

# Modern Algebra II

## Notes on MATH 6320

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## 1 January 13

A group  $G$  is called **cyclic** if  $G = \langle a \rangle$  for some  $a \in G$ , in which case  $a$  is a **generator**. For instance,

$$\begin{aligned}(\mathbb{Z}, +) &= \langle 1 \rangle = \langle -1 \rangle \\ &= \langle 2, 3 \rangle\end{aligned}$$

A group that is cyclic is Abelian. Rubiks cube group  $< S_{48}$ . It is generated by six elements,

$$\langle T, Bottom, L, R, F, Back \rangle$$

The group is

$$\left( \left( \frac{\mathbb{Z}^{11}}{2} \right) \times \left( \frac{\mathbb{Z}}{3} \right)^7 \right) \rtimes \left( (A_8 \times A_{12}) \rtimes \frac{\mathbb{Z}}{2} \right)$$

$A_8$  and  $A_{12}$  are alternating groups, and  $\rtimes$  denotes the semi-direct product, which we will define in the future.

If  $H < G$  we saw  $|G| = |H| \cdot (G : H)$ . Any infinite cyclic group is isomorphic to  $\mathbb{Z}$ . Otherwise, a cyclic group  $G$  is isomorphic to  $\frac{\mathbb{Z}}{|G|}$ . We use additive notation for a group only if it is abelian. If  $n$  is a positive integer,  $x \in G$ ,

$$n \cdot x = x + \dots + x \text{ (} n \text{ times)}$$

Likewise,

$$(-n) \cdot x = -x - x - \dots - x \text{ (} n \text{ times)}$$

Any Abelian group  $G$  is a  $\mathbb{Z}$ -module. If  $G$  is a finitely generated Abelian group, then the structure theorem for modules over a PID applies. We have

$$G \cong \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \oplus \frac{\mathbb{Z}}{m_1} \oplus \frac{\mathbb{Z}}{m_2} \oplus \dots \oplus \frac{\mathbb{Z}}{m_k}$$

Multiplicative notation: If  $x \in G$  and  $n$  is a positive integer, then

$$x^n = x \cdot x \cdot \dots \cdot x$$

$$x^{-n} = x^{-1} \cdot \dots \cdot x^{-1}$$

- Proposition 1.**
1. An infinite cyclic group has 2 possible generators,  $a$  and  $a^{-1}$ .
  2. If  $G = \langle a \rangle$  is a cyclic group of order  $n$ . Then  $\langle a^m \rangle = G$  if and only if  $m$  is relatively prime to  $n$ .
  3. If  $G = \langle a \rangle = \langle b \rangle$ , then  $G \rightarrow G$  defined by  $a \mapsto b$  is an automorphism.
  4. If  $G$  is cyclic of order  $n$ , and  $d|n$ , then  $H = \{x \in G : \text{order of } x \text{ divides } d\}$  is a subgroup of order  $d$ .

*Proof.* 1.

2.

3. The map is surjective, but since it is a map of finite sets, it is also injective.

4.  $H$  contains the identity. And  $x \in H$  implies that  $x^{-1} \in H$ . If  $x^d = 1$  and  $y^d = 1$ , then  $(xy)^d = 1$ , because  $G$  is abelian.

□

If  $G = \langle x \rangle$ ,  $x^n = 1$ ,  $n = de$ , then

$$H = \{x^m : x^{m \cdot d} = 1\}.$$

Note  $x^{md} = 1$  if and only if  $n = de|md$  if and only if  $e|m$ . Note that  $H$  is the subgroup defined by

$$\langle x^{\frac{n}{d}} \rangle$$

(Can be used to show that the multiplicative group of a finite field is cyclic. ) Suppose  $\varphi : G_1 \rightarrow G_2$  is a group homomorphism. Let  $H_2 < G_2$ . Then  $\varphi^{-1}(H_2)$  is a subgroup of  $G_1$ . We have

