Commutative Algebra
Notes on MATH 7830

Daniel Koizumi

March 15, 2023

Contents

1 January 18

Let R be a commutative Noetherian ring, and let M be an R-module. What does it mean for an element $r \in R$ to be a **zero-divisor**? It simply means that for some $m \neq 0$, $r \cdot m = 0$.

$$zdr_R(M) = \{r \in R | r \text{ is a zero divisor on } M\} = \bigcup_{\mathfrak{p} \in \operatorname{Ann}_R M} \mathfrak{p}.$$

We can say $r \in R$ is a non-zero divisor if it is not a zero divisor (abbrev. nzd). Fix a sequence $\mathbf{x} = x_1, \ldots, x_n \in R$.

Definition 1. We say that $\mathbf{x}$ is a **weakly M-regular** sequence on M if x_{i+1} is not a zero divisor on $\frac{M}{(x_1,\ldots,x_i)M}$ for all applicable i. It becomes a **regular sequence** if in addition $\frac{M}{\mathbf{x}M} \neq 0$.

Example. If $R = \Bbbk[x_1, \ldots, x_n]$, and note $\mathbf{x} = x_1, \ldots, x_n$ is a regular sequence on R.

We now introduce Koszul complexes. Given $r \in R$, we can write $K(r, R)$ to be the complex

$$0 \to R \to R \to 0.$$

there $R \to R$ is the homothetic map multiplication by r. The left first copy of R is labeled degree 1. Here, taking the homology functor of the sequence provides 0 on the left R if and only if r is a nzd. We have

$$K(\mathbf{x}, R) = \bigotimes_{i=1}^{n} K(x_i, R).$$

We will get

$$0 \to R \to R^n \to R^{\binom{n}{2}} \to \ldots \to R^{\binom{n}{2}} \to R^n \to R \to 0.$$

(exercise calculate the first and last maps). Given $M \in \mathcal{C}(R)$,

$$K(\mathbf{x}, M) = K(\mathbf{x}, R) \otimes_R M.$$

If M is just an R-module, it is merely replacing copies of R with copies of M. We denote $H_i(\mathbf{x}, M) = H_i(K(\mathbf{x}, M))$. Note

$$H_0(\mathbf{x}, M) = \frac{M}{\mathbf{x}M}.$$

$$H_1(\mathbf{x}, M) = \{m \in M | x_i \cdot M = 0 \forall i\} = (0 :_M (\mathbf{x})).$$

Remark: Note

$$K(\mathbf{x}, M) = K(x_1, R) \otimes K(x_2, R) \otimes \ldots \otimes K(x_n R) \otimes_R M$$

$$K(x_1, R) \otimes K(\mathbf{x}_{\geq 2}, M).$$

So we have

$$K(\mathbf{x}, M) = K(x_1, K(\mathbf{x}_{\geq 2}, M)).$$

In many proofs in this course, being able to decompose the Koszul complex in this way will allow us to do induction.

Remark:
We have $X, Y \in \mathcal{C}(R)$, we get the isomorphism

$$X \otimes_R Y \to Y \otimes_R X.$$

via $x \otimes_R y \mapsto (-1)^{(x)(y)} y \otimes_R x$ For any $\sigma \in S_n$,

$$K(x_1, \ldots, x_n) \cong K(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, R).$$

Also, we have a second perspective on Koszul complexes: that they are the iterated mapping cones. Given a morphism of complexes

$$f : X \to Y.$$

recall the **cone** is defined

$$\operatorname{cone}(f) = \left(Y \oplus \Sigma X, \begin{pmatrix} \partial^Y & f \\ 0 & \partial^X \end{pmatrix} \right).$$

We get that

$$0 \to Y \to \operatorname{cone}(f) \to \Sigma X \to 0.$$

$y \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}$, $\begin{pmatrix} y \\ \Sigma x \end{pmatrix} \mapsto \Sigma x$. The long exact sequence in homology yields

$$\ldots \to H_i(X) \to H_i(Y) \to H_i(\operatorname{cone}(f)) \to H_i(\Sigma X) \cong H_{i-1}(X) \to \ldots.$$

Where the connecting map $H_i(X) \to H_i(Y)$ is just $H_i(f)$.

Now consider $x \in R$, and the homothetic map $f : R \to R$.

Example. $\operatorname{cone}(f) = \left(R \oplus \Sigma R, \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = K(x, R)$. Ditto for the homothetic map on modules.

$$\operatorname{cone}(M \to M) = K(x, M).$$

Thus, $K(\mathbf{x}, M) = K(x_1, K(\mathbf{x}_{\geq 2}, M)$ is $\operatorname{cone}(K(\mathbf{x}_{\geq 2}, M) \to K(\mathbf{x}_{\geq 2}, M))$. This gives

$$H_i(\mathbf{x}_{\geq 2}, M) \to H_i(\mathbf{x}_{\geq 2}, M) \to H_i(\mathbf{x}, M) \to H_{i-1}(\mathbf{x}_{\geq 2}, M) \to \dots$$

where the connecting morphism is multiplication by x_1 up to sign. By looking at the images/cokernels/kernels of one segment in this sequence, we get induced SES

$$0 \to H_i(\mathbf{x}_{\geq 2}, M)/x_1 H_i(\mathbf{x}_{\geq 2}, M) \to H_i(\mathbf{x}, M) \to (0 :_{H_{i-1}(\mathbf{x}_{\geq 2}, M)} x_1) \to 0.$$

If M is an R-module, $\mathbf{x} = x_1, \dots, x_n \subset R$,

$$K(\mathbf{x}, M) \twoheadrightarrow H_0(\mathbf{x}, M) = \frac{M}{\mathbf{x}M}.$$

So,

$$K(\mathbf{x}, M) \to \frac{M}{\mathbf{x}M}.$$

is a weak equivalence if and only if

$$H_i(\mathbf{x}, M) = 0 \forall i \geq 1.$$

Lemma 1. When $\mathbf{x}$ is a weakly M-regular,

$$K(\mathbf{x}, M) \xrightarrow{\sim}\mathrel{\mkern-14mu}\rightarrow \frac{M}{\mathbf{x}M}.$$

Proof. When $n = 1$,

$$0 \to M \to M \to 0$$

has zero homology at degree 1 if and only if x is a nonzero divisor on M.

Now say when $n \geq 2$, we know that $K(\mathbf{x}, M) = K(x_n, K(\mathbf{x}_{\leq n-1}, M))$. By our induction hypothesis,

$$K(\mathbf{x}_{\leq n-1}, M) \twoheadrightarrow \frac{M}{(\mathbf{x}_{\leq n-1})M}.$$

We have

$$K(x, R) = (0 \to R \to R \to 0).$$

is semi-free.

$$K(\mathbf{x}, M) = K(x_n, R) \otimes_R K(\mathbf{x}_{\leq n-1}, M) \to K(x_n, \frac{M}{\mathbf{x}_{\leq n-1} M}).$$

Exercise 1

Prove this using the Koszul homology long exact sequence.

$\square$

Definition 2. $\mathbf{x}$ is **Koszi-regular** on M if

$$K(\mathbf{x}, M) \twoheadrightarrow^{\sim} \frac{M}{\mathbf{x}M}.$$

. Note that $x_1, \ldots, x_n$ is Koszi-regular on M if and only if any permutation

$$x_{\sigma(1)}, \ldots, x_{\sigma(n)}$$

is Koszi-regular on M for any $\sigma \in S_n$.

Exercise 2

(Weakly) regular sequences are senitive to permutations.

Theorem 1. Say $\mathbf{x} \subset J(R)$ and $M \neq 0$ is finitely generated as an R-module. Then the following are equivalent:

1. $\mathbf{x}$ is regular ($\equiv$ weakly regular).
2. $H_i(\mathbf{x}, M) = 0$ for all $i \geq 1$.
3. $H_1(\mathbf{x}, M) = 0$.

Our main application is when R is a local ring and $\mathbf{x} \subset \mathfrak{m}_R$. We use Nakayama's lemma: $J(R) \neq M$, so regularity is equivalent to weak regularity.

Proof. We know $1 \Rightarrow 2 \Rightarrow 3$. It remains to show $3 \Rightarrow 1$. We want to examine $H_*(x_1, \ldots, x_{n-1}, x_n, M)$. The module

$$K(\mathbf{x}, M) = K(x_n, K(\mathbf{x}_{\leq n-1}, M))$$

provides long exact sequence containing

$$0 \to H_i(\mathbf{x}_{\leq n-1}, M)/(x_n)H_i(\mathbf{x}_{\leq n-1}, M) \to H_i(\mathbf{x}, M) \to \left(0 :_{H_{i-1}(\mathbf{x}_{\leq n-1}, M)} x_n\right).$$

We have that

$$H_1(\mathbf{x}, M) = 0 \Rightarrow H_1(\mathbf{x}_{\leq n-1}, M) = (x_n)H_1(\mathbf{x}_{\leq n-1}, M).$$

so apply Nakayama's. We are doing the proof of equivalence by induction on n (it is already proven for $n = 1$), so we have

$$x_1, \ldots, x_{n-1}.$$

is M-regular. This implies further that

$$H_i(\mathbf{x}_{\leq n-1}, M) = 0.$$

for all $i \geq 1$. Moreover, applying this to our exact sequence above, $(0 : x_n) = 0$, so $H_0(\mathbf{x}_{\leq n}, M) = \ker\left(\frac{M}{\mathbf{x}_{\leq n-1}M} \to \frac{M}{\mathbf{x}_{\leq n-1}M}\right)$. ◻

Corollary 1. $\mathbf{x} \subset J(R)$, M finitely generated. The property that $\mathbf{x}$ is M-regular does not depend on the ordering of $\mathbf{x}$.

Lemma 2. Suppose we have a sequence $x_1, \ldots, x_n \subset R$ (now we drop the assumption regarding the Jacobson radical). Let M be an R-module. The following are equivalent:

1. $\mathbf{x}$ is Koszi-regular on M.
2. $\{x_1^{a_1}, \ldots, x_n^{a_n}\}$ is Koszi-regular on M for any choice $a_i \geq 1$.
3. $\mathbf{x}^{\mathbf{a}}$ is Koszi-regular on M for some $\mathbf{a} \geq (1, \ldots, 1)$.

Proof. It suffices to prove $x_1, \ldots, x_n$ is Koszi-regular on M if and only if $x_1^a, \ldots, x_n$ for some $a \geq 1$. Recall that Koszi-regularity means

$$K(x_1^a, x_2, \ldots, x_n, M) \to^{\sim} K(x_1^a, \frac{M}{(x_{\geq 2})M}).$$

Replacing M with $\frac{M}{(\mathbf{x}_{\geq 2})M}$, we are reduced to proving x is weakly M-regular if and only if x^a is weakly M-regular for some $a \geq 1$. x is not a zero divisor on M if and only if x^a is not a zero divisor on M for some or all $a \geq 1$. $\square$

Exercise 3

(this is also a theorem, called the rigidity of Koszul homology). If we take $\mathbf{x} \subset J(R)$ and M a finitely generated R-module, then $H_i(\mathbf{x}, M) = 0$ for some $i \geq 0$ implies that $H_j(\mathbf{x}, M) = 0$ for all $j \geq i$.

2 January 23

Let R be a commutative and Noetherian ring, and $M, N \in \mathcal{C}(R)$. Note

$$\mathrm{RHom}_R(M, N) = \mathrm{Hom}_R(pM, N).$$

where $pM \xrightarrow{\sim} M$ is a K-projective resolution. Recall

$$\mathrm{Ext}_R^*(M, N) = H^*(\mathrm{RHom}_R(M, N)).$$

For any $M, N, P \in \mathcal{C}(R)$, there exists

$$\theta : \mathrm{RHom}_R(M, N) \otimes_R^L P \to \mathrm{RHom}_R(M, N \otimes_R^L P).$$

Lemma 3. This is a weak equivalence when P is **perfect**. In particular

$$P \xrightarrow{\sim} (0 \to P_b \to \ldots \to P_c \to 0)\,.$$

Where P_i is finitely generated as a projective R-module. We get a morphism of complexes

$$\mathrm{Hom}_R(pM, N) \otimes_R p(P) \to \mathrm{Hom}_R(pM, N \otimes_R p(P)).$$

Defined by

$$f \otimes x \mapsto \left(m \mapsto (-1)^{|x||m|} f(m) \otimes x\right).$$

In the category of modules over R, if we look at

$$\mathrm{Hom}_R(M, N) \otimes_R P \to \mathrm{Hom}_R(M, N \otimes_R P)$$

to prove this when P is a finitely generated projective.

Lemma 4. Rees' Lemma. Let $\mathbf{x} \subset R$ be a finite subset. Let M, N be R-modules. Let N be an R-module such that $\mathbf{x}N = 0$. And let M be an R-module such that $\mathbf{x}$ is Koszi-regular on M. This means that

$$K(\mathbf{x}, M) \xrightarrow{\sim} \frac{M}{\mathbf{x}M}.$$

Lemma 5.

$$\mathrm{RHom}_R(N, \frac{M}{\mathbf{x}M} \xrightarrow{\sim} \mathrm{RHom}_R(N, M) \otimes_R \bigwedge^{*}(\Sigma R^c).$$

In particular,

$$\mathrm{Ext}^*_R(N, \frac{M}{\mathbf{x}M} \cong \mathrm{Ext}^*_R(N, M) \otimes_R \bigwedge^{*}(\Sigma R^c).$$

Where c denotes the rank of the free module.

Corollary 2.

$$\inf \mathrm{Ext}^*_R(N, M) = \inf \mathrm{Ext}^*_R(N, \frac{M}{\mathbf{x}M} + c.$$

We also have

$$\mathrm{Ext}^*_R(N, M) \cong \mathrm{Ext}^{*+c}_R(N, \frac{M}{\mathbf{x}M}).$$

Recalling the alternating product complex will have zero differentials.

Proof. We want to compute

$$\mathrm{RHom}_R(N, \frac{M}{\mathbf{x}M} \xrightarrow{\sim} \mathrm{RHom}_R(N, K(\mathbf{x}, M)) \xrightarrow{\sim} \mathrm{RHom}_R(N, M \otimes^L_R K(\mathbf{x}, R)).$$

$$\xrightarrow{\sim} \mathrm{RHom}_R(N, M) \otimes^L_R K(\mathbf{x}, R).$$

since $K(\mathbf{x}, R)$ is perfect. Since $\mathbf{x} \cdot N = 0$, $\mathbf{x} \cdot \mathrm{Ext}^*_R(N, M) = 0$ (Exercise, show this is true). Using this and long exact sequence associated to Koszul complexes, one can calculate the isomorphism at the level of Ext.

