\hat{H}$. GIven $g \in G$, g acts on $\hat{H}$ via

$$\nu \xrightarrow{g} \nu^g \in \hat{H}$$

Example. $G = S_3$. Its dual $\hat{G}$ has three irreducibles by counting conjugacy classes.

$$\hat{G} = \{\text{trivial}, \text{sgn}, \sigma\}$$

and σ is 2-dimensional. There are three conjugacy classes, and triv, sgn are both 1-dimensional. σ is 2-dimensional.

$$1^2 + 1^2 + 2^2 = 6 = 3! = |G|$$

Burnside? theorem. We would like to see that σ is an induced representation. Let's examine $A_3 < S_3$. In this case

$$\left[\frac{S_3}{A_3}\right] = 2$$

$$[A_3] = 3$$

A_3 is a cyclic subgroup since it has order 3. $\hat{A}_3$ consists of characters which are determined by the value on a chosen generator a. Viewing the group as the third roots of unity, the representations can be viewed as rotations by $e^{2\pi i/3}$

$$\hat{A}_3 = \{\text{triv}, \nu, \nu^2\}$$

We now look at

$$\text{Ind}_{A_3}^{S_3}(\nu)$$

We have $\hat{A}_3 = \{\text{triv}, \nu, \nu^{-1}\}$. Given $g \in G$, the orbit of triv is triv by itself. Conjugating ν by an element of $A_3 \ni g$ (see the above exercise) we get back ν. Otherwise we get ν^{-1}. If an element of G commutes with an element of A_3, we could have trouble, because S_3 is not abelian. It would only make sense that given $g \notin A_3$ it just permutes ν and ν^{-1}. Using this, we can calculate the character of the induced. Note

$$\text{ch}\left(\text{Ind}_H^G(\nu)\right)(g) = \begin{cases} 0 & g \notin H \\ \frac{1}{[H]}\sum_{g' \in G} \text{ch}(\nu)(g'gg'^{-1}) & g \in H \end{cases}$$

By our above calculation. When $H = A_3$, $G = S_3$, the second formula above is

$$\frac{1}{3}\sum_{g' \in A_3} \text{ch}(\nu)(g) + \frac{1}{3}\sum_{g \notin A_3} \text{ch}(\nu^{-1})(g) = \text{ch}(\nu)(g) + \text{ch}(\nu^{-1})(g)$$

in particular $\text{ch}(\text{Ind}_H^G(\nu))(1) = 2$. And $\text{ch}(\nu^{-1})(g) = \overline{\nu(g)}$. So we have

$$2\text{Re}\left(\nu(a)\right) = -1$$

$\nu(a) = e^{2\pi i/3}$. If the character is irreducible, we would have

$$\text{ch}(\text{Ind}_H^G(\nu)) = \text{ch}(\pi_1) + \ldots + \text{ch}(\pi_k)$$

where the π_i 's are in $\hat{G}$. But this is also $\text{ch}(\sigma)$. Work with norms of the characters, but we will get $\text{Ind}_H^G(\nu) = \sigma$ is irreducible.

Exercise 6

Equivalence of characters implies equivalence of representations

We will prove the Frobenius reciprocity using this formula on Friday, and then try to generalize the results to compact groups.

14 February 11

Character version of Frobenius Reciprocity. If G is a finite group, and $H < G$, and if ν is a finite dimensional representation of H, then we can construct $\mathrm{Ind}_H^G(\nu)$. This is also finite dimensional. It thus has character

$$\mathrm{ch}\left(\mathrm{Ind}_H^G(\nu)\right)(g) = \frac{1}{[H]} \sum_{g' \in G} \chi_\nu(g' g g'^{-1})$$

where χ_ν is an extension by zero of $\mathrm{ch}(\nu)$ on H. We will take a representation π which is finite dimensional of G. We calculate

$$\begin{aligned}
&\left(\mathrm{ch}(\mathrm{Ind}_H^G(\nu)) | \mathrm{ch}(\pi)\right) \\
&= \frac{1}{[G]} \sum_{g \in G} \left(\mathrm{ch}(\mathrm{Ind}_H^G(\nu))(g) \overline{\mathrm{ch}(\pi)(g)}\right) \\
&= \frac{1}{[G][H]} \sum_{g, g' \in G} \chi_\nu(g' g g'^{-1}) \overline{\mathrm{ch}(\pi)(g)} \\
&= \frac{1}{[H][G]} sum_{g, g' \in G} \chi_\nu(g) \overline{\mathrm{ch}(\pi)(g'^{-1} g g')}
\end{aligned}$$

Now π is constant on conjugacy classes so we have

$$= \frac{1}{[H][G]} \sum_{g, g' \in G} \chi_\nu(g) \overline{\mathrm{ch}(\pi)(g)}$$

$$= \frac{1}{[H]} \sum_{g \in G} \chi_\nu(g) \overline{\mathrm{ch}(\pi)(g)}$$

$$= \frac{1}{[H]} \sum_{h \in H} \mathrm{ch}(\nu)(h) \overline{\mathrm{ch}(\pi)(h)}$$

since χ_ν is an extension by 0 of ν.

$$= \left(\mathrm{ch}(\nu) | \mathrm{ch}\left(\mathrm{Res}_H^G(\pi)\right)\right)$$

where the inner product is taken over H. Note the previous inner product was taken over G. We mentioned that this was the dimension of $\mathrm{Hom}_H(\nu, \mathrm{Res}_H^G(\pi))$. The starting inner product was $\dim\left(\mathrm{Hom}_H(\mathrm{Ind}_H^G(\nu), \pi)\right)$.

We now look at generalizing results to compact groups. Let G be a topological group. This means that G is a group and a Hausdorff topological space with the extra condition that the group operation $G \times G \to G$ is continuous as well as the inversion operation $G \to G$. If we don't have the Hausdorff condition, take the closure of e. Then we can take

$$\frac{G}{\overline{\{e\}}}$$

which is Hausdorff. We can also assume G is locally compact. We need local compactness because of Riesz-Markov-Kakutani's theorem. We can take a σ-algebra generated by open sets. From there we can define measures. To each measure we can define a linear form $\int$. To restate the theorem:

Theorem 10. $C_0(G)$ defined to be the continuous real functions on G with compact support. We can define a linear form

$$f \mapsto \int_G f(g) d\mu(g)$$

The statement of the theorem says that this determines a measure μ.

If we have $C_0(G)$ positive functions $f(g) \geq 0$. We can examine $\varphi : C_0(G) \to \mathbb{R}$ which is a linear form and $\varphi(f) \geq 0$ for $f \geq 0$. If we consider a Lie group G, we can replace the condition that G is a topological space with manifold structure. The multiplication operation and inversion function are smooth maps. Differentiable manifolds are locally compact and Hausdorff so all is well. Locally compact groups generalize Lie groups.

If we have a topological group G, we can talk about $C_0(G)$ (real or complex valued functions with certain conditions) as above. Of course, if G is compact, then $C_0(G) = \mathcal{C}(G)$ all continuous functions $G \to \mathbb{R}$ or $\mathbb{C}$. If G is finite, the only topology we can put on G to make it Hausdorff is the discrete topology. The space we have is $\mathcal{C}(G) = \mathbb{C}[G]$. The main idea in our theory was to consider

$$\mu : f \mapsto \frac{1}{[G]} \sum_{g \in G} f(g)$$

We have $\mu : \mathbb{C}[G] \to \mathbb{C}$ a linear form. What are its properties? We have $\mu : \mathbb{C}[G] \to \mathbb{C}$ a linear form and μ is positive. If f is a positive function, then $f(g) \geq 0$ for all $g \in G$, which means $\mu(f) \geq 0$. We have

1. above, μ is **positive**.
2. $\mu(R(g)f) = \mu(f)$ for any $f \in \mathbb{C}[G], g \in G$. We have

   $$\mu(R(g)f) = \frac{1}{[G]} \sum_{g' \in G} f(gg') = \frac{1}{[G]} \sum_{g' \in G} f(g')$$

   We call μ **right-invariant**.
3. $\mu(1) = 1$. We call μ **normalized**.

Now assume μ is a linear form with properties $2, 3$. We can define

$$\delta_g(g') = \begin{cases} 1 & g' = g \\ 0 & g' \neq g \end{cases}$$

If we examine definitions, $R(g)\delta_1 = \delta_{g^{-1}}$. We have $\mu(\delta_g) = \mu(\delta_1)$. We call this value $c = \mu(\delta_1)$. We have

$$1 = \mu(1) = \mu(\sum_{g \in G} \delta_g) = \sum_{g \in G} \mu(\delta_g) = \sum_{g \in G} \mu(\delta_1) = [G]c$$

So $c = \frac{1}{[G]}$. Now we have

$$f = \sum_{g \in G} \delta_g f(g)$$

So

$$\mu(f) = \frac{1}{[G]} \sum_{g \in G} f(g)$$

Hence μ is completely determined by properties $2, 3$. In the case of bigger groups, $2, 3$ also determine a unique measure, which is called the Haar measure. It is determined by Riesz's.

15 February 14

Last time, we considered the compact groups G. $\varphi(G)$ = space of all continuous functions $G \to \mathbb{C}$. We have that $\varphi(G)$ is a Banach space with norm

$$\|f\| = \max_{g \in G} |f(g)|$$

since f is from a compact set, it indeed has a maximum. We saw that $|f(g)|$ is complete as a metric space with metric $d(f,g) = \|f - g\|$. We show that there exists a unique linear form μ on $\varphi(G)$ such that

1. μ is positive. $f \geq 0$ implies $\mu(f) \geq 0$. As a separate remark, $\varphi(G)$ has a natural right representation on G defined by $R(g)f(h) = f(hg)$.
2. $\mu(R(g)f) = \mu(f)$ for any $f \in \varphi(G)$, $g \in G$. This property is called **right-invariance.**
3. μ is **normalized**. In other words $\mu(1) = 1$.

Last time, we proved that if G is finite with the discrete topology, then $\mathbb{C}[G]$ is the space $\varphi(G)$. We proved uniqueness in that case last class. On our space $\mathcal{C}(G)$ (it's actually $\mathcal{C}$ and not φ), we have a representation

$$(L(g)f)(h) = f(g^{-1}h)$$

$L(gg') = L(g)L(g')$.

Exercise 7

In the sense of continuous representations we will discuss later, R above is a continuous representation.

We calculate

$$(R(g)L(g')f)(h) = L(g')f(hg) = f(g'^{-1}hg)$$
$$(L(g')R(g)f)(h) = f\left(g'^{-1}hg\right)$$

In other words, $R(g)$ and $L(g')$ commute for any $g, g' \in R$. Right and left regular representations commute. For fixed $g \in G$, let's consider the linear form $\nu_g(f) = \mu(L(g)f)$. We have ν_g is positive and right invariant.

$$\nu_g(R(g')f) = \mu(L(g)R(g')f) = \mu(R(g')L(g)f) = \mu(L(g)f)$$

by right invariance of μ. We have $\nu_g(1) = \mu(1) = 1$, so $\nu_g = \mu$. But also note that by calculation μ is left invariant. μ is thus **bi-invariant.**

We will also see some other properties of μ. We first see an example of a Haar measure.

Example. Let us consider S_1 as a group. Or $\frac{\mathbb{R}}{2\pi\mathbb{Z}}$. For a continuous function f on S_1, we have

$$\mu(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi}) d\varphi$$

μ is positive, μ is linear, μ is normalized. We also haveright invariance:

$$\mu(R(e^{i\theta})f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi + i\theta}) d\varphi$$

By change of variables

$$\mu(R(e^{i\theta} f)) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\varphi})$$

(do rest of work as an exercise). Note that $\varphi \mapsto e^{in\varphi}$ is an orthonormal set.

By Riesz's theorem, if such a function μ, which is a linear form $\mathcal{C}(G) \to \mathbb{C}$, then μ is represented by a measure which by abuse of notation we also denote by μ,

$$\mu(f) = \int_G f(g) d\mu(g)$$

1. μ is a positive measure on G.
2. If A is a measurable set in G, $\mu(Ag) = \mu(A)$
3. $\mu(G) = 1$. Consequently we also have $\mu(gA) = \mu(A)$.

Such a measure μ with the above properties is called a **Haar measure on G.**

Remark. Assume that G is a locally compact group. An example is $(\mathbb{R}, +)$. We can consider

$\mathcal{C}_0(G)$- Continuous functions with compact support

What about $\mu : \mathcal{C}_0(G) \to \mathbb{C}$. If the function satisfies

1. μ is positive
2. $\mu(R(g)f) = \mu(f)$ for all $g \in G$, $f \in \mathcal{C}_0(G)$
3. μ is nonzero

So the list of criteria is almost all of the properties for compact groups except property 3, since 1 is not compactly supported.

Theorem 11. Such a linear form exists, and it is unique up to a positive multiple.

16 February 16

We now construct Haar measures on compact groups. Consider compact group G and $\mathcal{C}_\mathbb{R}(G)$, the space of real valued continuous functions on G. We would like to construct a linear form

$$\mu : \mathcal{C}_\mathbb{R}(G) \to \mathbb{R}$$

satisfying

1. $\mu(f) \geq 0$ if $f \geq 0$
2. $\mu(R(g)f) = \mu(f)$ for any $g \in G$ and $f \in \mathcal{C}_\mathbb{R}(G)$.
3. $\mu(\text{id}) = 1$. (so it is normalized)

We talked about $G = \mathbb{R}$, which isn't compact but nonetheless gives a measure that satisfies the first two properties. By Riesz-Markov-Kakutani's theorem, μ gives a measure. By the theorem there is a positive borel measure (abuse of notation again call it μ) such that

$$\mu(f) = \int_G f(g) d\mu(g)$$

Proof. (Sketch): Take a function $f \in \mathcal{C}_\mathbb{R}(G)$. Construct a **right mean**. Take a finite sequence $\overline{a} = (a_1, \ldots, a_n)$, $a_i \in G$. We construct

$$\mu(f, \overline{a})(g) = \frac{1}{n} \sum_{i=1}^{n} f(ga_i)$$

$$\mu_f = \{\mu(f, \overline{a}) | \text{ all finite sequences } \overline{a} \text{ in } G\}$$

This set has compact closure. Remember that $\|f\| = \max_{g \in G} |f(g)|$ makes $\mathcal{C}_\mathbb{R}(G)$ a Banach space. Claim: $\overline{\mu_f}$ contains one constant function. Its value is $\mu(f)$. We will prove Arzela Ascoli in the setting of compact groups.

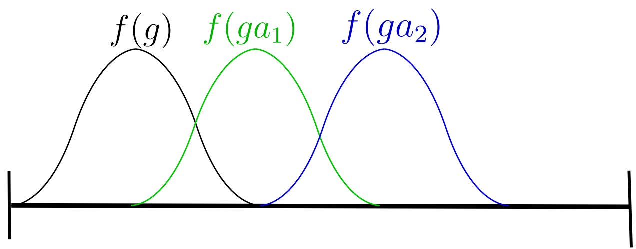

Figure 4: We will look at translates of the same function and somehow add them together

Lemma 2. Take $f \in \mathcal{C}_{\mathbb{R}}(G)$. $M(f) = \max_{g \in G} f(g)$. $m(f) = \min_{g \in G} f(g)$. We can define

$$V(f) = M(f) - m(f)$$

The claim is that $V(f) = 0$ if and only if f is constant.

The above lemma is directly provable from techniques of Foundations. The function V is called the **variation of** f.

Lemma 3. $V : \mathcal{C}_{\mathbb{R}}(G) \to \mathbb{R}$ is a continuous map.

Proof. Suppose we have $\varepsilon > 0$ and $f, f' \in \mathcal{C}_{\mathbb{R}}(G)$ and $\|f - f'\| < \varepsilon$. We have

$$|f(g) - f'(g)| < \varepsilon \forall g \in G$$

$$-\varepsilon < f(g) - f'(g) < \varepsilon$$

which implies $f'(g) - \varepsilon < f(g) < f'(g) + \varepsilon$. But this gives

$$m(f') - \varepsilon < f(g) < M(f') + \varepsilon$$

for all $g \in G$. We have

$$m(f') - \varepsilon < m(f) \leq M(f) < M(f') + \varepsilon$$

From this, we can calculate $V(f)$:

$$V(f) = M(f) - m(f) < M(f') + \varepsilon - m(f) < M(f') - m(f') + 2\varepsilon = V(f') + 2\varepsilon$$

$V(f) - V(f') < 2\varepsilon$. We also have similarly by symmetry that

$$V(f') - V(f) < 2\varepsilon$$

Hence
$$|V(f) - V(f')| < 2\varepsilon$$
proving continuity. □

We now study the means
$$\mu(f,\overline{a})(g) = \frac{1}{n}\sum_{i=1}^{n} f(ga_i)$$
What is its norm?
$$\|\mu(f,\overline{a})\| \leq \frac{1}{n}\sum_{i=1}^{n}\max_{g\in G}|f(g)| = \|f\|$$
by the triangle inequality. If a is fixed, $f \mapsto \mu(f,\overline{a})$ is a linear map. Now boundedness as above implies that the linear map is continuous. If we look at
$$M(\mu(f,\overline{a})) \leq M(f)$$
We also have
$$m(\mu(f,\overline{a})) \geq m(f)$$
We get $V(\mu(f,\overline{a})) = M(\mu(f,\overline{a})) - m(\mu(f,\overline{a})) \leq M(f) - m(f) = V(f)$. The conclusion?

Lemma 4. $V(\mu(f,\overline{a})) \leq V(f)$.

Remember now that we defined
$$\mu_f = \{\mu(f,\overline{a})|\overline{a}\}$$
We have
$$m(f) \leq \mu(f,\overline{a})(g) \leq M(f)$$
We will need the following lemma to finish our work:

Lemma 5. Let $f \in \mathcal{C}_{\mathbb{R}}(G)$ such that $V(f) > 0$. Then $\exists\ \overline{a}$ such that $V(\mu(f,\overline{a})) < V(f)$.

We write a box here, but the proof of the existence of μ will actually be done on Friday. □

17 February 23

Last time, for a continuous function $f \in \mathcal{C}_{\mathbb{R}}(G)$, we have defined $\mu(f,\overline{a})(g) = \frac{1}{n}\sum_{i=1}^{n} f(ga_i)$ and also
$$\nu(f,\overline{a})(g) = \frac{1}{n}\sum_{i=1}^{n} f(a_i g)$$
and we defined
$$\mu_f = \text{set of all right means}$$
$$N_f = \text{set of all left means}$$
There related to each other in terms of the opposite group. The last thing we proved was that $\overline{\mu_f}$ has a constant function. We also have $\overline{N_f}$ contains a constant function. We would like to show that if we take one constant function in one set, it corresponds to a constant function in the other. Take two sequences
$$\overline{a}, \overline{b}$$
so we can calculate
$$\nu(\mu(f,\overline{a}),\overline{b})(g) = \frac{1}{m}\sum_{i=1}^{m}\mu(f,\overline{a})(b_i g) = \frac{1}{m}\sum_{i=1}^{m}\frac{1}{n}\sum_{j=1}^{n} f(b_i g a_j)$$

$$= \mu(\nu(f, \overline{b}), \overline{a})(g)$$

Let's say $\overline{\mu_f}$ contains constant function φ and $\overline{\nu_f}$ contains constant function ψ. Taking $\varepsilon > 0$, choose $\overline{a}, \overline{b}$ where,

$$\|\mu(f, \overline{a}) - \varphi\| < \varepsilon/2$$

$$\|\nu(f, \overline{b}) - \psi\| < \varepsilon/2$$

First, let's consider

$$\|\nu(\mu(f, \overline{a}), \overline{b}) - \psi\|$$

$$= \|\nu(\mu(f, \overline{a}), \overline{b}) - \nu(\varphi, \overline{b})\| = \|\nu(\mu(f, \overline{a}) - \varphi, \overline{b})\|$$

$$\leq \|\mu(f, \overline{a}) - \varphi\| < \varepsilon/2$$

We also get by a similar argument,

$$\|\mu(\nu(f, \overline{b}), \overline{a}) - \psi\| = \|\mu(\nu(f, \overline{b})\overline{a}) - \mu(\psi, \overline{a})\| = \|\mu(\nu(f, \overline{b}) - \psi, \overline{a})\| < \varepsilon/2$$

We estimate $\|\varphi - \psi\|$, whic must be

$$\|\varphi - \nu(\mu(f, \overline{a}), \overline{b}) + \mu(\nu(f, \overline{b}), \overline{a}) - \psi\|$$

$$< \varepsilon$$

by the triangle inequality and our two inequalities above. If we show that μ_f contains two constant functions φ, φ', $\varphi = \psi = \varphi'$. So uniqueness of the constant function follows.

If $f \in \mathcal{C}_{\mathbb{R}}$, then we proved that $\overline{\mu_f}, \overline{N_f}$ contain a unique constant function φ. We call the value of φ the **mean value of** f, $\mu(f)$. We would like to prove the following: $\mu : \mathcal{C}_{\mathbb{R}}(G) \to \mathbb{R}$ is a positive linear form on $\mathcal{C}_{\mathbb{R}}(G)$. Once we prove this, then we will have the existence and uniqueness of a Haar measure by Riesz-Markov-Kakutani's theorem.

1. If we take $\alpha \in \mathbb{R}$, αf, then

$$\mu(\alpha \cdot f, \overline{a})(g) = \frac{1}{n} \sum_{i=1}^{n} \alpha f(ga_i) = \alpha \cdot \mu(f, \overline{a})(g)$$

   So the observation is that $\mu(\alpha \cdot f, \overline{a}) = \alpha \cdot \mu(f, \overline{a})$. Hence we have $\mu_{\alpha \cdot f} = \alpha \cdot \mu_f$ and $\overline{\mu_{\alpha \cdot f}} = \alpha \cdot \overline{\mu_f}$. Hence $\mu(f) \cdot \alpha = \mu(\alpha \cdot f)$. This tells us that μ is homogeneous.

2. We have positivity. If $f(g) \geq 0$ for all $g \in G$. Then

$$\mu(f, \overline{a})(g) \geq 0$$

   since all translates are non-negative. This implies that all functions in μ_f are positive, and so all functions in $\overline{\mu_f}$ are positive, so $\mu(f) \geq 0$ (note the deception of the word positive).

3. Taking $f, f' \in \mathcal{C}_{\mathbb{R}}(G)$. We have

$$\mu(f + f') = \mu(f) + \mu(f')$$

   (obtain a sequence for $\mu(f)$ and $\mu(f')$ each, and take the sum of the sequences). We apply a lemma:

> **Lemma 6.** Let $f \in \mathcal{C}_{\mathbb{R}}(G)$. Call $\overline{a}$ a finite sequence in G. We have
>
> $$\mu(\mu(f, \overline{a})) = \mu(f)$$

Proof. Let φ be the constant function with value $\mu(f)$. For $\varepsilon > 0$, there exists $\overline{b}$ such that $\|\nu(f, \overline{b}) - \varphi\| < \varepsilon$. Since $\nu(\varphi, \overline{b}) = \varphi$,

$$\|\nu(f - \varphi, \overline{b})\| < \varepsilon$$

We also have

$$\|\mu(\nu(f - \varphi, \overline{b}), \overline{a})\| \leq \|\nu(f - \varphi, \overline{b})\| < \varepsilon$$

We have

$$\|\nu(\mu(f - \varphi, \overline{a}), \overline{b})\| < \varepsilon$$

$$\|\nu(\mu(f, \overline{a}), \overline{b}) - \varphi\| < \varepsilon$$

For any $\overline{a}$, $\overline{N_{\mu(f,\overline{a})}}$ contains φ. So $\mu(\mu(f, \overline{a})) = \mu(f)$, completing the proof. $\square$

Back to additivity, take $\alpha = \mu(f), \beta = \mu(f')$. Take the constant functions $\varphi(g) = \mu(f)$ and $\varphi'(g) = \mu(f')$. We have

$$\|\mu(f, \overline{a}) - \varphi\| < \varepsilon/2$$

Take any $\overline{b}$, and look at

$$\|\mu(\mu(f, \overline{a}) - \varphi, \overline{b})\| \leq \|\mu(f, \overline{a}) - \varphi\| < \varepsilon/2$$

The former expression is

$$\|\mu(\mu(f, \overline{a}), \overline{b}) - \varphi\| < \varepsilon/2$$

What we calculated last week was that this is

$$\|\mu(f, \overline{a} \cdot \overline{b}) - \varphi\|$$

This tells us

$$\|\mu(f, \overline{a} \cdot \overline{b}) - \varphi\| < \varepsilon/2$$

This holds for any $\overline{b}$. We also have

$$\mu(f') = \mu(\mu(f', \overline{a}))$$

by the lemma. We can find $\overline{b}$ such that $\|\mu(\mu(f', \overline{a})\overline{b}) - \varphi'\| < \varepsilon/2$. This is equal to

$$\mu(f', \overline{a} \cdot \overline{b}).$$

We have

$$\mu(f + f', \overline{a} \cdot \overline{b}) = \mu(f, \overline{a} \cdot \overline{b}) + \mu(f, \overline{a} \cdot \overline{b})$$

The former differs from φ by less than $\varepsilon/2$ and likewise for the latter from φ'. Hence the entire expression differs from $\varphi + \varphi'$ by less than ε. This tells us $\varphi + \varphi'$, a constant function, is in $\overline{\mu_{f+f'}}$. Hence we have additivity.

We will show on Friday that this is right invariant and unique.

18 February 25

The sup norm on real valued functions on a compact group G forms a Banach space $\mathcal{C}_{\mathbb{R}}(G)$. Then from last class we constructed a linear form

$$\mu : \mathcal{C}_{\mathbb{R}}(G) \to \mathbb{R}$$

such that μ is also positive. That is, if $f(g) \geq 0$ for all $g \in G$. If we take any function $f \in \mathcal{C}_{\mathbb{R}}(G)$, we can take

$$\mu_f = \{\mu(f, \overline{a})\}$$

then $\overline{\mu_f}$ contains a constant function whose value we denoted $\mu(f)$. We proved its uniqueness in the last class! We would like to show that the positive condition implies continuity. We proved that $\mu(1) = 1$. We apply this to the following result:

Proposition 2. Let $\nu : \mathcal{C}_{\mathbb{R}}(G) \to \mathbb{R}$ be a linear form such that ν is positive and $\nu(1) = 1$. Then $|\nu(f)| \leq \|f\|$ for all $f \in \mathcal{C}_{\mathbb{R}}(G)$. In particular, ν is continuous.

Proof. For all $g \in G$,

$$-\|f\| \leq f(g) \leq \|f\|$$

Note that if $f' \leq f$, $\nu(f') \leq \nu(f)$. Hence

$$-\nu(\|f\|) \leq \nu(f) \leq \nu(\|f\|).$$

but note that

$$\nu(\|f\| \cdot 1) = \|f\| \cdot \nu(1) = \|f\|$$

hence we have

$$-\|f\| \leq \nu(f) \leq \|f\|$$

□

In particular, μ is continuous. Boundedness?

We would like to calculate

$$\mu(R(g)f)$$

Note $R(g)f(h) = f(hg) = \mu(f, \{g\})(h)$. By a result from last class,

$$\mu\left(\mu(f, \{g\})\right) = \mu(f)$$

μ is right invariant. We prove uniqueness. Let's assume ν is a linear form that is positive, normalized, and right-invariant. By our proposition above, ν is continuous. Let's calculate $\nu(\mu(f, \overline{a}))$ for any finite sequence $\overline{a}$ in G. Remember this is

$$\nu\left(\frac{1}{n}\sum_{i=1}^{n} f(\bullet \cdot a_i)\right) = \nu\left(\frac{1}{n}\sum_{i=1}^{n} R(a_i)f\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n} \nu\left(R(a_i)f\right) = \frac{1}{n}\sum_{i=1}^{n} \nu(f) = \nu(f)$$

Hence $\nu\left(\mu(f, \overline{a})\right) = \nu(f)$. This tells us ν on μ_f has value equal to $\mu(f)$. By the continuity of ν, we also have that ν has constant value $\nu(f)$ on $\overline{\mu_f}$. It contains a unique constant function φ which has value $\mu(f)$. We have $\mu(f)$ is the value of ν on $\overline{\mu_f}$, which must be $\nu(f)$. There exists a unique positive, normalized, right invariant linear form $\mu : \mathcal{C}_{\mathbb{R}}(G) \to \mathbb{R}$. μ determines a unique positive measure on G such that

$$\mu(f) = \int_G f(g) d\mu(g)$$

The measure μ (which we use as an abuse of notation) is called the Haar measure on G. If we look at the linear form $f \mapsto \mu(L(g)f)$, since left translation commutes with right translations, we have that this is right invariant and satisfies the other properties, so it must be equal to μ. The Haar measure is thus also left invariant.

If G is just locally compact, there exists a nonzero positive linear form $\mu : \mathcal{C}_0(G) \to \mathbb{R}$ (space of real compactly supported functions) which is right invariant. This is called a right Haar measure, since Riesz-Markov-Kakutani's theorem says

$$\mu(f) = \int_G f(g) d\mu(g)$$

Of course, uniqueness is different. If ν is another right Haar measure, it follows that $\nu = c \cdot \mu$ where $c > 0$. Normalization doesn't make sense in this context, so we are ignoring it and thus allowing scalar multiples. The proof of left invariance also fails to work in this context, since we are only guaranteed that μ is same up to constant multiple when we left translate it. That is, $\mu(L(g)f) = c_g\mu(f)$ for some c_g positive real. $g \mapsto c_g$ is a homomorphism. We have $\Delta_G : G \to \mathbb{R}_+^*$ by $g \mapsto c_g$.

Exercise 8

Δ_G is continuous.

G is called unimodular if $\Delta_G \equiv 1$. Unimodular is equivalent to G having a biinvariant Haar measure. Why are compact groups unimodular? There is a reason: We have $\Delta_G(G)$ is compact. This implies $\Delta_G(G) = \{1\}$, hence G is unimodular.

Example. An example of a unimodular group is $GL_2(\mathbb{R})$. It is a locally compact group. Then take a closed subgroup

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

for $a > 0$ which form a closed subgroup (right half plane). If you look at it as acting on vectors $(x, 1)$, it forms a group of affine transformations of $\mathbb{R}$, $x \mapsto ax + b$. Calculate the Haar measure on this group. By the uniqueness we discussed, it should be a multiple of the Lebesgue measure with a function on a, b. We can calculate the function, and thus get the Haar measure on the group.

Lemma 7. $\overline{\mu_f}$ is compact.

This follows from the Arzela Ascoli with the compactness criterion for $\mathcal{C}_{\mathbb{R}}(G)$. We will discuss this monday.

19 February 28

To complete the proof of the existence of the Haar measure, we would like to show $\overline{\mu_f}$ is compact. Remember the norm

$$\|f\| = \max_{g \in G} |f(g)|$$

Let X be a compact space, and consider the space of continuous functions on X, call it $\mathcal{C}(X)$ (perhaps complex valued) with norm

$$\|f\| = \max_{x \in X} |f(x)|$$

Let's assume that $\mathcal{S} \subset \mathcal{C}(X)$. We have $\mathcal{S}$ is pointwise bounded if for any $x \in X$, the set $\{f(x) | f \in \mathcal{S}\}$ is bounded.

We call $\mathcal{S}$ equicontinuous if for $x \in X$, $\varepsilon > 0$, there exists neighborhood U of x such that $y \in U$ implies $|f(x) - f(y)| < \varepsilon$.

Theorem 12. (Arzela-Ascoli) Let $\mathcal{S} \subset \mathcal{C}(X)$ be a pointwise bounded and equicontinuous. Then $\overline{\mathcal{S}}$ is compact.

We postpone the proof, but apply this theorem to μ_f. We merely have to check that

1. μ_f is pointwise bounded
2. μ_f is equicontinuous.

To prove 1, we proved that

$$-\|f\| \leq \mu(f, \overline{a}) \leq \|f\|$$

For all sequences $\overline{a}$. Hence μ_f is uniformly bounded. To prove 2, let G be a topological group, $f \in \mathcal{C}(G)$, f is a right uniformly continuous if for any $\varepsilon > 0$, there exists a neighborhood U of 1 such that

$$|f(g) - f(h)| < \varepsilon$$

if $gh^{-1} \in U$. That is, $g \in Uh$. We define uniform continuity on the left similarly but by swapping g, h. For the condition $h^{-1}g \in U$.

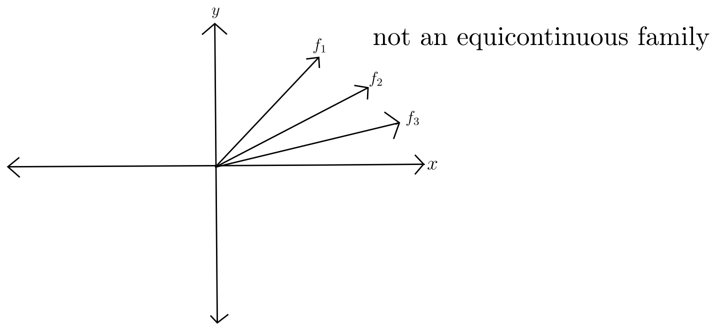

Figure 5: We have three functions forming a family which are **not** equicontinuous

Lemma 8. Let G be a compact group. Let $f \in \mathcal{C}(G)$. Then f is right and left uniformly continuous.

(this is a generalization of the statement that a continuous function on a compact set is uniformly continuous from real analysis.)

We will show this lemma implies 2, and then prove the lemma. Recall

$$\mu_f = \{\mu(f, \overline{a}) | \overline{a} \text{ finite subsets of } G\}$$

We claim that for any $f \in \mathcal{C}(G)$, $\{\mu(f, \overline{a}), \overline{a}\}$ are an equicontinuous set. Take $\varepsilon > 0$, $f \in \mathcal{C}(G)$. By the lemma, f is uniformly continuous. Accordingly choose a neighborhood U of 1, where

$$|f(g) - f(h)| < \varepsilon$$

if $gh^{-1} \in U$. Remember

$$\mu(f, \overline{a})(g) = \frac{1}{n} \sum_{i=1}^{n} f(ga_i)$$

If we look at $ga_i(ha_i)^{-1} = gh^{-1} \in U$, we have

$$|f(ga_i) - f(ha_i)| < \varepsilon$$

Now note

$$|\mu(f, \overline{a})(g) - \mu(f, \overline{a})(h)| \leq \frac{1}{n} \sum_{i=1}^{n} |f(ga_i) - f(ha_i)| < \frac{1}{n} \sum_{i=1}^{n} \varepsilon = \varepsilon$$

Hence we have shown equicontinuity since we have chosen h that works regardless of our choice of $\overline{a}$. Arzela Ascoli+the lemma we just used imply that $\overline{\mu_f}$ is compact. We prove the lemma:

Proof. If we look at $G \times G$, then for any U, an open neighborhood of $1 \in G$, we can form $B_U = \{(g, h) | hg^{-1} \in U\}$. Note that

$$(g, h) \mapsto hg^{-1}$$

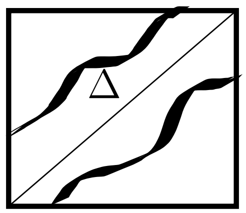

Figure 6: diag

is a continuous map $G \times G \to G$. Hence B_U is open in $G \times G$. Also note that B_U contains the diagonal set $\Delta = \{(g, g) \in G \times G | g \in G\}$. We claim: if A is an open set in $G \times G$ containing Δ, then there exists U such that $B_U \subset A$. We define map $\kappa : G \times G \to G \times G$, the shear map

$$\kappa(g, h) = (g, hg^{-1})$$

which is a continuous map. We can then consider the inverse

$$\kappa^{-1}(g, h) = (g, hg)$$

which is also continuous. Hence κ is a homomorphism. We have that $\kappa(\Delta) = G \times \{1\}$. Hence $\kappa(A)$ is an open set containing $G \times \{1\}$. We have $(g, 1) \in \kappa(A)$. Since $\kappa(A)$ is an open set and we are working with the product topology, V_g is an open neighborhood of g, U_g is an open neighborhood of 1, then $V_g \times U_g \subset \kappa(A)$. Hence V_g form an open cover of G, but the compactness of G implies there exists $V_{g_1}, \ldots, V_{g_n}$ a finite subcover, we can take

$$U = \bigcap U_{g_i}$$

a neighborhood of the identity. Hence $(\bigcup_{i=1}^n V_{g_i}) \times U \subset \kappa(A)$. In particular, $G \times U \subset \kappa(A)$ (since the V_{g_i}'s form a cover). But $\kappa^{-1}(G \times U) \subset A$ is B_U. Let's take our continuous $f \in \mathcal{C}(G)$, and $\varepsilon > 0$. Then the set $\{(g, h) || f(g) - f(h)| < \varepsilon\}$ is open since

$$(g, h) \mapsto f(g) - f(h)$$

is continuous. Call this open set A, which contains Δ, and apply our claim, which gives uniform continuity. On Wednesday, we will prove Arzela Ascoli. $\square$

Now if we have shown Arzela Ascoli, we have shown $\mathcal{S}$ is compact.

20 March 2

Today, we will prove Arzela Ascoli to complete our argument that $\overline{\mu_f}$ is compact.

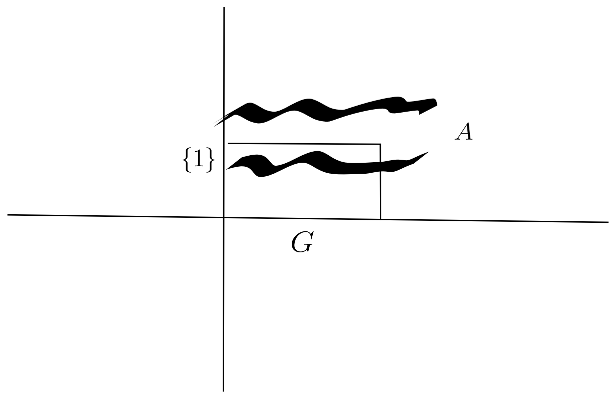

Figure 7: shear

Theorem 13. Let X be a compact space, and $\mathcal{C}(X)$ be $\{f : X \to \mathbb{C} | f \text{ is continuous}\}$.

$$\|f\| = \max_{x \in X} |f(x)|$$

The metric defined by this norm is complete, so $\mathcal{C}(X)$ forms a Banach space. If $\mathcal{S} \subset \mathcal{C}(X)$, and $\mathcal{S}$ is pointwise bounded and equicontinuous, then $\overline{\mathcal{S}}$ is compact.

Proof. We show some things.

1. $\mathcal{S}$ is a bounded set in $\mathcal{C}(X)$. That is,

$$\mathcal{S} \subset \{f | \|f\| \leq M\}$$

for some large M. To prove this, take $\varepsilon > 0$. Then for any $x \in X$, a neighborhood U_x of x exists such that for $y \in U_x$,

$$|f(y) - f(x)| < \varepsilon$$

for all $f \in \mathcal{S}$ by equicontinuity. We have that $(U_x, x \in X)$ is an open cover of X. We have that the compactness of X implies that there are $x_1, \ldots, x_m \in X$ such that $(U_{x_1}, \ldots, U_{x_m})$ is an open subcover of X. The set

$$\{f(x_i) | f \in \mathcal{S}, i = 1, \ldots, m\}$$

is a bounded set in $\mathbb{C}$. It is $\leq C$ for some C. This implies that if we take an arbitrary $x \in X$, then $x \in U_{x_i}$, and we have

$$|f(x) - f(x_i)| < \varepsilon$$

We have

$$|f(x)| \leq |f(x) - f(x_i)| + |f(x_i)| < C + \varepsilon$$

for all $f \in \mathcal{S}$. Since this holds for all $x \in X$,

$$\|f\| \leq C + \varepsilon = M$$

This shows $\mathcal{S}$ is bounded by $M = C + \varepsilon$.

2. For any $\delta > 0$, the set $\mathcal{S}$ can be covered with finitely many open balls of radius δ centered at elements of $\mathcal{S}$. So given $\varepsilon > 0$, we define a certain cover $(U_{x_1}, \dots, U_{x_m})$ of X in the following discussion. Denote by D the disk in $\mathbb{C}$,
$$D = \{z||z| \leq M\}$$
Consider D^n, a product of n copies of D, this is compact. Then we can consider
$$\Phi(f) = (f(x_1), \dots, f(x_m))$$
which is a map $\Phi : \mathcal{S} \to D^n$. Recall the topology on D^n can be defined with the metric
$$d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$
There exists a finite cover of D^n with balls of radius ε. Call the balls $B_1, \dots, B_m$ which intersect $\Phi(\mathcal{S})$. Then $B_1, \dots, B_m$ is an open cover of $\Phi(\mathcal{S})$. For each i, select an arbitrary $\Phi(f_i)$ in $B_i \cap \Phi(\mathcal{S})$,

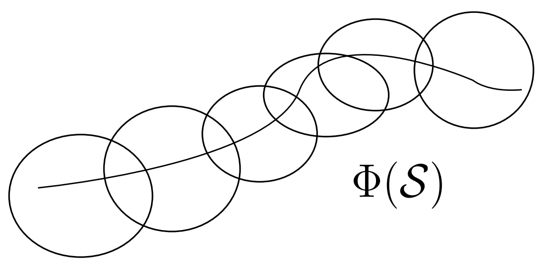

Figure 8: We cover $\Phi(\mathcal{S})$ in D^n.

$f_1, \dots, f_m \in \mathcal{S}$. Select C_i, open balls of radius 2ε centered at $\Phi(f_i)$. This implies $B_i \subset C_i$ for each $1 \leq i \leq m$. This implies $C_1, \dots, C_m$ is a cover of $\Phi(\mathcal{S})$. Select cover U_{x_i} in a following way. For $x \in X$, $x \in U_{x_i}$ for $1 \leq i \leq n$,
$$|g(x) - g(x_i)| < \varepsilon$$
for all $g \in \mathcal{S}$. This cover exists by compactness of X.
$$|f(x) - f_j(x)| \leq |f(x) - f(x_i)| + |f(x_i) - f_j(x_i)| + |f_j(x_i) - f_j(x)| < 2\varepsilon + |f_j(x_i) - f(x_i)|$$
for $f \in \mathcal{S}$. We have $\Phi(f) \in C_j$ for some j (the bound is regardless of choice of x_i), so choosing some appropriate j, we have $|f(x_i) - f_j(x_i)| < 2\varepsilon$. As a result,
$$|f(x) - f_j(x)| < 4\varepsilon$$
for all $x \in X$.

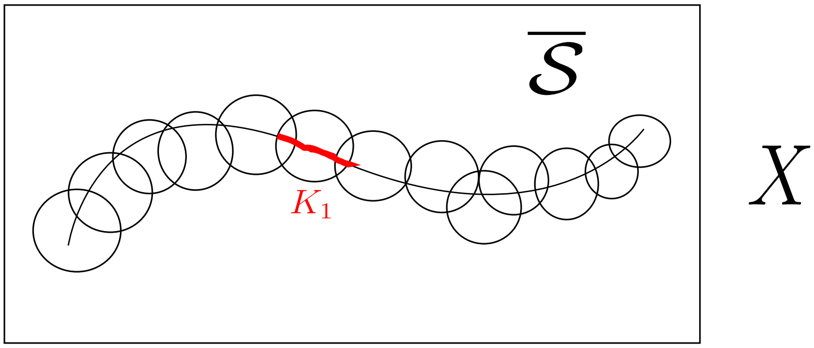

Figure 9: picSbar

3. We can now prove compactness. Assume that $\overline{\mathcal{S}}$ is not compact for contradiction. By definition of compactness, there exists an open cover $\mathcal{U}$ such that it has no finite open subcover of $\overline{\mathcal{S}}$. Cover $\overline{\mathcal{S}}$ by closed balls of radius 1, which we can do by 2. We can choose $K_1 = \overline{\mathcal{S}} \cap Z_1$, where Z_1 is one of the closed balls of radius 1 such that K_1 is not covered by a finite subcover of $\mathcal{U}$. We also examine 2 with $\delta = \frac{1}{2}$. We can cover K_1 with open balls of radius $\frac{1}{2}$. We can again construct a subset K_2 inside a ball of radius $\frac{1}{2}$ that is not covered by a finite subcover of $\mathcal{U}$. We obtain a decreasing sequence

$$K_1 \supset K_2 \supset K_3 \supset \dots$$

   Pick $F_i \in K_i$. This gives us a sequence (F_n) is a sequence in $\mathcal{C}(X)$ that is Cauchy. Since $\mathcal{C}(X)$ is complete, F_n converge to a function F in $\mathcal{C}(X)$. Note that $F \in \overline{\mathcal{S}}$ by closedness.

□

21 March 4

Last time, we were proving the Arzela Ascoli theorem. Given a subset $\overline{\mathcal{S}}$, we would like to show $\overline{\mathcal{S}}$ is compact. If it is not, there exists an open cover $\mathcal{U}$ of $\overline{\mathcal{S}}$ such that there is no finite subcover. We were doing an argument in which we found a descending sequence of open sets for which the compactness assumption fails. In particular, we found

$$Z_1 \cap \overline{\mathcal{S}} = K_1$$

covered by $\mathcal{U}$ that isn't covered by a finite subcover. Repeat the argument for radius $\frac{1}{2}$, proceed inductively to get

$$K_1 \supset K_2 \supset K_3 \supset \dots$$

We chose $F_i \in K_i$, which then converged to some $F \in \overline{\mathcal{S}}$ since they form a Cauchy sequence. This allows us to choose neighborhood V of F from $\mathcal{U}$, and then some n such that $K_n \subset V$, a contradiction. Let G be a compact group. There exists a unique **Haar** measure on G, positive biinvariant and normalized

$$f \mapsto \frac{1}{n} \sum_{g \in G} f(g)$$

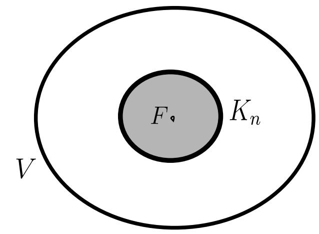

Figure 10: Ball

(akin to that for finite groups as in the above equation.) μ is positive, $\mu(G) = 1$.

Lemma 9. Let U be a nonempty open set in G. Then $\mu(U) > 0$.

Proof. Let's examine U. Since $U \neq \emptyset$, $g \in U$ for some $g \in G$. This implies that

$$\{Uh | h \in G\}$$

covers G. There exists $h_1, \ldots, h_n$ such that $Uh_1, \ldots, Uh_n$ form an open covering of G. Finite subadditivity of the measure implies

$$\begin{aligned} 1 = \mu(G) \leq \sum_{i=1}^{n} \mu(Uh_i) &= \sum_{i=1}^{n} \mu(U) \\ &= n\mu(U) \end{aligned}$$

(the second equality is from biinvariance) which hence cannot be 0. Note this gives us $\mu(U) \geq \frac{1}{n}$ □

Lemma 10.

$$\int_G f(g^{-1}) d\mu(g) = \int_G f(g) d\mu(g)$$

for any $f \in \mathcal{C}_{\mathbb{R}}(G)$.

Proof. Consider the following linear form:

$$f \mapsto \int_G f(g^{-1}) d\mu(g)$$

which is defined by the integral. Applying this to the identity function yields 1. The image of

$$R(h)f \mapsto \int_G R(h)f(g^{-1}) d\mu(g) = \int_G f((h^{-1}g)^{-1}) d\mu(g)$$

$$= \int_G f(g^{-1}) d\mu(g)$$

by change of variables. Note this is also positive, so we can then apply uniqueness of the Haar measure to get equality of the linear forms. □

Note $\mathcal{C}(G)$, continuous functions on G. This is a Banach space with norm

$$\|f\| = \max_{g \in G} |f(g)|$$

We can also look at $L^2(G, \mu)$ (abbreviated often $L^2(G)$), the class of measurable functions which are square integrable. Define

$$(f|f') = \int_G f(g)\overline{f'(g)} d\mu(g)$$

We can then define

$$\|f\|_2 = (f|f)^{\frac{1}{2}}$$

which we can use to define a Hilbert space. We can then define

$$i : \mathcal{C}(G) \to L^2(G, \mu)$$

What is its class in the space of measurable functions? The above is of course a continuous linear map. If we take $f \in \mathcal{C}(G)$, we can calculate $\|f\|_2 = \int |f(g)|^2 d\mu(g) \leq \|f\|^2$. This yields continuity. i is also injective. Suppose $i(f) = 0$, this means that $\|f\|_2 = 0$, meaning f is zero almost everywhere. Does this mean f is zero?

Proposition 3. i is injective

Proof. Assume that $f \neq 0$. Then there is some $g \in G$ for which $f(g) \neq 0$. Then f is nonzero on some neighborhood U in G. We can restrict this neighborhood so that we may assume $|f(g)| > c$ for some $c > 0$. In particular, $|f(g)|^2 > c^2$. We have that the positivity of $|f|^2$ implies

$$\int_G |f|^2 d\mu(g) \geq \int_U |f|^2 d\mu(g) \geq c^2 \int_U d\mu(g) > 0$$

So i must be injective (the last inequality follows from the lemma). □

Moving toward the Peter-Weyl theorem, take complex vector space V.

Definition 4. V is a **topological vector space** if V is a vector space with a topology such that

$$V \times V \to V$$

by $(v, w) \mapsto v + w$ is continuous,

$$\mathbb{C} \times V \to V$$

by $(c, v) \mapsto cv$ is continuous.

We will prove the following lemma.

Lemma 11. A topological vector space V is Hausdorff if and only if $\overline{\{0\}}$ is $\{0\}$.

Example. For $V = \mathbb{C}^n$, V is a topological vector space. $\mathbb{C}^n$ is Hausdorff because it inherits its topology from a metric. Let V be a finite dimensional Hausdorff topological vector space (long word :( ). Let $e_1, \ldots, e_n$ be a basis for V. The map

$$\mathbb{C}^n \to V$$

given by $(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1 e_1 + \ldots + \alpha_n e_n$ is an isomorphism of topological vector spaces. Any linear endomorphism is automatically continuous.

Definition 5. A representation of G on a Hausdorff topological vector space E is a homomorphism of G into $\mathrm{GL}(E)$ (linear maps $E \to E$ which are continuous with continuous inverse). This requires

$$G \times E \to E$$

by $(g, v) \mapsto \pi(g)v$ is continuous. In the finite dimensional case, this is the same as a representation as defined before.

22 March 14

Topological vector spaces

Definition 6. Let E be a topological vector space over $\mathbb{C}$ if

- It is a vector space over $\mathbb{C}$
- it is a topological space
- $E \times E \xrightarrow{+} E$ by addition is continuous.
- $\mathbb{C} \times E \xrightarrow{m} E$ is continuous.

Lemma 12. Let E be a topological vector space over $\mathbb{C}$. Then the following are equivalent:

1. E is Hausdorff
2. $\{0\}$ is closed

Proof. We show 1 implies 2. (see figure hausdorff). Hence $\{0\}$ is closed.

We show 2 implies 1. Suppose $\{0\}$ is closed, so $E \setminus \{0\}$ is open. We have

$$F : (u, v) \mapsto E$$

by $F(u, v) = u - v$ is continuous. We have

$$F^{-1}(E \setminus \{0\}) = E \times E \setminus \Delta$$

is open, where $\Delta = \{(w, w) | w \in E\}$. So given $u, v \in E$ no on the diagonal, there exists U containing u, V of v, such that

$$U \times V \subset E \times E \setminus \Delta$$

and $U \times V \cap \Delta = \emptyset$, $U \cap V - \emptyset$. $\square$

Example. 1. Let E be a normed space with norm $\| \cdot \|$, so we can define a metric $d(u, v) = \|u - v\|$. It defines a topology on E which is Hausdorff by the above lemma.

2. Consider $\mathbb{C}^n$ with the Euclidean norm.

$$\|c\| = \sqrt{\sum_{i=1}^{n} |c_i|^2}$$

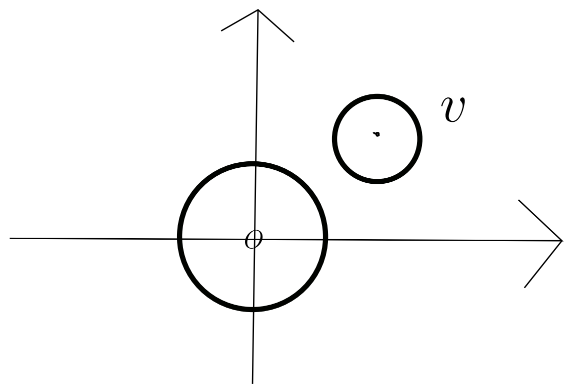

Figure 11: Take intersections of all complements of neighborhoods of v not containing 0.

for $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$. This is also a Hausdorff vector space by the above lemma (both examples can also be checked directly, metric spaces are nice).

Lemma 13. Let E be a finite dimensional Hausdorff topological vector space. Let $v_1, \ldots, v_n$ be a basis of E. The map $\phi : \mathbb{C}^n \to E$ defined by

$$\phi(c) = \sum_{i=1}^{n} c_i v_i$$

is an isomorphism of topological vector spaces.

Proof. By definition, ϕ is a linear isomorphism. It is continuous by our topology on $\mathbb{C}^n$. ϕ is continuous. We last show ϕ is open. We look at $\mathbb{C}^n$, which we denote by S the unit sphere. S is closed and bounded implies S is compact. Consider the open unit ball B_1. $\phi(S)$ is compact, E is Hausdorff, implies $\phi(S)$ is closed. Hence $E \setminus \phi(S)$ is open, and note $0 \in E \setminus \phi(S)$. Hence $E - \phi(S)$ is a neighborhood of 0. Now consider the multiplication

$$\mathbb{C} \times E \xrightarrow{m} E$$

and that it sends

$$(0, 0) \mapsto 0$$

Now $\exists \varepsilon > 0$, $U \ni 0$ such that

$$|z| \leq \varepsilon \Rightarrow z \cdot U \subset E \setminus \phi(S)$$

Pick $v \in U \setminus \{0\}$. Now there exists $c \neq 0$ mapping to v under ϕ,

$$v = \phi(c)$$

Now note

$$\frac{c}{\|c\|} \in S$$

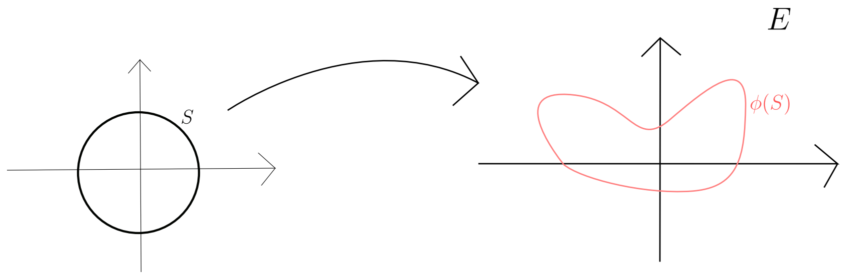

Figure 12: The unit sphere and its image

Hence $\phi\left(\frac{c}{\|c\|}\right) \in \phi(S)$. We can rewrite this as

$$\frac{1}{\|c\|}\phi(c) = \frac{1}{\|c\|} \cdot v$$

This implies

$$\frac{1}{\|c\|} > \varepsilon$$

We get $\|c\| < \frac{1}{\varepsilon}$. Hence we get

$$v \in \phi\left(\frac{1}{\varepsilon}B_1\right)$$

for any $v \in U$. This tells us that $U \subset \phi(\frac{1}{\varepsilon}B_1) = \frac{1}{\varepsilon}\phi(B_1)$. So $\phi(B_1)$ contains a neighborhood of 0, and so is a neighborhood of 0. Consider a neighborhood O od d, there exists $r > 0$ hwere $d + rB_1 \subset O$. Hence $\phi(d + rB_1)$ is a neighborhood of $\phi(d)$. Hence $\phi(O)$ is a neighborhood of $\phi(d)$. $\square$

Corollary 1. Let E and F be finite dimensional Hausdorff vector spaces. Let $A : E \to F$ be a linear map. Then A is continuous.

Proof. Say $E \cong \mathbb{C}^n$, $F \cong \mathbb{C}^m$ via the preceding lemma.

$$\begin{tikzcd}
\mathbb{C}^n \arrow[d] & E \arrow[l, "\sim"'] \arrow[d, "A"] \\
\mathbb{C}^m & F \arrow[l, "\sim"]
\end{tikzcd}$$

yields a continuous map $\mathbb{C}^n \to \mathbb{C}^m$. Since the top and bottom maps are topological isomorphisms, A is continuous as desired. $\square$

We can consider the category of finite dimensional Hausdorff topological vector spaces, and that of finite dimensional vector spaces. You have a forgetful functor from the former.

Theorem 14. The forgetful functor above is an equivalence of categories.

Next time, we will define representations of compact groups on Hausdorff topological vector spaces.

23 March 16

Last time, we were examining Hausdorff topological vector spaces. If E is finite dimensional and Hausdorff, then

$$E \cong \mathbb{C}^n$$

as topological vector spaces. The forgetful functor from topological vector spaces to the category of vector spaces is an equivalence of categories. Given E we consider $GL(E)$ the group of automorphisms of E has Hausdorff topological vector spaces. Let's assume that E is a Hausdorff topological vector space, and $F \subset E$. There is the induced subspace topology on F, in which F is Hausdorff. Note it is also a topological vector space.

Lemma 14. Let E be a Hausdorff topological vector space, and F be a finite dimensional vector subspace. Then F is closed.

Proof. Assume that F is not closed. Take a basis $e_1, \dots, e_n$ of F. Take $e_{n+1} \in \overline{F} \setminus F$, $e_{n+1} \neq 0$. The family $\{e_1, \dots, e_{n+1}\}$ is linearly independent. We can discuss

$$\phi : \mathbb{C}^{n+1} \to E$$

given by $\phi(\overline{c}) = \sum_{i=1}^{n+1} c_i e_i$. This is a map

$$\phi : \mathbb{C}^{n+1} \to F'$$

where $F' = \{\text{span of } e_1, \dots, e_{n+1}\}$. By a theorem from last time this induces an isomorphism of topological vector spaces. We have that ϕ restricts nicely:

$$\phi : \mathbb{C}^n \times \{0\} \to F$$

to an isomorphism. But note that $(0, \dots, 0, 1)$ mapping to e^{n+1} is not in the closure of $\mathbb{C}^n \times \{0\}$, a contradiction. □

If G is a compact group, and E is a Hausdorff topological vector space, we can look at representations

$$\pi : G \to GL(E)$$

defined by $(g, u) \mapsto \pi(g)v$. We add the condition that this map $G \times E \to E$ is continuous. If E, is a Banach space, its topology is given by norm $\|\cdot\|$, and metric

$$d(u, v) = \|u - v\|$$

and also it is a complete metric space. Say we look at $C(G)$, the **continuous complex-valued functions on** G . Then $C(G)$ is a Banach space with norm

$$\|f\| = \max_{g \in G} |f(g)|$$

. Consider $g \mapsto \pi(g)v$, $G \to E$. We check this map is continuous, in order to show that $G \times E \to E$ is continuous (see definition of representation over compact groups). We would have

$$\{\pi(g)v | g \in G\}$$

is compact in E. We have

$$\{\|\pi(g)v\| | g \in G\} \subset \mathbb{R}$$

is compact.

$$\|\pi(g)v\| \leq M_v$$

for $g \in G$ is bounded. (See Banach Steinhaus theorem). This shows

$$\|\pi(g)\| \leq M_C.$$

We have

$$\|\pi(g)\| = \sup_{\|v\| \leq 1} \|\pi(g)v\|$$

(see texed notes for more details on the theorem). Consider $g, g' \in G$, $v, v' \in E$.

$$\|\pi(g)v - \pi(g')v'\| = \|\pi(g)v - \pi(g')v + \pi(g')v - \pi(g')v'\|$$

$$\leq \|\pi(g)v - \pi(g')v\| + \|\pi(g')(v - v')\| \leq \|\pi(g)v - \pi(g')v\| + \underbrace{\|\pi(g')\|}_{\leq C}\|v - v'\|$$

We take a neighborhood U of 1, $g' \in gU$. The former norm is $\leq \frac{\varepsilon}{2}$. We can choose v, v' to have distance less than $\frac{\varepsilon}{2C}$, and note $\|\pi(g')\| \leq C$.

If E is a finite dimensional Hausdorff, we know $E \cong \mathbb{C}^n$. The natural isomorphism $E \cong \mathbb{C}^n$ comes from the Euclidean norm. We have

$$g \mapsto \pi(g)v$$

It is enough to check that $g \mapsto \pi(g)e_i$ is continuous for a basis e_i of E. We have

$$\pi(g)e_i = \sum_{j=1}^{n} \pi(g)_{ji} e_j$$

The continuity of the representation thus occurs if and only if the matrix coefficients of $\pi(g)$ are continuous.

We can prove Maschke's theorem, the orthogonality relations. We can also define characters. A lot of machinery from finite groups transfers to this setting. Where does the analogy break down?

If we take the circle group S^1, a compact abelian group. From this group, there may be infinitely many irreducible representations that are non-isomorphic, which is different from the finite group case.

24 March 18

Let G be a compact group, $C(G)$ be the continuous complex functions on G. Write

$$\|f\| = \max_{g \in G} |f(g)|$$

We have

$$(R(g)f)(h) = f(hg)$$

$h \in G$. We note

$$\|R(g)f\| = \|f\|$$

Hence $R(g)$ is a continuous linear map on $C(G)$ to itself with inverse $R(g^{-1})$, hence $R(g)$ is a automorphism of topological vector spaces. We have

$$R(g) \in GL(C(G)).$$

We have $R : G \to GL(C(G))$. Using Banach-Steinhaus, we proved that this is a representation if for any $f \in C(G)$, the function

$$g \mapsto R(g)f$$

is continuous. Note from last class we can assert uniform continuity on f. Take $\varepsilon > 0$, so there exists U, a neighborhood of 1, such that

$$|f(g) - f(g')| < \varepsilon$$

for all $g' \in gU$. We also have

$$|f(hg) - f(hg')| < \varepsilon$$

for all $h \in G$.

$$|(R(g)f)(h) - (R(g')f)(h)| < \varepsilon$$

so we have

$$\|R(g)f - R(g')f\| < \varepsilon$$

R is the **right regular representation** of G. Similarly we can define

$$(L(g)f)(h) = f(g^{-1}h)$$

To show this is a representation, you can either work with G^{opp} or prove directly by a similar argument.

Definition 7. $f \in C(G)$ is **right G-finite** if $\mathrm{span}\{R(g)f|g \in G\}$ is finite dimensional. Similarly, we can define **left G-finite** functions.

Let's suppose f is right G-finite. We have

$$F = \mathrm{span}\{R(g)f|g \in G\}$$

is finite dimensional. This means F is an invariant subspace. F is closed, and we have a representation on F. We can also take a basis $e_1, \ldots, e_n$ of F. For any g we can write

$$R(g)f = \sum_{i=1}^{n} a_i(g)e_i$$

where a_i are linear combinations of matrix coefficients of the representation on F. By last class, a_i 's are continuous, so they are in $C(G)$. We have

$$f(hg) = (R(g)f)(h) = \sum_{i=1}^{n} a_i(g)e_i(h)$$

We can write this function on $G \times G$ defined by $(g, h) \mapsto f(gh)$.

$$f(gh) = \sum_{j=1}^{m} a_i(g)b_i(h)$$

where $a_i, b_i \in C(G)$,

$$(R(g)f)(h) = f(hg) = \sum_{j=1}^{m} a_i(h)b_i(g)$$

implying

$$R(g)f = \sum_{j=1}^{m} b_i(g)a_i$$

so we see f is right G-finite.

Theorem 15. Let $f \in C(G)$. The following conditions are equivalent:

1. f is right G-finite.
2. f is left G-finite.
3. There exists $a_i, b_i \in C(G)$ such that $f(gh) = \sum_{i=1}^{n} a_i(g)b_i(h)$..

Proof. We have just shown 1 and 3 are equivalent. Note that 2 and 3 are also equivalent. □

We call f G-finite in this case.

Denote by $R(G)$ the set of all G-finite functions in $C(G)$. Note $R(G) \subset C(G)$. We prove some properties of $R(G)$.

We first prove $R(G)$ is a vector subspace of $C(G)$. Multiplying a G-finite function yields another by property 3, and similarly for adding G-finite functions.

We also prove $C(G)$ is a Banach algebra. If we have two functions $f, f' \in C(G)$, then we can define $(f \cdot f')(g) = f(g) \cdot f'(g)$ for all $g \in G.C(G)$ is in this case a commutative algebra with unit 1. This algebra structure has nothing to do with the group structure of G.

The second property we prove with regards to $R(G)$ is that $R(G) \subset C(G)$ is a subalgebra with identity 1. We can note

$$\begin{aligned}(f \cdot f')(gh) &= \left(\sum_{i=1}^{n} a_i(g)b_i(h)\right) \cdot \left(\sum_{j=1}^{m} c_j(g)d_j(h)\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i c_j)(g)(c_i \cdot d_j)(h)\end{aligned}$$

this is a finite sum with $n \cdot m$ terms, and by 3 $f \cdot f'$ is in $R(G)$. Also note $1 \in R(G)$.

How big is this subalgebra?

Theorem 16. (One version of Peter-Weyl): $\overline{R(G)} = C(G)$ with respect to the topology on $C(G)$ given by its norm. Any function can hence be uniformly approximated by a G-finite function.

Proof. Assume that (π, V) is a finite dimensional representation of G. We can pick a basis $v_1, \ldots, v_n$ a basis of G. We can write down $\pi(g)$ in this basis, and they are given by a matrix

$$\pi(g)v = \sum_{j=1} \pi(g)_{ji} v_j$$

From last class, we know $\pi(g)_{ji}$ are continuous. Hence $\pi(\bullet)_{ji} \in R(G)$. Note

$$\pi(gh)_{ij} = \sum_{k=1}^{n} \pi(g)_{ik}\pi(h)_{kj}$$

Now assume $G \subset GL(n, \mathbb{C})$. For example, we could say $G = SO(n), O(n)$, or even the unitary groups $SU(n), U(n)$. Our group acts naturally on $\mathbb{C}^n$. If we have $g \in G$, we can look at $g \mapsto g_{ij}$ obtained by taking the matrix coefficient corresponding to g. This is a function in $R(G)$. Recall the Stone-Weierstrass theorem. We have $C(G)$ is a complex algebra of functions on a compact space by pointwise multiplication. □

25 March 21

Last time, we defined the subalgebra $R(G)$ of $C(G)$ of G finite functions. They are functions of $C(G)$ which span a finite dimensional subspace. $R(G)$ is self-adjoint. If we have $f \in R(G)$, then we also have $\overline{f}$ is in $R(G)$. If $f \in R(G)$, then $f(gh) = \sum_{i=1}^{n} a_i(g)b_i(h)$ where a_i, b_i are in $C(G)$. If we take the complex conjugate, then $\overline{a_i}, \overline{b_i}$ satisfy the same requirement. We proved that this requirement was equivalent to the function being in $R(G)$.

We have that $R(G)$ satisfies the conditions of Stone-Weierstrass except separation of points. That is, for any two points, is there is a function that takes different values at each? This is not clear. But, if we take $G \subset GL(n, \mathbb{C})$, this is true. We got a Peter-Weyl theorem:

Theorem 17. Peter-Weyl. $\overline{R(G)} = C(G)$.

Let's assume we have a finite dimensional representation (π, V) of G. This means that if we take $e_1, \ldots, e_n$ of V,

$$\pi(g)e_i = \sum_{j=1}^{n} \pi(g)_{ji} e_j$$

where $\pi(g)_{ji}$ are continuous. $v \in V$, we can take $v^* \in V^*$, where

$$\langle \pi(g)v, v^* \rangle$$

(application of $\pi(g)$ to v^*) creates a function from G to the inner product in $\mathbb{C}$, which is $c_{v,v^*} \in C(G)$. We have $c_{v,v^*} \in R(G)$. We can say $R(G)$ is an algebra of matrix coefficients.

Theorem 18. Let $f \in C(G)$, then TFAE:

- $f \in R(G)$
- f is a matrix coefficient

We just proved that if f is a matrix coefficient, then $f \in R(G)$. We proved this last time. We show the converse: If $f \in R(G)$, then f is a matrix coefficient. Let's suppose $f \in R(G)$. We know that right translates of f span a finite dimensional subspace

$$F = \text{span}\{R(g)f; g \in G\}$$

We know $b_1, \ldots, b_n$ form a basis of F. We know F is invariant for R. The representation on F is continuous.

$$R(g)f = \sum a_i(g)b_i$$

is a linear combination of matrix coefficients, where a_i are continuous. Note

$$R(g)f(h) = f(hg) = \sum_{i=1}^{n} a_i(g)b_i(h)$$

$v^* \in F^*$ can be chosen in such a way that $\langle b_i, v^* \rangle = b_i(1)$ for each i. Now

$$\langle R(g)f, v^* \rangle = \sum a_i(g)b_i(1)$$

We can then say $\langle R(g)f, v^* \rangle = \sum_{i=1}^{n} a_i(g)b_i(1) = f(g)$ by our calculation of $f(hg)$. This shows that f is a matrix coefficient. Before we go towards the proof of the Peter-Weyl theorem. One of the properties of $R(G)$ was that if you had $f \in R(G)$, then $f(gh) = \sum_{i=1}^{n} a_i(g)b_i(h)$ where a_i, b_i are in $C(G)$. We have the following claim:

$$f(gh) = \sum_{i=1}^{n} a_i(g)b_i(h)$$

where $a_i, b_i \in R(G)$. Because of the theorem, we know that f is a matrix coefficient. In particular, this means that f is a linear combination of functions $\pi(\bullet)_{ij}$ of some finite dimensional representations. But note that each one of those by calculation is equal to

$$\pi(gh)_{ij} = \sum_{k=1}^{n} \pi(g)_{ik}\pi(h)_{kj}$$

which is of the desired form.

We need some functional analysis to prove the Peter-Weyl theorem. We discuss compact operators on Hilbert space. Let E be a Hilbert space, so it has an inner product $(\bullet|\bullet)$ and norm $\| \bullet \|$ that makes a complete space. Let $\mathcal{L}(E)$ be the set of all continuous linear maps $E \to E$. The continuous linear map is equivalent to bounded linear maps. That is, a linear map $T : E \to E$ is continuous if and only if $T(B_1)$ is bounded where B_1 is the unit ball. We define

$$\|T\| = \sup_{v \in B_1} \|Tv\|$$

called the **operator norm**. Note $\mathcal{L}(E)$ is an algebra with unit.

We say $T : E \to E$ is **compact** if $\overline{T(B_1)}$ is compact in E. Since the norm is a continuous function, $\overline{T(B_1)}$ is bounded. In this case,

1. T is continuous. T, S compact imply $(T + S)(B_1) \subset \overline{T(B_1)} + \overline{S(B_1)}$ is compact. Hence $T + S$ is compact.

   If T is compact and S is bounded, $T \cdot S(B_1) \subset T(\|S\| \cdot B_1) = \|S\| \cdot T(B_1) \subset \|S\| \cdot \overline{T(B_1)}$ is compact.

Hence the compact operators form an ideal in $\mathcal{L}(E)$.

26 March 23

Compact Operators on Hilbert space. We want to show K-finite functions are dense in $C(G)$. The idea is to look at

$$C(G) \subset L^2(G, \mu)$$

The latter is a Hilbert space. Basically what we need is some results about compact linear operators on a Hilbert space, which wouldn't necessarily apply to $C(G)$. Let us note that if E is finite dimensional, we can define an inner product by choice of basis and declaring that the basis is orthonormal. The inner product makes E a Hilbert space. If we have a linear map

$$T : E \to E$$

then we proved that T is continuous and so T is bounded. $T(B_1)$ is contained in some large ball $B_{\|T\|}$. If we take the ball's closure, it is compact by Heine Borel on finite dimensional vector space with its topology induced by inner product. Since we are dealing with a Hilbert space, we have a notion of adjoint linear maps. If $T : E \to E$ is a continuous linear map, then we can define

$$(T^* u|v) = (u|Tv)$$

for any $u, v \in E$. T^* is called the adjoint of T. We can conclude

$$\|T^*\| = \|T\|$$

Therefore T^* is bounded and T^* is continuous. We note some properties of T^* :

1. $(T^*)^* = T$
2. $(T + S)^* = T^* + S^*$.
3. $(TS)^* = S^* T^*$.
4. $I^* = I$
5. $(\alpha T)^* = \overline{\alpha} T^*$ for $\alpha \in \mathbb{C}$.

Theorem 19. Let T be a self-adjoint compact operator on E. Then either $\|T\|$ or $-\|T\|$ is an eigenvalue for T.

Issue: If you have $T : E \to E$ and λ is an eigenvalue, then $\exists v \in E$, $v \neq 0$ such that

$$Tv = \lambda v$$

so that v is the eigenvector for the eigenvalue λ. Because we can't define a determinant on an infinite dimensional Hilbert space, there need not be *any* eigenvalues. Some properties:

1. If the eigenvalue λ exists for E,
$$\|Tv\| = |\lambda|\|v\|$$
So $|\lambda|\|v\| \leq \|T\|\|v\|$ so $|\lambda| \leq \|T\|$.

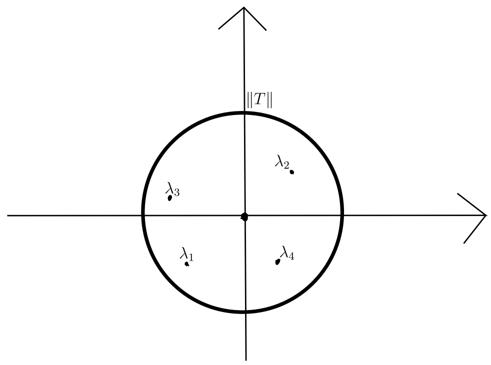

Figure 13: eigenvalue

2. If we take any compact operator T, then T^* is also a compact operator (haven't proved this but maybe exercise). We know T^*T is compact, and this is self-adjoint, since
$$(T^*T)^* = T^*(T^*)^* = T^*T$$

3. If T is self-adjoint, and λ is an eigenvalue of T, λ is a real number. Let's assume $v \in E$, $v \neq 0$, $\lambda v = Tv$. We have
$$\lambda(v|v)(Tv|v) = (v|Tv) = \overline{\lambda}(v|v)$$
so $\lambda = \overline{\lambda}$.

Before the proof, we need a "converse" of the Cauchy-Schwarz inequality.

Lemma 15. If $|(u|v)| = \|u\|\|v\|$ for $u, v \in E$, then u, v are proportional.

Proof. Let us prove this by inner-product methods. If u or v is zero, this is true. Hence we can assume $u, v \neq 0$. We can write $u = \mu \cdot v + w$, $w \perp v$. We can write by the Pythagorean
$$\|u\|^2 = |\mu|^2\|v\|^2 + \|w\|^2$$
$$(u|v) = \mu(v|v) + (w|v) = \mu(v|v)$$
Hence $\mu\|v\|^2 = (u|v)$. Hence
$$|(u|v)| = |\mu|\|v\|^2 = \|u\| \cdot \|v\|$$
So
$$|\mu| = \frac{\|u\|}{\|v\|}$$

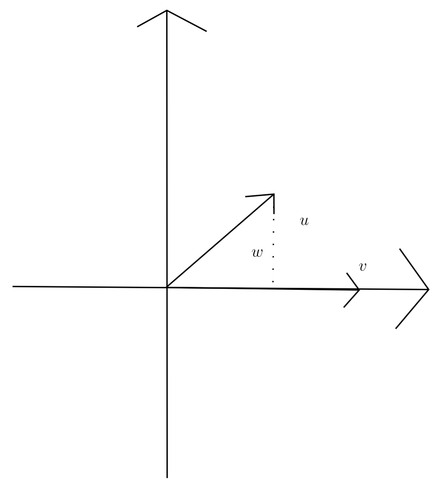

Figure 14: plane

We have

$$\|u\|^2 = \frac{\|u\|^2}{\|v\|^2}\|v\|^2 + \|w\|^2$$
$$= \|u\|^2 + \|w\|^2$$

implying $\|w\|^2 = 0$, so $w = 0$. $\square$

Theorem 20. Let T be a self-adjoint operator with nonzero norm. Then T has an eigenvalue.

Proof. By scaling we can assume that $\|T\| = 1$. We claim that either 1 or -1 is an Eigenvalue of T. Since

$$\|T\| = 1,$$

the supremum of $\|Tv\| = 1$ for $\|v\| \leq 1$. We can pick a sequence v_n where $\|v_n\| \leq 1$ and $\|Tv_n\| \to 1$. On the other hand, $v_n \in B_1$ in our space, so $Tv_n \in T(B_1) \subset \overline{T(B_1)}$. Since T is compact, $\overline{T(B_1)}$ is compact. Can assume that (v_n) is a sequence in B_1 such that Tv_n converges to some vector $u \in \overline{T(B_1)}$. This implies $\|u\| = 1$. Also

$$T^2 v_n \to Tu$$

First of all,

$$1 = \|T\| \cdot \|u\| \geq \|Tu\|$$

Note that $\|Tu\| = \lim_{n\to\infty} \|T^2 v_n\| \geq \limsup \|T^2 v_n\| \|v_n\|$. Now we can rewrite this using the Cauchy-Schwarz inequality:

$$\limsup \|T^2 v_n\| \cdot \|v_n\| \geq \limsup |(T^2 v_n | v_n)|$$

and $(T^2 v_n | v_n) = (Tv_n | Tv_n)$, so we can drop the absolute value:

$$\limsup \|T^2 v_n\| \cdot \|v_n\| \geq \limsup \|Tv_n\|^2$$
$$= \lim_{n\to\infty} \|Tv_n\|^2 = \|u\|^2 = 1$$

We get a chain of inequalities ending and starting with 1, so everything in between must be equal. In particular, $\|Tu\| = 1$. $\square$

27 March 25

Last time, we established two important facts: If T was a self-adjoint compact linear operator, with $\|T\| = 1$, we constructed a vector u on the unit sphere ($\|u\| = 1$) such that

$$\|Tu\| = 1$$

We have

$$\begin{aligned} 1 = \|Tu\|^2 = (Tu|Tu) = (T^*Tu|u) = (T^2u|u) \\ \leq \|T^2u\| \cdot \|u\| \\ \leq \|T^2\|\|u\|^2 = \|T\|^2 = 1 \end{aligned}$$

since $\|u\| = 1$. We have

$$(T^2u|u) = \|T^2u\| \cdot \|u\|$$

We can finally conclude that $T^2u = \mu u$. (The vectors are proportional to each other by the Cauchy schwarz equality from last class). We calculate

$$\mu(u|u) = (T^2u|u) = (Tu|Tu) = \|Tu\|^2 = 1$$

$T^2u = u$. We have a finite dimensional vector space F spanned by u, Tu, which is T-invariant. T has an eigenvalue on this space. To prove this, we've two cases:

1. $Tu = u$. It has eigenvalue 1 in this case.
2. $Tu \neq u$. Set $v = \frac{1}{2}(Tu - u)$. $Tv = \frac{1}{2}(T^2u - Tu) = \frac{1}{2}(u - Tu) = -v$. In this case $Tv = -v$, so -1 is an eigenvalue.

This tells us that if we have a compact self-adjoint operator that has nonzero norm, then there is a nonzero eigenvalue.

Theorem 21. Let T be a self-adjoint compact operator and $\lambda \in \mathbb{R} \setminus \{0\}$ be an eigenvalue of T. Then the eigenspace F of λ is finite-dimensional.

Proof. $F = \{v | Tv = \lambda v\}$. Assume that F is infinite dimensional. We can construct a sequence $e_1, e_2, \ldots$ ($Te_1, \ldots$ sits inside a compact set so we may pick a convergent subsequence) such that $(e_i|e_j) = \delta_{ij}$. In this sequence, you can calculate the distance between vectors:

$$\|e_i - e_j\|^2 = (e_i - e_j|e_i - e_j) = 2$$

in particular when $i \neq j$. So $Te_n = \lambda e_n$ is not a Cauchy sequence.

□

We may also use this to show that the eigenvalues for such an operator converge to 0. Since eigenspaces are orthogonal, we can pick an orthonormal sequence $e_1, \ldots,$ and proceed with the rest of the proof.

Let G be a compact group, μ be a Haar measure. $C(G)$ is a Banach space, with norm

$$\|f\| = \max_{g \in G} |f(g)|$$

If $f \in C(G)$, $f \in L^2(G) = L^2(G, \mu)$. We can make an inner product

$$(f|f') = \int_G f(g)\overline{f'(g)} d\mu(g)$$

We have this map $i : C(G) \to L^2(G)$.

$$\|f\| = (f|f)^{\frac{1}{2}}$$

We have

$$\|f\|^2 = (f|f) = \int_G |f(g)|^2 d\mu(g)$$

noting $|f(g)|^2 \leq \|f\|^2$. So

$$\int_G |f(g)|^2 d\mu(g) \leq \|f\|^2$$

since the measure of G is 1.

$$\|i(f)\|_2 \leq \|f\|$$

i is a continuous linear map. Let's assume f is in $C(G)$ such that $i(f) = 0$. This means

$$\|f\|_2 = \int_G |f(g)|^2 d\mu(g) = 0$$

Assume that $f \neq 0$. There exist $g \in G$ with $f(g) \neq 0$. $|f(g)|^2 > 0$. There exists an open neighborhood U of g such that $|f(h)|^2 \geq M$ for $h \in U$ and some $M > 0$.

$$\int_G |f(g)|^2 d\mu(g) \geq \int_U |f(h)|^2 d\mu(h) \geq M\mu(U) > 0$$

as we proved that the Haar measure is positive on open sets as before. If $f \neq 0$, the L^2 norm must be positive. We have proved we can consider $C(G)$ as a subspace of $L^2(G)$. Let's assume that $f \in L^2(G)$ so $|f| \in L^2(G)$. Since $\mu(G)$ is 1, $1 \in L^2(G)$. Therefore the inner product is defined, which we may calculate

$$(|f||1) = \int_G |f(g)| d\mu(g) = \|f\|_1$$

$L^1(G)$ is the space of integrable functions on G with the norm as above. We have

$$\|f\|_1 \leq \|f\|_2 < \infty$$

So $L^2(G) \subset L^1(G)$. Let's take $f \in C(G)$. We can define $(R(f)\phi)(h) = \int_G f(g)\phi(hg)d\mu(g)$. First, we have to show that this is well defined. We will talk about this next time.

28 March 28

Let G be a compact group and $C(G)$ be the Banach space of continuous functions. Then there is a natural map

$$C(G) \xrightarrow{i} L^2(G)$$

Here's the definition of $i:$ Take a function and associate it to a class of functions in L^2. So i is linear, and we proved last time that this is injective. Each class for an element in $L^2(G)$ has at most one continuous function.

We defined an operator on $L^2(G)$ which we will prove to be compact. Given $f \in C(G)$, we have

$$(R(f)\varphi)(g) = \int_G f(h)\varphi(gh)d\mu(h)$$

$$= \int_G f(h) \cdot (R(h)\varphi)(g) d\mu(h)$$

So

$$R(f) : L^2(G) \to L^2(G)$$

(more precisely its image is in $L^2(G)$) and it is a compact operator. We make it self-adjoint by considering $R(f)^*R(f)$. $\varphi \in L^2(G) \subset L^1(G)$. By some arguments, $f(h) \cdot \varphi(gh)$ is integrable over $h \in G$ with the measure μ. Consider

$$|(R(f)\varphi)(g)| = \left|\int_G f(h)\varphi(gh)d\mu(h)\right|$$

$$\leq \int_G |f(h)| \cdot |\varphi(gh)| d\mu(h)$$

$$\leq \|f\| \cdot \int_G |\varphi(gh)| d\mu(h) = \|f\| \cdot \int_G |\varphi(h)| d\mu(h)$$

$$\leq \|f\| \|\varphi\|_1 \leq \|f\| \cdot \|\varphi\|_2$$

since integrals over μ are bi-invariant. We further have a claim that $R(f)\varphi$ is in $C(G) \subset L^2(G)$. The integral depends on φ, but it only depends on the class of φ.

$$\|R(f)\varphi\| \leq \|f\| \cdot \|\varphi\|_2$$

If we grant the claim that $R(f)$'s image lands in $C(G)$, it would be a continuous map into $C(G)$, which continuously includes into $L^2(G)$ by last lecture. Let us prove the claim. Picking $\varepsilon > 0$, there exists a neighborhood U of 1 such that

$$\left|f(g^{-1}h) - f(g'^{-1}h)\right| < \varepsilon$$

for $(g^{-1}h)(g'^{-1}h)^{-1} \in U$, or alternatively $g^{-1}g' \in U$, so h may be arbitrary in G. Let us compare the function

$$|(R(f)\varphi)(g) - (R(f)\varphi)(g')| = \left|\int_G f(h)\varphi(gh)d\mu(h) - \int_G f(h)\varphi(g'h)d\mu(h)\right|$$

Now by bi-invariance this is

$$\left|\int_G f(g^{-1}h)\varphi(h)d\mu(h) - \int_G f(g'^{-1}h)\varphi(h)d\mu(h)\right|$$

$$= \left|\int_G \varphi(h)\left(f(g^{-1}h) - f(g'^{-1}h)\right) d\mu(h)\right|$$

$$\leq \int_G |\varphi(h)| \cdot \left|f(g^{-1}h) - f(g'^{-1}h)\right| d\mu(h)$$

Our condition $g^{-1}g' \in U$ is the same as saying $g' \in gU$. Given this, we have

$$\leq \int_G |\varphi(h)|\varepsilon d\mu(h) \leq \varepsilon\|\varphi\|_1 \leq \varepsilon\|\varphi\|_2$$

This shows $R(f)\varphi$ is continuous, so in $C(G)$.

We know $R(f)$ is continuous; we would like to show the composition

$$L^2(G) \xrightarrow{R(f)} C(G) \xrightarrow{i} L^2(G)$$

is compact. We will consider

$$\overline{\{R(f)\varphi | \|\varphi\|_2 \leq 1\}} \subset C(G)$$

and claim that the set is compact.

$$\|R(f)\varphi\| \leq \|f\| \cdot \|\varphi\|_2$$
$$\leq \|f\|$$

This set is hence pointwise bounded. We proved that $|(R(f)\varphi)(g) - (R(f)\varphi)(g')| \leq \varepsilon\|\varphi\|_2$ when $g' \in gU$ for a chosen neighborhood U of 1. This is the meaning of equicontinuous. THe set is hence pointwise bounded and equicontinuous. By Arzela Ascoli, the set is compact. The image of the closed unit ball is compact in $C(G)$ by $R(f)$. Hence the further image by i in $L^2(G)$ is compact. In particular,

$$R(f) : L^2(G) \to L^2(G)$$

is a compact operator. If we take $R(f) : L^2(G) \to L^2(G)$, then we can define $R(f)^*$, the adjoint of $R(f)$.

1. $f^*(g)$ is defined to be $\overline{f(g^{-1})} \in C(G)$ for $f \in C(G)$. We have $(f^*)^*$. We also have $(f+f')^* = f^* + f'^*$. We also have
$$(\alpha f)^* = \overline{\alpha} f^*$$
so the operation $f \mapsto f^*$ is anti-linear.

We are going to prove next time that $R(f)^* = R(f^*)$. $R(f^*) \cdot R(f)$ is self adjoint:
$$R(f^*) \cdot R(f) = R(f)^* \cdot R(f)$$
So,
$$(R(f^*) \cdot R(f))^* = R(f)^* R(f)^{**} = R(f)^* R(f) = R(f^*) R(f)$$
The operator $R(f^*)R(f) : L^2(G) \to L^2(G)$ is hence a self-adjoint compact operator.

29 March 30

$R(f) : L^2(G) \to L^2(G)$. And we defined
$$(R(f)\phi)(g) = \int_G f(h)\phi(gh)d\mu(h)$$
$R(f)\phi \in C(G)$. We have
$$(R(f)\varphi|\psi) = \int_G (R(f)\varphi)(g)\overline{\psi(g)}d\mu(g)$$
$$= \int_G \left(\int_G f(h)\varphi(gh)d\mu(h)\right)\overline{\psi(g)}d\mu(g)$$
We use Fubini's theorem to swap the order of integration:
$$\int_G \int_G f(h_0\varphi(gh)\overline{\psi(g)}d\mu(g)d\mu(h)$$
$$= \int_G \int_G f(h)\varphi(g)\overline{\psi(gh^{-1})}d\mu(g)d\mu(h)$$
$$= \int_G \left(\int_G f(h)\varphi(g)\overline{\psi(gh^{-1})}d\mu(g)\right)d\mu(h)$$
$$= \int_G \left(\int_G f(h)\varphi(g)\overline{\psi(gh^{-1})}d\mu(h)\right)d\mu(g)$$
$$= \int_G \varphi(g)\overline{\left(\int_G \overline{f(h^{-1})}\psi(gh)d\mu(h)\right)}d\mu(g)$$
Now recall $\overline{f(h^{-1})} = f^*(h)$. But this integral is
$$(R(f^*)\psi)(g)$$
so we have
$$(\varphi|R(f^*)\psi)$$
which implies $R(f)^* = R(f^*)$. If we take $R(f^*)$, $R(f)$,
$$R(f^*)R(f) = R(f)^*R(f)$$

Now if $R(f^*)R(f) \neq 0$, then the operator has a real eigenvalue $\lambda \neq 0$.
$$R(f^*)R(f) = 0 \Rightarrow R(f^*)R(f)\varphi = 0, \varphi \in L^2(G)$$

$$\Rightarrow (R(f)^* R(f)\varphi|\varphi) = 0$$
$$\|R(f)\varphi\|^2 = 0$$

implying $R(f)\varphi = 0$, or $R(f) = 0$. Assuming there is a nonzero eigenvalue, we want to show it is strictly positive:

$$R(f)^* R(f)\varphi = \lambda\varphi$$
$$(R(f)^* R(f)\varphi|\varphi) = \lambda\|\varphi\|_2^2$$

but the former is $\|R(f)\varphi\|_2^2$. Hence $\lambda > 0$.

Proposition 4. Let F be the eigenspace of $R(f^*)R(f)$ for eigenvalue λ. Then F is finite dimensional. Note $F \subset C(G)$, since $R(f)$ maps $C(G)$ into $C(G)$.

Take $\varphi \in L^2(G)$. We take

$$(R(g)\varphi)(h) = \varphi(hg)$$

If we calculate the L^2 norm, we get

$$\|R(g)\varphi\|_2^2 = \int_G |\varphi(hg)|^2 \, d\mu(h)$$
$$= \|\varphi\|_2^2$$

$R(g)$ is a continuous linear map on $L^2(G)$. Moreover, it preserves the norm, so $R(g)$ is unitary. $R(g)^* = R(g^{-1})$, so the same result holds for the adjoint. We use the observation that we have

$$\begin{tikzcd} C(G) \arrow[r, "R(g)"] \arrow[d, dashed, "i"] & C(G) \arrow[d, dashed, "i"] \\ L^2(G) \arrow[r, "R(g)"] & L^2(G) \end{tikzcd}$$

If $\varphi \in C(G)$, then $g \mapsto R(g)\varphi$ is continuous with respect to the supremum norm. We first take $\psi \in L^2(G)$. There is a fact: $C(G)$ is dense in $L^2(G)$ with respect to the topology on $L^2(G)$. Take $\varepsilon > 0$. Then there exists $\varphi \in C(G)$ such that

$$\|\psi - \varphi\| < \varepsilon/3$$

We check continuity:

$$\|R(g)\psi - R(g')\psi\|_2 \leq \|R(g)\psi - R(g)\varphi\|_2 + \|R(g)\varphi - R(g')\varphi\|_2 + \|R(g')\varphi - R(g)'\psi\|$$

We have that this is

$$< \frac{2\varepsilon}{3} + \|R(g)\varphi - R(g')\varphi\|_2$$

$f \in C(G)$, then we can let $\|f\|_2^2 < 2\varepsilon/3 \max \|R(g)\varphi - R(g')\varphi\| < \varepsilon$. The same argument applies to left regular representation. So R is the right regular representation on $L^2(G)$. R is the **right regular representation on $L^2(G)$**, and L is the**left regular representation.** So the next lemma:

Lemma 16. For any $f \in C(G)$ and $g \in G$, $R(f)L(g)$ is $L(g)R(f)$. The conclusion is that $F \subset C(G)$ is contained in $R(G)$. Let's take $\varphi \in L^2(G)$, then

$$L(g)\varphi(h) = \varphi(g^{-1}h)$$

.

$$R(f)L(g)\varphi(h) = \int_G f(k)(L(g)\varphi)(hk)d\mu(k)$$

and $L(g)\varphi(hk) = \varphi(g^{-1}hk)$. The above integral is thus

$$(R(f)\varphi)g^{-1}h = L(g)R(f)(h).$$

$R(f^*)R(f)l(g) = L(g)R(f^*)R(f)$ so

$$\begin{aligned} &R(f)^*R(f)L(g)\phi \\ &= L(g)(R(f^*)R(f)\phi) \\ &= \lambda\phi \end{aligned}$$

and $L(g)\phi \in F$. $F \subset R(G)$.

30 April 1

Recall that last time we proved $R(f^*)R(f)$ is a self-adjoint compact operator on $L^2(G)$. Given eigenvalue $\lambda > 0$, F its corresponding eigenspace, then F is finite dimensional contained in $R(G)$.

$R(G)$ is $R(f)$-invariant. Given $\varphi \in R(G)$,

$$\varphi(gh) = \sum_{i=1}^{n} a_i(g)b_i(h), a_i, b_i \in R(G)$$

Let us calculate the action:

$$\begin{aligned} R(f)(\varphi)(g) &= \int_G f(h)\varphi(gh)d\mu(h) \\ &= \sum_{i=1}^{n} a_i(g)\int_G f(h)b_i(h)d\mu(h) \end{aligned}$$

We see that $R(f)(\varphi)$ is a linear combination of a_i 's. Hence $R(f)\varphi \in R(G)$.

Theorem 22. (Peter-Weyl) $R(G)$ is dense in $L^2(G)$.

$$\overline{R(G)} = L^2(G)$$

Proof. First remark: $\overline{R(G)}$ is $R(f)$ invariant for any $f \in C(G)$. Hence $R(f^*)R(f)$-invariant. If we denote $E = \overline{R(G)}^{\perp}$, $L^2(G) = \overline{R(G)} \oplus E$. Peter Weyl follows from $E = 0$. If $\overline{R(G)}$ is $R(f^*)R(f)$ invariant, then E is also $R(f^*)R(f)$-invariant by self-adjointness of the operator. Now we want to consider $R(f^*)R(f)$ restricted to E. We have two cases:

1. $R(f^*)R(f)|_E = 0$
2. $R(f^*)R(f)|_E \neq 0$

Let us show that we get a contradiction in the second case. In the second case, $R(f^*)R(f)$ is a self-adjoint compact operator on E. Given that it is nonzero, the norm

$$\lambda = \|R(f^*R(f)\|$$

is an eigenvalue of the operator. If we consider any eigenvector $\varphi \in E$ nonzero for λ, $\varphi \in R(G)$. But remember $\varphi \in \overline{R(G)}^{\perp} = E$, so $\varphi = 0$.

Let us take $\psi \in E$. Then $R(f^*)R(f)\psi = 0$.

$$0 = (R(f^*)R(f)\psi|\psi) = (R(f)\psi|R(f)\psi) = \|R(f)\psi\|_2$$

which implies that $R(f)\psi = 0$. Note $R(f)\psi \in C(G)$. We know that

$$\begin{aligned} 0 = (R(f)\psi)(1) &= \int_G f(h)\overline{\psi(h)}d\mu(h) \\ &= \int_G f(h)\overline{\overline{\psi}(h)}d\mu(h) \end{aligned}$$

and $(f|\overline{\psi})$, $f \in C(G)$ is arbitrary. $\overline{\psi} = 0$ in $L^2(G)$. This implies $\psi = 0$. $\square$

We give a second version of Peter-Weyl, which is stronger. The continuous version of Peter-Weyl:

Theorem 23. (Peter-Weyl) $R(G)$ is dense in $C(G)$.

Let us take $g \in G$, $g \neq 1$. Since G is Hausdorff, we can take a neighborhood of the identity, U, and a neighborhood of g, V, such that

$$U \cap V = \emptyset$$

We can moreover pick a single neighborhood U of 1 such that Ug and U are disjoint. Then we can take a continuous function φ on G such that

$$\varphi|_U = 0, \varphi|_{Ug} = 1$$

$$\begin{aligned} \|R(g)\varphi - \varphi\|_2^2 &= \int_G |R(g)\varphi(h) - \varphi(h)|^2 \, d\mu(h) \\ &= \int_G |\varphi(hg) - \varphi(h)|^2 \, d\mu(h) \\ &= \int_U |\varphi(hg) - \varphi(h)|^2 \, d\mu(h) + \int_{G\setminus U} \cdots \end{aligned}$$

The latter is positive. But note $\varphi(h) = 0$ in the previous integral and $\varphi(hg) = 1$. Hence the former integral is $\mu(U) > 0$. Hence

$$R(g)\varphi \neq \varphi$$

hence $R(g)$ is not the identity operator on $L^2(G)$. However the map $g \mapsto R(g)$ makes R a faithful representation (the kernel is trivial). We just proved that $R(G)$ is dense in $L^2(G)$. Hence by continuity, $R(g) \neq I$ on $R(G)$. There exists $\varphi \in R(G)$ where $R(g)\varphi \neq \varphi$. This means that

$$h \mapsto \varphi(hg)$$

and

$$h \mapsto \varphi(h)$$

are different.

Lemma 17. Take $g, g' \in G$ distinct. There exists $\varphi \in R(G)$ such that $\varphi(g) \neq \varphi(g')$.

Proof. Look at $h = g^{-1}g' \neq 1$. By the previous discussion, there exists $\psi \in R(G)$ such that

$$R(h)\psi \neq \psi$$

$R(g')\psi(k) = \psi(kg') \neq R(g)\psi(k) = \psi(kg)$. We write $R(g^{-1})R(g')\psi \neq \psi$. Define $\varphi = L(k^{-1})\psi$. $\varphi(g) \neq \varphi(g')$. $\square$

Stone Weierstrass yields the second Peter-Weyl!

31 April 4

$\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$ forms a topological space similar to $\mathbb{R}_+^* \times \mathbb{R}$, $a > 0, b \in \mathbb{R}$. Find the f associated to the Haar measure, which applied to measurable sets is

$$\int_E f(a,b)d\mu(a)d\lambda(b)$$

Last time, if we looked at a compact group G, $g, g' \in G$, $g \neq g'$. $\varphi \in R(G)$, $\varphi(g) \neq \varphi(g')$. φ is a matrix coefficient by definition of (π, V).

$$\pi(g) \neq \pi(g')$$

In particular, for $g \neq 1$, there exists a finite dimensional representation (π, V) such that $\pi(g) \neq I$. So $g \notin \ker \pi$.

Theorem 24. Let $g \in G$, $g \neq I$. Then $\exists$ a finite dimensional representation (π, V) of G such that $\pi(g) \neq I$.

We can prove a much stronger result by generalizing this theorem:

Theorem 25. Let U be an open neighborhood of 1 in G. Then there exists a finite dimensional representation (π, V) such that $\ker \pi$ is contained in U.

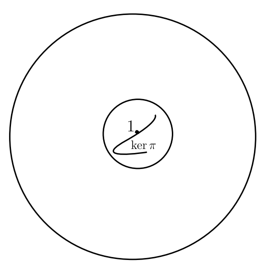

Figure 15: koolkernel

Proof. Suppose we have group G and neighborhood U of 1. $G \setminus U$ is closed. Since G is compact, $K = G \setminus U$ is compact. For any $g \in K$, $g \neq 1$ and there exists a matrix coefficient φ_g such that $\varphi_g(g) \neq \varphi(1)$. Now the following set is open:

$$V_g = \{h \in G | \varphi_g(h) \neq \varphi_g(1)\}$$

containing g and not containing 1. Then

$$\mathcal{U} = \{V_g | g \in K\}$$

is an open cover of K, so that there must be a finite subcover. Pick $g_1, g_2, \ldots, g_n$ that forms this finite subcover. Now we attach φ_i to each V_{gi}. (π_i, V_i) is a finite representation with matrix coefficient φ_{g_i}. $\pi_i(h) \neq I$ if $h \in V_{g_i}$.

Let's take

$$\pi = \bigoplus_{i=1}^{n} \pi_i$$

Then $\pi(h) \neq \mathrm{id}$ for any $h \in \bigcup V_{g_i}$. □

Let G be a topological group.

Definition 8. G has **no small subgroups** if there exists a neighborhood U of 1 such that $H < G$ and $H \subset U$ implies $H = \{1\}$. In other words, there exists a minimal element of the partially ordered set that are open neighborhoods of 1 of G.

Example.
1. Finite groups have no small subgroups.
2. $GL(n, \mathbb{C})$ has no small subgroups.
3. Any closed subgroup of $GL(n, \mathbb{C})$ has no small subgroup by 2.

Consider

$$\prod_I (\frac{\mathbb{Z}}{2\mathbb{Z}})$$

where I is infinite. By Tychonoff's theorem, the space is compact. In the product topology, we say U is open if U_i is an open subset of $\mathbb{Z}_2$ for $i \in$ some finite set F, and $U_i = \mathbb{Z}_2$ otherwise. This is an example of a compact group with small subgroups.

Similarly, $(S^1)^I$ has small subgroups.

We reformulate the theorem we proved in terms of small subgroups:

Theorem 26. Let G be a compact group without small subgroups. There would then be a neighborhood U of 1 such that $H \subset U$, H a subgroup implies $H = \{1\}$. We constructed a finite dimensional representation π with $\ker \pi \subset U$. So $\ker \pi = 1$.

32 April 6

Why does $\mathrm{GL}(n, \mathbb{C})$ have no small subgroups? The general linear group is an open subset of $M_n(\mathbb{C})$. If we took any subgroup G, then G would also have no small subgroups if we showed the general linear group has no small subgroups. First of all, we define the exponential function for matrices. We define

$$\exp(A) = e^A$$

for $\sum_{n=1}^{\infty} \frac{1}{n!} A^n$. Does this series converge? Yes. We can take the metric attached to the operator norm

$$\|A\| = \max_{\|x\| \leq 1} \|Ax\|$$

We have

$$\|A \cdot B\| \leq \|A\| \cdot \|B\|$$

In particular, we see that

$$\|A^n\| \leq \|A\|^n$$

want to show

$$N \mapsto \sum_{n=0}^{N} \frac{1}{n!} A^n$$

is convergent. It suffices to show that this is a Cauchy sequence. We can find any $\varepsilon > 0, n_0$ such that

$$\left\| \sum_{i=n+1}^{m} \frac{1}{i!} A^i \right\| < \varepsilon$$

for $n, m \geq n_0$.

$$\left\| \sum_{i=n+1}^{m} \frac{1}{i!} A^i \right\| \leq \sum_{i=n+1}^{m} \frac{1}{i!} \|A\|^i$$

But note that the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|A\|^n = e^{\|A\|}$$

does converge. Hence the above is a Cauchy sequence. We have

$$e^0 = I$$

and

$$e^A \cdot e^{-A} = I$$

and

$$e^{-A} \cdot e^A = I$$

by calculation. The map $A \mapsto e^A \in \mathrm{GL}(n, \mathbb{C})$. $A \in M_n(\mathbb{C})$. This function is a smooth function. Its derivative at the identity:

$$\frac{\|e^A - e^0 - I \cdot A\|}{\|A\|} = \frac{\|e^A - I - A\|}{\|A\|}$$

$$e^A - I - A = A^2 \cdot \sum \bullet$$

$$\frac{\|A\|^2 \cdot (\text{convergent stuff})}{\|A\|} \to 0$$

as $A \to 0$. We showed the derivative of e^A at $A = I$ is the identity transformation. There exists V, an open ball around 0 in $M_n(\mathbb{C})$ and U, an open neighborhood of I in $\mathrm{GL}_n(\mathbb{C})$ such that

$$\exp : V \to U$$

is a diffeomorphism. To show this, we applied the inverse function theorem but needed extra conditions because $\mathrm{GL}(n, \mathbb{C})$ is a Lie group (technical details with showing that e^A was smooth). Assume that $H \subset \exp(\frac{1}{2}V)$. We claim $H = \{1\}$, showing that $\mathrm{GL}_n(\mathbb{C})$. Since H is in an image of an exponential image. Taking $h \in H, h = e^T, T \in \frac{1}{2}V$.

$$h^2 = (e^T)^2 = e^{2T}$$

We see that $2T \in V$. But note $h^2 \in H$. $h^2 = \exp(S)$ for some $S \in \frac{1}{2}V$. We get

$$h^2 = e^{2T} = e^S$$

recall that $2T, S$ are elements of V, so $2T = S$ by the construction of V above. We say $T = \frac{1}{2}S$ for $S \in \frac{1}{2}V$. Hence $T \in \frac{1}{4}V$. If T had nonzero norm, we have a contradiction by reapplying this argument however many times to say $T \in \frac{1}{2^n}V$. We show that if we have a compact group with no small subgroups, then it is a matrix group. Next time, we will discuss a projective system.

33 April 8

Exercise 9

Consider the projection $G \to G/K$ for closed normal subgroup K of compact group G. G/K has the quotient topology.

$$\left\{U \subset G/K \text{ is open if } p^{-1}(U) \text{ is open in } G\right\}$$

Show that this is a topology on G/K, G/K is a topological group, and $p : G \to G/K$ is continuous. G/K is a compact group.

Denote by $\mathcal{F} = \{K|G/K \text{ is a Lie group}\}$. We have that $\mathcal{F}$ is a directed set, ordered by inclusion. If $K, H \in \mathcal{F}$, then $K \cap H \in \mathcal{F}$. Consider

$$G \to \frac{G}{H} \times \frac{G}{K}$$

by $g \mapsto (gH, gK)$. The product above is compact since each quotient is, and it is also a Lie group. The map factors through by the first isomorphism theorem:

$$G/\left(H \cap K\right) \to \frac{G}{H} \times \frac{G}{K}$$

(exercise, show that the above is continuous). Note that the above map is also a homeomorphism. It is injective and continuous, also surjective. It is open (exercise purely in topology). A continuous map of Lie groups is also a differentiable map. This yields what is called a projective system. Attached to the projective system

$$\frac{G}{K} \to \frac{G}{H}$$

if $K \subset H$ in $\mathcal{F}$. We can define a limit

$$\lim_{K \in \mathcal{F}} G/K$$

The group G is isomorphic to the projective limit. The construction of this limit: take the product

$$\prod_{K \in \mathcal{F}} G/K$$

which is a compact group by Tychonoff's but not necessarily a manifold (what is its dimension?).

Let's assume that (π, V) is a finite dimensional representation of G.

Proposition 5. There exists an inner product on V such that (π, V) is unitary.

Proof. We can pick a basis for V and define an inner product which is the Euclidean inner product with respect to the basis. We define a new inner product

$$(u|v) = \int_G \langle \pi(g)u, \pi(g)v \rangle \, d\mu(g)$$

μ is a Haar measure for V. The integrand is a continuous function of g. We have

$$\overline{(u|v)} = (v|u)$$

and this inner product is multilinear. Note

$$(u|u) = \int_G \langle \pi(g)u, \pi(g)u \rangle \, d\mu(g)$$

since the integrand is based on an inner product, it is positive. Hence $(u|u) > 0$ if $u \neq 0$. The invariance of the Haar measure implies the unitary-ness of the inner product. If we have

$$(\pi(g)u|v) = \int_G \langle \pi(h)\pi(g)u|\pi(h)v \rangle \, d\mu(h)$$

the former is equal to $\pi(hg)u$. We send $h \mapsto hg^{-1}$. We have

$$\int_G \langle \pi(h)u|\pi(h)\pi(g^{-1})v\rangle \, d\mu(h)$$

$$= (u|\pi(g^{-1})v)$$

Hence $\pi(g)$ is unitary. Let's say $U \subset V$ is invariant for π. We have that U is invariant for π. $U^\perp$ is invariant for $\pi(g)^* = \pi(g)^{-1}$, so that $U^\perp$ is also invariant under $\pi(g)$.We have

$$V = U \oplus U^\perp$$

□

Theorem 27. Any finite dimensional representation of G is a direct sum of irreducibles.

We have that $R(G)$ is spanned by the matrix coefficients of irreducible representations as a corollary.

Definition 9. Say $\hat{G}$ is the set of isomorphism classes of finite dimensional irreducible representations. To each class $[\pi] \in \hat{G}$, we can attach $M(\pi)$.

34 April 11

Given a compact group G, we defined a group $\hat{G}$, the set of isomorphism classes of irreducible finite dimensional representations.

$$R(G) = \bigoplus_{[\pi]\in\hat{G}} M(\pi)$$

and $M(\pi)$ was defined to be the space spanned by matrix coefficients of π. Remember that if $[\pi] \neq [\nu]$, $M(\pi) \perp M(\nu)$. The proof of this is similar to the finite group case. Given $A \in \mathrm{Hom}_{\mathbb{C}}(U, V)$, we can define

$$B = \int_G \nu(g)A(\pi(g^{-1}))d\mu(g)$$

$$Bu = \int_G \nu(g)A\pi(g^{-1})ud\mu(g)$$

and

$$\langle \varphi, Bu\rangle = \int_G \langle \varphi, \nu(g)A\pi(g^{-1})u\rangle \, d\mu(g)$$

Using the invariance of the Haar measure,

$$B\pi(g) = \int_G \nu(h)A\pi(h^{-1}g)d\mu(h)$$

doing a change of variables $h \mapsto gh$, we get

$$\int_G \nu(gh)A\pi(h^{-1})d\mu(h) = \nu\,(g)\,B$$

$B \in \mathrm{Hom}_G(U, V)$. The kernel of B is an invariant subspace of U. It is either the entire space U (by irreducibility of π), in which case $B = 0$, or 0, in which case B is injective. Its image is also an invariant subspace of V, so it must be the whole space V. Hence B is zero or B is an isomorphism. The isomorphism case is impossible as in the finite group case. We can examine matrix coefficients:

$$0 = \int_G \sum_{j,k} \nu(g)_{i,j}A_{j,k}\pi(g^{-1})_{k,\ell}$$

and $A_{j,k} = \delta_{j,a}\delta_{k,b}$. So

$$\int_G \nu(g)_{i,a}\pi(g^{-1})_{b,\ell}d\mu(g)$$

and so $M(\pi) \perp M(\nu)$ (it is the same proof as that for finite groups). Recall $R(G)$, now $\bigoplus_{[\pi]\in\hat{G}} M(\pi)$ is dense in $C(G)$ with respect to the supremum norm. Since $C(G)$ is dense in $L^2(G)$,

$$\bigoplus_{[\pi]\in\hat{G}} M(\pi) \subset L^2(G)$$

is dense. If we accordingly define

$$\widehat{\bigoplus_{[\pi]\in\hat{G}}} M(\pi) = L^2(G)$$

where the direct sum hat symbol means the orthogonal Hilbert space sum,

$$\sum_{[\pi]\in\hat{G}} \pi \to$$

in $L^2(G)$. If we apply this to the case when $G = S^1 = T$ (1 dimensional torus), we get a statement on Fourier series. If G is finite, then $\hat{G}$ is finite. We want to show that if G is infinite, then $\hat{G}$ is infinite.

Lemma 18. The following statements are equivalent:

1. G is finite
2. $C(G)$ is finite dimensional

Proof. $1 \Rightarrow 2$: In this case, the topology on G is discrete. $C(G)$ is the space of complex valued functions on G, which has dimension $|G|$.

$2 \Rightarrow 1$: Suppose G is an infinite set. See the attached figure. We prove the caption statement by induction. $n = 2$ is from the Hausdorff condition. We can pick V_i neighborhoods of g_i, and W_i of g_n where $V_i \cap W_i = \emptyset$ by the Hausdorff condition. Then we take an appropriate intersection with the neighborhoods we got from hypothesis to obtain the desired case for n given the $n-1$ case. Now let us look at g_i and its neighborhood A_i from the caption. There exists $\varphi_i \in C(G)$, such that $\varphi_i(g_i) = 1$, and $\mathrm{supp}(\varphi_i) \subset A_i$. Now the space spanned by φ_i in $C(G)$ has dimension n (the φ_i are linearly independent). Since n was arbitrary, $C(G)$ is infinite dimensional. $\square$

The inclusion $C(G) \to L^2(G)$ is injective. Hence $L^2(G)$ has larger dimension than $C(G)$. Hence the same statement follows for $L^2(G)$. G is finite is equivalent to $\dim(L^2(G)$ is finite. Since $L^2(G) = \bigoplus_{[\pi]\in\hat{G}} M(\pi)$, if $\hat{G}$ is infinite, $L^2(G)$ is an infinite dimensional space, whence G is infinite.

Theorem 28. If G is a compact group, the following are equivalent:

1. G is finite
2. $\hat{G}$ is finite.

We will see next that if G is abelian, as in the case of finite groups, $\hat{G}$ is also a group, and $\hat{G}$ is infinite. Pontryagin duality for locally compact abelian groups maps compact groups into discrete groups. Now what happens in the case of abelian groups? Because of Schur's lemma, if we take π as an irreducible finite dimensional representation of G (compact and abelian), all $\pi(g)$'s commute! Then the dimension of π is 1. $\pi(g)$ is $\varphi(g)\cdot I$ for the identity operator I (in 1-dimensional space). φ here is a character $\varphi : G \to \mathbb{C}^*$ which is continuous. We can compose with the norm function to get a function

$$\varphi : G \to \mathbb{C}^* \xrightarrow{|\bullet|} \mathbb{R}_+^*$$

$|\varphi(G)| = 1$. Hence we have a unitary character, which maps into the unit circle.

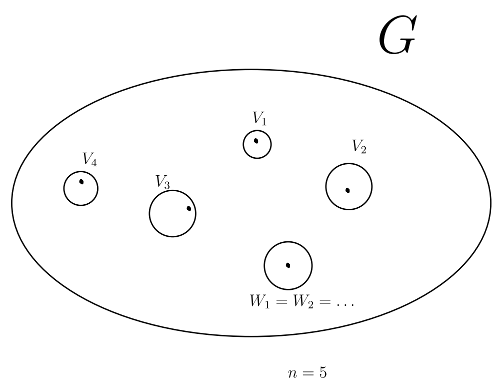

Figure 16: We would like to pick $g_1, \dots, g_n \in G$ as above that are mutually different, and such that there are some neighborhoods U_i of g that are disjoint.

35 April 13

We spend more time talking about Abelian compact groups and their representations. Consider such a group G. We can describe $\hat{G}$. If (π, V) is an irreducible finite dimensional representation, then (π, V) can be described as a character multiplied by the identity operator. $\pi(g)$ commutes with the representation. Any eigenspace for any eigenvalue is invariant for $\pi(h)$.

$$\pi(g) = \varphi(g)I$$

using Schur's lemma. We get $\dim\pi = 1$. The only thing we get for $\varphi : G \to \mathbb{C}$ is that it is continuous. Since it is a representation, $\varphi(gh) = \varphi(g)\varphi(h)$. $\varphi(1) = 1$. We called this function a **character**. $\varphi(1) = \varphi(g)\varphi(g^{-1})$ for any $g \in G$, so $\varphi(g) \neq 0$. Hence φ maps to $\mathbb{C}^*$.

We get composition

$$G \xrightarrow{\varphi} \mathbb{C}^* \xrightarrow{|\bullet|} \mathbb{R}_+^*$$

We get that G is a subgroup of $\mathbb{R}_+^*$, but the only subgroups of $\mathbb{R}_+^*$ is $\mathbb{R}_+^*$. Hence $|\varphi(g)| = 1$. Now we can view $\hat{G}$ as the group of characters.

$$\varphi(g)(\varphi|\psi) = \varphi(g)\int_G \varphi(h)\overline{\psi(h)}d\mu(h)$$

$$= \int_G \varphi(gh)\overline{\psi(h)}d\mu(h) = \int_G \varphi(h)\psi(g^{-1}h)d\mu(h)$$

$$= \overline{\psi(g^{-1})}\int_G \varphi(h)\overline{\psi(h)}d\mu(h)$$

$$= \overline{\psi(g^{-1})}(\varphi|\psi)$$

Hence

$$\psi(g^{-1}) = \overline{\psi(g^{-1})}$$

$$= \psi(g)$$

so we get

$$\varphi(g)(\varphi|\psi) = \psi(g)(\varphi|\psi)$$

If $(\varphi|\psi) \neq 0$, then $\varphi = \psi$. Hence either characters are equal or they are perpendicular. In the former case, $(\varphi|\psi) = \|\varphi\|^2$. We also have

$$(\varphi|\varphi) = \int_G \varphi(g)\overline{\varphi(g)}d\mu(g)$$

$$= \int_G \varphi(g)\varphi(g^{-1})d\mu(g)$$

$$= \int_G d\mu(g) = 1$$

If $\varphi = \psi$, $\|\varphi\| = 1$. The orthonormal characters thus form an orthonormal system in $L^2(G)$. We can use the Peter-Weyl theorem, which states that if we have an irreducible representation corresponding to φ

$$L^2(G) = \widehat{\bigoplus_{\varphi\in\hat{G}}}\mathbb{C}\cdot\varphi$$

which is $M(\pi)$ (or each term in the sum? check).

$f \in L^2(G)$,

$$f = \sum_{\varphi\in\hat{G}} a_\varphi \cdot \varphi$$

(abstract Fourier transform of f). Now $(f|\psi) = a_\psi$. Hence we have Bessel's equality:

$$\|f\|^2 = \sum_{\varphi\in\hat{G}} |a_\varphi|^2$$

(an infinite version of Pythagorean theorem of sorts). Now we can rewrite

$$a_\varphi = (f|\varphi) = \int_G f(g)\overline{\varphi(g)}d\mu(g)$$

(abstract Plancherel formula).

$$\int_G |f(g)|^2 d\mu(g) = \|f\|^2$$

$$\mathcal{F}f(\varphi) = a_\varphi$$

Example. $G = T = S^1$ is a compact abelian group. We can parametrize it by

$$\{e^{i\varphi}|\varphi \in \mathbb{R}\}$$

What are its characters, and what is $\hat{T}$? The identity character is in $\hat{T}$, call it φ_1. Now consider

$$\varphi_n(e^{i\varphi}) = e^{in\varphi}$$

for $n \in \mathbb{Z}$. What is

$$\sum_{n\in\mathbb{Z}} \mathbb{C}\cdot\varphi_n$$

Note $\varphi_n \cdot \varphi_m = \varphi_{n+m}$. Hence the above vector space is a subalgebra of $C(T)$. It is also selfadjoint: $\overline{\varphi_n} = \varphi_{-n}$. By Stone-Weierstrass, it is dense in $C(T)$. It is dense in $L^2(T)$, which is

$$\widehat{\bigoplus_{n\in\mathbb{Z}}}\mathbb{C}\cdot\varphi_n$$

By what we proved, this is also

$$\widehat{\bigoplus_{\varphi\in\hat{T}}}\mathbb{C}\cdot\varphi$$

We can conclude that $\hat{T}=\mathbb{Z}$ with $n\mapsto\varphi_n$.

In the case G is compact and abelian, $\hat{G}$ is called the **dual group**. In the previous example, we can write, where $\hat{\mu}$ is the counting measure,

$$\int_{\hat{G}}|a_\varphi|^2 d\hat{\mu}(\varphi)$$