$$\varphi^{-1}(\{e_2\}) < G_1$$

But this is the kernel of  $\varphi$ . In fact, it is a normal subgroup, which we denote  $\ker \varphi \triangleleft G_1$ . We say that a subgroup is normal if  $gHg^{-1} = H$  for all  $g \in G$ . Recall  $xH = \{xh : h \in H\}$ . If  $A, B$  are subsets of a group  $G$   $AB = \{ab : a \in A, b \in B\}$ . Set  $K = \ker(G_1 \rightarrow G_2)$ . If  $k \in K$  and  $g \in G_1$ , then

$$\varphi(gkg^{-1}) = \varphi(g)\varphi(g^{-1}) = \varphi(e_1) = e_2.$$

Given  $K \triangleleft G$ , there is a canonical surjection  $G \twoheadrightarrow \frac{G}{K}$   $g \mapsto gK$  which has kernel  $K$ . The group above is defined

$$\frac{G}{K} = \{gK : g \in G\}$$

We want to give this group its structure. So we define

$$(g_1K)(g_2K) = g_1g_2K$$

This is well defined (check this). It also makes  $\frac{G}{K}$  a group. A group  $G$  acts on a set  $S$  if there is a group homomorphism  $\pi : G \rightarrow \text{Perm}(S)$ . For each  $g \in G$  we have  $\pi_g : S \rightarrow S$  a bijection. Now note that the identity element of the group must be the identity of the permutation of the group. In other words,

$$\pi_{g_1 g_2} = \pi_{g_1} \pi_{g_2}$$

For shorthand, we often write  $\pi_g(x) = gx$  for  $x \in S$ .

**Example.** If  $H < G$ , then  $G$  acts on  $\frac{G}{H}$  (the set of left cosets). It acts via

$$x \mapsto (gH \mapsto xgH)$$

The kernel of this homomorphism, call it  $\pi$ , is

$$\{x \in G \mid xgH = gH \ \forall g \in G\} \triangleleft G$$

If  $x \in K$ , then  $g^{-1}xg \in H$  for all  $g \in G$ . In particular, it happens when  $g = id$ , so  $x \in H$ . So  $K < H < G$ . Since  $K \triangleleft G$ , we also have  $K \triangleleft H$ .

**Proposition 2.** Suppose  $G$  is a finite group and  $H < G$  such that

$$p = (G : H)$$

is the smallest prime dividing  $|G|$ . Then  $H \triangleleft G$ .

*Proof.* We can define  $G \xrightarrow{\pi} \text{Perm}(\frac{G}{H})$ . Note the latter group is the symmetric group on  $p$  symbols, which is of order  $p!$ . Then we can factor this map through and get

$$\frac{G}{K} \rightarrow \text{Perm}(\frac{G}{H})$$

an injective map. So  $(G : K)$  divides  $p!$ . But

$$(G : K) = (G : H)(H : K)$$

So  $(G : H)(H : K)$  divides  $p!$ . Now  $(G : H) = p$ , so  $(H : K)$  divides  $(p - 1)!$ . Since  $p$  is the smallest integer dividing  $|G|$ ,  $(H : K) = 1$ . Hence  $H = K \triangleleft G$ .  $\square$

**Proposition 3.** If  $\varphi : G_1 \rightarrow G_2$  has kernel  $K$ , then  $\varphi$  factors through  $\frac{G_1}{K}$ , meaning there is a suitable map  $i : \frac{G_1}{K} \rightarrow G_2$  making the following diagram commute:

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_2 \\ & \searrow \pi & \uparrow i \\ & & G_1/K \end{array}$$

where  $i$  is an injective homomorphism, and  $\pi$  is the canonical surjection.

*Proof.* Given  $gK \in \frac{G_1}{K}$ , define

$$i(gK) = \varphi(g)$$

this is well defined, for given any other representative  $hK = gK$ , so that  $g^{-1}h \in K$ , we have

$$i(hK) = \varphi(h)$$

but  $\varphi(g)^{-1}\varphi(h) = \varphi(g^{-1}h) = e_2$ , so  $\varphi(g) = \varphi(h)$ . (Check this gives a homomorphism)  $\square$

**Example.** (Group Action) Suppose  $G$  is a group. We have

$$\text{Aut}(G) \text{ (the group of automorphisms of } G\text{)}$$

We have a representation  $G \mapsto \text{Aut}(G) < \text{Perm}(G)$  defined by  $x \mapsto c_x \in \text{Aut}(G)$  where  $c_x$  is the conjugation by  $x$ .

