Representation Theory

Notes on MATH 6260

Daniel Koizumi

February 23, 2022

Contents

1 January 10

Course structure: We will look at representation theory on finite groups first, and then proceed to look at it over compact groups.

Textbook: J.P. Serre: Linear Representations of Finite Groups

Definition 1. Let V be a vector space over $\mathbb{C}$, and G be a group. A representation (π, V) of G on V is a homomorphism

$$\pi : G \to GL(V)$$

$\dim V = \dim \pi$. V is the representation space of (π, V).

Denote by $Rep(G)$ all representations of G. We would like to say that it is a category. Take two representations (π, V) and (ν, U). Look at a linear map $\varphi : V \to U$. If

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\pi(g)"'] & U \arrow[d, "\nu(g)"] \\ V \arrow[r, "\varphi"'] & U \end{tikzcd}$$

commutes, for all $g \in G$, we call φ an intertwining operator. We can also define the sum of intertwining operators, which would be an intertwining operator. $\mathrm{Hom}_{\mathbb{C}}(V, U)$ of linear maps is a complex vector space. The intertwining maps form a linear subspace of the Hom. We denote this space $\mathrm{Hom}_G(V, U)$.

Assume we have a representation (π, V). A linear subspace $U \subset V$ is G-invariant if $\pi(g)(U) \subset U$. We can define $\nu(g)$ as $\pi(g)|_U : U \to U$. (ν, U) is a representation of G. This constructs **subrepresentations of** (π, V). In this case, the inclusion $i : U \to V$ is an intertwining map. We define quotient representations. Look at a quotient space $\frac{V}{U}$. We can define $\rho(g)(v + U) = \pi(g)(v) + U$. This is well defined. If $\varphi : (\pi, V) \to (\nu, U)$ is a morphism of representations, $\ker \varphi = \{v \in V | \varphi(v) = 0\}$. Look at $v \in \ker \varphi$,

$$\varphi(\pi(g)v) = \nu(g)\varphi(v) = 0$$

so $\pi(g)v \in \ker \varphi$. $\ker \varphi$ is a subrepresentation of (π, V). Can look at the subspace $\mathrm{im}\varphi$, which is also an invariant under G. $\mathrm{im}\varphi$ is a subrepresentation of (ν, U).

$$\nu(g)u = \nu(g)\varphi(v) = \varphi(\pi(g)v) \in \mathrm{im}\varphi$$

If we have two representations (π, V), (ν, U). We can define $V \oplus U = W$. For any $g \in G$, we can define $\pi \oplus \nu(g)(v, u) = (\pi(g)(v), \nu(g)(u))$. This is a **direct sum of** (π, V), (ν, U). If we have a morphism $V \xrightarrow{\varphi} U$,

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \\ V/ker\varphi \arrow[r, "\sim"] \arrow[ur, dashed] & im\varphi \arrow[u] \end{tikzcd}$$

The resulting map $\frac{V}{\ker \varphi} \to U$ is a linear isomorphism, but it is also an isomorphism in the category of representations.

2 January 12

Take (π, V) a representation of G. Take nonzero $v \in V$.Consider $\{\pi(g)v | g \in G\}$, which is a finite set since we are assuming that G is finite.

We can look at $U = \mathrm{span}(\{\pi(g)v | g \in G\})$, which is finite dimensional since it is spanned by the vectors. Note $v \in U$. U is a G-invariant subspace of V.

$$u = \sum c_g \pi(g)v \in \mathbb{C}$$

$$\pi(h)\mu = \sum c_g \pi(hg)v$$

U is a subrepresentation of V.

The representation (π, V) is irreducible if it is not the zero representation and the only G-invariant subspaces of V are V and $\{0\}$. First remark: irreducible representations have to be finite dimensional. If V is irreducible, $v \in V$ nonzero, $U = span\{\pi(g)v | g \in G\}$ contains $v \neq 0$. By irreducibility, $U \neq 0$ implies $U = V$. Every representation of a finite group contains an irreducible subrepresentation. This is because looking at one $span\{\pi(g)v | g \in G\}$, which is finite dimensional, we can continue taking subrepresentations until either the dimension is 1 or we hit an irreducible.

Assume we take (π, V), U invariant subspace. π induces a subrepresentation (ν, U).

Theorem 1. (Mascke) There exists a subrepresentation (ρ, W) such that $\pi = \nu \oplus \rho$. This tells us $V = U \oplus W$ and they are both G-invariant.

Proof. Pick a linear complement W of U. Then $V = U \oplus W$ and we can define a linear projection $p : V \to U$. We say $[G] = \text{Card(G)}$. Write

$$Q = \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1}) p \pi(g).$$

We would like to show that Q is a projection. Note that becuase U is invariant, $\text{im}(Q) \subset U$. We have Qu for $u \in U$,

$$Qu = \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1}) p \pi(g) u = u.$$

Hence $\text{im}(Q) = U$. Note $Q^2 = Q$. $Qv \in U$, so $Q^2 v = Qv$. Hence Q is a projection on $\text{im}(Q)$ along $\ker(Q)$. Let us calculate $Q\pi(h)v = \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1}) p \pi(g) \pi(h) v = \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1}) p \pi(gh) v$.

$$= \frac{1}{[G]} \sum_{g \in G} \pi(hg^{-1}) p \pi(g) v = \pi(h) \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1}) p \pi(g).$$

So $Q\pi(h) = \pi(h)Q$, so Q is an intertwining map. $\text{im}(Q)$ is an invariant subspace. We also have $\ker(Q)$ is an invariant subspace. $\ker(Q)$ is the desired W. $\square$

(π, V) is a finite dimensional representation.

Theorem 2. Any finite dimensional representation is a finite direct sum of irreducible subrepresentations.

Suppose we have a morphism $(\pi, V) \xrightarrow{\varphi} (\nu, U)$.

1. (π, V) is irreducible. Two cases occur:
   - $\ker \varphi = \{0\} \Rightarrow \varphi$ injects.
   - $\ker \varphi = V \Rightarrow \varphi = 0$
2. (ν, U) is irreducible representation. Two cases occur again:
   - $\text{im}(\varphi) = \{0\} \Rightarrow \varphi = 0$
   - $\text{im}(\varphi) = U \Rightarrow \varphi$ surjects.

Proposition 1. If (π, V) and (ν, U) are two irreducible representations, then either $\text{Hom}_G(V, U) = \{0\}$ or (π, V) and (ν, U) are isomorphic.

Lemma 1. (Schur's) If (π, V) is an irreducible representation of finite group G, then $\mathrm{Hom}_G(V,V) = \mathbb{C}\cdot I$ (the set of complex multiples of the identity map). Here, we use the fact that we're working over $\mathbb{C}$ since we need eigenvalues.

Given $\varphi \in \mathrm{Hom}_G(V,V)$. Note that φ has an eigenvalue λ with eigenvector v. $\varphi - \lambda I$ has nonzero kernel containing v, which implies the kernel is the entire space V.

3 January 14

Theorem 3. Any representation of a finite group G is a direct sum of irreducible representations.

Proof. (for infinite dimensional representations V): Denote by $\mathcal{S}$ the set of all irreducible subrepresentations of (π, V). If we have a family (ν_i, U_i), $i \in I$, a family of representations, we can form the direct sum of the representations. We can form

$$U = \bigoplus_{i\in I} U_i$$

which is the set of families of vectors parametrized by I, $(u_i : i \in I)$ where $u_i = 0$ except for finitely many i. We define the sum and scalar products accordingly. We can then define the representation $\nu = \bigoplus_{i\in I} \nu_i$ by

$$\nu(g)(u_i : i \in I) = (\nu_i(g)u_i : i \in I)$$

We can look at $\mathcal{T}$, the set consisting of subsets $I \subset \mathcal{S}$ such that

$$\bigoplus_{i\in I} \nu_i$$

is a subrepresentation of (π, V). Now this set is ordered by inclusion. To apply Zorn's lemma, we need to show that totally ordered subsets are bounded. Suppose we have a totally ordered subset $\mathcal{C}$. It is an exercise to show that the union over elements of $\mathcal{C}$ is a bound in $\mathcal{T}$. Zorn's lemma would then show that there exists $M \in \mathcal{T}$ that is maximal under the partial order. Now the subrepresentation

$$\bigoplus_{i\in M} \nu_i$$

which is either the whole representation π or a strict subrepresentation. Assume the latter holds. Then by a theorem from a previous class (Maschke's) the representation can be written

$$\left(\bigoplus_{i\in M} \nu_i\right) \oplus \rho$$

but note that ρ has an irreducible subrepresentation ω, which we can move over to the left representation. □

Definition 2. Denote by $\mathbb{C}[G] = \{f : G \to \mathbb{C}\}$. Note

$$\dim(\mathbb{C}[G]) = [G] = \mathrm{Card}(G)$$

We call the **right regular representation** $R : G \to \mathrm{Aut}(\mathbb{C}[G])$

$$(R(g)f)(h) = f(hg)$$

The resulting automorphisms are linear, and we have the following calculations:

$$(R(gg')f)(h) = f(hgg') = (R(g')f)(hg) = (R(g)R(g')f)(h)$$

and $R(1) = I$. Similarly, we have the **left regular representation** $L : G \to \mathrm{Aut}(\mathbb{C}[G])$ by

$$(L(g)f)(h) = f(g^{-1}h)$$

We can consider group G and G^{opp}, which is G with the reversed operation $g * h = h \cdot g$. We have a homomorphism $G \to G^{opp}$ by

$$g \mapsto g^{-1}$$

which is an isomorphism. Under this isomorphism, right and left regular representations are isomorphic. Let's examine a basis of $\mathbb{C}[G]$, which consists of functions

$$\delta_g : G \to \mathbb{C}$$

which is equal to 1 at g and 0 elsewhere. Of course $(\delta_g : g \in G)$ is a basis of $\mathbb{C}[G]$. The action

$$(R(g)\delta_1)(k) = \delta_1(kg)$$

which is 0 if $k \neq g^{-1}$ and 1 if $k = g^{-1}$, so $R(g)\delta_1 = \delta_{g^{-1}}$. We have hence proved $R(g)\delta_1 = \delta_{g^{-1}}$. Now assume $g \neq 1$. Then $R(g)\delta_1 = \delta_{g^{-1}} \neq \delta_1$, which shows $R(g)$ is not the identity operator. The kernel of $R : G \to \mathrm{GL}(\mathbb{C}[G])$ is hence trivial.

The regular representation $R = \bigoplus_{i=1}^k \rho_i$ and $R(g) \neq I$ if and only if $rho_i(g) \neq I$ for some $i \in \{1, \ldots, k\}$. We have the following theorem

Theorem 4. Let $g \in G$ be a non-identity element. Then there exists an irreducible representation (π, V) of G such that $\pi(g) \neq 1$.

Now we see an application of this theorem to an example. Consider an Abelian finite group G. Let's look at what we can say about an irreducible representation (π, V) of G. $\pi(a)\pi(b) = \pi(b)\pi(a)$ for any $a, b \in G$. Then note $\pi(b) \in \mathrm{Hom}_G(V, V)$ (it is an intertwining map). Then by Schur's lemma,

$$\pi(b) = \lambda(b) \cdot I$$

for some complex number $\lambda(b)$. We have $dim(\pi) = 1$, and all irreducible representations are 1-dimensional. We have

$$b \mapsto \lambda(b) \xrightarrow{|\cdot|} \mathbb{R}_+^*$$

Since b has finite order, we have that some power is equal to 1, implying that some power of $|lambda(b)|$ is 1. But this implies $|lambda(b)| = 1$. We call

$$\varphi : G \to \{z \in \mathbb{C} \mid |z| = 1\}$$

the homomorphism a **character of the group G.** We claim that if G is abelian, then all irreducible representations are characters. But we also have the converse. Given $a, b \in G$, we have the group commutator $aba^{-1}b^{-1} \in G$. If φ is a character, then $\varphi(aba^{-1}b^{-1})$ is $\varphi(a)\varphi(b)\varphi(a)^{-1}\varphi(b)^{-1} = 1$, which implies that $aba^{-1}b^{-1}$ is the identity. If it weren't, by a previous result an irreducible representation would detect it.

4 January 19

We proved last time that G is Abelian if and only if all irreducible representations are 1-dimensional. Any irreducible representation (π, V), with

$$\pi(g) = \varphi(g) \cdot I$$

where $\varphi : G \to \mathbb{C}^*$ is a **character**, with $|\varphi(g)| = 1$. Such a map factors through to get map $\varphi : G \to \{z \in \mathbb{C}^* || z| = 1\}$. If we have two characters $\varphi, \psi \in \hat{G}$,

$$(\varphi \cdot \psi)(g) = \varphi(g) \cdot \psi(g)$$

The resulting product is also a character. The character defined by $\varphi : G \to \{1\}$ also serves as the identity. Likewise, the other properties of a group hold for characters of a group (including inverses). Note that $\hat{G}$ is an abelian group, which we will call the **dual group of** G .

Now assume that G is finite (not necessarily abelian). On the space of functions $G \to \mathbb{C}$, which we call $\mathbb{C}[G]$, we define

$$(f|f') = \frac{1}{[G]} \sum_{g \in G} f(g)\overline{f'(g)}$$

The above is an inner product of f with f'.

We calculate the product of the regular representation with this inner product:

$$(R(g)f|f') = \frac{1}{[G]} \sum_{h \in G} (R(g)f)(h)f'(h)$$

$$= \frac{1}{[G]} \sum_{h \in G} f(hg)f'(h)$$

$$= \frac{1}{[G]} \sum_{h \in G} f(h)\overline{f'(hg^{-1})}$$

$$= \frac{1}{[G]} \sum_{h \in G} f(h)(R(g^{-1})f')(h)$$

$$= (f|R(g^{-1})f')$$

which implies $R(g) = R(g^{-1})^*$ (its adjoint), so R is unitary. In this case we call the representation R unitary.

Suppose we have two characters $\varphi, \psi \in \hat{G}$. We calculate

$$\varphi(g)(\varphi|\psi)$$

for a given $g \in G$. We have

$$\varphi(g)(\varphi|\psi) = \varphi(g)\frac{1}{[G]} \sum_{h \in G} \varphi(h)\overline{\psi(h)}$$

We can then move $\varphi(g)$ inside:

$$\frac{1}{[G]} \sum_{h \in G} \varphi(gh)\overline{\psi(h)}$$

We can then swap g and h?

$$\frac{1}{[G]} \sum_{h \in H} \varphi(hg)\overline{\psi(h)} = \frac{1}{[G]}\varphi(h)\overline{\psi(hg^{-1})}$$

$$\psi(hg^{-1}) = \psi(h)\psi(g^{-1}) = \psi(h)\psi(g)^{-1} = \psi(h)\overline{\psi(g)}.$$

We have above

$$= \left(\frac{1}{[G]} \sum_{h \in G} \varphi(h)\overline{\psi(h)}\right)\psi(g) = (\varphi|\psi) \cdot \psi(g)$$

If $\varphi \neq \psi$, then there is some $g \in G$ where $\varphi(g) \neq \psi(g)$. This implies $(\varphi|\psi) = 0$, so $\varphi \perp \psi$. This means $[\hat{G}] \leq [G]$, so the dual group is finite. Note

$$\|\varphi\|^2 = (\varphi|\varphi) = \frac{1}{[G]}\sum_{g\in G}\varphi(g)\overline{\varphi(g)} = 1.$$

We have $\mathbb{C}[G]$, and inner product $(\cdot|\cdot)$, with $R(g)$ unitary operators on V. Since the group is Abelian, the unitary operators $R(g)$ commute.

Exercise 1

There exists a basis for V $(e_i; 1 \leq i \leq [G])$ such that the unitary operators are diagonal matrices. To prove this, do this first for a single operator, then inductively do this for all of the operators. In other words $R(g)e_i = \varphi_i(g)e_i$ where $|\varphi_i(g)| = 1$. We have

$$V_i = \mathbb{C} \cdot e_i$$

is invariant for G. As we have proved from last class, $\varphi_i(g)$ is a character of G.

We have

$$(R(g)e_i)(1) = e_i(g)$$

which is also

$$\varphi_i(g)e_i(1)$$

By rescaling the orthonormal basis, we may assume $e_i(1) = 1$. (what does the notation $e_i(g)$ mean). We have $\{\varphi|\varphi \in \hat{G}\}$ is an orthonormal basis of $\mathbb{C}[G]$. Hence we have shown $[\hat{G}] = [G]$. From the structure theorem of finite abelian groups, $\hat{G} \cong G$ (even though there is not necessarily a natural isomorphism). We can then construct

$$\hat{\hat{G}} = (\hat{G})\hat{}$$

and a map $\alpha : G \to \hat{\hat{G}}$ defined by

$$g \mapsto (\varphi \mapsto \varphi(g))$$

the evaluation morphism. We have

$$(\alpha(gg'))(\varphi) = (\alpha(g)\alpha(g'))(\varphi)$$

so it is a group homomorphism. If $\alpha(g) = 1$, then $\varphi(g) = 1$ for all irreducible representations, so $g = 1$. So α is an inclusion (injection) of groups of the same finite order, implying α is isomorphic.

This is a special case of Pontryagin duality, which states there is a natural isomorphism $\hat{\hat{G}} \cong G$, which we will show holds for all abelian locally compact groups. We define a Fourier transform in this context. We can define characters on $\mathbb{R}$ by $x \mapsto e^{i\lambda x}$. This yields an isomorphism $\mathbb{R} \cong \hat{\mathbb{R}}$. This way we can define a Fourier transform on finite abelian group G in an analogous manner: Given a function $f \in \mathbb{C}[G]$, we can define

$$(\mathcal{F}f)(\varphi) = \frac{1}{[G]}\sum_{g\in G} f(g)\overline{\varphi(g)}$$

Note that this is $(f|\varphi)$. Since φ form an orthonormal basis, we can write

$$f = \sum_{\varphi\in\hat{G}}(f|\varphi)\varphi$$

and

$$f(g) = \sum_{\varphi\in\hat{G}}(\mathcal{F}f)(\varphi)\varphi(g) \text{ this is called the inverse Fourier transform on } \mathcal{F}f$$

We can use Bessel's equality, which says that

$$\|f\|^2 = \sum_{\varphi \in \hat{G}} |(f|\varphi)|^2$$

$$= \sum_{\varphi \in \hat{G}} |(\mathcal{F}f)(\varphi)|^2$$

This is called the Plancherel theorem. We will in the future generalize this to abelian locally compact groups. On Friday, we will see how to generalize these ideas to nonabelian groups. Irreducible representations may not be characters.

5 January 21

Last time, we discussed for a finite abelian group G, dual group $\hat{G}$, the set $\{\varphi : G \to \{z \in \mathbb{C} \big| |z| = 1\}\}$. We proved that $[\hat{G}] = [G]$ and that $R = \bigoplus_{\varphi \in \hat{G}} \varphi$. What can we hope for in general?

Example. Let's look at $G = S_n$, the permutation group on n elements, also known as the symmetric group.

1. $n = 2$. We have $[S_2] = 2$, $\mathbb{Z}_2$. We have two characters: the trivial character sending all elements of the group to 1, and **the sign character** sending even permutations to 1 and odd permutations to -1.
2. Now if we look at S_3, which has order 6, the group is not abelian and it has two characters as above *triv* and *sgn*. We proved last time that if all irreducible representations have dimension 1, then the group is abelian. So there is an irreducible representation σ of S_3 that is not 1 dimensional.

   Claim: there exists a 2-dimensional irreducible σ unique up to isomorphism.

   We are generalizing $R = \bigoplus_{\varphi \in \hat{G}} \varphi$ in that each representation will appear with multiplicity equal to its dimension. Write $\hat{G}$ to be the set of isomorphism classes of irreducible representation of G. $\hat{G}$ is a finite set, but it is difficult to define a group structure on the set (can't compose a 2-dimensional representation with a 1-dimensional). We can, however, introduce additional structure on $\hat{G}$. If we have two irreducible representations, we can take their tensor product. Even though the tensor product may not be irreducible, it can be written as a direct sum of irreducibles. There is a theorem in Alg Geo/Num Theory called Tanaka Duality related to this, which (generalizes Pontryagin duality?).

Remark. S_n has no other characters except *triv* and *sgn*.

If $g \in S_n$ is a transposition, then it satisfies

$$g^2 = 1$$

So ψ is a character of S_n, we have $\psi(g)^2 = 1$ implies $\psi(g) = \pm 1$. Now all transpositions are conjugate. That is, if g, g' are transpositions, then $g' = hgh^{-1}$ for some $h \in S_n$. This implies

$$\psi(g') = \psi(h)\psi(g)\psi(h)^{-1} = \psi(g)$$

since multiplication on $\mathbb{C}$ is commutative. So $\psi(g) = 1$ on all transpositions, or $\psi(g') = -1$ on all transpositions. Since the transpositions generate the group, this determines ψ. So this implies our earlier claim that there is a 2-dimensional representation somehow.

Theorem 5. Let (π, V) be a finite dimensional representation of a finite group G. There exists an inner product $(\cdot|\cdot)$ on V such that (π, V) is a unitary representation. Note we did not say that this inner product is unique, and it need not be.

Proof. We can define an inner product on V by fixing a basis e_i, so we can write any $u \in V$ as

$$u = \sum_{i=1}^{n} u_i e_i$$

and then we can write

$$\langle u|v\rangle = \sum_{i=1}^{n} u_i \overline{v_i}$$

We can then define a new inner product

$$(u|v) = \frac{1}{[G]} \sum_{g\in G} \langle \pi(g)u|\pi(g)v\rangle$$

What we get is linear in the first variable because it is a linear combination of functions which are linear in the first variable. Likewise for the second variable. We also have

$$(u|v) = \overline{(v|u)}$$

We have

$$(u|u) = \frac{1}{[G]} \sum_{g\in G} \langle \pi(g)u|\pi(g)u\rangle \geq 0$$

Now if the inner product is zero, because each of the terms of the sum are nonnegative reals, each term in the sum is also zero, implying $\langle u|u\rangle = 0$.

We can now calculate $(\pi(g)u|\pi(g)v) = \frac{1}{[G]} \sum_{h\in G} \langle \pi(h)\pi(g)u|\pi(h)\pi(g)v\rangle$. Now this is

$$\frac{1}{[G]} \sum_{h\in G} \langle \pi(hg)u, \pi(hg)v\rangle$$

$$= \frac{1}{[G]} \sum_{h\in G} \langle \pi(h)u|\pi(h)v\rangle = (u|v)$$

so that $\pi(g)$ is unitary. So if (π, V) is finite dimensional, (ν, U) is a subrepresentation, we can introduce inner product $(\cdot|\cdot)$ and write $V = U \oplus U^{\perp}$. This gives an alternate proof of Theorem 1 on January 14. $\square$

We now discussion **Orthogonality relations**. Recall

$$\mathbb{C}[G]$$

which has natural product

$$(f|f') = \frac{1}{[G]} \sum_{g\in G} f(g)\overline{f'(g)}$$

We would like to compare representations (π, V), (ν, U). We look at a linear map $A \in \mathrm{Hom}_{\mathbb{C}}(U, V)$, and examine

$$B = \frac{1}{[G]} \sum_{g\in G} \pi(g)A\nu(g)^{-1}$$

which is essentially an average on A after twisting before and afterwards by the two representations. We have $B \in \mathrm{Hom}_{\mathbb{C}}(U, V)$, of course, but we moreover have

$$\pi(g)B = \frac{1}{[G]} \sum_{h\in G} \pi(g)\pi(h)A\nu(h^{-1})$$

$$= \frac{1}{[G]} \sum_{h\in G} \pi(gh)A\nu(h^{-1})$$

$$= \frac{1}{[G]} \sum_{h \in G} \pi(h) A \nu(h^{-1} g) = \frac{1}{[G]} \sum_{h \in G} \pi(h) A \nu(h^{-1}) \nu(g)$$
$$= B\nu(g)$$

(by changing variables in sum). We have shown that the average is an intertwining map over the representations π, ν, so $B \in \mathrm{Hom}_G(U, V)$. Let's relate this to Schur's lemma. If (π, V), (ν, U) are irreducible representations, then there are two cases (we examine the second next class):

1. If $(\pi, V), (\nu, U)$ are not isomorphic
2. $(\nu, U) = (\pi, V)$.

We have 1 implies $\mathrm{Hom}_G(U, V) = \{0\}$. Here's why we have the two cases above: Given an intertwining map $U \to V$, the kernel of the map is either 0 or the entire set. In the former case, the image must be nonzero so the entire set V, which implies isomorphism. In the latter, the map is the zero map.

Back to the first case, we have

$$\sum_{g \in G} \pi(g) A \nu(g^{-1}) = 0$$

for all A in case 1. Now A can be represented as a matrix if we fix orthonormal bases $(e_1, \ldots, e_n)$ of U, $(f_1, \ldots, f_m)$ of V. We plug in A_{pq}, which is zero everywhere except where it is 1 in the p-th row and q-th column. We have

$$\sum_{j,k} \sum_{g \in G} \pi(g)_{ij} (A_{pq})_{jk} \nu(g^{-1})_{kl} = 0$$

We have $A_{jk} = \delta_{jp} \delta_{kq}$, so the sum is

$$\sum_{g \in G} \pi(g)_{ip} \nu(g^{-1})_{ql} = 0$$

If we have a unitary matrix, we have $\nu(g^{-1})_{ql} = \overline{\nu(g)_{lq}}$. We have

$$(\pi_{ip} | \nu_{lq}) = 0$$

6 January 24

Let G be an Abelian group. Schur's lemma says that if (π, V) is an irreducible representation of G, and a linear map $A \in \mathrm{Hom}_G(V, V)$ such that $\pi(g)A = A\pi(g)$, then Schur's lemma implies $A = \lambda I$. For any $g \in G$, Schur's lemma implies that $\pi(g) = \varphi(g) I$. Assume that the dimension of V is greater than 1. We can take a strict nonzero subspace $U \subsetneq V$ (by choice which is dimension 1). It is an invariant subspace, and so we can get a smaller representation, which contradicts the irreducibility of $\pi(g)$.

We stopped last time at where G is a finite group, and took two representations

$$(\pi, V) \ncong (\nu, U)$$

are irreducible. Then we can take a basis $e_1, \ldots, e_n$ in V and $f_1, \ldots, f_m$ in U which are orthonormal. We can then represent $\pi(g)$ as a matrix $(\pi(g)_{ij})$ and $\nu(g) = (\nu(g)_{k\ell})$ which are $n \times n$ and $m \times m$ matrices, respectively. Then

$$(\pi_{ij} | \nu_{k\ell}) = 0$$

Here the matrix coefficients are characters.

We now provide an invariant interpretation. Define

$$M(\pi) = \text{span of } \pi_{ij}, 1 \leq i, j \leq n$$

which we call the matrix coefficient subspace attached to π. Let us take $v \in V$, linear form $\varphi \in V^*$. The matrix coefficient corresponding to G and φ is

$$g \mapsto \varphi\left(\pi(g)v\right)$$

We can then write

$$\pi(g)e_i = \sum_{j=1}^{n} \pi(g)_{ji} e_j$$

And we can write

$$v = \sum_{i=1}^{n} c_i e_i$$

$$\pi(g)v = \sum_{i=1}^{n} c_i \pi(g) e_i$$

$$= \sum_{i,j=1}^{n} c_i \pi(g)_{ji} e_j$$

Applying φ we get

$$\varphi(\pi(g)v) = \sum_{i,j=1}^{n} c_i \pi(g)_{ji} \varphi(e_j)$$

We have just shown that $M(\pi)$ is independent of choice of basis. We can then reformulate $(\pi_{ij}, \nu_{k\ell}) = 0$. It says: if (π, V) is not isomorphic to (ν, U), then $M(\pi) \perp M(\nu)$. Now we can make a definition

$$\hat{G} = \text{the set of equivalence classes of irreducible reps of } G$$

So $M(\pi)$ depends only on the equivalence class of the representation, and we denote the equivalence class of π as $[\pi]$. This makes a well defined map

$$[\pi] \mapsto M(\pi)$$

If we look at the sum

$$\sum_{[\pi]\in\hat{G}} \dim(M(\pi)) \leq \dim(\mathbb{C}[G]) = [G]$$

We have argued $\hat{G}$ is a finite set, which we called the **dual of G.** We don't know how large $M(\pi)$ is. We do have a bound $|\hat{G}| \leq |G|$. If the group is Abelian then, as we have shown before, we have equality. Otherwise, the inequality is strict because some $M(\pi)$ must be of dimension greater than 1. We do need to show that if a representation is not one dimensional, then $M(\pi)$ is not of dimension 1.

We are looking at the situation $(\pi, V) = (\nu, U)$. In this case, given a linear map $A \in \mathrm{Hom}_{\mathbb{C}}(V, V)$, we have an average

$$B = \frac{1}{[G]} \sum_{g\in G} \pi(g) A \pi(g^{-1})$$

This implies $B \in \mathrm{Hom}_G(V, V)$. Since π is irreducible, $B = \lambda I$ by Schur's lemma. The variant that we get from our argument is the following: that

$$\frac{1}{[G]} \sum_{g\in G} \pi(g) A \pi(g^{-1}) = \lambda I$$

We now calculate λ. The trick is to take the trace of this expression:

$$tr(B) = \lambda \cdot tr(I) = \lambda \cdot \dim(\pi)$$

But from the left hand side for the average expression, we get

$$tr(B) = \frac{1}{[G]} \sum_{g \in G} tr(\pi(g) A \pi(g^{-1})$$

We can conduct cyclic permutation on the product to move $\pi(g)$ to the back, making

$$tr(B) = \frac{1}{[G]} \sum_{g \in G} tr(A\pi(g^{-1})\pi(g)) = \frac{1}{[G]} \sum_{g \in G} tr(A) = tr(A)$$

This yields

$$\begin{aligned} \lambda &= \frac{tr(A)}{\dim(\pi)} \cdot \frac{1}{[G]} \sum_{g \in G} \pi(g) A \pi(g^{-1}) \\ &= \frac{tr(A)}{\dim(\pi)} \cdot I \end{aligned}$$

We would like to see what happens if we plug in matrices. That is, looking at coefficients, we get for the second to last expression above,

$$\frac{tr(A)}{\dim(\pi)} \cdot \frac{1}{[G]} \sum_{j,k=1}^{n} \pi(g)_{ij} A_{jk} \pi(g^{-1})_{k\ell}$$

and

$$\frac{tr(A)}{\dim(\pi)} \frac{tr(A)}{\dim(\pi)} \delta_{i\ell}$$

We pick a matrix that is one at index (p, q) and zero elsewhere. We have

$$A_{jk} = (\delta_{jp}\delta_{kq})_{pq}$$

Looking at the above summation, we have

$$\frac{1}{[G]} \sum_{g \in G} \pi(g)_{ip} \overline{\pi(g)_{ql}^{-1}} = (\pi_{ip} | \pi_{\ell q})$$

We have an orthonormal basis that is invariant under the group action. We also have $tr(A) = \delta_{pq}$, and

$$(\pi_{ip} | \pi_{\ell q}) = \frac{1}{\dim \pi} \delta_{i\ell} \delta_{pq}$$

This is the celebrated **Schur orthogonality relation**. The ultimate conclusion is that

$$\dim(M(\pi)) = (\dim \pi)^2$$

which gives us what we wanted to show from earlier in the lecture. Adding to our earlier result, we have

$$\sum_{[\pi] \in \hat{G}} \dim(\pi)^2 = \sum_{[\pi] \in \hat{G}} \dim(M(\pi)) \leq [G]$$

We examine an example:

Example. Look at S_3, the first nonabelian symmetry group. It has two representations $triv$, sgn. It must have at least one representation that is not one dimensional by the nonabelianness of S_3. We call one such representation σ (it could have more). First, let us calculate

$$\sum_{[\pi] \in \hat{S_3}} \dim(\pi)^2 = 1 + 1 + (\dim(\sigma)) \leq 6 = [S_3]$$

But $\dim(\sigma)$ is at least 2, meaning that $\dim(\sigma) = 2$. This shows there is only one two dimensional representation up to equivalence. In the next two classes, we will figure out how to find σ.

On Wednesday, we will figure out how to show the above inequality before the example is actually an equality.

7 January 26

We have a finite group G. For any class $[\pi] \in \hat{G}$, we have $M(\pi) \subset \mathbb{C}[G]$, matrix representations of π. We can consider the direct sum

$$M = \bigoplus M(\pi) \subset \mathbb{C}[G]$$

Our next step is to prove equality. We have

1. $M(\pi)$ are G-invariant for R. Fixing a basis $e_1, \ldots, e_n$ of V. In this case, $\pi(g)e)i = \sum \pi(g)_{ji}e_j$.

$$(\pi(g)_{ji})\text{span}(M(\pi))$$

   Let's examine $\pi(\bullet)_{ij}$, and apply $R(g)$. We have

$$(R(g)\pi_{ij})(h) = \pi_{ij}(hg) = \sum_{k=1}^{n} \pi(h)_{ik}\pi(g)_{kj}$$

$$R(g)\pi_{ij} = \sum_{k=1}^{n} \pi(g)_{kj}\pi_{ik} \in M(\pi)$$

   (we are concluding it is in $M(\pi)$.

2. M is G-invariant for R. Suppose that $M \neq \mathbb{C}[G]$. This implies that $M^{\perp} \neq 0$. The orthogonal complement of an invariant subspace is invariant, so $M^{\perp}$ is G-invariant. It hence contains some G-invariant subspace U such that its representation (ν, U) is irreducible. Let's look at (ν, U). We pick a basis $f_1, \ldots, f_m$ of U. We look at

$$\nu(g)f_i = \sum_{j=1}^{m} \nu(g)_{ji}f_j$$

   We can write

$$(\nu(g)f_i)(h) = \sum_{j=1}^{m} \nu(g)_{ji}f_j(h)$$

   Now this is equal to $f_i(hg)$ because ν is a restriction of R. We now evaluate this at $h = 1$. We have

$$f_i(g) = \sum_{j=1}^{m} \nu(g)_{ji}f_j(1)$$

   We can then conclude that f_i is a linear combination of $\nu_{ji} \in M(\nu)$. The conclusion is $f_i \in M(\nu)$. This implies $(f_i|f_i) = 0$ since f_i are in a subspace of $M^{\perp}$, so $f_i = 0$. This shows $M = \mathbb{C}[G]$.

What is $M(\pi)$? Take the vector space V of π, and its dual V^*. We have a natural map

$$V^* \otimes V \xrightarrow{\alpha} M(\pi)$$

defined by

$$f \otimes v \mapsto f(\pi(\bullet)v) \in \mathbb{C}[G]$$

On V^* we have the dual representation $V \to V^*$ defined by

$$g \mapsto \pi(g)^*$$

which is not a representation because taking adjoints flips order. We do have the representation $\pi^*(g) = \pi(g^{-1})^*$. If we have a representation π we thus have a natural representation π^*. We can thus view $V^* \otimes V$ as being acted on by $G \times G$. Note $M(\pi)$ is invariant for the left regular representation. This yields an isomorphism of irreducible representations. Here we have a decomposition of irreducible representations of M. The dimension of $M(\pi)$ is hence $(\text{dim}\pi)^2$.

Theorem 6.

$$\bigoplus_{[\pi]\in\hat{G}} M(\pi) = \mathbb{C}[G]$$

and

$$[G] = \sum_{[\pi]\in\hat{G}} \dim(\pi)^2$$

Now we introduce another important tool, the notion of character.

Definition 3. Let (π, V) be finite dimensional. We define the **character** of the representation to be

$$\mathrm{ch}(\pi) = \mathrm{tr}(\pi(\bullet)) \in \mathbb{C}[G]$$

Take a basis $e_1, \ldots, e_n$ of V. We have

$$\pi(g) \to (\pi(g)_{ij})$$

and

$$\mathrm{tr}(\pi(g)) = \sum_{i=1}^{n} \pi(g)_{ii}$$

Let's list some properties of the character.

1. $\mathrm{ch}(\pi)(1) = \mathrm{tr}(id) = \dim(\pi)$
2. $g, h \in G$, can calculate hgh^{-1}. So $\mathrm{ch}(\pi)(hgh^{-1}) = \mathrm{tr}(\pi(hgh^{-1})) = \mathrm{tr}(\pi(h)\pi(g)\pi(h)^{-1})$ and

   $$\mathrm{tr}(\pi(g)\pi(h^{-1})\pi(h)) = \mathrm{tr}(\pi(g))$$

   by the fact that you can do cyclic permutations in products without changing the trace. That is, $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$. We proved the fact that characters are constant on conjugacy classes.
3. If we have a representation $\pi = \nu \oplus \rho$, then

   $$\mathrm{ch}(\pi) = \mathrm{ch}(\nu) + \mathrm{ch}(\rho)$$

   Let's look at the matrix for π:

   $$\begin{bmatrix} \nu & 0 \\ 0 & \rho \end{bmatrix}$$

   by choosing a basis for V. The trace is accordingly the sum of traces. By Maschke's theorem, the characters of irreducible representations determine those for all representations.

Now let's take two irreducible representations π, ν. Let's assume that $[\pi] \neq [\nu]$. Then we can look at

$$(\mathrm{ch}(\pi)|\mathrm{ch}(\nu))$$

The former character is in $M(\pi)$, the latter is in $M(\nu)$ and $M(\pi) \perp M(\nu)$, so the inner product is 0. Characters of nonisomorphic irreducible representations are orthogonal to each other. The second question is what happens with the norm of a character? What is

$$(\mathrm{ch}(\pi)|\mathrm{ch}(\pi)) = \sum_{i=1}^{n}\sum_{j=1}^{n}(\pi_{ii}|\pi_{jj})$$

Now note that $(\pi_{ii}|\pi_{jj}) = 0$ unless $i = j$ by the orthogonality relation. We get a single term

$$\sum_{i=1}^{n}(\pi_{ii}|\pi_{ii}) = \sum_{i=1}^{n}\frac{1}{\dim(\pi)} = 1$$

Hence

$$(\mathrm{ch}(\pi) : [\pi] \in \hat{G})$$

is an orthonormal family in $\mathbb{C}[G]$.

Exercise 2

Look at S_3. You can two irreducible characters $\mathrm{triv}, \mathrm{sgn}$. There is a third irreducible character. From there, you can find an orthonormal basis for $\mathbb{C}[S_3]$. From there, you can find the third irreducible character.

We will see later how to do this in other ways.

8 January 28

Reminder of last time: Let G be a finite group, $\hat{G}$ be its dual, which is the isomorphism classes of irreducible representations. Each element $[\pi]$ is attached to its character $\mathrm{ch}(\pi) \in \mathbb{C}[G]$. We proved last time the Schur orthogonality relations, which say that

$$(\mathrm{ch}(\pi)|\mathrm{ch}(\nu)) = \delta_{\pi\nu} \text{ for } [\pi], [\nu] \in \hat{G}$$

And $\{\mathrm{ch}(\pi)|\pi ß\hat{n}G\}$ is an orthogonal family. So let's describe the first reformulation of the Schur Orthogonality relations.

Assume that π, ν are finite dimensional representations on spaces V, U respectively. We can look at the space of morphisms

$$\mathrm{Hom}_G(\pi, \nu)$$

which are linear maps $V \to U$ that commute with the action. Since the space is finite dimensional, Hom is a finite dimensional vector space. We have the following theorem:

Theorem 7. $\dim_{\mathbb{C}} \mathrm{Hom}(\pi, \nu) = (\mathrm{ch}(\pi)|\mathrm{ch}(\nu))$. The former gives homological information about the category of representations.

Proof. Let's check what happens when π, ν are irreducible. We know from Schur's lemma there are two options:

1. $[\pi] \neq [\nu]$, in which case $\mathrm{Hom}_G(\pi, \nu) = \{0\}$.
2. $[\pi] = [\nu]$, in which case $\mathrm{Hom}_G(\pi, \pi) = \mathbb{C}I$.

In case 1, the dimension is 0, and in the latter case, the dimension is 1. These also match with $(\mathrm{ch}(\pi)|\mathrm{ch}(\nu))$. The rest is based on Maschke's theorem. Suppose we have $\pi = \pi_1 \oplus \pi_2$. This means

$$\mathrm{Hom}_G(\pi, \nu) = \mathrm{Hom}_G(\pi_1 \oplus \pi_2, \nu)$$

$$= \mathrm{Hom}_G(\pi_1, \nu) \oplus \mathrm{Hom}_G(\pi_2, \nu)$$

(exercise: prove this equality). If we prove this,

$$\dim(\mathrm{Hom}_G(\pi_1 \oplus \pi_2, \nu)) = \dim(\mathrm{Hom}_G(\pi_1, \nu)) + \dim(\mathrm{Hom}_G(\pi_2, \nu))$$

We have

$$\dim(\mathrm{Hom}_G(\pi_1 \oplus \pi_2 \oplus \ldots \oplus \pi_k, \nu)$$

$$= \sum_{i=1}^{n} \dim\left(\mathrm{Hom}_G(\pi_i, \nu)\right)$$

We can do the second argument on the second variable. An analogous formula:

$$\dim\left(\mathrm{Hom}_G(\pi, \nu_1 \oplus \ldots \oplus \nu_k)\right)$$

$$= \sum_{j=1}^{k} \dim\left(\mathrm{Hom}_G(\pi, \nu_j\right)$$

We know that the character of a direct sum is a direct sum of characters, and additivity applies to the right side $(\mathrm{ch}(\pi)|\mathrm{ch}(\nu))$. ☐

The characters describe the structure of the category. How big is $\hat{G}$? If π is a finite dimensional representation of G, then $\mathrm{ch}(\pi)$ is constant on conjugacy classes. We can look at the vector subspace of all functions on $\mathbb{C}[G]$ constant on conjugacy classes. These are called the **central functions**.

1. The dimension of the space of central functions is equal to the number of conjugacy classes.
2. $\{\mathrm{ch}(\pi)|\pi \in \hat{G}\}$ implies $\mathrm{Card}(\hat{G}) \leq$ the number of conjugacy classes. We would like to show that this is an equality. If G is abelian, then note the number of conjugacy classes is $[G]$, but this is actually $[\hat{G}]$. This result is a generalization of that result.

Theorem 8. If we take $f \in M(\pi)$, $[\pi] \in \hat{G}$, we can form

$$\frac{1}{[G]} \sum_{h \in G} f(hgh^{-1})$$

$$= \frac{f(1)}{\dim(\pi)} \cdot \mathrm{ch}(\pi)(g)$$

We have to assume that we can take an orthonormal basis for $M(\pi)$.

Proof. Take $f = \pi_{ij}$ (matrix coefficients in some orthonormal basis $(e_1, \ldots, e_n)$ with respect to a G-invariant inner product). It's enough to prove the formula for all such f 's by linearity. Plugging this in,

$$\frac{1}{[G]} \sum_{h \in G} \pi(hgh^{-1})_{ij} = \frac{1}{[G]} \sum_{h \in G} (\pi(h)\pi(g)\pi(h)^{-1})_{ij}$$

$$\frac{1}{[G]} \sum_{h \in G} \sum_{\ell, k=1}^{n} \pi(h)_{ik} \pi(g)_{k\ell} \pi(h)^{-1}_{\ell j}$$

Since $\pi(h)$ is unitary, $\pi(h)^{-1}_{\ell j} = \overline{\pi(h)_{\ell j}}$. Now we have

$$\sum_{k,\ell=1}^{n} \pi(g)_{k\ell} \frac{1}{[G]} \sum_{h \in G} \pi(h)_{ik} \overline{\pi(h)_{j\ell}}.$$

$$= \sum_{k,\ell=1}^{n} \pi(g)_{k\ell} \cdot \frac{1}{\dim(\pi)} \delta_{ij} \delta_{k\ell}.$$

$$= \frac{\delta_{ij}}{\dim(\pi)} (\mathrm{ch}(\pi))(g)$$

and note

$$\delta_{ij} = I_{ij} = \pi(1)_{ij}$$

giving us the theorem result. ☐

We want to show that characters span the space of central functions. Let's take φ a central function on conjugacy classes. Take $f \in M(\pi)$. Assume that $\varphi \perp \mathrm{ch}(\pi), \pi \in \hat{G}$. We show

$$(\varphi|f) = 0$$

which gives that $\varphi \perp M(\pi)$ for any $\pi \in \hat{G}$. This implies $\varphi \perp \mathbb{C}[G]$, which gives $\varphi = 0$. In other words, $\{\mathrm{ch}(\pi)|\pi \in \hat{G}\}$ is an orthonormal basis of the space of central functions. We calculat e

$$(\varphi|f) = \frac{1}{[G]} \sum_{g\in G} \varphi(g)\overline{f(g)}$$

The first step is to split the sum over conjugacy classes, and φ is constant over each mini-sum. We can then apply the earlier theorem.

9 January 31

We would like to complete a proof from last class. We wanted to show that the space of central functions on G is spanned by characters of irreducible representations $\mathrm{ch}(\pi)$ for $\pi \in \hat{G}$. To prove this, we know that characters

$$\{\mathrm{ch}(\pi)|\pi \in \hat{G}\}$$

is an orthonormal set in $\mathbb{C}[G]$. It is enough to show that if φ is a central function on G such that $(\varphi|\mathrm{ch}(\pi)) = 0$, then $\varphi = 0$. The proof will be based on a formula we proved last time. If we take $f \in M(\pi)$ and average it over conjugacy classes:

$$\frac{1}{[G]} \sum_{h\in\hat{G}} f(hgh^{-1}) = \frac{f(1)}{\dim(\pi)} \cdot \mathrm{ch}(\pi)(g)$$

We want to calculate $(\varphi|f) = \frac{1}{[G]} \sum_{g\in G} \varphi(g)\overline{f(g)}$. This is also

$$\frac{1}{[G]^2} = \sum_{g,h\in G} \varphi(hgh^{-1})\overline{f(hgh^{-1})}$$

Our function f is central, so conjugation by h is irrelevant:

$$\frac{1}{[G]^2} \sum_{g\in G} \varphi(g) \sum_{h\in G} \overline{f(hgh^{-1})}$$

$$= \frac{1}{[G]} \sum_{g\in G} \varphi(g) \frac{1}{[G]} \sum_{h\in G} \overline{f(hgh^{-1})}$$

$$\frac{1}{[G]} \sum_{g\in G} \varphi(g) \frac{\overline{f(1)}}{\dim(\pi)} \overline{\mathrm{ch}(\pi)(g)}$$

$$= \frac{\overline{f(1)}}{\dim(\pi)} \frac{1}{[G]} \sum_{g\in G} \varphi(g)\overline{\mathrm{ch}(\pi)(g)}$$

$$= \frac{\overline{f(1)}}{\dim(\pi)} (\varphi|\mathrm{ch}(\pi)) = 0$$

We have $\varphi \perp M(\pi)$, $\pi \in \hat{G}$, so $\varphi \perp \bigoplus_{[\pi]\in\hat{G}} M(\pi) = \mathbb{C}[G]$ (this equality is from a previous theorem (Bernsie's theorem?)). So $\varphi = 0$.

We have ultimately shown that the dimension of the space of central functions is equal to the number of conjugacy classes which is equal to $\mathrm{Card}\left(\hat{G}\right)$.

Example. We look at $G = S_3$. We have remarked that there are three irreducible representations, so $\text{Card}(\hat{G}) = 3$. We have the trivial representation, the sgn representation, and then an unknown representation σ which is two dimensional (see last class). We have $[G] = 6$. We have $\{1\}$ which is one conjugacy class, the conjugacy class of transpositions,

$$\{(1\ 2), (2\ 3), (3\ 1)\} \text{ (in cyclic notation)}$$

and the conjugacy class of everything else:

$$\{(3\ 1\ 2), (3\ 2\ 1)\}$$

The third and first ones are even, while the second is odd. The set of even permutations is a normal subgroup, which is kernel of sgn. Note that the character of triv is equal to 1 everywhere. We have

$$\text{ch}(\text{sgn}) = \begin{cases} 1 & \text{even} \\ -1 & \text{odd} \end{cases}$$

If we look at

$$\text{ch}(\text{triv}) - \text{ch}(\text{sgn}) = \begin{cases} 0 & \text{even} \\ 2 & \text{odd} \end{cases}$$

We note that this difference is orthogonal to $\text{ch}(\sigma)$. So

$$\begin{aligned} &(\text{ch}(\text{triv}) - \text{ch}(\text{sgn}|\text{ch}(\sigma)) \\ &= \frac{1}{6}\sum_{g\in S_3}\cdots \\ &= \frac{1}{3}\left(3\text{ch}(\sigma)((1\ 2))\right) = 0 \end{aligned}$$

so $\text{ch}(\sigma) = 0$ on transpositions (it is constant on transpositions). We also know $\text{ch}(\sigma)(1) = 2$.

$$\text{ch}(\sigma) = \begin{cases} 2 & 1 \\ 0 & \text{transposition} \\ x & \text{3-cycles} \end{cases}$$

Now note

$$(\text{ch}(\sigma)|\text{ch}(\sigma)) = \frac{1}{6}\left(2\cdot 1 + 0\cdot 3 + 2x\right) = \frac{1}{3}\left(1+x\right) = 0$$

so $x = -1$. We have σ is an induced representation.

The next step: one technique of constructing representations is to look at group G and its subgroups $H < G$. We first define a functor

$$\text{Rep}(G)$$

and given object (π, V), it satisfies $\pi : G \to \text{GL}(V)$, so if we have a subgroup, we can use the inclusion $\iota : H \hookrightarrow G$ and the induced $\text{Res}_H^G(\pi) : H \to \text{GL}(V)$.

$$(\pi, V) \to \text{Res}_H^G(\pi)$$

We get a functor Res_H^G. If we have two representations $(\pi, V), (\pi', V')$, and the map is an induced map of representations, so $\pi'(g)\circ\varphi = \varphi\circ\pi(g)$ for all $g \in G$.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\pi(g)"'] & V \arrow[d, "\pi'(g)"] \\ V \arrow[r, "\varphi"'] & V \end{tikzcd}$$

so $\varphi \in \mathrm{Hom}_H(V, V')$. We have an adjoint functor which goes in the opposite direction. We would like to construct a functor

$$\mathrm{Rep}(H) \to \mathrm{Rep}(G)$$

which we call induction. Let us recall regular representations, defined on $\mathbb{C}[G]$. Define $(R(g)f)(g') = f(g'g)$. Let's take a representation (ν, U) on $H < G$ (we are not assuming anything about their size, so we will use this when we work on compact groups). Define $\mathrm{Ind}(U) = \{f : G \to U | f(hg) = \nu(h)(f(g)) \forall h \in H, g \in G\}$. We have

$$\mathrm{Ind}(U) \subset \mathbb{C}[G, U]$$

We point out that if we take triv $= \{1\}$, the only representation is $\mathbb{C}$ with the trivial action. In this case, $\mathrm{Ind}(U) = \mathbb{C}[G]$. We define a map $\rho(g) : \mathrm{Ind}(U) \to \mathrm{Ind}(U)$ by

$$(\rho(g)f)(g') = f(g'g)$$

Note that

$$(\rho(g)f)(hg') = f(hg'g) = \nu(h)f(g'g) = \nu(h)(\rho(g)f)(g')$$

We also have

$$\rho(g)f \in \mathrm{Ind}(U)$$

We have that $\rho(g) : \mathrm{Ind}(U) \to \mathrm{Ind}(U)$ where $\rho(1) = \mathrm{id}$. We have $\rho(gg') = \rho(g)\rho(g')$. Therefore $(\rho, \mathrm{Ind}(U))$ is a representation of G defined by $\mathrm{Ind}_H^G(\nu)$, which is called the **induced representation.** The regular representation R is nothing else than $\mathrm{Ind}_{\{1\}}^G(\mathrm{triv})$.

We point out that Ind is a functor, so we if we have (ν, U) and (ν', U') representations of H, let's assume we have $\varphi : U \to U'$ an intertwining map. This means $\nu'(h) \circ \varphi(h) = \varphi \circ \nu(h)$. We have for $f \in \mathrm{Ind}(U)$

$$\Phi(f)(g) = \varphi(f(g))$$

Exercise 3

φ is the intertwining map of $\varphi \mapsto \Phi : \mathrm{Ind}_H^G \to \mathrm{Ind}_H^G$. Check $\Phi = \mathrm{Ind}_H^G(\varphi)$.

Next time, we will see Ind_H^G is a right adjoint of the restriction functor.

10 February 2

Suppose we have a group G and $H < G$. We can first consider categories

$$\mathrm{Rep}(G)$$

and

$$\mathrm{Rep}(H)$$

We of course have the restriction functor

$$\mathrm{Rep}(G) \to \mathrm{Rep}(H)$$

$\mathrm{Res}_H^G(\pi)$ is a restriction of $\pi \in (\mathrm{Rep}(G))$. We also constructed an induction functor, which takes $\nu \in \mathrm{Rep}(H)$, and defines $\mathrm{Ind}_H^G(\nu) \in \mathrm{Rep}(G)$.

Theorem 9. We claim that Ind_H^G is a right adjoint of the functor Res_H^G. This means that if we look at $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$ for $\pi \in \mathrm{Rep}(G), \nu \in \mathrm{Rep}(H)$,

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu)) = \mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu)$$

via natural isomorphism.

Let's assume we want to construct an irreducible representation of G. If we have such a representation, any homomorphisms in the preceding Hom set must be injective if not 0. In other words, π is isomorphic to a subrepresentation in $\mathrm{Ind}_H^G(\nu)$. (Frobenius reciprocity is mentioned, the functorial form)

Exercise 4

If a representation (ν, V) is a direct sum of two representations $\nu_1 \oplus \nu_2$, then in $\mathrm{Ind}_H^G(\nu) = \mathrm{Ind}_H^G(\nu_1) \oplus \mathrm{Ind}_H^G(\nu_2)$. This implies that the functor Ind_H^G is exact. If G, H are finite, then we look at

$$0 \to \nu' \to \nu \to \nu'' \to 0$$

Because we are working with a semi-simple category, $\nu \cong \nu' \oplus \nu''$. We hence get

$$0 \to \mathrm{Ind}_H^G(\nu') \to \mathrm{Ind}_H^G(\nu') \oplus \mathrm{Ind}_H^G(\nu'') \to \mathrm{Ind}_H^G(\nu'') \to 0$$

First, let us consider $\mathrm{Ind}_H^G(\nu)$, in particular its space $\mathrm{Ind}_H^G(U) = \{f : G \to U | f(hg) = \nu(h)f(g), h \in H, g \in G\}$. So we can take a function $f \in \mathrm{Ind}_H^G(U)$, and take it to U by evaluating it at $1 \in G$. So $e(f) = f(1)$. What can we say about the map? Taking the induced representation ρ on $\mathrm{Ind}_H^G(U)$,

$$e(\rho(h)f) = (\rho(h)f)(1) = f(1 \cdot h) = f(h) = \nu(h)f(1)$$

$$= \nu(h)e(f)$$

We see that $e \circ \rho = \nu(h) \circ e$. Hence $e \in \mathrm{Hom}_H(\mathrm{Res}_H^G(\mathrm{Ind}_H^G(\nu)), \nu)$. Take $\psi \in \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$, $\mathrm{Res}_H^G(\psi) \in \mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \mathrm{Ind}_H^G(\nu))$. $e \circ \psi \in \mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu)$. We would like to show that this map A is an isomorphism. Let's define this map $A(\psi) = e \circ \psi$. We construct the inverse map to A. Suppose we have an element $\mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu)$. We would like to create an element $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$. We have $\varphi : V \to U$, $\varphi \in \mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu)$.. Take $v \in V$. Let's consider $F_v : G \to U$ given by

$$F_v(g) = \varphi(\pi(g)v).$$

Let's check $F_v(hg) = \varphi(\pi(h)\pi(g)v) = \nu(h)\varphi(\pi(g)v)$, hence F_v is in the induced representation space $\mathrm{Ind}_H^G(U)$. Hence this gives us

$$v \mapsto F_v$$

a map $V \to \mathrm{Ind}(U)$. In fact, this is a linear map. Now what happens with the G action?

$$\pi(g)v \mapsto F_{\pi(g)v}$$

but

$$F_{\pi(g)v}(g') = \varphi(\pi(g')\pi(g)v) = \varphi(\pi(g'g)v) = F_v(g'g) = \rho(g')F_v(g).$$

Hence the bigger map

$$\varphi \mapsto (v \mapsto F_v)$$

$Hom_H(\mathrm{Res}_H^G(\pi), \nu) \to \mathrm{Ind}_H^G(U)$ is in

$$\mathrm{Hom}_G\Big(\mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu), \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu)\Big)$$

We have $\Phi \in \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$. This gives $B(\varphi) = \Phi$. Claim: A and B are inverses to each other.

$$(A \circ B)\varphi(v) = A(\Phi)(v) = e \circ \Phi(v) = e(F_v) = F_v(1) = \varphi(v)$$

Now

$$(B \circ A)(\psi)(v)(g) = (B\,(A(\psi))\,(v))(g) = A(\psi)(\pi(g)v) = e \circ \psi(\pi(g)v) = (\psi(\pi(g)v))\,(1) = (\rho(g)\psi(v))(1) = \psi(v)(g)$$

so $B \circ A(\psi) = \psi$. This shows the natural isomorphism. This is the proof of Frobenius reciprocity. This is a very on-the-nose proof, but it will show up constantly throughout the course.

11 February 4

Fact: $\text{Ind}_K^G(\nu) \cong \text{Ind}_H^G\left(\text{Ind}_K^H(\nu)\right)$. Frobenius Reciprocity:

$$\text{Hom}_G\left(\pi, \text{Ind}_K^G(\nu)\right) = \text{Hom}_K(\text{Res}_K^G(\pi), \nu)$$

$$= \text{Hom}_K(\text{Res}_K^H \circ \text{Res}_H^G(\pi), \nu)$$

$$= \text{Hom}_H\left(\text{Res}_H^G(\pi), textInd_K^H(\nu)\right)$$

$$= \text{Hom}_G\left(\pi, \text{Ind}_H^G \circ \text{Ind}_K^H(\nu)\right)$$

Thus this gives us a natural isomorphism

$$\text{Ind}_K^G(\nu) \cong \text{Ind}_H^G\left(\text{Ind}_K^H(\nu)\right)$$

The general theorem: Two adjoint functors are isomorphic. Yoneda's lemma

Induction on stages is useful for calculations. The dimension of induced representations? Assume G is finite. By induction from identity to G of

$$\text{Ind}_{\{1\}}^G(\mathbb{C}) = R(G) = \mathbb{C}[G]$$

If G is finite,

$$\text{Ind}_H^G(U) = \mathbb{C}[G, U] = \mathbb{C}[G] \otimes_{\mathbb{C}} U(\text{space of functions from } G \text{ to } U\ )$$

We have

$$\dim\, \text{Ind}_H^G(U) \leq [G] \cdot \dim U$$

Remember that

$$F(hg) = \nu(h)F(g)$$

These functions are determined on each (right) coset Hg by its value at g. Given group G, we have cosets Hence $\text{Ind}(U)$ is the direct sum of functions supported on one coset only. Pick coset C to be an H right-coset. Pick

$$Hg_0 = C$$

Pick $(e_i; i \in I)$ a basis of U.

$$e_{c,i}(g) = \begin{cases} 0 & g \in C \\ \nu(gg_C^{-1})e_i & g \in C = Hg_C \end{cases}$$

We see that $\left(e_{c,i} : c \in \frac{G}{H}, i \in I\right)$.

$$\dim\text{Ind}(U) = [\frac{G}{H}] \cdot \dim(U)$$

This gives us the precise formula.

Example. We finish the example of $G = S_3$. $\hat{G}$ consists of $\text{triv}, \text{sgn}, \sigma$. σ is 2-dimensional, and we know its character. We have $A_3 \subset S_3$, the set of even permutations. We have $a = (2\ 3\ 1)$. Since A_3 is cyclic group, it has irreducible representations that are one dimensional. That is, $\hat{A}_3 = \{1, \varphi, \varphi^2\}$. $\varphi \in \hat{A}_3$ which is nontrivial, $\varphi(a) = i \cdot \frac{2\pi}{3}$. The other nontrivial element is φ^2. Let's calculate $\text{Res}_{A_3}^{S_3}(\sigma)$. We know

$$\text{ch}\left(\text{Res}_{A_3}^{S_3}(\sigma)\right) = \text{ch}(\sigma)|_{A_3}$$

$$= \begin{cases} 2 & \text{at } 1 \\ 0 & \text{on odd perms} \\ -1 & \text{at } A_3 \setminus \{1\} \end{cases}$$

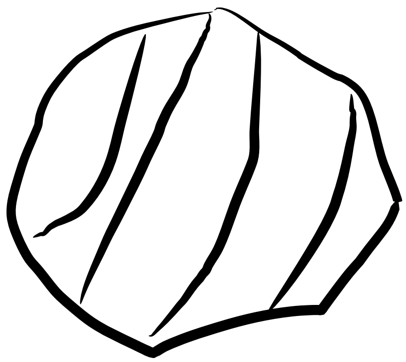

Figure 1: Split G into cosets

Now this character must be a sum of irreducible characters (since A_3 is abelian, its irreducibles are dimension 1). We see

$$\mathrm{Res}^{S_3}_{A_3} = \varphi \oplus \varphi^{-1}$$

We can calculate $\dim \mathrm{Hom}_{A_3}(\mathrm{Res}^{S_3}_{A_3}(\sigma), \varphi) = 1$. By Frobenius reciprocity, this is

$$\dim \mathrm{Hom}_{S_3}(\sigma, \mathrm{Ind}^{S_3}_{A_3}(\varphi)) = 1$$

The conclusion is $\sigma \cong \mathrm{Ind}^{S_3}_{A_3}(\varphi)$. This gives us the last remaining representation of S_3. But you may ask about φ^{-1}. There is no distinction. By symmetry, $\sigma \cong \mathrm{Ind}^{S_3}_{A_3}(\varphi^{-1})$.

12 February 7

Let's assume G is finite. Then if we take $H < G$, and let (ν, U) be an irreducible representation of H. Then it's finite dimensional, and from what we proved last time, $\mathrm{Ind}^G_H(\nu)$ is finite dimensional. By Maschke's theorem,

$$\mathrm{Ind}^G_H(\nu) = \bigoplus_{i=1}^m \pi_i$$

where π_i's are irreducible representations of G. Therefore if (π, V) is an irreducible representation of G. As we have discussed,

$$\dim(\mathrm{Hom}_G(\pi, \mathrm{Ind}^G_H(\nu))) = \dim(\mathrm{Hom}_G(\pi, \bigoplus_{i=1}^m \pi_i)$$

$$= \sum_{i=1}^n \dim\left(\mathrm{Hom}_G(\pi, \pi_i)\right)$$

We know from a previous lecture that each term is zero if $\pi \not\cong \pi_i$ and 1 if $\pi \cong \pi_i$.

$$= \text{multiplicity of } \pi \text{ in } \mathrm{Ind}^G_H(\nu)$$

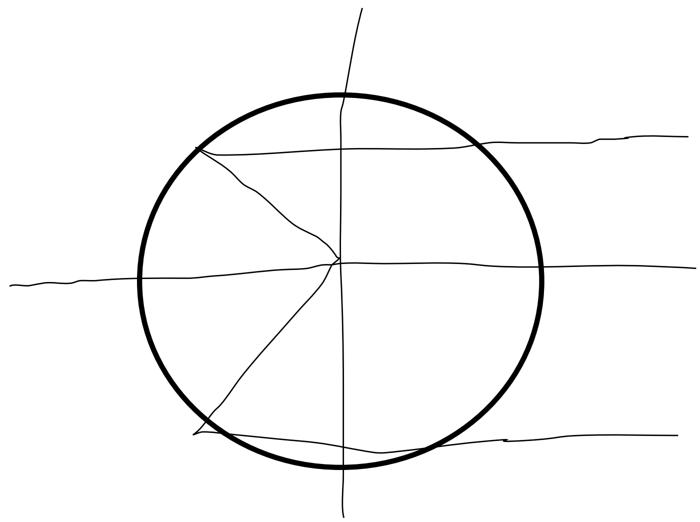

Figure 2: testcirc

By Frobenius reciprocity,

$$\dim(\mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu)) = \text{mult. of } \nu \text{ in } \mathrm{Res}_H^G(\pi)$$

and $\mathrm{Res}_H^G(\pi) \cong \bigoplus_{j=1}^k \nu_j$ where ν_j are irreducible representations of H. Then the multiplicity of π in Ind_H^G =the multiplicity of ν in $\mathrm{Res}_H^G(\pi)$. We would like to calculate the character of induced representations. $\mathrm{Ind}_H^G(\nu)$. For a character to exist, its representation must be finite dimensional. We must assume H, G are finite, and that ν is finite dimensional. As noted before, we have found a basis for $\mathrm{Ind}_H^G(\nu)$. We look at the right coset space $H \setminus G \ni C$. And the space

$$\mathrm{Ind}_H^G(U) = \bigoplus_{C \in H\setminus G} \Big\{\text{space of functions in } \mathrm{Ind}_H^G(U) \text{ supported on } C\Big\}$$

$f(hg) = \nu(h)f(g)$. $\{e_i, 1 \leq i \leq n\}$ is a basis of U.

$$e_{c,i}(g) = \begin{cases} 0 & g \notin C \\ \nu(gg_c^{-1})e_i & \end{cases}$$

where g_c is a representative of C. We have

$$\{e_{c,i} | 1 \leq i \leq n, c \in H \setminus G\}$$

is a basis for $\mathrm{Ind}_H^G(U)$. In particular, $\mathrm{Ind}_H^G(\nu)$ is finite dimensional. Let's take $g \in G$. We are going to calculate how g acts on the basis. We have denoted by ρ the action of the induced representation.

$$\left(\rho(g)e_{c,i}\right)(g') = e_{c,i}(g'g)$$

$$g'g \notin C \Rightarrow \rho(g)e_{c,i} = 0$$

$g' \notin C_g^{-1} = D$, a coset. We conclude that $\rho(g)$ is a linear combination of such cosets. We have

$$\sum \rho(g)(D,j)(c,i)\rho_{D,j}(g')$$

$(\rho(g)e_{c,i})(g_D) = \nu(g_D g g_C^{-1})e_i$.

$$\sum_{j=1}^{n} \nu(g_D g g_C^{-1})_{ji} e_j$$

$$= \sum_{j=1}^{n} \nu(g_D g g_C^{-1})_{ji} e_{D,j}(g_D)$$

$C \cdot g^{-1} = D$ is a relation we are using. $D = C$ (there is a critical error nearby that you should check written notes for). $Hg_C \cdot g^{-1} = Hg_C$, which means $g_C g^{-1} g_C^{-1} \in H$.

$$\mathrm{ch}(\mathrm{Ind}_H^G(\nu))(g) = \sum_{C \cdot g = c} \mathrm{ch}(\nu)(g_C g g_C^{-1})$$

So the above equation is the first observation. Now $g_C g g_C^{-1} \in H$, so we have

$$\sum_{g_C g g_C^{-1} \in H} \mathrm{ch}(\nu)(g_C g g_C^{-1}).$$

Since $\mathrm{ch}(\nu)$ is constant on conjugacy classes, we can conjugate the elemnt in its parameters by some $h \in H$, which we can then tak the average with:

$$\frac{1}{[H]} \sum_{h \in H} \sum_{g_C g g_C^{-1} \in H} \mathrm{ch}(\nu)\left(h g_C g g_C^{-1} h^{-1}\right)$$

Now we have

$$\frac{1}{[H]} \sum_{g' \in G \ni g' g g'^{-1} \in H} \mathrm{ch}(\nu)(g' g g'^{-1})$$

(the difference between this sum and the last sum is that the former only sums over coset representatives g_C, one for each coset C, while here we are summing over all $g' \in G$). We have $\mathrm{ch}(\nu) : H \to \mathbb{C}$. We can extend $\mathrm{ch}(\nu)$ to a function χ_ν on G. It is

$$\chi_\nu(g') = 0, g' \notin H$$

$$\chi_\nu(g') = \mathrm{ch}(\nu)(g'), g' \in H$$

$\mathrm{ch}(\mathrm{Ind}_H^G(\nu))(g) = \frac{1}{[H]} \sum_{g' \in G} \chi_\nu(g' g g'^{-1})$ (this is the formula that will imply Frobenius reciprocity directly, as we will discuss on Wednesday). First observe that the character of Ind_H^G vanishes on all conjugacy classes that don't intersect H.

Example. Note the regular representation $R = \mathrm{Ind}_{\{1\}}^G(\mathrm{triv})$. The character of the trivial representation is 1 at the origin and zero everywhere else.

$$\mathrm{ch}(R(g)) = \begin{cases} [G] \cdot 1 & \text{at } 1 \\ 0 & \text{at } g \neq 1 \end{cases}$$

13 February 9

This marks the last class in which we review representation theory of finite groups. Let G be a finite group and H be a subgroup. We have (ν, U) a finite dimensional representation, we can then talk about $\mathrm{ind}_H^G(U)$. We can then obtain

$$\mathrm{ch}\left(\mathrm{Ind}_H^G(U)\right)$$

We have the visualization We take the average

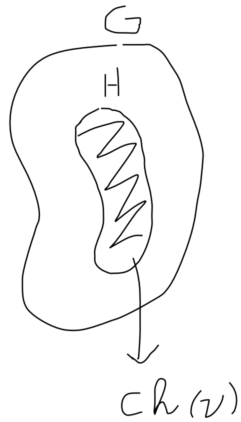

Figure 3: We can extend $\mathrm{ch}(\nu)$ to $\chi_\nu : G \to \mathbb{C}$ with $\chi_\nu|_H = \mathrm{ch}(\nu)$.

$$\mathrm{ch}\left(\mathrm{Ind}_H^G(\nu)\right)(g) = \frac{1}{[G]} \sum_{g' \in G} \chi_\nu\left(g' g g'^{-1}\right)$$

We have $\mathrm{ch}(\mathrm{Ind}_H^G(\nu))(1) = \dim\left(\mathrm{Ind}_H^G(\nu)\right) = [H \backslash G] - \dim(U)$. The more interesting case arises when $H \triangleleft G$. In this case, the support of $\mathrm{ch}\left(\mathrm{Ind}_H^G(\nu)\right)$ is in H.

$$\mathrm{ch}\left(\mathrm{Ind}_H^G(\nu)\right)(g) = \begin{cases} 0 & g \notin H \\ \frac{1}{[H]} \sum_{g' \in G} \mathrm{ch}(\nu)(g' g g'^{-1}) & g \in H \end{cases}$$

Exercise 5

If you look at the representation $h \mapsto \nu(h)$, $g \in G$, and then if you look at

$$h \mapsto \nu(ghg^{-1})$$

since $ghg^{-1} \in H$, we can see that this is a representation of the group H. Call it $\nu^g(h) = \nu(ghg^{-1})$, the representation **twisted by** g. If ν is irreducible, ν^g stay irreducible. You can have the action of G on the dual space of H, $\hat{H}$. GIven $g \in G$, g acts on $\hat{H}$ via

$$\nu \xrightarrow{g} \nu^g \in \hat{H}$$

Example. $G = S_3$. Its dual $\hat{G}$ has three irreducibles by counting conjugacy classes.

$$\hat{G} = \{\mathrm{trivial}, \mathrm{sgn}, \sigma\}$$

and σ is 2-dimensional. There are three conjugacy classes, and $\mathrm{triv}, \mathrm{sgn}$ are both 1-dimensional. σ is 2-dimensional.

$$1^2 + 1^2 + 2^2 = 6 = 3! = |G|$$

Burnside? theorem. We would like to see that σ is an induced representation. Let's examine $A_3 < S_3$. In this case

$$\left[\frac{S_3}{A_3}\right] = 2$$

$$[A_3] = 3$$

A_3 is a cyclic subgroup since it has order 3. $\hat{A}_3$ consists of characters which are determined by the value on a chosen generator a. Viewing the group as the third roots of unity, the representations can be viewed as rotations by $e^{2\pi i/3}$

$$\hat{A}_3 = \{\text{triv}, \nu, \nu^2\}$$

We now look at

$$\text{Ind}_{A_3}^{S_3}(\nu)$$

We have $\hat{A}_3 = \{\text{triv}, \nu, \nu^{-1}\}$. Given $g \in G$, the orbit of triv is triv by itself. Conjugating ν by an element of $A_3 \ni g$ (see the above exercise) we get back ν. Otherwise we get ν^{-1}. If an element of G commutes with an element of A_3, we could have trouble, because S_3 is not abelian. It would only make sense that given $g \notin A_3$ it just permutes ν and ν^{-1}. Using this, we can calculate the character of the induced. Note

$$\text{ch}\left(\text{Ind}_H^G(\nu)\right)(g) = \begin{cases} 0 & g \notin H \\ \frac{1}{[H]}\sum_{g' \in G} \text{ch}(\nu)(g' g g'^{-1}) & g \in H \end{cases}$$

By our above calculation. When $H = A_3$, $G = S_3$, the second formula above is

$$\frac{1}{3}\sum_{g' \in A_3} \text{ch}(\nu)(g) + \frac{1}{3}\sum_{g \notin A_3} \text{ch}(\nu^{-1})(g) = \text{ch}(\nu)(g) + \text{ch}(\nu^{-1})(g)$$

in particular $\text{ch}(\text{Ind}_H^G(\nu))(1) = 2$. And $\text{ch}(\nu^{-1})(g) = \overline{\nu(g)}$. So we have

$$2\text{Re}\left(\nu(a)\right) = -1$$

$\nu(a) = e^{2\pi i/3}$. If the character is irreducible, we would have

$$\text{ch}(\text{Ind}_H^G(\nu)) = \text{ch}(\pi_1) + \ldots + \text{ch}(\pi_k)$$

where the π_i 's are in $\hat{G}$. But this is also $\text{ch}(\sigma)$. Work with norms of the characters, but we will get $\text{Ind}_H^G(\nu) = \sigma$ is irreducible.

Exercise 6

Equivalence of characters implies equivalence of representations

We will prove the Frobenius reciprocity using this formula on Friday, and then try to generalize the results to compact groups.

14 February 11

Character version of Frobenius Reciprocity. If G is a finite group, and $H < G$, and if ν is a finite dimensional representation of H, then we can construct $\text{Ind}_H^G(\nu)$. This is also finite dimensional. It thus has character

$$\text{ch}\left(\text{Ind}_H^G(\nu)\right)(g) = \frac{1}{[H]}\sum_{g' \in G} \chi_\nu(g' g g'^{-1})$$

where χ_ν is an extension by zero of $\text{ch}(\nu)$ on H. We will take a representation π which is finite dimensional of G. We calculate

$$\left(\text{ch}(\text{Ind}_H^G(\nu))|\text{ch}(\pi)\right)$$

$$= \frac{1}{[G]} \sum_{g \in G} \left(\mathrm{ch}(\mathrm{Ind}_H^G(\nu))(g) \overline{\mathrm{ch}(\pi)(g)} \right)$$

$$= \frac{1}{[G][H]} \sum_{g,g' \in G} \chi_\nu(g' g g'^{-1}) \overline{\mathrm{ch}(\pi)(g)}$$

$$= \frac{1}{[H][G]} sum_{g,g' \in G} \chi_\nu(g) \overline{\mathrm{ch}(\pi)(g'^{-1} g g')}$$

Now π is constant on conjugacy classes so we have

$$= \frac{1}{[H][G]} \sum_{g,g' \in G} \chi_\nu(g) \overline{\mathrm{ch}(\pi)(g)}$$

$$= \frac{1}{[H]} \sum_{g \in G} \chi_\nu(g) \overline{\mathrm{ch}(\pi)(g)}$$

$$= \frac{1}{[H]} \sum_{h \in H} \mathrm{ch}(\nu)(h) \overline{\mathrm{ch}(\pi)(h)}$$

since χ_ν is an extension by 0 of ν.

$$= \left(\mathrm{ch}(\nu) | \mathrm{ch} \left(\mathrm{Res}_H^G(\pi) \right) \right)$$

where the inner product is taken over H. Note the previous inner product was taken over G. We mentioned that this was the dimension of $\mathrm{Hom}_H(\nu, \mathrm{Res}_H^G(\pi))$. The starting inner product was $\dim \left(\mathrm{Hom}_H(\mathrm{Ind}_H^G(\nu), \pi) \right)$.

We now look at generalizing results to compact groups. Let G be a topological group. This means that G is a group and a Hausdorff topological space with the extra condition that the group operation $G \times G \to G$ is continuous as well as the inversion operation $G \to G$. If we don't have the Hausdorff condition, take the closure of e. Then we can take

$$\frac{G}{\overline{\{e\}}}$$

which is Hausdorff. We can also assume G is locally compact. We need local compactness because of Riesz's theorem. We can take a σ-algebra generated by open sets. From there we can define measures. To each measure we can define a linear form $\int$. To restate the theorem:

Theorem 10. $C_0(G)$ defined to be the continuous real functions on G with compact support. We can define a linear form

$$f \mapsto \int_G f(g) d\mu(g)$$

The statement of the theorem says that this determines a measure μ.

If we have $C_0(G)$ positive functions $f(g) \geq 0$. We can examine $\varphi : C_0(G) \to \mathbb{R}$ which is a linear form and $\varphi(f) \geq 0$ for $f \geq 0$. If we consider a Lie group G, we can replace the condition that G is a topological space with manifold structure. The multiplication operation and inversion function are smooth maps. Differentiable manifolds are locally compact and Hausdorff so all is well. Locally compact groups generalize Lie groups.

If we have a topological group G, we can talk about $C_0(G)$ (real or complex valued functions with certain conditions) as above. Of course, if G is compact, then $C_0(G) = \mathcal{C}(G)$ all continuous functions $G \to \mathbb{R}$ or $\mathbb{C}$. If G is finite, the only topology we can put on G to make it Hausdorff is the discrete topology. The space we have is $\mathcal{C}(G) = \mathbb{C}[G]$. The main idea in our theory was to consider

$$\mu : f \mapsto \frac{1}{[G]} \sum_{g \in G} f(g)$$

We have $\mu : \mathbb{C}[G] \to \mathbb{C}$ a linear form. What are its properties? We have $\mu : \mathbb{C}[G] \to \mathbb{C}$ a linear form and μ is positive. If f is a positive function, then $f(g) \geq 0$ for all $g \in G$, which means $\mu(f) \geq 0$. We have

1. above, μ is **positive**.
2. $\mu(R(g)f) = \mu(f)$ for any $f \in \mathbb{C}[G], g \in G$. We have

$$\mu(R(g)f) = \frac{1}{[G]} \sum_{g' \in G} f(gg') = \frac{1}{[G]} \sum_{g' \in G} f(g')$$

   We call μ **right-invariant**.
3. $\mu(1) = 1$. We call μ **normalized**.

Now assume μ is a linear form with properties $2, 3$. We can define

$$\delta_g(g') = \begin{cases} 1 & g' = g \\ 0 & g' \neq g \end{cases}$$

If we examine definitions, $R(g)\delta_1 = \delta_{g^{-1}}$. We have $\mu(\delta_g) = \mu(\delta_1)$. We call this value $c = \mu(\delta_1)$. We have

$$1 = \mu(1) = \mu(\sum_{g \in G} \delta_g) = \sum_{g \in G} \mu(\delta_g) = \sum_{g \in G} \mu(\delta_1) = [G]c$$

So $c = \frac{1}{[G]}$. Now we have

$$f = \sum_{g \in G} \delta_g f(g)$$

So

$$\mu(f) = \frac{1}{[G]} \sum_{g \in G} f(g)$$

Hence μ is completely determined by properties $2, 3$. In the case of bigger groups, $2, 3$ also determine a unique measure, which is called the Haar measure. It is determined by Riesz's.

15 February 14

Last time, we considered the compact groups G. $\varphi(G) =$ space of all continuous functions $G \to \mathbb{C}$. We have that $\varphi(G)$ is a Banach space with norm

$$\|f\| = \max_{g \in G} |f(g)|$$

since f is from a compact set, it indeed has a maximum. We saw that $|f(g)|$ is complete as a metric space with metric $d(f, g) = \|f - g\|$. We show that there exists a unique linear form μ on $\varphi(G)$ such that

1. μ is positive. $f \geq 0$ implies $\mu(f) \geq 0$. As a separate remark, $\varphi(G)$ has a natural right representation on G defined by $R(g)f(h) = f(hg)$.
2. $\mu(R(g)f) = \mu(f)$ for any $f \in \varphi(G)$, $g \in G$. This property is called **right-invariance.**
3. μ is **normalized**. In other words $\mu(1) = 1$.

Last time, we proved that if G is finite with the discrete topology, then $\mathbb{C}[G]$ is the space $\varphi(G)$. We proved uniqueness in that case last class. On our space $\mathcal{C}(G)$ (it's actually $\mathcal{C}$ and not φ), we have a representation

$$(L(g)f)(h) = f(g^{-1}h)$$

$L(gg') = L(g)L(g')$.

Exercise 7

In the sense of continuous representations we will discuss later, R above is a continuous representation.

We calculate

$$(R(g)L(g')f)(h) = L(g')f(hg) = f(g'^{-1}hg)$$
$$(L(g')R(g)f)(h) = f\left(g'^{-1}hg\right)$$

In other words, $R(g)$ and $L(g')$ commute for any $g, g' \in R$. Right and left regular representations commute. For fixed $g \in G$, let's consider the linear form $\nu_g(f) = \mu(L(g)f)$. We have ν_g is positive and right invariant.

$$\nu_g(R(g')f) = \mu(L(g)R(g')f) = \mu(R(g')L(g)f) = \mu(L(g)f)$$

by right invariance of μ. We have $\nu_g(1) = \mu(1) = 1$, so $\nu_g = \mu$. But also note that by calculation μ is left invariant. μ is thus **bi-invariant.**

We will also see some other properties of μ. We first see an example of a Haar measure.

Example. Let us consider S_1 as a group. Or $\frac{\mathbb{R}}{2\pi\mathbb{Z}}$. For a continuous function f on S_1, we have

$$\mu(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})d\varphi$$

μ is positive, μ is linear, μ is normalized. We also haveright invariance:

$$\mu(R(e^{i\theta})f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi+i\theta})d\varphi$$

By change of variables

$$\mu(R(e^{i\theta}f)) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\varphi})$$

(do rest of work as an exercise). Note that $\varphi \mapsto e^{in\varphi}$ is an orthonormal set.

By Riesz's theorem, if such a function μ, which is a linear form $\mathcal{C}(G) \to \mathbb{C}$, then μ is represented by a measure which by abuse of notation we also denote by μ,

$$\mu(f) = \int_G f(g)d\mu(g)$$

1. μ is a positive measure on G.
2. If A is a measurable set in G, $\mu(Ag) = \mu(A)$
3. $\mu(G) = 1$. Consequently we also have $\mu(gA) = \mu(A)$.

Such a measure μ with the above properties is called a **Haar measure on G.**

Remark. Assume that G is a locally compact group. An example is $(\mathbb{R}, +)$. We can consider

$$\mathcal{C}_0(G)\text{- Continuous functions with compact support}$$

What about $\mu : \mathcal{C}_0(G) \to \mathbb{C}$. If the function satisfies

1. μ is positive
2. $\mu(R(g)f) = \mu(f)$ for all $g \in G$, $f \in \mathcal{C}_0(G)$
3. μ is nonzero

So the list of criteria is almost all of the properties for compact groups except property 3, since 1 is not compactly supported.

Theorem 11. Such a linear form exists, and it is unique up to a positive multiple.

16 February 16

We now construct Haar measures on compact groups. Consider compact group G and $\mathcal{C}_{\mathbb{R}}(G)$, the space of real valued continuous functions on G. We would like to construct a linear form

$$\mu : \mathcal{C}_{\mathbb{R}}(G) \to \mathbb{R}$$

satisfying

1. $\mu(f) \geq 0$ if $f \geq 0$
2. $\mu(R(g)f) = \mu(f)$ for any $g \in G$ and $f \in \mathcal{C}_{\mathbb{R}}(G)$.
3. $\mu(\mathrm{id}) = 1$. (so it is normalized)

We talked about $G = \mathbb{R}$, which isn't compact but nonetheless gives a measure that satisfies the first two properties. By Riesz's theorem, μ gives a measure. By Riesz's theorem there is a positive borel measure (abuse of notation again call it μ) such that

$$\mu(f) = \int_G f(g) d\mu(g)$$

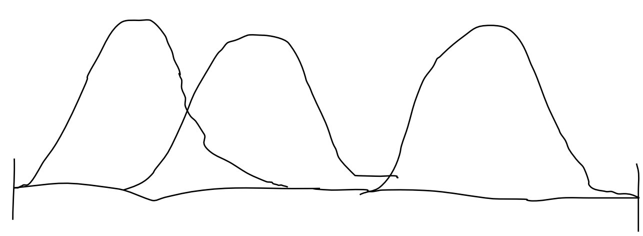

Figure 4: We will look at translates of the same function and somehow add them together

Proof. (Sketch): Take a function $f \in \mathcal{C}_{\mathbb{R}}(G)$. Construct a **right mean**. Take a finite sequence $\overline{a} = (a_1, \ldots, a_n)$, $a_i \in G$. We construct

$$\mu(f, \overline{a})(g) = \frac{1}{n} \sum_{i=1}^{n} f(g a_i)$$

$$\mu_f = \{\mu(f, \overline{a}) | \text{ all finite sequences } \overline{a} \text{ in } G\}$$

This set has compact closure. Remember that $\|f\| = \max_{g \in G} |f(g)|$ makes $\mathcal{C}_{\mathbb{R}}(G)$ a Banach space. Claim: $\overline{\mu_f}$ contains one constant function. Its value is $\mu(f)$. We will prove Arzela Ascoli in the setting of compact groups.

Lemma 2. Take $f \in \mathcal{C}_{\mathbb{R}}(G)$. $M(f) = \max_{g\in G} f(g)$. $m(f) = \min_{g\in G} f(g)$. We can define

$$V(f) = M(f) - m(f)$$

The claim is that $V(f) = 0$ if and only if f is constant.

The above lemma is directly provable from techniques of Foundations. The function V is called the **variation of** f.

Lemma 3. $V : \mathcal{C}_{\mathbb{R}}(G) \to \mathbb{R}$ is a continuous map.

Proof. Suppose we have $\varepsilon > 0$ and $f, f' \in \mathcal{C}_{\mathbb{R}}(G)$ and $\|f - f'\| < \varepsilon$. We have

$$|f(g) - f'(g)| < \varepsilon \forall g \in G$$

$$-\varepsilon < f(g) - f'(g) < \varepsilon$$

which implies $f'(g) - \varepsilon < f(g) < f'(g) + \varepsilon$. But this gives

$$m(f') - \varepsilon < f(g) < M(f') + \varepsilon$$

for all $g \in G$. We have

$$m(f') - \varepsilon < m(f) \leq M(f) < M(f') + \varepsilon$$

From this, we can calculate $V(f)$:

$$V(f) = M(f) - m(f) < M(f') + \varepsilon - m(f) < M(f') - m(f') + 2\varepsilon = V(f') + 2\varepsilon$$

$V(f) - V(f') < 2\varepsilon$. We also have similarly by symmetry that

$$V(f') - V(f) < 2\varepsilon$$

Hence

$$|V(f) - V(f')| < 2\varepsilon$$

proving continuity. □

We now study the means

$$\mu(f, \overline{a})(g) = \frac{1}{n}\sum_{i=1}^{n} f(ga_i)$$

What is its norm?

$$\|\mu(f, \overline{a})\| \leq \frac{1}{n}\sum_{i=1}^{n} \max_{g\in G} |f(g)| = \|f\|$$

by the triangle inequality. If a is fixed, $f \mapsto \mu(f, \overline{a})$ is a linear map. Now boundedness as above implies that the linear map is continuous. If we look at

$$M(\mu(f, \overline{a})) \leq M(f)$$

We also have

$$m(\mu(f, \overline{a})) \geq m(f)$$

We get $V(\mu(f, \overline{a})) = M(\mu(f, \overline{a})) - m(\mu(f, \overline{a})) \leq M(f) - m(f) = V(f)$. The conclusion?

Lemma 4. $V(\mu(f,\overline{a})) \leq V(f)$.

Remember now that we defined

$$\mu_f = \{\mu(f,\overline{a})|\overline{a}\}$$

We have

$$m(f) \leq \mu(f,\overline{a})(g) \leq M(f)$$

We will need the following lemma to finish our work:

Lemma 5. Let $f \in \mathcal{C}_{\mathbb{R}}(G)$ such that $V(f) > 0$. Then $\exists\ \overline{a}$ such that $V(\mu(f,\overline{a})) < V(f)$.

We write a box here, but the proof of the existence of μ will actually be done on Friday. □

17 February 23

Last time, for a continuous function $f \in \mathcal{C}_{\mathbb{R}}(G)$, we have defined $\mu(f,\overline{a})(g) = \frac{1}{n}\sum_{i=1}^{n} f(ga_i)$ and also

$$\nu(f,\overline{a})(g) = \frac{1}{n}\sum_{i=1}^{n} f(a_i g)$$

and we defined

$$\mu_f = \text{set of all right means}$$

$$N_f = \text{set of all left means}$$

There related to each other in terms of the opposite group. The last thing we proved was that $\overline{\mu_f}$ has a constant function. We also have $\overline{N_f}$ contains a constant function. We would like to show that if we take one constant function in one set, it corresponds to a constant function in the other. Take two sequences

$$\overline{a}, \overline{b}$$

so we can calculate

$$\nu(\mu(f,\overline{a}),\overline{b})(g) = \frac{1}{m}\sum_{i=1}^{m} \mu(f,\overline{a})(b_i g) = \frac{1}{m}\sum_{i=1}^{m}\frac{1}{n}\sum_{j=1}^{n} f(b_i g a_j)$$

$$= \mu(\nu(f,\overline{b}),\overline{a})(g)$$

Let's say $\overline{\mu_f}$ contains constant function φ and $\overline{\nu_f}$ contains constant function ψ. Taking $\varepsilon > 0$, choose $\overline{a}, \overline{b}$ where,

$$\|\mu(f,\overline{a}) - \varphi\| < \varepsilon/2$$

$$\|\nu(f,\overline{b}) - \psi\| < \varepsilon/2$$

First, let's consider

$$\|\nu(\mu(f,\overline{a}),\overline{b}) - \psi\|$$

$$= \|\nu(\mu(f,\overline{a}),\overline{b}) - \nu(\varphi,\overline{b})\| = \|\nu(\mu(f,\overline{a}) - \varphi,\overline{b})\|$$

$$\leq \|\mu(f,\overline{a}) - \varphi\| < \varepsilon/2$$

We also get by a similar argument,

$$\|\mu(\nu(f,\overline{b}),\overline{a}) - \psi\| = \|\mu(\nu(f,\overline{b})\overline{a}) - \mu(\psi,\overline{a})\| = \|\mu(\nu(f,\overline{b}) - \psi,\overline{a})\| < \varepsilon/2$$

We estimate $\|\varphi - \psi\|$, whic must be

$$\|\varphi - \nu(\mu(f,\overline{a}),\overline{b}) + \mu(\nu(f,\overline{b}),\overline{a}) - \psi\|$$

$$< \varepsilon$$

by the triangle inequality and our two inequalities above. If we show that μ_f contains two constant functions φ, φ', $\varphi = \psi = \varphi'$. So uniqueness of the constant function follows.

If $f \in \mathcal{C}_{\mathbb{R}}$, then we proved that $\overline{\mu_f}, \overline{N_f}$ contain a unique constant function φ. We call the value of φ the **mean value of** f, $\mu(f)$. We would like to prove the following: $\mu : \mathcal{C}_{\mathbb{R}}(G) \to \mathbb{R}$ is a positive linear form on $\mathcal{C}_{\mathbb{R}}(G)$. Once we prove this, then we will have the existence and uniqueness of a Haar measure by Riesz's theorem.

1. If we take $\alpha \in \mathbb{R}$, αf, then

$$\mu(\alpha \cdot f, \overline{a})(g) = \frac{1}{n} \sum_{i=1}^{n} \alpha f(ga_i) = \alpha \cdot \mu(f, \overline{a})(g)$$

   So the observation is that $\mu(\alpha \cdot f, \overline{a}) = \alpha \cdot \mu(f, \overline{a})$. Hence we have $\mu_{\alpha \cdot f} = \alpha \cdot \mu_f$ and $\overline{\mu_{\alpha \cdot f}} = \alpha \cdot \overline{\mu_f}$. Hence $\mu(f) \cdot \alpha = \mu(\alpha \cdot f)$. This tells us that μ is homogeneous.

2. We have positivity. If $f(g) \geq 0$ for all $g \in G$. Then

$$\mu(f, \overline{a})(g) \geq 0$$

   since all translates are non-negative. This implies that all functions in μ_f are positive, and so all functions in $\overline{\mu_f}$ are positive, so $\mu(f) \geq 0$ (note the deception of the word positive).

3. Taking $f, f' \in \mathcal{C}_{\mathbb{R}}(G)$. We have

$$\mu(f + f') = \mu(f) + \mu(f')$$

   (obtain a sequence for $\mu(f)$ and $\mu(f')$ each, and take the sum of the sequences). We apply a lemma:

Lemma 6. Let $f \in \mathcal{C}_{\mathbb{R}}(G)$. Call $\overline{a}$ a finite sequence in G. We have

$$\mu(\mu(f, \overline{a})) = \mu(f)$$

Proof. Let φ be the constant function with value $\mu(f)$. For $\varepsilon > 0$, there exists $\overline{b}$ such that $\|\nu(f, \overline{b}) - \varphi\| < \varepsilon$. Since $\nu(\varphi, \overline{b}) = \varphi$,

$$\|\nu(f - \varphi, \overline{b})\| < \varepsilon$$

We also have

$$\|\mu(\nu(f - \varphi, \overline{b}), \overline{a})\| \leq \|\nu(f - \varphi, \overline{b})\| < \varepsilon$$

We have

$$\|\nu(\mu(f - \varphi, \overline{a}), \overline{b})\| < \varepsilon$$

$$\|\nu(\mu(f, \overline{a}), \overline{b}) - \varphi\| < \varepsilon$$

For any $\overline{a}$, $\overline{N_{\mu(f,\overline{a})}}$ contains φ. So $\mu(\mu(f, \overline{a})) = \mu(f)$, completing the proof. □

Back to additivity, take $\alpha = \mu(f), \beta = \mu(f')$. Take the constant functions $\varphi(g) = \mu(f)$ and $\varphi'(g) = \mu(f')$. We have

$$\|\mu(f, \overline{a}) - \varphi\| < \varepsilon/2$$

Take any $\overline{b}$, and look at

$$\|\mu(\mu(f, \overline{a}) - \varphi, \overline{b})\| \leq \|\mu(f, \overline{a}) - \varphi\| < \varepsilon/2$$

The former expression is

$$\|\mu(\mu(f, \overline{a}), \overline{b}) - \varphi\| < \varepsilon/2$$

What we calculated last week was that this is

$$\|\mu(f, \overline{a} \cdot \overline{b}) - \varphi\|$$

This tells us

$$\|\mu(f, \overline{a} \cdot \overline{b}) - \varphi\| < \varepsilon/2$$

This holds for any $\overline{b}$. We also have

$$\mu(f') = \mu(\mu(f', \overline{a}))$$

by the lemma. We can find $\overline{b}$ such that $\|\mu(\mu(f', \overline{a})\overline{b}) - \varphi'\| < \varepsilon/2$. This is equal to

$$\mu(f', \overline{a} \cdot \overline{b}).$$

We have

$$\mu(f + f', \overline{a} \cdot \overline{b}) = \mu(f, \overline{a} \cdot \overline{b}) + \mu(f, \overline{a} \cdot \overline{b})$$

The former differs from φ by less than $\varepsilon/2$ and likewise for the latter from φ'. Hence the entire expression differs from $\varphi + \varphi'$ by less than ε. This tells us $\varphi + \varphi'$, a constant function, is in $\overline{\mu_{f+f'}}$. Hence we have additivity.

We will show on Friday that this is right invariant and unique.