Alternatively,

$$\mathrm{RHom}_R(N, M) \cong \mathrm{Hom}_R(N, I).$$

where $M \cong I$ is an injective resolution ($\cong$ denotes weak equivalence in $M \cong I$). Now

$$\mathbf{x} \cdot \mathrm{Hom}_R(N, I) = 0.$$

$$\begin{aligned} &\mathrm{RHom}_R(N, M) \otimes_R K(\mathbf{x}, R) \\ &\cong \mathrm{Hom}_R(N, I) \otimes_R K(\mathbf{x}, R) \\ &\cong \mathrm{Hom}_R(N, I) \otimes_R K(\mathbf{0}, R). \end{aligned}$$

where $\mathbf{0}$ is a zero sequence of length c. To get the in particular part of lemma 3, take homology. The details are an exercise. $\square$

If we want to compute $\mathrm{Ext}^n_R(N, \frac{M}{\mathbf{x}M})$ it would be

$$\left(\mathrm{Ext}_R(N, M) \otimes \bigwedge (\Sigma R^c)\right)^n.$$

$$= \bigoplus_i \mathrm{Ext}^i_R(N, M) \otimes_R \left(\bigwedge (\Sigma R^c)\right)^{n-i}.$$

So

$$\mathrm{Ext}^n_R(N, \frac{M}{\mathbf{x}M}) \cong \bigoplus_i \mathrm{Ext}^i_R(N, M) \otimes_R R^{cchoosei-n}.$$

If we had a $\mathbb{Z}$-graded object V, we think of it having upper and lower gradings via

$$V^i = V_{-i}.$$

Notation-wise, the supremum of the graded object V,

$$\mathrm{sup} V^* = \sup\{i \mid V^i \neq 0\}.$$

$$\inf V^* = \inf\{i \mid V^i \neq 0\}.$$

We brought all of this up to discuss **depth**. Now fix $I \subset R$ an ideal. We can define for any $M \in \mathcal{C}(R)$,

$$\mathrm{depth}_R(I, M) = \mathrm{inf}\mathrm{Ext}^*_R\left(\frac{R}{I}, M\right).$$

This is called the I-depth of M. We could get a few important properties.

Remark. We have the following.

1. Given an exact sequence $0 \to L \to M \to N \to 0$ of complexes, we get a long exact sequence in Ext. If the Ext groups for L and N vanish, then so too must those of M. Hence we get

$$\mathrm{depth}_R(I, M) \geq \min\{\mathrm{depth}_R(I, L), \mathrm{depth}_R(I, N)\}.$$

This is all from that exact sequence

$$\mathrm{Ext}^i_R\left(\frac{R}{I}, L\right) \to \mathrm{Ext}^i_R\left(\frac{R}{I}, M\right) \to \mathrm{Ext}^i_R\left(\frac{R}{I}, N\right) \to \mathrm{Ext}^{i+1}_R\left(\frac{R}{I}, L\right) \to \dots.$$

Theorem 2. Let $\mathbf{x} = x_1, \dots, x_c$ be a generating set for the ideal I. Then we can compute

$$\mathrm{depth}_R(I, M) = c - \mathrm{sup} H_*(\mathbf{x}, M).$$

This is true for any $M \in \mathcal{C}(R)$.

If we look at $K(\mathbf{x}, R) \to \frac{R}{\mathbf{x}R} = \frac{R}{I}$, we +-+. We prove this theorem when M is a module. Koszul complexes revisited. We started by introducing it as a tensor product as short complexes. Instead, we could start with an exterior algebra, end up with the differential. It is the same as giving a map $f : F \to R$ where F is a finite free R-module and with fixed chosen basis of rank c. One can choose a Koszul complex attached to f. Look at

$$K(f) = \left(\bigwedge^*(\Sigma F), \partial\right).$$

The former module is an exterior algebra on F. Taking a differential of a typical element, it has form $\partial(e_{i_1} \wedge \ldots \wedge e_{i_n}) = \sum_{j=1}^n (-1)^{j-1} f(e_{i_j}) e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_n}$.

For example,

$$e_1 \wedge e_2 \xrightarrow{\partial} f(e_1)e_2 - e_1 f(e_2).$$

Lemma 6. Suppose we have $\mathbf{x} = x_1, \ldots, x_c \subset R$. For any $y \in (\mathbf{x})$, then

$$K(\mathbf{x}, y; M) \cong K(\mathbf{x}, 0, M).$$

The above is isomorphism as R-complexes. The latter is just

$$K(\mathbf{x}, M) \otimes K(0, R).$$

Proof. We stare at the following picture: We get

$$\begin{tikzcd} R^{n+1} \arrow[r, "{[x_1 x_2 \ldots x_c y]}"] & R \arrow[d, equal] \\ R^{n+1} \arrow[u, "\sim"] \arrow[r, "{[x_1 x_2 \ldots x_c 0]}"'] & R \end{tikzcd}$$

In particular,

$$\sup H_*(\mathbf{x}, y; M) = 1 + \sup H_*(\mathbf{x}, M).$$

Thus,

$$c + 1 - \sup H_*(\mathbf{x}, y; M) = c - \sup H_*(\mathbf{x}, M).$$

$\square$

Corollary 3. (Check this corollary.) The quantity

$$c - \sup H_*(\mathbf{x}, M).$$

is independent of the choice of generating set for the ideal I.

Theorem 3. We have

$$\operatorname{depth}_R(I, M) = c - \sup H_*(\mathbf{x}, M)$$

where $\mathbf{x} = x_1, \ldots, x_c$ generates the ideal I.

Proof. We prove this when M is a module. What does it mean for

$$\operatorname{depth}_R(I, M) = 0?.$$

It precisely means that

$$\operatorname{Hom}_R(\frac{R}{I}, M) \neq 0.$$

This is because the zeroth Ext group is the homology. The depth zero is if and only if

$$I \subset \operatorname{zdr}_R(M).$$

which holds if and only if $H_c(\mathbf{x}, M) \neq 0$. This is also if and only if

$$\sup H_*(\mathbf{x}, M) = c.$$

So we can assume that $\operatorname{depth}_R(I, M) \geq 1$. In particular, there exists $y \in I$ which is nonzero divisor on M. Then in particular y is Koszi-regular on M. We would like to compute

$$\operatorname{Ext}_R^*\left(\frac{R}{I}, \frac{M}{\mathbf{x}M}\right).$$

What is the supremum of the above complex? Rees's lemma (that corollary afterwards) says

$$\inf \operatorname{Ext}_R^*\left(\frac{R}{I}, \frac{M}{yM}\right) = \inf \operatorname{Ext}_R^*\left(\frac{R}{I}, M\right) - 1.$$

This applies because

$$y \cdot \left(\frac{R}{I}\right) = 0.$$

In terms of depth, it tells us that

$$\operatorname{depth}_R(I, M) = 1 + \operatorname{depth}_R(I, \frac{M}{\mathbf{x}M}) = 1 + c - \sup H_*(.$$

We also have

$$H_*(\mathbf{x}, \frac{M}{yM}) = H_*(\mathbf{x}, K(y, M)) = H_*(\mathbf{x}, y; M).$$

We just saw that this is exactly

$$H_*(\mathbf{x}, 0; M).$$

because $y \in (\mathbf{x})$. If we calculate the supremum, the supremums are the same. In particular,

$$\sup H_*(\mathbf{x}, \frac{M}{yM}) = \sup H_*(\mathbf{x}, 0; M) = 1 + \sup H_*(\mathbf{x}, M).$$

So this justifies the proof of the theorem by completing an induction step. □

One huge takeaway from the story: we have that the depth is the longest Koszi-regular sequence in I. Next time, we discuss depth in the context of local.

3 January 25

Let R be a commutative ring and $I \subset R$ be an ideal. Let M be an R-complex. Recall

$$\begin{aligned}\operatorname{depth}_R(I, M) &= \inf \operatorname{Ext}_R^*\left(\frac{R}{I}, M\right) \\ &= c - \sup H_*(\mathbf{x}, M) \\ &= \text{length of any maximal } M\text{-Koszi regular sequence} \\ &= \text{length of any maximal } M\text{-regular sequence in } I.\end{aligned}$$

For the second to last equality, we need M to be a module. For the last equality, we further assume M is finitely generated and $I \subset J(R)$.

Example. Say $\mathbf{x} = x_1, \ldots, x_c$ and $\mathbf{y} = y_1, \ldots, y_\alpha$. We have

$$\sup H_*(\mathbf{x}, \mathbf{y}, M) \leq \sup H_*(\mathbf{x}, M) + d..$$

We have $K(\mathbf{x}, \mathbf{y}, M) = K(\mathbf{y}, K(\mathbf{x}, M))$. This implies that $I \subset J$ implies $\operatorname{depth}_R(I, M) \leq \operatorname{depth}_R(J, M)$.

Exercise 4

Show depth does not change across two ideals when their radicals are the same.

Today, we look at local rings. Say (R, m, k) is a local ring with $k = \frac{R}{m}$. There is a natural notion of depth by just selecting the maximal ideal.

$$\text{depth}(m, M) = \text{inf}\text{Ext}^*_R(k, M).$$

Can compute with $\mathbf{x}$ a system of parameters for R. That is, the radical of the ideal they generate is m and it is of minimal length among such ideals. As R is Noetherian, we note $m = (x_1, \ldots, x_n)$. The above depth quantity is

$$n - \sup H_*(\mathbf{x}, M)..$$

We also note that when $d = \dim(R)$, the system of parameters has length d.

We move towards the Ausland Buchsbaum equality.

Definition 3. Let F be an R-complex, then F has **finite flat dimension**. If F is weakly equivalent to

$$0 \to F_b \to \ldots \to F_a \to 0$$

where F_i is flat, will write $\text{flatdim}_R F < \infty$.

Example. Perfect examples satisfy the equality and so do Koszul complexes. Any flat module as well.

If the flat dimension of some F is finite, then

$$\text{Tor}_i^R(\bullet, F) = 0 \ \forall |i| \gg 0$$

on Mod R. This is because

$$\text{Tor}_i^R(M, F) = H_i(M \otimes_R (0 \to F_b \to \ldots \to F_a \to 0)),.$$

in particular $\text{Tor}_i^R(M, F) = 0$ for $i \notin [a, b]$. Fact: This property cahracterizes the flat dimension being finite.

Theorem 4. (Auslander-Buchsbaum equality). Say (R, m, k) is local. When the flat dimension of F is finite,

$$\text{depth}_R(M \otimes^\ell_R F) = \text{depth}_R(M) - \sup H_*(k \otimes^\ell_R F).$$

For any R-complex M.

Let us look at the case $M = R$. It says

$$\text{depth}_R(F) = \text{depth}_R(R \otimes^\ell_R F) = \text{depth}(R) - \sup H_*(k \otimes^\ell_R F).$$

Suppose we take a finitely generated module. Let N be a finitely generated R-module. We can write down the resolution of N. Such an N has a **minimal free resolution**. Take a minimal generating set for N, $R^{b_0} \twoheadrightarrow \varepsilon N$. It has kernel which is also minimally generated, say by R^{b_1}. We can continue the process to get a complex G of free modules.It is called a semi-free resolution of N. Weak equivalence from the first degree to N.

$$\partial G \subset mG.$$

Definition 4. G is called **the free minimal free resolution of N.**

If we look at

$$\text{Tor}_i^R(k, N) = H_i(k \otimes_R G) = (k \otimes_R G)_i.$$

So we have

$$\text{Tor}_i^R(k, N) = 0.$$

if and only if $(G)_i = 0$. So flat dimension is finite if and only if N has a finite free resolution. In particular, precisely when N is perfect.

$$\sup \operatorname{Tor}_*^R(k, N) = \text{length of } G = \text{proj dim}_R(N).$$

Back to AB-equality, a second special case we could look at is as follows: If N is a finitely generated module with finite projective dimension, then

$$\operatorname{depth}_R(N) = \operatorname{depth}(R) - \operatorname{projdim}_R(N).$$

Corollary 4. (of the above special case). If the projective dimension of N is finite,

$$\operatorname{depth}_R(N) \leq \operatorname{depth}_R(R)..$$

Equality holds if and only if N is projective (or precisely when it is free in this case). In general, when the projective dimension is infinite, the inequality is false in general. Mention example to Sri in next lecture. Hint: start with ring of depth 0.

Subtracting variants of AB equalities, we get

$$\operatorname{depth}_R(F) - \operatorname{depth}_R(M \otimes_R^\ell F) = \operatorname{depth}(R) - \operatorname{depth}(M)..$$

When the Koszul complex is acyclic, ie $H_*(\mathbf{x}, M) = 0$, then the $\operatorname{depth}_R(\mathbf{x}, M) = \infty$.

Proof. (proof of the AB-equality). We would like to compute the depth. So we compute

$$\operatorname{RHom}_R(k, M \otimes_R^\ell F).$$

Note we have a map from $\operatorname{RHom}_R(k, M) \otimes_R^\ell F$. That map is a quasi-isomorphism because k is a finitely generated module, so $G \xrightarrow{\sim} k$ G_i is finite free, also (using?) flat dimension of F is finite.

$$\operatorname{Hom}_R(N, M) \otimes F \xrightarrow{\sim} \operatorname{Hom}_R(N, M \otimes_R F).$$

N is finitely generated R-module and F flat. Verify that the above weak equivalence indeed exists. Key observation: $\operatorname{RHom}_R(k, M)$ is weakly equivalent to the complex of k-vector spaces.

$$\operatorname{RHom}_R(k, M) \otimes_R^k \xrightarrow{\sim} \operatorname{RHom}_R(k, M) \otimes_k^\ell \left(k \otimes_R^\ell F\right).$$

$$\begin{aligned}\operatorname{Ext}_R^*(k, M \otimes_R F) &= H^*(\operatorname{RHom}(k, M) \otimes_k^\ell \left(k \otimes_R^\ell F\right)) \\ &= \operatorname{Ext}_R^*(k, M) \otimes_k H_*(k \otimes_R^\ell F).\end{aligned}$$

If the reader wishes to find a reference, this is Foxby's proof. □

Take any R-complex, M. Let $s = \sup H_*(M)$. Say $s < \infty$. Then

$$\operatorname{depth}_R(M) \geq -s$$

with equality if and only if $\operatorname{depth}_R H_s(M) = 0$. If M was a module, recall the depth is 0 if and only if

$$\inf \operatorname{Ext}_R^*(k, M) = 0.$$

In particular, $H_R(k, M) \neq 0$ or $k \xhookrightarrow{M}$, or m is an associated prime of M. One proof: if M as above,

$$\operatorname{Ext}_R^s(N, M) \cong \operatorname{Hom}_R(N, H_s(M)).$$

where N is any R-module. Check this (it is not that difficult, using a projective resolution of M). Key: M is isomorphic to M' with $M_i' = 0$ for all $i > s$.

$$\begin{tikzcd}
\dots \arrow[r] & M_{s+1} \arrow[r, "\partial"] & M_s \arrow[r] \arrow[d] & M_{s-1} \arrow[r] \arrow[d, equal] & \dots \arrow[d, equal] \\
& 0 \arrow[r] & M_s/\partial(M_{s+1}) & M_{s-1} \arrow[r] & \dots \\
& H_s(M) \arrow[r, equal] & Z_s(M)/\partial(M_{s-1}) \arrow[u, hook] & &
\end{tikzcd}$$

In particular,

$$0 \to \Sigma^s H_s(M) \xhookrightarrow{M}\to M'' \to 0.$$

where the second map is isomorphism in homology in degrees $\leq s-1$ and $H_i(M'') = 0$ for $i \geq s$. Let $\mathbf{x} = x_1, \dots, x_n$ be a generating set for m. We get

$$H_{i+1}(\mathbf{x}, M'') \to H_i(\mathbf{x}, \Sigma^s H_s(M)) \to H_i(\mathbf{x}, M) \to H_i(\mathbf{x}, M'') \to .$$

We get

$$H_j(M'') = 0.$$

for all $j \geq s-1$. So $M'' \xrightarrow{\sim} M'''$ with M_j''' for $j \geq s$. If we look at $K(\mathbf{x}, M''')$, how far does the complex go? The complex is zero for degrees $j \geq s+n+1$. Thus $H_j(\mathbf{x}, M''') = 0$ for $j \geq s+n+1$. Now we know

$$H_i(\mathbf{x}, \Sigma^s H_s(M)).$$

Thus

$$H_j(\mathbf{x}, M) = 0.$$

for $j \geq n+s+1$ implies that

$$\operatorname{depth}_R(M) \geq -s.$$

And

$$\begin{aligned} H_{n+s}(\mathbf{x}, M) &\cong H_{n+s}(\mathbf{x}, \Sigma^s H_s(M)) \\ &\cong H_n(\mathbf{x}, H_s(M)). \end{aligned}$$

Where these are isomorphisms. We get

$$H_n(\mathbf{x}, H_s(M)) \neq 0$$

implying $\operatorname{depth} H_s(M) = 0$. This completes the proof of the fact we wanted to prove above.

1. Say $\operatorname{flatdim}_R(F) < \infty$. Then $\forall\ M$,

$$\operatorname{depth}_R(M \otimes_R^\ell F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^\ell F)..$$

2. $s = \sup H_*(M)$ is finite.

$$depth_R(M) \geq -s.$$

   with equality if and only if $\operatorname{depth}_R(H_s(M)) = 0$.

APplication. Say that F is a finite free complex,

$$0 \to F_d \to \dots \to F_0 \to 0$$

where the length of the homology modules are all finite and nonzero.

$$0 < \operatorname{length}(H_*(F)) < \infty.$$

The claim is that for any M,

$$\operatorname{depth}_R(M) = d - \sup H_*(F \otimes_R M).$$

Normally, we would apply this in the case when we are looking at a system of parameters and the Koszul complex associated to it. This states a generalization of that fact.

When $mM \neq M$, one can check that $H_*(F \otimes_R M) \neq 0$ (one of the modules is nonzero). Then one gets that $d \geq \operatorname{depth}_R(M)$. Then one gets that

$$d \geq \operatorname{depth}_R(M).$$

Over any local ring R, there exists M such that $M \neq mM$ and $\operatorname{depth}_R(M) = \dim(R)$. So $d \geq \dim(R)$. So $d \geq \dim(R)$. This is called the new intersection theorem. Hochster in the 70s, Andre in 2016, Bhatt in 2021. In the last five minutes, we sketh the proof.

Proof. Say $s = \sup H_*(F \otimes_R M) < \infty$. The infinite and zero cases are an exercise in homological algebra. If we look at prime $p \neq m$, look at

$$H_i(F \otimes_R M)_p = H_i(F_p \otimes_{R_p} M_p).$$

We have

$$H(F_p) = 0$$

because the length is finite. In particular,

$$H_*(F_p \otimes_{R_p} M_p) = 0.$$

Thus

$$H_i(F \otimes_R M).$$

is m-$***$ torsion. ie, each $a \in H_i(F \otimes_R M)$ is such that $m^n \cdot a = 0$ for some n. IN particular,

$$\operatorname{depth} H_s(F \otimes_R M) = 0$$

so $s = -\operatorname{depth}(F \otimes_R M) = -[\operatorname{depth}_R(M) - \sup H(k \otimes_R^{\ell} F)]$. Say $\partial F \subset mF$. Solving for the depth, we get

$$\operatorname{depth}_R(M) = \sup H(k \otimes_R^{\ell} F) - s.$$

The supremum is d. This concludes the proof. □

4 January 30

Recap: Let $I \subset R$ be an ideal in a commutative Noetherian ring, and let M be and R-complex. We recall depth of I in M is given by choosing $\mathbf{x} = x_1, \ldots, x_c$ a generating set, and then

$$\operatorname{depth}_R(I, M) = c - \sup H_*(\mathbf{x}, M).$$

For reference, the definition was that depth was

$$\inf \operatorname{Ext}_R^*(R/I, M).$$

Let $(R, \mathfrak{m}, k)$ be local.

$$\operatorname{depth}_R(M) = \operatorname{depth}(\mathfrak{m}, M).$$

We discussed last time that

$$\operatorname{depth}_R(M) \geq -\sup H_*(M).$$

With equality if and only if $\mathfrak{m} \in \operatorname{Ass} H_s(M)$ where $s = \sup H_*(M)$.

Exercise 5

Say M i bounded,

$$0 \to M_b \to \ldots \to M_a \to 0.$$

$$\operatorname{depth}_R(M) \geq \inf\{\operatorname{depth}(M_i) - i | a \leq i \leq b\}$$

with an analogous statement for homology:

$$\operatorname{depth}_R(M) \geq \inf\{\operatorname{depth}(H_i(M)) - i | \inf H_*(M) \leq i \leq \sup H_*(M)\}.$$

The latter statement is a strengthening of the inequality above with minus sup because $\operatorname{depth}(H_i(M)) \geq 0$.

R not necessarily local

Remark. Let us assume that $H_*(M)$ is bounded. Take $\mathbf{x} = x_1, \ldots, x_c$. We have

$$\sup H_*(M) \leq \sup H_*(\mathbf{x}, M) \leq \sup H_*(M) + c.$$

Call the former inequality (1) and the latter (2).

Lemma 7. (a) Inequality (2) always holds and equality holds if and only if $\operatorname{depth}_R(\mathbf{x}, H_s(M)) = 0$. (b) Inequality (1) holds when $(\mathbf{x}) \subset J(R)$ and $H_i(M)$ is finitely generated $\forall i$. Equality holds when $\mathbf{x}$ is $H_s(M)$-regular where $s = \sup H_*(M)$.

Proof. We can reduce the proof to when $c = 1$ and apply induction. Recall

$$H(\mathbf{x}, M) = H(x_1, K[x_2, \ldots, x_c; M]).$$

When $c = 1$, the exact sequence looks like

$$H_i(M) \xrightarrow{x} H_i(M) \to H_i(x, M) \to H_{i-1}(M).$$

The exactness gives us inequality 2 immediately. More precisely, for $i \geq s+1$, one gets $H_i(x, M) = 0$. Moreover

$$H_{s+1}(x, M) \neq 0 \Rightarrow x \text{ is a zero } \text{——} .$$

This settles (a). We have

$$H_i(M) \neq 0 \Rightarrow H_i(x, M) \neq 0.$$

by Nakayama's lemma. Moreover, this implies

$$\sup H_*(x, M) \geq \sup H_*(M).$$

We have

$$0 \to H_{s+1}(x, M) \to H_s(M) \xrightarrow{x} H_s(M).$$

□

Corollary 5. (2) implies that

$$\operatorname{depth}_R(\mathbf{x}, M) \geq -\sup H_*(M).$$

with equality if and only if $(\mathbf{x}) \subset \mathfrak{p} \in \operatorname{Ass} H_s(M)$. Or $\operatorname{depth}(\mathbf{x}, H_s(M)) = 0$ (check the former?).

Proposition 1. Let R be a local ring, and M be any complex with $H_*(M)$ bounded. Then for any $I \subset R$,

$$\operatorname{depth}_R(M) \leq \operatorname{depth}_R(I, M) + \dim(\frac{R}{I}).$$

In particular, if M is any finitely generated module,

$$\operatorname{depth}_R(M) \leq \inf\{\dim\left(\frac{R}{\mathfrak{p}}\right) \mid \mathfrak{p} \in \operatorname{Ass}(M)\}$$

$$\leq \dim_R(M).$$

As we will get into later, Cohen-Macaulay modules satisfy equality.

Proof. Let $I = (y_1, \ldots, y_c)$. Let $x_1, \ldots, x_d$ be such that they generate an ideal whose radical is the maximal ideal $\mathfrak{m}/I = \mathfrak{m}_{R/I}$. They are a system of parameters. Thus $\sqrt{(\mathbf{y}, \mathbf{x})} = \mathfrak{m}_R$. So we can apply the lemma (b) to $K(\mathbf{y}, M)$. We get that

$$\sup H_*(\mathbf{y}, M) \leq \sup H_*(\mathbf{x}, K(\mathbf{y}, M)).$$

The latter is $\sup H_*(\mathbf{x}, \mathbf{y}; M)$. Thus

$$d + c - \sup H_*(\mathbf{y}, M) \geq d + c - \sup H_*(\mathbf{x}, \mathbf{y}, M).$$

The former is $d + \operatorname{depth}(I, M) \geq \operatorname{depth}_R(M)$. $\square$

Let R be a local ring with residue field $k = \frac{R}{\mathfrak{m}_R}$. The Auslander-Buchsbaum equality states that if F is an R-complex with $\operatorname{flatdim}_R(F) < \infty$, then for any R-complex M,

$$\operatorname{depth}(M \otimes_R^{\ell} F) = \operatorname{depth}_R(M) - \sup H_*(k \otimes_R^{\ell} F).$$

Proposition 2. If $I \subset R$ ideal,

$$\operatorname{depth}(I, M) = \inf\{\operatorname{depth} M_{\mathfrak{p}} \mid I \subset \mathfrak{p}\}.$$

Proof. $I \subset \mathfrak{p}$ implies

$$\operatorname{depth}(I, M) \leq \operatorname{depth}(\mathfrak{p}, M) \leq \operatorname{depth}_{R_{\mathfrak{p}}}(\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}}).$$

Check the last inequality using Koszul homology. Say $I = (x_1, \ldots, x_c)$, $s = \sup H_*(\mathbf{x}, M), \mathfrak{p} \in \operatorname{Ass} H_s(\mathbf{x}, M)$, then the depth of $H_s(\mathbf{x}, M)_{\mathfrak{p}}$ is zero in the ring $R_{\mathfrak{p}}$.

Consider $K(\mathbf{x}, M)_{\mathfrak{p}} = K(\mathbf{x}, M_{\mathfrak{p}})$. We have

$$\sup H_*(K(\mathbf{x}, M_{\mathfrak{p}})) = \sup H_*(\mathbf{x}, M) = s.$$

$$\operatorname{depth}_{R_{\mathfrak{p}}} K(\mathbf{x}, M_{\mathfrak{p}}) = -s.$$

$$\operatorname{depth}_{R_{\mathfrak{p}}}(K(\mathbf{x}, R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}}).$$

$$= \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \sup H_*\left(k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} K(\mathbf{x}, R_{\mathfrak{p}})\right).$$

implying

$$-s \geq \operatorname{depth} M_{\mathfrak{p}} - c.$$

$$\operatorname{depth}_R(I, M) \geq \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}.$$

Using Auslander-Buchsbaum. $\square$

Remark. The proof shows

$$\operatorname{depth}_{\mathfrak{p}}(I, M) = \operatorname{depth}_{\mathfrak{p}} M_{\mathfrak{p}}.$$

$\forall \mathfrak{p} \in \operatorname{Ass} H_s(\mathbf{x}, M)$.

Theorem 5. Suppose we have $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ a local homomorphism of local rings. Let M be an R-complex, and N be an S-module such that N is flat as an R-module. Then

$$\operatorname{depth}_S(N \otimes_R M) = \operatorname{depth}_R(M) + \operatorname{depth}_{(S/\mathfrak{m}_R S}(N/\mathfrak{m}_R N)\,.$$

We apply this when $M = R$ and $N = S$. In this case,

Corollary 6.

$$\operatorname{depth}_S(S) = \operatorname{depth}_R(R) + \operatorname{depth}_{S/\mathfrak{m}_R S}(S/\mathfrak{m}_R S)\,.$$

We had an exact

$$R \to S \to \frac{S}{\mathfrak{m}_R S}.$$

Compare: Under the same hypotheses,

$$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}_R S)\,.$$

Later on, Cohen-Macaulayness behaves nicely with exact sequences.

Proof. Let $\mathbf{x} = x_1, \ldots, x_c$ be a generating set for $\mathfrak{m}_R$. Pick $\mathbf{y} = y_1, \ldots, y_d$ in $\mathfrak{m}_S$ such that

$$\mathbf{y}\left(\frac{S}{\mathfrak{m}_R S}\right).$$

is the maximal ideal of $\frac{S}{\mathfrak{m}_R S}$. Then $\mathbf{x}S, \mathbf{y}$ generates $\mathfrak{m}_S$. We want to compute depth, so let us consider

$$K(\mathbf{x}, \mathbf{y}, N \otimes_R M) \cong K(\mathbf{y}, N) \otimes_R K(\mathbf{x}, M).$$

(Commutativity of tensor products). N flat over R implies $K(\mathbf{y}, N)$ has finite flat dimension over R. We can apply Auslander-Buchsbaum:

$$\operatorname{depth}_R(K(\mathbf{x}, \mathbf{y}, N \otimes_R M)) = \operatorname{depth}_R(K(\mathbf{x}, M)) - \sup\left(k \otimes_R K(\mathbf{y}, N)\right).$$

Note

$$(\mathbf{x}, \mathbf{y}) \cdot H\left(\mathbf{x}, \mathbf{y}, N \otimes_R M\right) = 0.$$

So

$$\mathfrak{m}_S \cdot H(\mathbf{x}, \mathbf{y}, N \otimes_R M) = 0.$$

$$\mathfrak{m}_R H(\mathbf{x}, \mathbf{y}, N \otimes_R M) = 0.$$

Similarly,

$$\mathfrak{m}_R H(\mathbf{x}, M) = 0.$$

Above

$$-\sup(k \otimes_R K(\mathbf{y}, N)) = -\sup(K\left(\mathbf{y}, \frac{N}{\mathfrak{m}_R N}\right).$$

This implies that

$$\operatorname{depth}_S(N \otimes_R M) = -\sup H_*(\mathbf{x}, \mathbf{y}, N \otimes_R M) = -\sup H_*(\mathbf{x}, M) - \sup H_*(\mathbf{y}, \frac{N}{\mathfrak{m}_R N}).$$

∎

Exercise 6

Let M be a finitely generated R-module with R local. We have a sequence of inequalities

$$\operatorname{depth}(R) - \operatorname{depth}_R(M) \leq \operatorname{grade}_R(M) \leq \operatorname{codim}_R(M) \leq \dim(R) - \dim(M) \leq \operatorname{pdim}_R(M)$$

where $\operatorname{grade}_R(M) = \operatorname{depth}_R(\operatorname{ann}_R(M), R)$. Codimension is $\operatorname{ht}(\operatorname{ann}_R(M)) = \inf\{\dim R_{\mathfrak{p}} \mid \mathfrak{p} \supset \operatorname{ann}_R(M)\}$. The last inequality is the most nontrivial inequality, following from the intersection theorem, recalled below:

Theorem 6. If we have a finite free complex

$$0 \to F_d \to \ldots \to F_0 \to 0.$$

where $0 < \operatorname{length}(H_*(F)) < \infty$. Then $d \geq \dim(R)$.

This is a consequence of Auslander Buchsbaum and the existence of finitely generated Cohen Macaulay modules.

Corollary 7. Let R be a local ring, M be a nontrivial finitely generated R-module with $\operatorname{pdim}_R(M) < \infty$. Then for any finitely generated R-module N,

$$\dim_R(N) - \dim_R(M \otimes_R N) \leq \operatorname{pdim}_R(M).$$

Deducing this corollary from the theorem above is an exercise.

Inspired by a result by Serre: Let R be regular local ring. If we take any finite R-modules M, N, then

$$\dim_R(N) - \dim_R(M \otimes_R N) \leq \dim(R) - \dim(M).$$

We discuss Cohen-Macaulay rings on Wednesday.

5 February 6

Let R be a local ring $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring. Recall that

$$\operatorname{depth}(R) \leq \dim R \leq \operatorname{edim}(R).$$

where the last number is **embedding dimension,** defined as

$$\operatorname{edim}(R) = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2).$$

So we have two inequalities we label 1 and 2. Recall when equality holds for 1, R was said to be **Cohen-Macaulay.** 2 holds by Krull's height theorem. We are interested in Cohen Macaulay rings where equality holds throughout. The embedding codepth of R is

$$\operatorname{codepth}(R) = \operatorname{edim}(R) - \operatorname{depth}(R).$$

Definition 5. We say R is regular if $\operatorname{codepth}(R) = 0$.

Exercise 7

R is regular if and only if $\mathfrak{m}$ is generated by a regular sequence. Either also holds if and only if codepth is 0.

Example.
1. Let's say we have $k[[x_1, \ldots, x_n]]$ with k is a field. This is regular.
2. $k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ also is regular.
3. $\mathbb{Z}_{(p)}$ is regular.
4. Regular implies Cohen-Macaulay. Contrapositive is true. Recall $k[x,y,z]/(xz,yz)$ is non-CM. It is thus non-regular.
5. Let $R = k[[x,y]]/(xy)$ has codepth $1 > 0$. R is not regular.

We want to understand how regularity behaves under flat maps. Let $\varphi : (R, \mathfrak{m}) \to S$ be a local flat extension. Last week, we proved that Cohen-Macaulayness is determined as follows. S is CM iff R and $\frac{S}{\mathfrak{m}S}$ are CM. Does the same thing hold for regularity? This turns out to fail frequently for regularity. Consider $k[[x^2]] \hookrightarrow k[[x]]$. The fiber has codepth 1.

Also recall that if R is CM, then every localization $R_\mathfrak{p}$ is CM for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Question: If R is regular, must $R_\mathfrak{p}$ be regular? Since regularity corresponds to smoothness geometrically, we expect this to be true. Homological characterization of regularity. We recap some stuff on minimal free resolutions. Recall for a finitely generated R-module M, a **minimal free resolution** of M over $(R, \mathfrak{m}, k)$ is a free resolution

$$F \xrightarrow{\sim} M$$

(weak equivalence) with $\partial(F) \subset \mathfrak{m}F$.

Exercise 8

These exist and are unique up to isomorphism of complexes.

Betti numbers: The i-th Betti number of M is $\beta_i^R(M)$ is the k-vector space rank

$$\operatorname{rank}_k \operatorname{Tor}_i^R(M,k) = \operatorname{rank}_k \operatorname{Ext}_R^i(M,k) = \operatorname{rank}_R F_i$$

with F a minimal free resolution of M.

Example.
1. $R = k[[x_1, \ldots, x_n]]$ with $M = \frac{R}{f}$, $f \in (\mathbf{x})$ nontrivial. Then the minimal free resolution of M is the Koszul complex

   $$0 \to R \xrightarrow{f} R \to 0.$$

   $\beta_i(M) = 1$ if $i = 0, 1$ and 0 otherwise. The minimal free resolution of k is

   $$\operatorname{Kos}(\mathbf{x}) \xrightarrow{\sim} k.$$

   $$R^{\binom{n}{2}} \to R^n \xrightarrow{(x_1 \ldots x_n)} R.$$

2. We saw $R = k[[x,y]]/(xy)$ is not regular. What if $M = R/x$ has minimal free resolution

   $$\ldots \to R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \to 0.$$

   The Betti numbers are all 1.

For k, it has minimal free resolution

$$\ldots \to R^2 \to R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{(x\ y)} R \to 0.$$

Betti numbers are all 2, except for at index 0.

One can guess from these examples is that the termination of the resolution indicates regularity.

Fact: As long as $M \neq 0$ and finitely generated, the projective dimension of M the length of the shortest projective resolution of M.

$$\mathrm{pdim}(M) = \sup\{i \geq 0 | \beta_i^R(M) \neq 0\}.$$

Theorem 7. Auslander-Buchsbaum-Serre theorem. Let $(R, \mathfrak{m}, k)$ be local. Then TFAE:

1. R is regular
2. $\mathrm{pdim}(M) < \infty$ for all M finitely generated.
3. $\mathrm{pdim}_R(k) < \infty$

Today we sketch the proof of this theorem.

Proof. 1 implies 2. Let $\mathbf{x}$ be a minimal generating set for $\mathfrak{m}$.

$$\mathrm{Kos}^R(\mathbf{x}) \xrightarrow{\sim} k.$$

$$\beta_i^R(M) = \mathrm{rank} Tor_i^R(M, k).$$
$$= \mathrm{rank}_k H_i(M \otimes_R Kos^R(\mathbf{x}) = 0.$$

for $i > d$ by the fact stated as definition earlier, we have that the projective dimension is finite. We will come back to 3 implies 1.

2 implies 3 is literally just an application of 2. See the next proof given by Serre. $\square$

Lemma 8. Let $(R, \mathfrak{m}, k)$ local. Then $\beta_i(k) \geq \binom{\mathrm{edim}(R)}{i}$ for $i \geq 0$.

Proof. Proof. Let $F \xrightarrow{\sim} k$ be the minimal free resolution. Let also $x_1, \ldots, x_e$ be a minimal generating set of $\mathfrak{m}$. $K = \mathrm{Kos}(\mathbf{x}) \to k$. We can lift this map along $F \to k$, call it

$$\varphi : K \to F.$$

Claim: φ is a split injection. If so, the Betti numbers of k

$$\beta_i(k) = \mathrm{rank}_R F_i \geq \mathrm{rank}_R K_i = \begin{pmatrix} e & i \end{pmatrix}.$$

If we prove the claim, we're good. Note $\varphi_i : K_i \to F_i$ is split injective if and only if its tensor with k is an injection of k-vector spaces.

$$\varphi_i \otimes_R k.$$

Nakayama's lemma.

By induction, show φ_i is split injective. Let $i > 0$. If $a \in K_i$, with $\varphi_i(a) \in \mathfrak{m}F_i$. Want to show $a \in \mathfrak{m}K_i$. Use commutative diagram arising from complex map φ. By splitting hypothesis, $\partial(a) \in \mathfrak{m}^2 K_{i-1}$ because of applying the inverse to the split F_{i-1} which is an R-module map.

Note by the definition of ∂ and since $\mathbf{x}$ is a minimal generating set for $\mathfrak{m}$ we have that the above implies $a \in \mathfrak{m}K_i$. $\square$

Serre's proof continues as follows. We prove $\mathrm{pdim}_R k < \infty$ implies R is regular. From Serre's inequality (the lemma above), the projective dimension of k. We note

$$\mathrm{pdim}(k) \geq \mathrm{edim}(R).$$

Auslander Buchsbaum tells us that the former quantity is the depth. And the opposite inequality is true (see previous lecture). ☐

We return to the localization problem.

Corollary 8. If R is regular, then so it $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. Since R is regular, $pdim(\frac{R}{\mathfrak{p}})$ is finite. Hence its localization is finite. ☐

Proposition 3. Let's say we have a local flat extension $\varphi : (R, \mathfrak{m}) \to S$.

1. If R and $S/\mathfrak{m}S$ are regular, then S is regular.
2. If S is regular, R is regular. (We saw earlier that $S/\mathfrak{m}S$ may not be regular.)

Proof. Denote $\overline{S} = S/\mathfrak{m}S$.

1. Given $\mathrm{depth}(S) = depth(R) + depth(\overline{S}) = \mathrm{edim}(R) + \mathrm{edim}(\overline{S})$. First equality by flatness and second by assumption. There is always a right exact sequence

$$\mathfrak{m}_R/\mathfrak{m}_R^2 \otimes \ell \to \mathfrak{m}_S/\mathfrak{m}_S^2 \to \mathfrak{m}_{\overline{S}}/\mathfrak{m}_{\overline{S}}^2 \to 0.$$

   We have

$$\mathrm{edim}(S) \leq \mathrm{edim}(R) + \mathrm{edim}(\overline{S}) = \mathrm{depth}(S).$$

   Hence S is regular.

2. Let $F \xrightarrow{\sim} k_R$ be minimal, since φ is flat and local,

$$F \otimes_R S \xrightarrow{\sim} \overline{S}..$$

   This resolution has finite length because S is regular. Hence R is regular.

☐

6 February 8

Last time recall that a local ring R is regular if $\mathrm{codepth}(R) = 0$. In particular, $\mathfrak{m}_R$ is generated by a regular sequence. We proved the Auslander-Buchsbaum-Serre theorem. We provide a second proof of 3 implies 1 without Serre's lemma.

Theorem 8. (Nagata). $(R, \mathfrak{m}, k)$ local and $x \in \mathfrak{m}/\mathfrak{m}^2$ nonzero divisor. Set $\overline{R} = R/x$. Then for any finitely generated R-module M there is an isomorphism

$$\mathrm{Tor}_x^R(M, k) \cong \mathrm{Tor}_x^R(M, k) \otimes_k \bigwedge \Sigma k.$$

Also, $\bigwedge \Sigma k = \mathrm{Tor}^R(\overline{R}, R)$. Hence,

$$\beta_i(M) = \beta_i^{\overline{R}}(M) + \beta_{i-1}^{\overline{R}}(M).$$

In fact, one can show the minimal R-free resolution of M has a very specific form. We get

$$\ldots \to G_2 \oplus G_1 \to G_1 \oplus G_0 \to G_0.$$

With expicitly described maps as follows: $G_1 \oplus G_0 \to G_0$ defined by

$$(a, b) \mapsto (\alpha_1 a, b \cdot x).$$

The map $G_2 \oplus G_1 \to G_1 \oplus G_0$ is defined by the matrix

$$\begin{pmatrix} \alpha_2 & x \\ \beta_2 & -\alpha_1 \end{pmatrix}.$$

If you cut off the bottom row,

$$\ldots \to \overline{G_3} \xrightarrow{\alpha_3} \overline{G_2} \xrightarrow{\alpha_2} \overline{G_1} \xrightarrow{\alpha_1} \overline{G_0} \to 0.$$

Example. Consider $k[[x, y]]/(xy)$ which is a non-regular ring. The minimal R-free resolution of the residue field k has specific form

$$\ldots \to R^2 \to R^2 \to R^2 \to R \to 0.$$

This example showed up in the previous lecture in more detail. Consider $y - x^2 \in \mathfrak{m}/\mathfrak{m}^2$ a nonzero divisor on R. The minimal resolution above is isomorphism to

$$\ldots \to R^2 \to R^2 \to R^2 \to R \to 0.$$

where $R^2 \to R$ is $\begin{pmatrix} x & y \cdot x^2 \end{pmatrix}$. The map $R^2 \to R^2$ next is

$$\begin{pmatrix} y & y - x^2 \\ 0 & -x \end{pmatrix}.$$

The next map is

$$\begin{pmatrix} x & y - x^2 \\ 0 & -y \end{pmatrix}.$$

According to the stronger result, we should be able to mod out by $y - x^2$ and examine

$$\ldots \to \overline{R} \xrightarrow{x} \overline{R} \xrightarrow{y} \overline{R} \xrightarrow{x} \overline{R} \to 0.$$

$$\ldots \xrightarrow{x} k[x]/x^3 \xrightarrow{x^2} k[x]/x^2 \xrightarrow{x} k[x]/x^3 \to 0.$$

We prove Nagata's theorem. From a course of homological algebra, one will have learned that a nonzero divisor $a \in R$ would yield a long exact sequence on tor. If $\overline{R} = \frac{R}{a}$, M, N are $\overline{R}$-modules, then there is long exact sequence

$$\ldots \to \mathrm{Tor}^{\overline{R}}_{n-1}(M, N) \to \mathrm{Tor}^{R}_{n}(M, N) \to \mathrm{Tor}^{\overline{R}}_{n}(M, N) \xrightarrow{\chi} \mathrm{Tor}^{\overline{R}}_{n-2}(M, N) \to \ldots.$$

where χ is the connecting map (note one of the tor modules is over R and not $\overline{R}$). To specialize to our setting, we would use $N = k$. There is an ext version of this long exact sequence as well. To specialize in our setting, we get this diagram. We want to show $\chi = 0$ in every degree. One can compute χ in the following way. Take a minimal $\overline{R}$ resolution $\overline{F} \xrightarrow{\sim} M$. Lift this to a sequence of free R-modules:

$$F_{i+1} \xrightarrow{\partial} F_i \xrightarrow{\partial} F_{i-1} \to \ldots.$$

$\partial \otimes_R \overline{R} = \partial^{\overline{F}}$. This may not be a complex. $\partial^2 = x \cdot \theta$ where $\theta = \{\theta_i : F_i \to F_{i-1}\}$ is a chain map. ∂^2 may not be zero, but it does satisfy this property because its lift squared is zero. The following diagram commutes:

$$\begin{tikzcd}
F_i \otimes_R k \arrow[r, "\theta_i \otimes_R k"] \arrow[d, "\sim"'] & F_{i-2} \otimes_R k \arrow[d, "\sim"] \\
\overline{F_i} \otimes_{\overline{R}} k \arrow[d, equal] & \mathrm{Tor}^{\overline{R}}_{i-2}(M, k) \\
\mathrm{Tor}^{\overline{R}}_{i-2}(M, k) \arrow[ur, "\chi"'] &
\end{tikzcd}$$

We thus know that $\theta(F) \subset \mathfrak{m}F$. Hence $\chi = 0$.

Second proof of 3 implies 1. Assuming that $\text{pdim}_R k < \infty$, induct on $d = \text{depth}(R)$. When $d = 0$, Auslander Buchsbaum tells us that the projective dimension of k is 0, hence $R = k$, and R is regular.

When $d > 0$, you can always find a nonzero divisor. BBy prime avoidance there exists $x \in \mathfrak{m}/\mathfrak{m}^2$ nonzerodivisor.

$$\text{codepth}(R) = \text{codepth}(R/x).$$

By Nagata's theorem, you can say that

$$\text{pdim}_{R/x} k = \text{pdim}_R k - 1 < \infty.$$

Since $\text{depth}(R/x) < d$, by induction R/x is regular and hence so is R.