## 2 January 18

Suppose  $K < H < G$  and  $K \triangleleft G$  and  $H \triangleleft G$ . Then  $\frac{G}{K}$  and  $\frac{G}{H}$  are groups, and

$$\frac{G}{K} \rightarrow \frac{G}{H}$$

defined by  $gK \mapsto gH$ . With kernel  $\{gK : gH = H\} = \{hK : h \in H\} = \frac{H}{K}$ . Note that  $K \triangleleft H$  so  $\frac{H}{K}$  is a group. By the first isomorphism theorem

$$\frac{G/K}{H/K} \xrightarrow{\sim} \frac{G}{H}$$

Suppose  $K \triangleleft G$ . Then subgroups of  $\frac{G}{K}$  correspond to subgroups of  $G$  that contain  $K$ . Likewise, normal subgroups of  $G/K$  correspond to normal subgroups of  $G$  that contain  $K$ . Recall: If  $A, B$  are subsets of  $G$  then

$$AB = \{ab \mid a \in A, b \in B\}$$

Let  $S \subset G$ . Then  $N_S = \{x \in G \mid xSx^{-1} = S\} < G$  is a subgroup of  $G$ , called **normalizer** of  $S$  in  $G$ . Define

$$Z_S = \{x \in G \mid xsx^{-1} = s \forall s \in S\} < G$$

is the centralizer of  $S$  in  $G$ .

$Z_G$  is the center of the group  $G$ . If  $H < G$  then  $H < N_H < G$ . In fact,  $H \triangleleft N_H$  by definition. Let  $H, K$  be subgroups of  $G$  and  $H \subset N_K$ . Then  $H \cap K \triangleleft H$ . For  $s \in H \cap K$  and  $h \in H$ ,

$$hsh^{-1} \in H$$

it is also in  $K$  since  $h \in N_K, s \in K$ .

$H \subset N_K$  gives  $HK = KH$ , which is a group (check).

### Exercise 1

Define  $\varphi : H \rightarrow \frac{HK}{K}$  via  $x \mapsto xK$ . Check that this is a group homomorphism. Also check this is surjective and  $\ker \varphi = H \cap K$ . We also have by the first isomorphism theorem

$$\frac{H}{H \cap K} \cong \frac{HK}{K}.$$

We stopped last time at the action of a group  $G$  on itself via conjugation.  $G \rightarrow \text{Aut}(G)$  maps via  $x \mapsto (g \mapsto xgx^{-1})$ . (Note the distinction between automorphism and permutation: permutations are not necessarily homomorphisms). The kernel of the action is the center of the group. The image of  $G \rightarrow \text{Aut}(G)$  is the group of inner automorphisms denoted  $\text{Inn}(G)$ .

**Proposition 4.**  $\text{Inn}(G) \triangleleft \text{Aut}(G)$ .

*Proof.* Let  $\varphi \in \text{Aut}(G)$ ,  $c_x \in \text{Inn}(G)$ . We would like to check that

$$\varphi \circ c_x \circ \varphi^{-1} \in \text{Inn}(G)$$

$$\varphi \circ c_x \circ \varphi^{-1}(g) = \varphi(x\varphi^{-1}(g)x^{-1}) = \varphi(x)g\varphi(x^{-1}) = c_{\varphi(x)}(g)$$

□

We are obliged to construct

$$\frac{\text{Aut}(G)}{\text{Inn}(G)} = \text{Out}(G)$$

Suppose  $G$  acts on  $S$ . Let  $s \in S$ . Define

$$G_s = \{x \in G : xs = s\}.$$

This is called the **stabilizer of the isotropy subgroup**. This is not to be confused with  $G \cdot s = \{xs \mid x \in G\}$ , which is called the **orbit of  $s$** . Suppose  $t \in G \cdot s$ . Then we compare  $G_s$  and  $G_t$ . In fact, we have  $G_t$  is conjugate to  $G_s$ . We have

$$\begin{aligned} G_t &= \{x \in G \mid xt = t\} = \{x \in G \mid xys = ys\} \\ &= \{x \in G \mid y^{-1}xys = s\} = \{x \in G \mid y^{-1}xy \in G_s\} = yG_sy^{-1} \end{aligned}$$

Set  $K = \ker(G \rightarrow \text{Perm}(S))$ . We can write  $K = \bigcap_{s \in S} G_s$ . We say that the action of  $G$  on  $S$  is faithful if  $K = \{e\}$ . Fixed points in  $S$  are those such that  $xs = s \forall x \in G$ .

Let  $s \in S$ . We can define  $G \rightarrow Gs$  by  $x \mapsto xs$ . This yields a map  $\frac{G}{G_s} \rightarrow Gs$   $xG_s \mapsto xs$ . If  $xG_s = yG_s$ , then  $y^{-1}x \in G_s$ . So  $yx^{-1}s = s$  so  $xs = ys$ . Hence the map is well defined, surjective, and injective. Hence  $\frac{G}{G_s} \xrightarrow{\sim} Gs$  is a bijection. We have

$$|Gs| = (G : G_s)$$

Two orbits  $Gs$  and  $Gt$  are either equal or disjoint. If they shared an element  $gs = ht$ . But this implies  $Gs \subset Gt$ , since for any  $g's \in Gs$ ,  $g's = g'g^{-1}gs = g'g^{-1}ht \in Gt$ . Similarly,  $Gt \subset Gs$ . We can then write

$$S = \bigcup Gs_i$$

so  $|S| = \sum |Gs_i| = \sum (G : G_{s_i})$ . Class formula?

Let  $G$  act on a set  $S$ . The action is transitive if for some  $s \in S$   $Gs = S$ . Equivalently, we say the same if there is only one orbit.

The action of  $G$  on  $S$  restricts to an action on each orbit. On each orbit, the group is transitive.

**Theorem 1.** Cauchy's theorem: Suppose  $G$  is a finite group and  $p$  is a prime dividing  $|G|$ . Then  $G$  has an element of order  $p$ .

**Definition 1.** Let  $p$  be a prime integer. A group  $G$  is a  $p$ -group if  $|G| = p^n$  for some  $n \in \mathbb{N}$ .

**Lemma 1.** Suppose  $G$  is a  $p$ -group acting on a set  $S$ . Let  $F$  = fixed points in  $S = \{s \in S \mid xs = s \forall x \in G\}$ . Then  $|S| \equiv |F|$  modulo  $p$ .

*Proof.* Apply the class formula  $|S| = \sum (G : G_{s_i})$ . We have  $|S| = |F| + \sum_{\text{other } i} (G : G_{s_i})$ . For  $s_i \notin F$ ,  $G_{s_i} \subsetneq G$ , so  $p \mid (G : G_{s_i})$ . □

*Proof.* (of Cauchy's theorem): Let  $S = \{(x_1, \dots, x_p) \mid x_i \in G, x_1 \cdots x_p = e\}$ . We have

$$|S| = |G|^{p-1}.$$

Define  $\sigma \in \text{Perm}(S)$  by  $\sigma : (x_1, \dots, x_p) \mapsto (x_p, x_1, \dots, x_{p-1})$ . We have

$$x_1 \cdots x_{p-1} = x_p^{-1}$$

$$x_p x_1 \dots x_{p-1} = e$$

So  $\sigma$  maps elements of  $S$  to  $S$ .  $\sigma$  has order  $p$ . Hence  $\langle \sigma \rangle$  is a  $p$ -group acting on  $S$ , so  $|F| \equiv |S| \pmod{p}$  by the lemma. But  $|S| \equiv 0 \pmod{p}$ , since  $p$  divides  $|G|$  which divides  $|S|$ .  $e, \dots, e) \in F$ , so  $|F| \geq p$ . But elements of  $F$  are of form  $(x, \dots, x)$ , which implies that any nonidentity in  $F$  corresponds to a desired element.  $\square$

Suppose  $p$  divides  $|G|$  and  $p$  is a prime. Let  $p^n$  be the highest power of  $p$  dividing  $|G|$ .

**Theorem 2.**  $G$  has a subgroup of order  $p^n$ . Such a subgroup called a  $p$ -Sylow subgroup.

**Lemma 2.** Fix a prime  $p$  dividing  $|G|$ . Suppose  $G$  acts on  $S$  with the property that  $\forall s \in S$ , there exists a  $p$ -subgroup of  $G$  that fixes only  $s$ . Then the action of  $G$  is transitive.

*Proof.* Suppose  $P$  is a  $p$ -subgroup fixing only  $s \in S$ . We have that  $|S| \equiv 1 \pmod{p}$  by a preceding lemma. If  $S$  has multiple orbits, we can write

$$S = S_1 \cup S_2$$

a disjoint union. Each subset has the same property as  $S$  that satisfies the theorem hypotheses, so  $|S_1| \equiv 1 \pmod{p}$  and  $|S_2| \equiv 1 \pmod{p}$ . Hence

$$|S| \equiv |S_1| + |S_2| \pmod{p}$$

implies

$$|S| \equiv 2 \pmod{p}$$

Hence the action is transitive.  $\square$

### 3 January 20

Let  $G$  be a finite group,  $p$  be a prime dividing  $|G|$ . A  **$p$ -Sylow subgroup of  $G$**  is a subgroup of order  $p^n$  where  $p^n$  is the highest power of  $p$  dividing  $|G|$ .

**Theorem 3.** If a prime  $p$  divides  $|G|$ , then a  $p$ -Sylow subgroup exists.

*Proof.* We will work on induction on  $|G|$ . If  $|G| = p$ , then we are done, since  $G$  is the desired group. If  $H < G$  and  $p \nmid (G : H)$  (so that in this case the highest power of  $p$  dividing  $|G|$  is also the highest power for  $|H|$ ), then a  $p$ -Sylow subgroup of  $H$  is also a  $p$ -Sylow subgroup of  $G$ . We may therefore assume that for all subgroups  $H \subsetneq G$ , we have  $p \mid (G : H)$ .

Let  $G$  act on itself via conjugation

$$G \rightarrow \text{Aut}(G)$$

The kernel of the homomorphism is also the center of the group denoted  $Z$ . Use the class equation:

$$|G| = |Z| + \sum_i (G : G_{x_i})$$

Here  $G$  is the set,  $Z$  is the set of fixed points, and the last set is the size of larger orbits. Here  $G_{x_i}$  is an isotropy subgroup, so  $(G : G_{x_i})$  is the cardinality of the orbit of  $x_i$ . By our hypothesis,  $(G : G_{x_i})$  is divisible by  $p$  for all  $i$ .

So  $|G| \equiv |Z| \pmod{p}$ . This implies that  $p$  divides the order of  $Z$  since  $|G| \equiv 0 \pmod{p}$ . We have  $e \in Z$ , so that  $\exists a \in Z$  of order  $p$  by Cauchy's theorem. Now  $\langle a \rangle$  has order  $p$ , and  $a \in Z$  implies

$$\langle a \rangle \triangleleft G.$$

$\frac{G}{\langle a \rangle}$  has smaller order than  $G$ , so it must have some  $p$ -Sylow subgroup  $P$  by induction hypothesis. Then  $|P|$  is  $p^{n-1}$  where  $p^n$  is the highest order of  $p$  dividing  $|G|$ . Now note that  $P$  corresponds to a subgroup of  $G$  which must have order  $p^n$ . It is  $\varphi^{-1}(P)$  where

$$\varphi : G \rightarrow \frac{G}{\langle a \rangle}$$

is the natural map. □

**Lemma 3.** Suppose  $A, B$  are finite subgroups of  $G$ . Then  $AB$  is a set of products of elements of  $A, B$ . Then  $|AB| = \frac{|A| \cdot |B|}{|A \cap B|}$ .

*Proof.*  $A \cap B < A$ , so  $A$  can be written as a disjoint union of cosets

$$\bigsqcup_{i \in I} a_i(A \cap B)$$

for some  $a_i \in A$ . So

$$\begin{aligned} AB &= \bigcup_{i \in I} a_i(A \cap B)B \\ &= \bigcup_{i \in I} a_i B \end{aligned}$$

Claim:  $AB = \bigsqcup_{i \in I} a_i B$  (the union is disjoint)

If  $a_i b = a_j b'$  for  $i \neq j$ ,  $b, b' \in B$ , then

$$a_i = a_j b' b^{-1}$$

so that  $a_j^{-1} a_i = b' b^{-1} \in A \cap B$  so that

$$a_i \in a_j(A \cap B)$$

and  $i = j$ .

Claim yields

$$|AB| = \sum_{i \in I} |a_i B| = |I| \cdot |B| = |B| \cdot \frac{|A|}{|A \cap B|}$$

□

**Theorem 4.** Let  $p$  be a prime dividing  $|G|$ . Then:

1. Each  $p$ -subgroup is contained in a  $p$ -Sylow subgroup
2. The  $p$ -Sylow subgroups are conjugate.
3. Let  $s_p$  be the number of  $p$ -Sylow subgroups. Then  $s_p \mid |G|$  and  $s_p \equiv 1 \pmod{p}$ .

*Proof.* Let  $\mathcal{S}$  be the set of all  $p$ -subgroups. Then  $G$  acts on  $\mathcal{S}$  by conjugation, since  $|H| = |xHx^{-1}|$ . Let  $\mathcal{M}$  be the set of maximal elements of  $\mathcal{S}$  (under inclusion). Claim: The action restricts to an action on  $\mathcal{M}$ . Let  $p \in \mathcal{M}$ . Suppose  $xPx^{-1} \subset Q$  for some  $Q \in \mathcal{S}$ . Then  $P \subset x^{-1}Qx \in \mathcal{S}$ . But  $P$  was maximal, so  $P = x^{-1}Qx$ . This implies  $xPx^{-1} = Q$ , so that  $xPx^{-1}$  is maximal.

Now note that any  $p$ -Sylow subgroup must be in  $\mathcal{M}$ . We would like to prove that any  $P \in \mathcal{M}$  is a  $p$ -Sylow subgroup, giving property 1 above. We know  $G$  acts on  $\mathcal{M}$  by the above argument. If  $P$  is a  $p$ -Sylow subgroup, then  $P \in \mathcal{M}$ . Since  $G$  acts on  $\mathcal{M}$ , any subgroup also does. In particular,  $P$  acts on  $\mathcal{M}$  via

conjugation, and  $P$  fixes  $P$  since  $xPx^{-1} = P \forall x \in P$ . Suppose  $P$  fixes some  $Q \in \mathcal{M}$ . Then  $xQx^{-1} = Q$  for all  $x \in P$ , so  $P < N_Q$ , and  $PQ$  is a subgroup of  $G$ . So

$$|PQ| = \frac{|P| \cdot |Q|}{|P \cap Q|}$$

( $|P|, |Q|$  are powers of  $p$  and the quotient is an integer, so it must be a power of  $p$ ) so  $PQ$  is a  $p$ -group. Then  $P, Q$  are maximal, so  $P \subset PQ$  implies  $P = PQ$  and  $Q \subset PQ$  implies  $Q = PQ$ . So  $P = Q$ .

We have that  $P$  acts on  $\mathcal{M}$  and fixes **only** itself. This implies the action by  $P$  is transitive by a previous lemma from class, which we recall:

**Remark.** Fix prime  $p$ . If  $G$  acts on  $S$  with the property that  $\forall s \in S, \exists$  a  $p$ -subgroup fixing only  $s$ , then  $G$  is transitive on  $S$ .

In our context,  $G$  acts on  $\mathcal{M}$ . Each  $P \in \mathcal{M}$  is a  $p$ -group that fixes **only**  $P \in \mathcal{M}$ . Hence  $G$  is transitive on  $\mathcal{M}$ . But then  $\mathcal{M}$  is precisely the set of  $p$ -Sylow subgroups, and they are all conjugate. This also gives us property 2. Lastly, we figure out the deal with  $s_p$ . We know  $s_p = |\mathcal{M}|$ .  $P \in \mathcal{M}$  acts on  $\mathcal{M}$  with 1 fixed point, so  $|\mathcal{M}| \equiv (\text{number of fixed points} = 1) \pmod{p}$ . So  $s_p \equiv 1 \pmod{p}$ . Now also  $G$  is transitive on  $\mathcal{M}$ , so  $|\mathcal{M}| = (G : G_P)$  for  $P \in \mathcal{M}$ . Hence  $s_p = \frac{|G|}{|G_P|}$ . We have 3.  $\square$

**Example.** Suppose  $|G| = 15$ . We look at  $s_3, s_5$ . By the above theorem,  $s_3 \equiv 1 \pmod{3}$  and  $s_3 | 15$ , so  $s_3 = 1$ . We also have  $s_5 \equiv 1 \pmod{5}$ ,  $s_5 | 15$ , implies  $s_5 = 1$ .

In general  $s_p = 1$  if and only if a (the)  $p$ -Sylow is normal.

If  $Q$  is a 5-Sylow, then  $(G : Q) = 3$ , which is the smallest prime dividing  $|G| = 15$ . This implies that  $Q$  is normal. This is an alternative way to see that  $s_5 = 1$ .

Say  $|P| = 3, |Q| = 5$ . Then  $PQ = 15$ , so  $PQ = G$ . We can say more: let  $[P, Q] = \langle [p, q] : p \in P, q \in Q \rangle$  where  $[p, q] = pqp^{-1}q^{-1}$  (the commutator). If  $P \triangleleft G, Q \triangleleft G$ , then  $pqp^{-1}q^{-1} \in P \cap Q$  so that in particular elements of  $P$  commute with those of  $Q$  (see proposition ahead). So

$$P \times Q \xrightarrow{(p,q) \mapsto pq} G$$

is a group homomorphism. We have

$$(p_1, q_1) \mapsto p_1q_1$$

$$(p_2, q_2) \mapsto p_2q_2$$

$$(p_1, q_1)(p_2, q_2) = p_1q_1p_2q_2 = p_1p_2q_1q_2$$

We know that the only groups of order 3, 5 respectively are  $\frac{\mathbb{Z}}{3}, \frac{\mathbb{Z}}{5}$ . Hence  $G$  must be  $\frac{\mathbb{Z}}{15}$ .

**Proposition 5.** Suppose

1.  $P \triangleleft G, Q \triangleleft G$
2.  $PQ = G$
3.  $P \cap Q = \{e\}$

Then  $G \cong P \times Q$ .

*Proof.* Consider  $pqp^{-1}q^{-1}$  for  $p \in P, q \in Q$ . Now because  $P \triangleleft G, qp^{-1}q^{-1} \in P$  and so  $pqp^{-1}q^{-1} \in P$ . Likewise,  $Q \triangleleft G$  implies  $pqp^{-1} \in Q$  and  $pqp^{-1}q^{-1} \in Q$ . Hence

$$pqp^{-1}q^{-1} \in P \cap Q = \{e\}$$



so that  $pqp^{-1}q^{-1} = e$ . In other words,  $pq = qp$ . Now define  $P \times Q \rightarrow G$  by

$$(p, q) \mapsto pq$$

The fact that elements of  $P$  commute with those of  $Q$  ensures that this is a group homomorphism. It is surjective because of property 2. It is also injective. Given  $(p, q)$  mapping to  $e$ , we have  $pq = e$ . But  $p = q^{-1} \in P \cap Q = \{e\}$ , so that  $p = e = q$ . Hence it is also injective.  $\square$

## 4 January 25

Last time: Let  $G$  be a finite group and  $p$  be a prime dividing  $|G|$ . Then each subgroup is contained in a  $p$ -Sylow subgroup. Only 2  $p$ -Sylow subgroups  $P, Q$  are conjugate. If  $s_p$  is the number of  $p$ -Sylow subgroups, then

$$s_p \mid |G|$$

and  $s_p \equiv 1 \pmod{p}$ .

**Corollary 1.** Suppose that  $|G| = pq$  where  $p \neq q$  are primes. Suppose  $p < q$ , and that  $p \nmid q - 1$ . Then  $G$  is cyclic.

*Proof.* Let  $Q$  be a  $q$ -Sylow subgroup. Its index is  $p$ , which is the smallest prime dividing the order of  $G$ , which implies  $Q \triangleleft G$ . Alternatively,  $s_q \equiv 1 \pmod{q}$  and  $s_q \mid |G|$  so  $s_q = 1$ . Therefore  $xQx^{-1} = Q$  for all  $x \in G$ .

$Q$  is normal, so conjugation by any element of  $G$  takes  $Q$  to itself. That is  $\forall x \in G$

$$c_x : Q \rightarrow Q$$

An automorphism of  $Q$  is determined by one is sent in  $Q \cong \frac{\mathbb{Z}}{q}$ , so  $\text{Aut}Q \cong \frac{\mathbb{Z}}{q}^*$  (multiplicative group which is of order  $q - 1$ ).

We have a map  $G \rightarrow \frac{\mathbb{Z}}{q}^*$ . Since  $p$  doesn't divide  $q - 1$ , the map must be trivial.

Restating,  $G \rightarrow \text{Aut}Q$  must be trivial, ie,  $xyx^{-1} = y$  for all  $y \in Q$ . This means  $Q$  is a subgroup of  $Z(G)$ .

If we take an element  $x$  of order  $p$ ,  $y \in Q$  of order  $q$ , then  $xy$  has order  $pq$ . Hence  $xy$  generates  $G$ . Note that  $Q$  is cyclic because any element divides the order of  $Q$ , so there is a generator of  $Q$ .  $\square$

Suppose  $|G| = pq$  where  $p \neq q$  are primes. Say  $p < q$ . Then there exists  $q$ -Sylow  $Q$ ,  $Q \triangleleft G$ , and we can't say that  $G$  is necessarily cyclic, but we have  $G \triangleright Q \triangleright \{e\}$ . We have  $\frac{G}{Q}$  and  $\frac{Q}{\{e\}}$  is cyclic. We will eventually define  $G$  to be in this case a **solvable group**. We say  $G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \dots \triangleright G_n = \{e\}$  is a **normal tower** of subgroups of  $G$ . A normal tower as above is a **cyclic tower** if  $\frac{G_i}{G_{i+1}}$  is cyclic for all  $i$ . It is an **abelian tower** if  $\frac{G_i}{G_{i+1}}$  is abelian for all  $i$ . For instance  $\mathbb{Q} \triangleright \{1\}$  is an Abelian tower but not a cyclic tower.

A group  $G$  is **solvable** if it has an abelian tower. Note:

- Abelian groups are solvable. Because  $G \triangleright \{e\}$  works.
- If  $|G| = pq$  for primes  $p \neq q$  then  $G$  is solvable.
- Let's examine  $G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(\mathbb{R}) \right\}$ , which is solvable.

**Definition 2.**  $G \rightarrow \mathbb{R}^\times \times \mathbb{R}^\times$  where  $G$  is as above, and define

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

which is a group homomorphism. The group of the latter matrices is isomorphic to  $\mathbb{R}^\times \times \mathbb{R}^\times$ . The kernel is

$$K = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in \mathbb{R} \right\}$$

Since the map is surjective,

$$\frac{G}{K} \cong (\mathbb{R}^\times)^2$$

So  $G \triangleright K \triangleright \{1\}$ , which gives the abelian tower.

- (Feit-Thompson Theorem 1963, 255 pages :O) Any group of odd order is solvable.

We will see soon that for  $n \geq 5$ , the group  $S_n$  is **not** solvable.

**Corollary 2.** Overkill consequence of the Feit-Thompson theorem)  $|S_5| = 120$  is even.

Given a group  $G$ , what is the smallest normal subgroup you could mod out by to make it abelian? We define

$$G' = \langle [a, b] = aba^{-1}b^{-1} : a, b \in G \rangle$$

where  $[a, b]$  denotes a commutator. In other words  $G'$  is generated by the commutators of  $G$ . Then  $G' \triangleleft G$ . We prove normality: given  $x \in G$ , and  $aba^{-1}b^{-1} \in G'$ ,

$$xaba^{-1}b^{-1}x^{-1} = [xax^{-1}, xbx^{-1}]$$

In  $\frac{G}{G'}$ , we have that  $aG'bG'a^{-1}G'b^{-1}G' = eG'$ , so  $\frac{G}{G'}$  is commutative. