The following is some compiled results from Andrea Ferretti's Commutative Algebra necessary to prove Hilbert's Nullstellensatz, with some filled in details.

Lemma 1. Ideals of the form $I(\{x\})$ for $x \in k^n$ are maximal.

Theorem 1. (The Nullstellensatz) The following statements are equivalent:

1. $I(V(J)) = \sqrt{J}$.
2. $V(J) = \emptyset \Rightarrow J = A$
3. Maximal ideals are of the form $I(\{x\})$ for points $x \in k^n$.

Proof. $1 \Rightarrow 2$:

$$V(J) = \emptyset \Rightarrow I(V(J)) = A = \sqrt{J}$$

But $\sqrt{J} = A \Rightarrow J = A$.

$2 \Rightarrow 3$: Let $\mathfrak{m}$ be a maximal ideal. Then $V(\mathfrak{m})$ either has to be a single point $\{x\}$, in which case $I(\{x\}) = \mathfrak{m}$, so

$$f \in \mathfrak{m} \Rightarrow f \in I(\{x\})$$

so we have $\mathfrak{m} \subset I(\{x\})$. Hence $\mathfrak{m} = I(\{x\})$ from the lemma. Otherwise $V(\mathfrak{m})$ is empty, in which case $\mathfrak{m} = A$ by assumption, so we get a contradiction.

$3 \Rightarrow 2$: Suppose $V(J) = \emptyset$. Then $J \neq A$ implies $J \subset \mathfrak{m}$ for maximal ideal $\mathfrak{m}$, in which case we get a contradiciton.

$2 \Rightarrow 1$: Let $J \subset A$. Pick $g \in I(V(J))$. Inside $A[y]$ consider $\overline{J}$ generated by elements of J and $yg - 1$. $g = 0$ on $V(J)$, so $V(\overline{J})$ is empty. By 2 we say $\overline{J} = A[y]$. Hence we can find a combination

$$1 = h_0(yg - 1) + h_1 f_1 + h_2 f_2 + \ldots + h_r f_r$$

for $f_1, \ldots, f_r \in J$ and $h_0, \ldots, h_r \in A[y]$. By multiplying by a sufficiently large power of g (to clear denominators),

$$g^t = h_0'(yg - 1) + h_1' f_1 + \ldots + h_r' f_r$$

for $h_0' \in A[y]$, $h_1', \ldots, h_r' \in A$. Now consider the image of both sides in

$$\frac{A[y]}{(yg - 1)}$$

but remember that $h_1' f_1 + \ldots + h_r' f_r \in J \subset A$ and $g^t \in A$. Since A injects into $A[y]/(yg - 1)$, we get $h_1' f_1 + \ldots + h_r' f_r = g^t \in J$ as desired. $\square$

Lemma 2. (Noether Normalization) Let k be a field, A be a finitely generated k-algebra. There exists $n \geq 0$ and algebraically independent elements $x_1, \ldots, x_n \in A$ where A is finitely generated as a module over $k[x_1, \ldots, x_n]$.

Proof. Let $y_1, \ldots, y_m$ be a minimal set of generators over A as a k-algebra. We argue by induction over m. The base case $m = 0$ is true because it implies $A = k$. Assuming the case for $m - 1$, we may assume that $y_1, \ldots, y_m$ are algebraically dependent (the theorem is automatically true if they are). We get a polynomial f over k with

$$f(y_1, \ldots, y_m) = 0$$

Perform the change of variables $z_i = y_i - y_1^{r^{i-1}}$ where r is to be chosen soon. For each monomial in

$$f(y_1, z_2 + y_1^r, \ldots, z_m + y_1^{r^{i-1}})$$

of say multidegree $(\alpha_1, \ldots, \alpha_m)$, the degree in y_1 of the changed monomial is

$$\alpha_1 + \alpha_2 r + \ldots + \alpha_m r^{m-1}$$

This implies that y_1 is integral over $k[z_2, \ldots, z_m]$, which is already a finitely generated $k[x_1, \ldots, x_n]$-module for some algebraically independent $x_1, \ldots, x_n$. An integral extension of such a module is also finitely generated. $\square$

Lemma 3. (Zariski's lemma) Let A be a finitely generated k-algebra that is also a field. Then A is a finite extension of k.

Proof. Let A be a finitely generated k-algebra that is also a field. By the Noether normalization lemma, we can write $A = k[x_1, \ldots, x_r]$ where $x_1, \ldots, x_m$ are algebraically independent, and $x_{m+1}, \ldots, x_r$ are integral over $k[x_1, \ldots, x_m]$ for $m \leq r$. We would like to show $m = 0$. Suppose otherwise. Then since A is a field,

$$\frac{1}{x_1} \in A$$

so it must be integral over $k[x_1, \ldots, x_m]$. This gives a nontrivial polynomial relation over $x_1, \ldots, x_m$, contradicting the algebraic independence of the variables. $\square$

Theorem 2. (Nullstellensatz 3) The maximal ideals of A correspond to points of k^n.

Proof. Let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_r]$. Then the algebra $A = k[x_1, \ldots, x_r]/\mathfrak{m}$ is finitely generated as an algebra and it is also a field. By Zariski's lemma, it is a finite field extension of k, hence an algebraic extension. We have a natural inclusion $k \hookrightarrow A$. But k is algebraically closed, so this is actually an isomorphism

$$A \cong k.$$

If we let $\lambda_i \in k$ be the image of x_i under the isomorphism, $f_i = x_i - \lambda_i \in \mathfrak{m}$. Since the ideal generated by the f_i is maximal, we have $\mathfrak{m} = (f_1, \ldots, f_r)$. $\square$