Now we tie up some loose ends. Suppose $(R, \mathfrak{m}, k)$ is local.

Proposition 4. If $x \in \mathfrak{m}$ a nzd, R is CM if and only if R/x is CM. Because their cohen macaulay defects are the same.

Proposition 5. If $x \in \mathfrak{m}$ is a nzd, then

1. If R is regular, then R/x is regular if and only if $x \notin \mathfrak{m}^2$.
2. If R/x is regular, $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ and R is regular.

Proof. When you look at

$$\text{codepth}(R/x) = \begin{cases} \text{codepth}(R) & \text{when } x \notin \mathfrak{m}^2 \\ \text{codepth}(R) + 1 & x \in \mathfrak{m}^2 \end{cases}.$$

□

Global setting: Let R be a commutative, Noetherian ring which is not necessarily local.

Definition 6. R is **regular** if $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \text{Spec}(R)$.

Exercise 9

Under this definition of regularity, show that R is regular if and only if $R[x]$ is regular or $R[[x]]$ is regular.

Example.
1. $k[x_1, \ldots, x_n]$ is regular with k some field.
2. $\mathbb{Z}[x_1, \ldots, x_n]$ is also regular.
3. Nagata's example

Theorem 9. (Bass-Murthy) If R is some commutative Noetherian ring and M is some finitely generated R-module,

$$\text{pdim}_R(M) < \infty$$

implies that $\text{pdim}_R(M_{\mathfrak{p}}) < \infty$ for all $\mathfrak{p} \in \text{Spec}(R)$.

Corollary 9. For a commutative, Noetherian ring, R being regular is equivalent to $\operatorname{pdim}_R(M) < \infty$ for all finitely generated M.

This corollary allows us to talk about Gorenstein, Cohen-Macaulay spaces.

Proof. (of corollary) R is regular if and only if $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec}(R)$. By Auslander-Buchsbaum-Serre, we have

$$\operatorname{pdim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty.$$

for all $\mathfrak{p}$ and M finitely generted. In particular, $\operatorname{pdim}(M) < \infty$ for all M finitely generated. □

Proof. (of Bass-Murthy). The forward direction is clear. Assume $\operatorname{pdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty$ for all $\mathfrak{p}$. Let $F \xrightarrow{\sim} M$ be a free resolution of M with F_i finitely generated for all $i \geq 0$. For $n \geq 0$, define

$$D_n = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{pdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq n\}.$$

$$= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{im}(\partial_n^F)_{\mathfrak{p}} \text{ is free over } R_{\mathfrak{p}}\}.$$

Since $\operatorname{im}(\partial_n^F)$ is finitely generated for all n, D_n is open in $\operatorname{Spec}(R)$. The point is that

$$D_0 \subset D_1 \subset D_2 \subset \ldots \subset \bigcup_{n \geq 0} D_n.$$

is an ascending chain. Hence $\bigcup_{n \geq 0} D_n = \operatorname{Spec}(R)$. We have

$$\operatorname{Spec}(R) = D_n \text{ for some } n \geq 0.$$

Hence $\operatorname{im}(\partial_n^F)$ is locally free. Thus the image of ∂_n^F is projective. Then

$$0 \to \operatorname{im}(\partial_n^F) \to F_{n-1} \to \ldots \to F_1 \to F_0 \to 0$$

is a projective resolution of M. This shows $\operatorname{pdim}_R M < \infty$. □

7 February 13

Suppose once again R is commutative Noetherian. Now suppose we have a map

$$\varphi : F_1 \to F_0$$

where F_i are free of finite rank. We can choose a basis and write φ as a matrix $R^s \to R^r$, denoted (a_{ij}). $I_c(\varphi)$ is an ideal generated by size c minors of (a_{ij}). Suppose we have M finitely generated R-module and

$$F_1 \to F_0 \to M \to 0$$

and

$$G_1 \to G_0 \to M \to 0$$

are finite free presentations. Call $\operatorname{rank}(F_0) = r$, $\operatorname{rank}(S_0) = s$. Then for all c,

$$I_{r-c}(\varphi) = I_{s-c}(\psi).$$

Exercise 10

Show the above equality.

The ideals are thus invariant of the presentation of M.

Definition 7. We call the ideal $I_{r-c}(\varphi) = \mathrm{Fitt}_c(M)$ the c-th fitting ideal of M.

We have (set $I_0 = R$)

$$R = I_0(\varphi) \supset I_1(\varphi) \supset I_2(\varphi) \supset \ldots \supset I_i(\varphi) \supset 0$$

where $i = \min\{r, s\}$. We get

$$\mathrm{Fitt}_0(M) \subset \mathrm{Fitt}_1(M) \subset \ldots \subset .$$

Fitting ideals thus yield an ascending chian. $\mathrm{Fitt}_0(M) \neq 0$ if and only if $r \leq s$. We also have $\mathrm{Fitt}_c(M) = R$ for $c > \min\{r, s\}$.

Example.

1. Let k be a field and V be a vector space. We could take a presentation

$$0 \to k^n.$$

So $\mathrm{Fitt}_i(V) = k$ if and only if $i \geq n$.

2. When R is a PID, and M is a finitely generated module, we can do

$$0 \to R^s \to R^r \to M \to 0.$$

diagonal matrix with 0's appended below. We have that the 0th fitting ideal is nontrivial if and only if $r = s$, which holds if and only if M has torsion. In this case the zeroth fitting ideal is the determinant of that minor, $\prod d_i$ the product of diagonal elements.

Proposition 6. The following are properties for $R^r \xrightarrow{\varphi} R^s \to M \to 0$.

1. If $R \to S$ is any map of rings, the c-th fitting ideal over S of $S \otimes_R M$ is the extension $S \cdot \mathrm{Fitt}_c^R(M)$.
2. $\mathrm{ann}_R M^r \subset \mathrm{Fitt}_0(M) \subset \mathrm{ann}_R(M)$.
3. Fix $c \geq 0$ and $\mathfrak{p} \in \mathrm{Spec}(R)$. TFAE:
   (a) $\mathrm{Fitt}_c(M) \not\subset \mathfrak{p}$
   (b) $\mathrm{Im}(\varphi)_{\mathfrak{p}}$ contains a free summand of $R_{\mathfrak{p}}^r$ of rank $\geq r - c$.
   (c) $\nu_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq c$
4. Fix $c \geq 0$ and $\mathfrak{p} \in \mathrm{Spec}(R)$. TFAE:
   (a) $\mathrm{Fitt}_{c-1}(M)_{\mathfrak{p}} = 0$ and $\mathrm{Fitt}_c(M)_{\mathfrak{p}} = R_{\mathfrak{p}}$.
   (b) $M_{\mathfrak{p}}$ is free of rank c over $R_{\mathfrak{p}}$.
   (c) $\mathrm{Im}(\varphi)_{\mathfrak{p}}$ is a free summand of rank $r - c$.
5. Fix $c \geq 0$. Then M is projective of rank c if and only if $\mathrm{Fitt}_{c-1}(M) = 0$ and $\mathrm{Fitt}_c(M) = R$.

Proof. Sketch of property 2: Pick a $r \times r$ determinant minor a in φ. We need to show $a \cdot R^r \subset \mathrm{Im}(\varphi)$. One could assume it is the first r columns of the matrix for φ. Multiply it with the cofactor matrix with zeros below it. Laplace expansion tells us we end up with a diagonal matrix with a 's along the diagonal, which is $a \cdot R^r$. Fix $a \in \mathrm{ann}_R(M)$, the map $R^r \xrightarrow{a \cdot I_r} R^r$. We just showed its image lands in the image of φ, so we can lift the map $R^r \to R^r$ to a map $R^s \to R^r$ factoring using φ. Apply the r-th exterior product. We just showed that a^r lands in the fitting ideal, but the proof could be adjusted to arbitrary product by using a diagonal matrix with each entry being an element of the product.

Part 3: Take local ring $(R, \mathfrak{m}, k)$. property 1 says $\mathrm{Fitt}_c(M) = R$ and 3 says $\nu_R(M) \leq c$. The first property is equivalent to $\mathrm{Fitt}_c(M) \neq \mathfrak{p}$. The third property is equivalent to

$$\nu_k(k \otimes_R M) \leq c.$$

The latter two mentioned properties are equivalent, so 1 and 3 are equivalent. The second property involves working with linear algebra.

To prove part 4 of the proposition, can once again assume R is local. The first property essentially says that there are no relations. It further involves deduction from 3. □

Theorem 10. Hilbert-Burch Theorem: Given $I \subset R$ with resolution

$$0 \to R^n \xrightarrow{\varphi} R^{n+1} \to I \to 0,$$

then there exists a nzd a_0 such that $I = a_0 \mathrm{Fitt}_1(\varphi)$. If I is projective, then it is principal. Otherwise, $\mathrm{pdim}(I) = 1$, then $\mathrm{depth}(I_n(\varphi), R) \geq 2$ and $\frac{R}{I_n(\varphi)}$ is perfect. Conversely, if $\varphi : R^n \to R^{n+1}$ is such that $\mathrm{depth}_R(I_n(\varphi), R) \geq 2$, then

$$0 \to R^n \to R^{n+1} \to I_n(\varphi) \to 0$$

is a free resolution. The last map admits description as follows. φ is a $n \times n+1$ matrix, and $n \times n$ minors can be obtained by removing single rows. Call the minor obtained by removing the i-th row a_i. The last map is $(-a_1, a_2, -a_3, \ldots, (-1)^{n+1} a_{n+1})$. More on this theorem can be read from Bruns and Herzog.

This theorem is important to proving that regular local rings are UFDs.

Lemma 9. Suppose M is a projective module. If M has a finite free resolution, then it must have a free resolution of length 1.

Proof. Let's say we have a free resolution of M

$$0 \to F_s \to F_{s-1} \to \ldots \to F_1 \to F_0 \to 0,$$

where F_i are free, $s \geq 2$. Because M is projective, $\partial(F_i)$ is projective for all $i \geq 1$.

$$F_{s-1} \cong F_s \oplus \partial\left(F_{s-1}\right).$$

We get

$$0 \to \partial(F_{s-1}) \hookrightarrow F_{s-2} \to \ldots \to F_0 \to 0.$$

We can take direct sum with F_s with the first two components above to yield a shorter resolution. □

Corollary 10. If $I \subset R$ is a projective finite free resolution implies I is principal.

Proof. It has a resolution as follows

$$0 \to R^n \to R^{n+1} \to I \to 0.$$

Hilbert-Burch then says that I is principal! □

Theorem 11. Suppose R is a commutative Noetherian domain such that each finitely generated R-module has a finite free resolution. Then R is a UFD.

Corollary 11. Suppose regular local rings are UFDs. Caveat: not all regular rings are UFDs.

Dedekind domains with nontrivial class group, for example, do not satisfy the strengthened version of the corollary.

Proof. First, suppose R is local. Induction on dimension R. We can assume the dimension of R is at least two, since we know the case when dimension 1 is DVRs. Pick $w \in \mathfrak{m}_R \setminus \mathfrak{m}_R^2$. Then R/w is a regular local ring, so hence it is a UFD. Thus w is a prime element. It thus suffices to very that R_w is a UFD. $\dim(R_w) < \dim(R)$.

Pick $\mathfrak{p} \in \operatorname{Spec}(R_w)$ of height 1. We would like to show that it is principal. Note $\mathfrak{p}$ has a finite free resolution, for $\mathfrak{p}$ coming from R. It is enough to show that $\mathfrak{p}$ is projective, by a previous corollary. This property can be tested locally. The localizations are regular local rings strictly being of less dimension than R, hence UFDs. Hence $\mathfrak{p}$ is locally free and hence projective.

For the non-local case, fix $\mathfrak{p}$ of height 1. We want that $\mathfrak{p}$ is principal. Since $\mathfrak{p}$ has a finite free resolution, it suffices to prove it is projective. This reduces to the local case, and we are done. $\square$

Let's say R is a PID and M is a torsion module. It has a presentation

$$0 \to R^r \to R^r \to M \to 0.$$

The zeroth fitting ideal is the product of the diagonal elements of the diagonal matrix. The length of M is the length of $R/\operatorname{Fitt}_0(M)$.

8 February 15

Today, we learn about complete intersections. We do **Cohen structure theory.** Suppose we have a local ring $(R, \mathfrak{m}, k)$. R is **equicharacteristic** if it contains a field k as a subring. We can look at the structure map $\mathbb{Z} \to R$. If $\operatorname{char}(R) = p > 0$ is a prime, the map factors

$$\frac{\mathbb{Z}}{(p)} \to R,$$

then p is also the characteristic of the residue field of R. Another thing that can happen is that $\operatorname{char}(k) = 0$. In particular, the image of $\mathbb{Z}$ lands in the units of R and 0. The mixed characteristic case: when $\mathbb{Q} \not\subset R$, and $\operatorname{char}(k) = p > 0$, $\operatorname{char}(R) = 0$. We have

$$\mathbb{Z}_{(p)} \hookrightarrow R.$$

When R is $\mathfrak{m}$-adically complete. If R is equicharacteristic, then R contains a copy of k as a subring.

When R is mixed characteristic, then it contains a discrete valuation ring $\mathscr{O}$,

$$\mathscr{O} \to R \to k.$$

MOreover, R is a quotient of

$$k[[x_1, \ldots, x_n]]$$

in the equicharacteristic case, and in the mixed characteristic is a quotient of

$$\mathscr{O}[[x_1, \ldots, x_n]].$$

Thus R is of the form $\frac{S}{I}$, where S is a regular ring. We call it the **Cohen presentation of** R**.** We would like to obtain from this a **minimal Cohen presentation.** Let's say $\frac{S}{I}$ is a Cohen-presentation, so S is a regular local ring. $I \subset \mathfrak{m}_S$. Suppose $I \not\subset \mathfrak{m}_S^2$. Then take an $x \in I$, then $S \to R$ via quotient projection factors through $\frac{S}{xS}$. Now $\frac{S}{xS}$ is still regular. By trimming elements, we can ensure that $I \subset \mathfrak{m}_S^2$. This is equivalently that

$$\operatorname{edim}(R) = \operatorname{edim}(S).$$

(Recall the embedding dimension is the rank of $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ over the residue field for a local ring.) We say that $R = \frac{S}{I}$ in this case is a **minimal Cohen presentation**.

Definition 8. A local ring $(R, \mathfrak{m}, k)$ is said to be a complete intersection or abbreviated c.i. if in some Cohen presentation of the $\mathfrak{m}$-adic completion of R

$$\hat{R} = \frac{S}{I},$$

I can be generated by a regular sequence in S. Note that R need not admit a Cohen presentation.

Let K^R = the Koszul complex on a minimal generating set for the maximal ideal of R, $\mathfrak{m}_R$. This is well defined up to isomorphism of complexes. What is $K^R \otimes_R \hat{R}$? It is just the Koszul complex of the completion $K^{\hat{R}}$. The minimal generating set for $\mathfrak{m}_R$ also generates $\hat{R}\mathfrak{m}_R = \mathfrak{m}_{\hat{R}}$. Then

$$H(K^R) \to H(\hat{R} \otimes_R K^R).$$

can be rewritten

$$H(K^R) \to \hat{R} \otimes_R H(K^R).$$

because $R \to \hat{R}$ flat. We get

$$H(K^R) \xrightarrow{\cong} \hat{R} \otimes_R H(K^R).$$

Since $\mathfrak{m} \cdot H(K^R) = 0$, this map is an isomorphism. Summing up, $K^R \xrightarrow{\sim} K^{\hat{R}}$.

Lemma 10. Let's say $R = S/I$ is a minimal Cohen presentation. If you compute the rank $H_i(K^R)$ over residue field k, this is merely the Betti numbers $\beta_i^S(R)$. In particular, the right hand side is independent of the minimal Cohen presentation.

Proof. Let K^S be the Koszul complex of S. S regular, so $K^S \xrightarrow{\sim} k$. In particular, the Koszul complex over S is a resolution of the residue field over S. Auslander Buchsbaum. Now, since $I \subset \mathfrak{m}_S^2$, what happens when we tensor K^S by R over S? It is just K^R. This just tells us that $H_i(K^R) = \mathrm{Tor}^S(R, k)$. The Betti numbers record the ranks of these Tor modules, so we are done. □

If we look at $R = S/I$ a minimal Cohen presentation, Krull's principal height theorem tells us

$$\mathrm{ht}(I) \leq \nu_S(I).$$

where $\nu_S(I)$ is the minimal number of geenrators of I. Recall that I is generated by a regular sequence if and only if equality holds. Use $\mathrm{depth}_S(I, S) = \mathrm{ht}(I)$.

Thus I is generated by a regular sequence if and only if $\nu_S(I)$ can be rewritten $\dim(S) - \dim(S/I) = \mathrm{edim}(S) - \dim(R) = \mathrm{edim}(R) - \dim(R)$. Also $\nu_S(I) = \beta_1^S(R)$. Finally, this number is also $\mathrm{rank}_k(H_1(K^R))$.

Summary: $R = S/I$ a minimal Cohen presentation, then I is ci if and only if $\mathrm{rank}(H_1(K^R)) = \mathrm{edim}(R) - \dim(R)$.

Proposition 7. If $(R, \mathfrak{m}, k)$ is local, then R is ci if and only if

$$\mathrm{rank}(H_1(K^R)) = \mathrm{edim}(R) - \dim(R).$$

Proof. $\hat{R} = S/I$ is a minimal Cohen presentation if

$$H(K^R) = H(K^{\hat{R}}).$$

or

$$\mathrm{edim}(R) = \mathrm{edim}(\hat{R}).$$

or
$$\dim(R) = \dim(\hat{R}).$$
$\square$

Remark. Say $\hat{R} = S/I$ is some Cohen-presentation with I generated by a regular sequence. We have $S \to \hat{R}$. Say $I \not\subset \mathfrak{m}^2$. We can pick $x \in I \setminus \mathfrak{m}_S^2$. Then $x \notin \mathfrak{m}I$ and so so can be extended to a minimal generating set for I. Say $(x, \overline{y})$. Then $x, \overline{y}$ is a regular sequence. If we call $\hat{R} = S'/(\overline{y})S'$.

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & \hat{R} \\ & S/xS \arrow[u, "\exists"'] \end{tikzcd}$$

Theorem 12. Iff R is a complete intersection, $R_{\mathfrak{p}}$ is a complete intersection $\forall \mathfrak{p} \in \operatorname{Spec}(R)$.

The part of this theorem that is nontrivial is that $R_{\mathfrak{p}}$ generally has very different Koszul complex from that of R. This is clear if R has a Cohen presentation. If R is of the form S/I, where I is generated by a regular sequence. $\mathfrak{p} \in \operatorname{Spec}(R) \subset \operatorname{Spec}(S)$.
$$R_{\mathfrak{p}} \cong S_{\mathfrak{p}}/IS_{\mathfrak{p}}.$$
$S_{\mathfrak{p}}$ is a regular local ring, and $IS_{\mathfrak{p}}$ is again generated by a regular local sequence.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \hat{R} \arrow[d, "\nexists"] \\ R_{\mathfrak{p}} \arrow[r] & \hat{R}_{\mathfrak{p}}^{\mathfrak{p}R_{\mathfrak{p}}} \end{tikzcd}$$

Localization does not necessarily commute with completion. If we pic $q \in \operatorname{Spec}(\hat{R})$ such that $q \cap R = \mathfrak{p}$,

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \hat{R} \arrow[d] \\ R_{\mathfrak{p}} \arrow[r] & \hat{R}_q \end{tikzcd}$$

where the bottom map is flat and local.We can use the following theorem.

Theorem 13. (Gromov) $\varphi : R \to S$ is a flat local map. Then S is ci if and only if R and $\frac{S}{\mathfrak{m}_R S}$ are ci.

Proposition 8. If R is ci, then $R[[x_1, \dots, x_n]]$ is ci.

Let's say we have a local ring $(R, \mathfrak{m}, k)$. Let's say R is $\frac{S}{I}$ that is also a minimal presentation. Also write $\mathfrak{m}_S = (x_1, \dots, x_n)$ a minimal generating set. Then also write $I = (f_1, \dots, f_c)$ where $\overline{f}$ is a minimal generating set. We write
$$\mathfrak{m}_R = \overline{x} \cdot R.$$
What does the Koszul complex look like? Recall its end admitted form
$$\to \bigoplus_{i=1}^{n} Re_i \to R \to 0$$

where $\partial(e_i) = x_i$. Write $f_j = \sum_{j=1}^n s_{ij}x_i$, where $s_{ij} \in \mathfrak{m}_S$. Let $z_j = \sum_{i=1}^n \overline{s_{ij}}e_i$ where the bar denotes image in R. We claim z_j are cycles. The differentiation becomes

$$\partial z_j = \sum_{i=1}^n \overline{s_{ij}}x_i = \overline{f_j} = 0.$$

Claim: The classes $[z_1], \dots, [z_c] \in H_1(K^R)$ form a basis. This can be checked.

If we have a regular ring, the Koszul complex is a resolution of the residue field. (Tate) Take $(R, \mathfrak{m}, k)$ local, and $f : R^n \to R$ where $\mathrm{Im}(f) = \mathfrak{m}$, $n = \mathrm{edim}(R)$. Then we get

$$K^R = \left(\bigwedge^* (\Sigma R^n), \partial\right).$$

Here, K^R is a differential graded (DG) R-algebra, even commutative. This means $H(K^R)$ is a graded commutative k-algebra. Note the map of k-algebras

$$\chi^R : \bigwedge_* \left(\Sigma H_1(K^R)\right) \to H(K^R).$$

Theorem 14. (Tate) R is ci if and only if χ^R is an isomorphism. (asmus) R is ci if and only if

$$\bigwedge^2 H_1(K^R) \twoheadrightarrow H_2(K^R).$$

In preparation for the proof, we will look into things like multiplicative resolutions.

9 February 27

Wiebe's theorem. Let S be a commutative Noetherian ring, and $I \subset J$ where

$$I = (a_1, \dots, a_n).$$

and $J = (x_1, \dots, x_n)$ where both generating sets are regular sequences. Write $\overline{a} = U \cdot \overline{x}$

Lemma 11. $(I : J) = \{s \in S | sJ \subset I\} = I + (\det U)$, where $\det(U) \notin I$.

Example. Suppose you take $k[x_1, \dots, x_n]$. Then use the ideal

$$(x_1^\ell, x_2^\ell, \dots, x_n^\ell) \subset (x_1, \dots, x_n)..$$

Then we can use the matrix

$$U = \begin{bmatrix} x_1^{\ell-1} & 0 & \dots & 0 \\ 0 & x_2^{\ell-1} & \dots & 0 \\ \dots & 0 & x_n^{\ell-1} & \end{bmatrix}.$$

(Short exercise apply the lemma to get a conclusion). Given the parenthetical conconlusion, if we instead took a system of the parameters instead of the regular sequence, this yields the monomial conjecture.

Proof. Equivalently,

$$\left(0 :_{\frac{S}{I}} \frac{J}{I}\right) = (det(U)) \cdot \left(\frac{S}{I}\right).$$

The former is $H_n(\overline{x} : \frac{S}{I})$. The vector $x_1, \pm x_2, \ldots, x_n$ yields a map

$$0 \to S/I \to (S/I)^n\,.$$

This induces

$$\bigwedge^* U : K(\overline{a} : S) \to K(\overline{x} : S).$$

We get a map

$$K_n(\overline{a} : S) \xrightarrow{det(U)} K_n(\overline{x} : S).$$

This induces

$$\bigwedge^* U K(\overline{a} : S/I) \to K(\overline{x} : \frac{S}{I}).$$

further inducing

$$H_n(\overline{a}, S/I) \xrightarrow{\det(U)} H_n(\overline{x}, S/I).$$

It is enough to prove that the above map, $H_n(\bigwedge^* U)$, is onto. Moreover,

$$H_*(\overline{a}, S/I) \to H_*(\overline{x} : S/I).$$

The theorem of Tate from last time showed that the latter and former are exterior algebras over their degree 1 parts. Therefore it suffices to prove that

$$H_1(\overline{a}, S/I) \twoheadrightarrow H_1(\overline{x}, S/I).$$

$$K(\overline{a}, S) \otimes_S K(\overline{x} : S).$$

The latter is weakly equivalent to $K(\overline{x}, S/I)$ and the former to $K(\overline{a}, S/J)$. We note that $\bigwedge^* U$ induces a map $K(\overline{a}, \frac{S}{I}) \to K(\overline{x}, \frac{S}{I})$, and also a surjection exists $K(\overline{a}, S/I) \to K(\overline{a}, S/J)$. $\square$

Remark. Special case: $R = S/I$ where S is RLR and $I \subset \mathfrak{m}_S^2$. Also, $\dim(R) = 0$ and R is a complete intersection. So $I = (a_1, \ldots, a_n) \subset (x_1, \ldots, x_n) = \mathfrak{m}_S$. The former is a regular sequence and $n = \text{edim}(R)$. $\overline{a} = U \cdot \overline{x}$. By the lemma, $(0 : \mathfrak{m}_R) = \det(U)$.

Definition 9. If $(R, \mathfrak{m}, k)$ is local, then the **socle** of R is defined to be

$$\text{Soc}(R) = (0 : \mathfrak{m}_R) = \text{Hom}_R(k, R).$$

Corollary 12. Let R be a complete intersection ring with $\dim(R) = 0$. Then the socle satisfies

$$\text{Soc}(R) \subset (\text{any nonzero ideal})\,.$$

Exercise from the preceding remark. It says complete intersection rings are Gorenstein. In general, the socle of R is annihilated by $\mathfrak{m}_R$. So we can say

$$\text{rank}_k(\text{Soc}(R)).$$

which is called the **type of** R.

Corollary 13. Suppose

$$(R, \mathfrak{m}_R) \twoheadrightarrow T.$$

with R a complete intersection ring. If $\varphi(\text{Soc}(R)) \neq 0$, then φ is an isomorphism.

From the previous corollary. Look at the kernel of the map!

Returning to Wiebe's theorem. If we look at $R(,\mathfrak{m}_R)$ local, if the image of $\bigwedge^c H_1(K) \to H_c(K)$ is nonzero, and $c = \operatorname{edim}(R) - \operatorname{depth}(R)$, then R is a complete intersection ring.

It suffices to check when the depth of R is zero. In particular,

$$\bigwedge^n H_1(K^R) \xrightarrow{\neq 0} H_n(K^R)$$

for $n = \operatorname{edim}(R)$ implies that R is a complete intersection ring. It is equivalent to the zeroth fitting ideal of $\mathfrak{m}_R$ being nonzero. Recall that one can assume R is complete, and write a minimal Cohen-presentation, $R = S/I$. You can identify $H_1(K^R)$ with the minimal generators of $I/\mathfrak{m}_S I$.

If we took $z_1, \ldots, z_n \in Z_1(K^R)$, such that $z_1 \wedge \ldots \wedge z_n$ nonzero in K_n^R, then the hypothesis states that $z_1, \ldots, z_n$ must be linearly independent.

Let $a_1, \ldots, a_n \in I/\mathfrak{m}I$ be some representatives of $z_1, \ldots, z_n$.

Case 1: Suppose $\dim(R) = 0$. In particular, the height of I is n. Then the depth of $\operatorname{depth}(I, S) = n$.

Exercise 11

$\exists a_1', \ldots, a_n' \in \mathfrak{m}_S I$ such that

$$a_1 + a_1', \ldots, a_n + a_n'.$$

is a regular sequence in S. Then their images are in H_1. Then a_i and $a_i + a_i'$ define the same class in $H_1(K^R)$. Simplifying notation, we can therefore assume $a_1, \ldots, a_n$ is a regular sequence.

This means $R' = S/(a_1 + a_1', \ldots, a_n + a_n')$, we have a map $R' \twoheadrightarrow R$, inducing

$$\bigwedge^n H_1(K^{R'}) \to H_n(K^{R'}).$$

forming a commutative diagram with bottom row

$$\bigwedge^n H_1(K^R) \to H_n(K^R).$$

Image of the map $\bigwedge^n H_1(K^{R'}) \to \bigwedge^n H_1(K^R)$ contains $z_1 \wedge \ldots \wedge z_n$. Bottom map is nonzero implies the diagonal composition is nonzero, impling the right map is nonzero. But the right map induces $R' \twoheadrightarrow R$. Since $\operatorname{Soc}(R')R \neq 0$, $R' \cong R$ by the corollary earlier.

Case 2: Now suppose $\dim(R) > 0$. We would like to show this case is impossible. We know that $z_1 \wedge \ldots \wedge z_n \neq 0$ in $K_n^R \cong R$. Therefore there exists $s \gg 0$ such that $z_1 \wedge \ldots \wedge z_n \notin \mathfrak{m}_R^s$ by the Krull intersection theorem. We would like to stare at the following diagram: $R \to R/\mathfrak{m}^{s+1} \to R/\mathfrak{m}^s$. This yields the following commutative diagram:

$$\begin{tikzcd}
\bigwedge^n H_1(K^R) \arrow[r] \arrow[d] & H_n(R) \arrow[d] \\
\bigwedge^n H_1(K^{R/\mathfrak{m}^{s+1}}) \arrow[r] \arrow[d] & H_n(R/\mathfrak{m}^{s+1}) \arrow[d] \\
\bigwedge^n H_1(K^{R/\mathfrak{m}^s}) \arrow[r] & H_n(R/\mathfrak{m}^s)
\end{tikzcd}$$

The diagonal map being nonzero implies $H_n(R/\mathfrak{m}^{s+1}) \to H_n(\frac{R}{\mathfrak{m}^s})$ is nonzero. This implies the middle horizontal is nonzero. So $R/\mathfrak{m}^{s+1}$ is a complete intersection. Then the vertical map being nonzero implies $R/\mathfrak{m}^{s+1} \xrightarrow{\sim} R/\mathfrak{m}^s$. Hence $\mathfrak{m}^s = 0$ by Nakayama's lemma.

We have a map

$$\bigwedge^i H_1(K^R) \to H_i(K^R).$$

Tate's theorem said that if R is complete intersection, then this map is isomorphic for all i. Asmus: Onto for $i = 2$ implies that R is complete intersection. Wiebe's theorem: nonzero for $i = \mathrm{edim}(R) - \mathrm{depth}(R)$ implies R is complete intersection. Always, it is nonzero for $i > \mathrm{edim}(R) - \dim(R)$. This was proved by Bruns. The only known proof uses the homological conjectures. This in turn is connected to the canonical element conjecture.

10 March 1

Today, we discuss injective modules. Suppose A is any ring, not necessarily commutative or Noetherian. Fix E, any (left) A-module. Given any exact sequence

$$0 \to N \to M \to L \to 0,$$

we have the following exact sequence:

$$0 \to \mathrm{Hom}_A(L, E) \to \mathrm{Hom}_A(M, E) \to \mathrm{Hom}_A(N, E).$$

We say that E is injective if the sequence above is exact on the right. Equivalently, $\mathrm{Hom}_A(\bullet, E)$ is an exact functor on the category of A-modules. Also equivalently, given any map $N \hookrightarrow M$ and $f : N \to E$, there exists $\tilde{f} : M \to E$ extending f.

Baer's criterion. E is injective if and only if $I \hookrightarrow A$ (left ideal) has the extension property. So $\mathrm{Hom}_A(A, E) \twoheadrightarrow \mathrm{Hom}_A(I, E)$.

Proof. $\Rightarrow$ is clear (assuming E is injective and proving the extension property). To show the reverse, the hypothesis implies that if you had any collection $I_i \hookrightarrow A$, we also get the extension property for $\bigoplus_i I_i \hookrightarrow \bigoplus A$. Fix $N \hookrightarrow M$ and $x \in M$. We thus get $A \to M$ sending 1 to x. Take the pullback of the two maps as follows:

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d] & A \arrow[d] \\ N \arrow[r] & M \end{tikzcd}$$

Note that from the definition of the pullback, if the bottom map is injective so is the top map. Also the definition of the defined module was $I = \{(n, a) | n = ax\}$, which naturally then defines a submodule of A. Kernel of $I \to N$ is the kernel of $A \to M$, which is $\mathrm{ann}_A(x)$. We can construct the diagram

$$\begin{tikzcd} \bigoplus_{x \in M} I_X \arrow[r, hook] \arrow[d] & \bigoplus_{x \in M} A \arrow[d] \\ N \arrow[r, hook] \arrow[dr] & M \arrow[d] \\ & E \end{tikzcd}$$

This proof is unfortunately mistaken. Apply the more conventional proof that uses Zorn's lemma. ☐

Proof. Alternative proof using Zorn's lemma. Given $N \hookrightarrow M$ and $f : N \to E$, we would like to extend it. We look at the collection of submodules $N \subset U \subset M$ satisfying the extension property so $\tilde{f} : U \to E$ exists. We can apply Zorn's lemma with respect to this, provided $M \neq U$, so we get a contradiction if $M \neq U$. Extend the map from $Ax \cap U$ to Ax. ☐

Lemma 12. Assume R is a commutative domain. E injective implies that E divisible. The converse holds when R is a PID.

Proof. Divisible means that given $x \in E$ and $r \in R$ which is nonzero, there exists $y \in E$ such that $x = ry$. $R \to E$ and $R \xhookrightarrow{\cdot r} R$ gives map $R \to E$ from the latter copy of R which factors the map. Hence the first implication follows. For the second implication, R PID implies that ideals are of the form (r). Divisbility implies that principal ideals satisfy the extension property, but PID says that these are all of the ideals, hence E is injective. $\square$

Example. Given $\mathbb{Z}$, $\mathbb{Q}$ is divisible and therefore injective.

Lemma 13. Any Abelian group ($\mathbb{Z}$-module) can be embedded into an injective $\mathbb{Z}$-module.

Proof. $\mathbb{Q}$ divisible implies that $\bigoplus \mathbb{Q}$ is also divisible. Now F any free $\mathbb{Z}$-module can be embedded

$$F \hookrightarrow \bigoplus \mathbb{Q}.$$

Then if $U \subset F$ is a submodule, then $F/U \hookrightarrow \bigoplus \mathbb{Q}/U$. Any $\mathbb{Z}$ module is a quotient of a free module. $\square$

Lemma 14. Let $A \to B$ be any map of rings and E be any injective A-module. Then $\mathrm{Hom}_A(B, E)$ viewed as a B-module is injective.

Take $f : B \to E$ be A-linear. $b \in B$, we can define $b \cdot f(x) = f(xb)$. Associativity of gives the left-B-module structure on $\mathrm{Hom}(B, E)$.

Proof. We would like to check that $\mathrm{Hom}_B(\bullet, \mathrm{Hom}_A(B, E))$ is an exact functor. Adjunction yields it is just $\mathrm{Hom}_A(\bullet, E)$. But the latter being exact implies that the former is exact.

Exercise 12

Come up with counter examples to the lemma's converse.

$\square$

Proposition 9. Let A be any ring. Each A-module embeds into an injective A-module.

Proof. Any ring is a $\mathbb{Z}$-module, so we get $\mathbb{Z} \to A$. Suppose we are given M an A-module. We can think of it as a $\mathbb{Z}$-module via $M|v\mathbb{Z} \hookrightarrow E$. (|v indicates downward arrow denoting forgetful functor to $\mathbb{Z}$-modules).

$$M \xhookrightarrow{\lambda} \mathrm{Hom}(A, M|v\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(A, E).$$

$\square$

Any extension $N \hookrightarrow M$ is essential if $\forall U \subset M$ nonzero submodules, $U \cap N \neq 0$. It is proper if $N \neq M$.

Lemma 15. An A-module E is injective if and only if it admits no proper essential extensions.

Proof. $\Rightarrow$ is true. If E is injective, take extension $E \hookrightarrow M$. It must split, because

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, equal] & M \arrow[dl] \\ E & \end{tikzcd}$$

yields the projection back to the E factor in $M \cong E \oplus W$. If $E \hookrightarrow M$ is essential, then $W \cap E = 0$ implies $W = 0$, so $E = M$.

Conversely, suppose E admits no proper essential extensions. We would like to show it is injective, so look at $N \hookrightarrow M$ with map $N \to E$. We would like to show the existence of map $M \to E$ extending $f : N \to E$. Look at the pushforward of the maps that we do have:

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d] & M \arrow[d] \\ E \arrow[r] & W \end{tikzcd}$$

where $W = \frac{M \oplus E}{((n, -f(n)) | n \in N)}$. Let $U \subset W$ be a maximal submodule such that $U \cap E = 0$ by Zorn's lemma. We have properties of maps below:

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d] & M \arrow[d] \\ E \arrow[r] \arrow[dr, hook, "\sim"'] & W \arrow[d, two heads] \\ & W/U \end{tikzcd}$$

$E \to W/U$ is essential because U is maximal. We can invert the ismorphism to get the desired map $\tilde{f} : M \to E$. ☐

Definition 10. Let M be any A-module. Then an injective hull of M is an essential extension $M \hookrightarrow E$ with E injective. Thus E is unique up to isomorphism.

Proposition 10. Let M be an A-module.

- Take $M \hookrightarrow I$ with I injective. Then a maximal essential extension of M in I is injective and hence an injective hull of M.
- If $M \hookrightarrow I$ with I injective, and $M \hookrightarrow E$ is an injective hull, then $\exists\ \varphi : E \to I$ monomorphism such that

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, hook] & E \arrow[dl, "\varphi"] \\ I & \end{tikzcd}$$

- $M \hookrightarrow E$ and $M \hookrightarrow E'$ injective hulls, there exists diagram

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, hook] & E \arrow[dl, "\varphi"] \\ I & \end{tikzcd}$$

$E_A(M)$ can be written as **the** injective hull of M. Note the automorphism group of E tends to be nontrivial.

Proof. 2 and 3 are straightforward. For 2, look at the kernel of φ, which must intersect M trivially, hence is itself trivial by the essential extension property.

For 1, let $M \hookrightarrow E \hookrightarrow I$, where E is a maximal essential extension of M in I. It is enough to prove that E admits no proper essential extensions. Say $E \hookrightarrow U$ is an essential extension. We note we have the diagram with the existence of ι.

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, hook] & U \arrow[dl, "\iota"] \\ I & \end{tikzcd}$$

The kernel of ι is 0. Therefore, we can assume E/M essential and U/E essential implies U/M is essential, implying $U = E$. ☐

Key consequence: A any ring and M any module, then M admits a minimal injective resolution unique up to isomorphism. This is fairly surprising, as we have some fairly relaxed hypotheses! Take any M, which injects into its injective hull. Obtain cokernel, and take the injective hull of the cokernel. Continue the process along injective hulls to get the minimal injective resolution of M.

Remark. If we have any family $\{E_i\}$ of injectives, then $\prod E_i$ is injective. Baer's criterion implies that when A is left Noetherian, then $\bigoplus E_i$ is injective.

11 March 13

Recall structure of injectives: Let M, N be modules over a ring. Then

$$\mathrm{Ext}^i_R(M, N) = H^i(\mathrm{Hom}_R(P, N)).$$

If E is an R-module. Then E is injective if and only if $\mathrm{Ext}^1_R(\bullet, E) = 0$ on $\mathrm{Mod}(R)$ if and only if $\mathrm{Ext}^1_R(R/I, E) = 0$ for all $I \subset R$. If we get an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathrm{Mod}(R)$,

$$\mathrm{Hom}_R(Z, E) \to \mathrm{Hom}(Y, E) \to \mathrm{Hom}_R(X, E) \to \mathrm{Ext}^1_R(Z, E) \to \dots.$$

Note that the surjectivity of $\mathrm{Hom}(Y, E) \to \mathrm{Hom}(X, E)$ for any such short exact sequence implies that all ext groups $\mathrm{Ext}^i(\bullet, E)$ vanish.

Let R be commutative Noetherian.

1. $\{E_\lambda\}_\lambda$ injective implies that $\bigoplus_\lambda E_\lambda$ is injective. This is from Baer's criterion.
2. We would like to claim that any injective module can be written as a direct sum of indecomposable injectives. Recall indecomposable can't be written as a direct sum of modules.
3. Indecomposable injectives are precisely of the form $\{E_R(\frac{R}{\mathfrak{p}})\}_{\mathfrak{p} \in \mathrm{Spec}(R)}$.
4. $E(R/\mathfrak{p})$ is $\mathfrak{p}$-local and $\mathfrak{p}$-power torsion. M an R-module is $\mathfrak{p}$-local if $M \xrightarrow{\sim} M_\mathfrak{p}$ is isomorphic. Equivalently, $M \xrightarrow{\cdot r} M$ is isomorphic for all $r \in R \setminus \mathfrak{p}$. Equivalently, R action on M factors through $R \to R_\mathfrak{p}$.

   M is $\mathfrak{p}$-power torsion if $\Gamma_{V(p)}(M)$ (which is defined as $\bigcup (0 :_M \mathfrak{p}^n)$,) is $= M$. $E(R/\mathfrak{p})$ is $\mathfrak{p}$-local and $\mathfrak{p}$-power torsion.
5. $E(R/\mathfrak{p}) = E(\frac{R}{\mathfrak{q}})$ if and only if $\mathfrak{p} = \mathfrak{q}$.

All of this is actually true for any ring that is finite over a commutative Noetherian ring.

Lemma 16. Set $S = U^{-1}R$ for multiplicatively closed $U \subset R$. Look at $R \to S$, which must be flat.

1. If E is an injective R-module, then $S \otimes_R E$ is injective S-module.
2. An injective S-module is injective over R.
3. $M \hookrightarrow N$ is essential, then $S \otimes_R M \hookrightarrow S \otimes_R N$ is essential. In particular, $S \otimes_R E_R(M) = E_S(S \otimes_R M)$.

Proof. 2 holds for any flat map $R \to S$. If I is an injective S-module, we would like to examine $\mathrm{Ext}^1_R(M, I)$ (we claim it is zero, see remark at beginning). The ext group is equal to, by adjunction, $\mathrm{Ext}^1_S(S \otimes_R M, I) = 0$ since I is injective as an S-module.

For 1, any finitely generated S-module is of the form $S \otimes_R M$ with M finitely generated over R. Take a presentation and clear denominators to get this claim. Moving on, we have

$$\mathrm{Ext}^1_R(S \otimes_R M, S \otimes_R E) \cong \mathrm{Ext}^1_R(M, S \otimes_R E) \cong \mathrm{Ext}^1_R(M, E) \otimes S.$$

S can be pulled out because S is flat over R. But the latest expression is 0, so we are done. By Baer's criterion, it suffices to check ideals, but S Noetherian implies that all ideals are finitely generated S-modules, which all have the aforementioned form. So this is all cases.

□

The key property of localization is that $R \to S$ flat implies that

$$S \otimes_R S \xrightarrow{\sim} S.$$

Absolutely flat maps.

Caveat: $R \to S$ flat does not necessarily imply that $(S \otimes_R E)$ is injective. The key example here is $R \to \hat{R}$ for local ring R and its completion with respect to the maximal.

Let us finally approach the essential extension claim 3 in the lemma:

Proof. We want to show $U^{-1}M \hookrightarrow U^{-1}N$ is esssential. If we take $S \cdot (\frac{x}{u}) \cap U^{-1}M \neq 0$ for $\frac{x}{u} \in U^{-1}N$. Can assume $u = 1$ and $x \neq 0$ in N. This may seem to reduce to the problem of $M \hookrightarrow N$ being essential, but note that a module does not necessarily include into the localization. Consider $\{ann_R(xu) \mid u \in U\}$. R Noetherian implies that this has maximal elements. We can replace x by a suitable ux and assume $\mathrm{ann}_R(x)$ is a maximal element. We examine $M \cap Rx$. It must be of the form $I \cdot x$ for some ideal $I \subset R$. So $(r_1 x, \ldots, r_n x) \neq 0$. Say $r \in R$ is such that $rx = 0$ in the localization. This implies that $\exists u \in U$ such that $urx = 0$. Now $r \in \mathrm{ann}_R(ux) = \mathrm{ann}_R(x)$. $r_i x \neq 0$ and in $M \cap Rx$, so we are done! □

This shows that the localization of injective hulls are injective hulls.

Lemma 17. Fix $\mathfrak{p} \in \mathrm{Spec}(R)$.

1. The injective hull $E(R/\mathfrak{p})$ is indecomposable.
2. The associated primes of $E(R/\mathfrak{p})$ are $\{\mathfrak{p}\}$.
3. $E(\frac{R}{\mathfrak{p}})$ is $\mathfrak{p}$ local and $\mathfrak{p}$-power torsion.
4. $\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E(R/\mathfrak{p})) \cong k(\mathfrak{p})$.

This lemma takes care of 3,4, and 5 of the beginning remarks we wanted to show. 1 was taken care of in the last lecture, so we only need to work on 2 afterwards.

Proof. For 1 we will show the stronger claim that $M \neq 0$ and $N \neq 0$ in $E(R/\mathfrak{p})$, $M \cap N \neq 0$. It is equivalent to every submodule being indecomposable. We know that $R/\mathfrak{p} \to E(R/\mathfrak{p})$ is essential, so $M \cap R/\mathfrak{p}$ and $N \cap R/\mathfrak{p}$ are nonzero. But these are ideals in $R/\mathfrak{p}$, but ideals in a domain always have nontrivial intersection, so this implies the entire claim.

For 3, look at $R/\mathfrak{p}$. If we note $k(\mathfrak{p})$, the field of fractions, we have $R/\mathfrak{p} \hookrightarrow k(\mathfrak{p})$ is essential. We can further take

$$R/\mathfrak{p} \hookrightarrow k(\mathfrak{p}) \hookrightarrow E_{R_\mathfrak{p}}(k(\mathfrak{p})).$$

The latter is injective over $R_\mathfrak{p}$, hence injective over R by the lemma earlier.

$$E_{R_\mathfrak{p}}(k(\mathfrak{p})) \cong R_\mathfrak{p} \otimes_R E_R(R/\mathfrak{p}).$$

We also have $E_R(R/\mathfrak{p}) = E_{R_\mathfrak{p}}(k(\mathfrak{p}))$. Hence $E_R(R/\mathfrak{p})$ is $\mathfrak{p}$-local.

2 follows from: for all $M \in \mathrm{Mod}(R)$, $\mathrm{Ass}(M) = \mathrm{Ass}(E_R(M))$. Always we have $M \hookrightarrow E_R(M)$, so $\mathrm{Ass}_R(M) \subset \mathrm{Ass}(E_R(M))$. $R/\mathfrak{p} \cong U \hookrightarrow E_R(M)$. If we look at $\mathrm{Ass}_R(U) \supset \mathrm{Ass}_R(U \cap M) \subset \mathrm{Ass}(M)$. Hence $\mathrm{Ass}(U \cap M) = \{\mathfrak{p}\}$. This is also saying that $E(R/\mathfrak{p})$ is $\mathfrak{p}$-power torsion. This is an exercise and implies 3 completely.

For 4, we may assume that R is local and $\mathfrak{p}$ is the maximal ideal. We want to show

$$\mathrm{Hom}_R(k, E(k)) \cong k..$$

First, we know $k \subset \mathrm{Hom}_R(k, E(k))$. The map $k \hookrightarrow E(k)$ gives us that the Hom module is nonzero. Say $x \notin i(k)$ such that $\mathfrak{p} \cdot x = 0$. Then $Rx \cap i(k) = 0$. Recall the Hom module was $\mathrm{Soc}_R(E_R(k))$, a k-subspace of $E_R(k)$. This must be indecomposable, hence 1-dimensional. □

We show 2 from the properties at the beginning. Say E is an injective R-module. We look at collections of injective submodules of E such that their sum is E. Say the collection of collections is $\mathcal{U}$. Pick $\mathfrak{p}$ associated to E so that $\frac{R}{\mathfrak{p}} \hookrightarrow E$. Then we get a map $E(R/\mathfrak{p}) \to E$ since E is injective. The map is 1-1 because of the essential condition. You can of course continue splitting E because distinct primes yield trivially intersecting submodules. This also shows 3. Personal question: how does selecting primes allow us to split the module?

There is also a uniqueness of decompositions that we didn't discuss. $E \cong \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(R)} E(R/\mathfrak{p})^{\mu(\mathfrak{p})}.\mu(\mathfrak{p})$ is basically

$$\mathrm{rank}_{k(\mathfrak{p})} \mathrm{Hom}_{R_\mathfrak{p}}\left(k(\mathfrak{p}, E_\mathfrak{p}\right).$$

Hence the decomposition is unique. Key:

$$\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E(R/\mathfrak{q})_\mathfrak{p}) = \begin{cases} k(\mathfrak{p}) & \mathfrak{p} = \mathfrak{q} \\ 0 & \text{else} \end{cases}.$$

Already, 4 from the lemma shows when $\mathfrak{p} = \mathfrak{q}$. If $\mathfrak{p} \neq \mathfrak{q}$, $\mathfrak{q} \not\subset \mathfrak{p}$, $\mathfrak{q}$-power torsion implies that everything is killed by a power of $\mathfrak{q}$.

$$E(R/\mathfrak{q})_\mathfrak{p} = 0.$$

Alternatively, use definition of associated primes.

If we start with $M \hookrightarrow E_R(M)$, we know $E_R(M) = \bigoplus_{\mathfrak{p} \in \mathrm{Ass}(M)} E(R/\mathfrak{p})^{\mu(\mathfrak{p})}$ where $\mu(\mathfrak{p})$ is the $k(\mathfrak{p})$-dimension of $\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), E_R(M)_\mathfrak{p}) = \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$. In particular, if $M \in \mathrm{Mod}(R)$, $\mu(\mathfrak{p}) < \infty$.This gives rise to the i-th Bass number.

$$\mu^i_R(\mathfrak{p}, M) = \mathrm{rank}_{k(\mathfrak{p})}(\mathrm{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}).$$

Let $M \xrightarrow{\sim} I$ is a minimal injective resolution, then $I^i \cong \bigoplus E(R/\mathfrak{p})^{\mu^i(\mathfrak{p}, M)}$. We did the $i = 0$ case above, and higher i steps follow by similar reasoning.

12 March 15

We discuss Matlis duality today. Let R be a commutative Noetherian ring.

1. There is a natural map $M \otimes_R \mathrm{Hom}_R(N, E) \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, N), E)$. Given by $(x \otimes \alpha) \mapsto [f \mapsto \alpha(f(x))]$. This is the adjoint of $\mathrm{Hom}_R(M, N) \otimes_R M \otimes_R \mathrm{Hom}_R(N, E) \to N \otimes_R \mathrm{Hom}_R(N, E) \to E$. The natural map is bijective when E is injective and M is finitely generated. This is clear when $M = R$. Also when M is finite free. We can also approximate M with a finite free resolution. We get presentation
$$\mathcal{G} \to \mathcal{F} \to M \to 0.$$
where $\mathcal{F}$ and $\mathcal{G}$ are finite free. We can say
$$0 \to \mathrm{Hom}_R(M, N) \hookrightarrow \mathrm{Hom}_R(\mathcal{F}, N) \to \mathrm{Hom}_R(\mathcal{G}, N)$$
is exact. So
$$\mathrm{Hom}_R(\mathrm{Hom}_R(\mathcal{G}, N), E) \to \mathrm{Hom}_R(\mathrm{Hom}_R(\mathcal{F}, N), E) \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, N), E) \to 0.$$
$$\mathcal{G} \otimes_R \mathrm{Hom}_R(N, E) \to \mathcal{F} \otimes_R \mathrm{Hom}_R(N, E) \to M \otimes_R \mathrm{Hom}_R(N, E) \to 0.$$
with commutative diagram with bottom row
$$\mathrm{Hom}_R(\mathrm{Hom}_R(\mathcal{G}, N), E) \to \mathrm{Hom}_R(\mathrm{Hom}_R(\mathcal{F}, N), E) \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, N), E) \to 0.$$

Corollary 14. If E', E are injective, then $\mathrm{Hom}_R(E', E)$ is flat.

Proof. $\bullet \otimes_R \mathrm{Hom}_R(E', E) \cong \mathrm{Hom}(\mathrm{Hom}(\bullet, E'), E)$. The latter is exact on the category of finitely generated modules over R. Hence, the former is exact on the category of modules over R since every module over R is a colimit of a diagram of finitely generated modules. $\square$

Hence $\mathrm{Hom}_R(\bullet, E) : \mathrm{Inj}(R) \to \mathrm{Flat}(R)$. Also,
$$\mathrm{Hom}_R(\bullet, E) : \mathrm{Flat}(R) \to \mathrm{Inj}(R).$$

2. Say $(R, \mathfrak{m}, k) \twoheadrightarrow \mathcal{S}$ and we take the injective hull of k, $E_R(k)$. $\mathrm{Hom}_R(\mathcal{S}, E_R(k)) \cong E_{\mathcal{S}}(k)$. $(0 :_{E_R(k)} I)$ is injective because it is a cobase change of injective modules. We get a triangle from $k \hookrightarrow E_R(k)$ and $k \hookrightarrow (0 :_{E_R(k)} I) \hookrightarrow E_R(k)$. Fix local ring $(R, \mathfrak{m}, k)$. $E = E_R(k)$, and $(\bullet)^v = \mathrm{Hom}_R(\bullet, E)$. We get
$$(\bullet)^v : \mathrm{Mod}(R) \to \mathrm{Mod}(R).$$
Consider $M \to M^{vv} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$. Given $x \in M$, it sends to the evaluation map $f \mapsto f(x)$. We know $R^v = E$. Also, $R^{vv} = E^v = \mathrm{Hom}_R(E, E) = \mathrm{End}(E)$. $R \to \mathrm{End}(E)$ via $r \mapsto (E \xrightarrow{\cdot r} E)$. Say M is finitely generated. We have
$$M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)..$$
We have a commutative diagram with bottom row isomorphic to top row, with bottom row
$$M \otimes_R R \to M \otimes_R \mathrm{End}_R(E).$$

Lemma 18. (a) When N has finite length, (A) $\ell(N) = \ell_R(N^v)$ and $N \xrightarrow{\sim} N^{vv}$. For vector spaces, this is a similar statement to when you replace length with dimension. We also have (B) $\beta_0^R(N) = \mu^0(N)$. Also, (C) $N \cong N^{vv}$. $\beta_0^0(M) = \mathrm{rank}_k(\frac{M}{\mathfrak{m}M})$.
$$\mu_R^0(M) = \mathrm{rank}_k(\mathrm{Hom}_R(k, M)) = \mathrm{rank}_k(\mathrm{Soc}_R(M)).$$
(b) When R is Artinian, the duality is an equivalence of categories $(\mathrm{Mod}(R))^{\mathrm{op}} \to \mathrm{Mod}(R)$.

Proof. We have $\mathrm{Hom}_R(k,E)=k$. Hence $k^{vv}=k$. Also $k\to k^{vv}$ is an isomorphism (check).

(A) and (C) hold for $N=k$. Now induce on $\ell_R(N)$. We have base case $\ell_R(N)=1$. Say $\ell_R(N)\geq 2$. If $\ell_R(N')=\ell_R(N)-1$,

$$0\to k\hookrightarrow N\to N'\to 0.$$

we have

$$0\to N'^v\to N^v\to k\to 0.$$

and $\ell(N^v)=\ell(N'^v)+1=\ell(N)$. Use the five lemma on exact sequence with top row the first sequence above and bottom row

$$0\to k^{vv}\to N^{vv}\to N'^{vv}\to 0.$$

with column maps evaluation. (B) $\beta_0^R(N)=\mathrm{rank}_k(k\otimes_R N)=\mathrm{rank}_k(\mathrm{Hom}(k\otimes_R N,E))$

$$=\mathrm{rank}_k(\mathrm{Hom}_R(k,N^v)=\mu_R^0(N^v).$$

With regards to the second part of the proof, have a look at the functor applied to itself. $\square$

Theorem 15. Suppose R is complete with respect to $\mathfrak{m}$-adic topology. One has an equivalence of categories

$$\mathrm{mod}(R)\to\mathrm{Art}(R).$$

(the former indicates finitely generated modules) with both functors forward and back are $(\bullet)^v$.

Lemma 19. Say $(R,\mathfrak{m},k)$ is complete local. Then $R\xrightarrow{\sim}\mathrm{End}_R(E)$.

Proof. For the proof, we reduce to the Artinian case. For each $n\geq 1$, we look at $R\to\frac{R}{\mathfrak{m}^nR}$ and the map $R\to\mathrm{End}_R(E)$. $E_n=\mathrm{Hom}_R(R/\mathfrak{m}^nR,E)\subset E$. We claim we get a natural map $\mathrm{End}_R(E)\to\mathrm{End}_{R/\mathfrak{m}^nR}(E)$. We get a map $\frac{R}{\mathfrak{m}^nR}\to\mathrm{End}_{R/\mathfrak{m}^nR}(E)$ which makes a commutative square but is also an isomorphism itself due to the Artinian condition. We get $R\to\lim\limits_n\frac{R}{\mathfrak{m}^nR}\to\lim\limits_n\mathrm{End}(E_n)$. The first map is an isomorphism because R is complete and the last module is $\mathrm{End}_R(E)$. Have to check that $\mathrm{End}_R(E)=\lim\limits_n\mathrm{End}_R(E_n)$. $E=\bigcup_{n\geq 0}E_n$ implies our claim. $\square$

Proof. Of the theorem. Let's say E is artinian. Say

$$E\supset M_0\supset M_1\supset\dots$$

is a descending chain of submodules. Applying the Cech operator, we get

$$R=E^v\twoheadrightarrow M_0^v\twoheadrightarrow M_1^v\twoheadrightarrow\dots.$$

R is Noetherian implies that $M_n^v\twoheadrightarrow M_{n+1}^v$ is an isomorphism for $n\gg 0$. This implies $M_n=M_{n+1}$ for all $n\gg 0$. $\frac{M_{n+1}}{M_n}$ is $\mathfrak{m}$-power torsion. So $k\hookrightarrow\frac{M_{n+1}}{M_n}$. But this means

$$\left(\frac{M_{n+1}}{M_n}\right)^v\twoheadrightarrow k^v=k.$$

a contradiction.

E Artinian implies that M^v is Artinian for all finitely generated M.

$$R^n\twoheadrightarrow M.$$

gives

$$M^v\xhookrightarrow{surj}(R^n)^v=E^n.$$

M artinian implies $k^m \cong \mathrm{Soc}(M) \hookrightarrow M$ and the socle also embeds into E^m. By essentialness, $M \to E^m$ exists and is an embedding. When dualized, $R^m \twoheadrightarrow M^v$ by finite generation of M.

Now we do know that the double dual is an isomorphism. $M \to M^{vv}$. $M^{vv} \xrightarrow{\sim} M \otimes_R \mathrm{End}_R(E)$. The lemma implies we get a commutative diagram with the aforementioned maps and $M \otimes_R R \to M \otimes_R \mathrm{End}_R(E)$ and $M \otimes_R R \cong M$.

Now if N is artinian, $N \cong N^{vv}$. Now for general artinian embedded $0 \to N \hookrightarrow E^m \to E^r$ □

The last thing we need to check above is that for M artinian, $\mathrm{Soc}(M) \to M$ is essential. We will just do this later.

Given map $R \twoheadrightarrow k$, we have in $\mathrm{mod}(k\ )$ the notion of duality. Can we lift the vector space duality to R? In particular, is there a module I such that

$$\mathrm{Hom}_R(\bullet, I) \cong \mathrm{Hom}_k(\bullet, k) \text{ on } \mathrm{mod}(k)?$$

Matlis duality finishes the job! $I = E$ can be used.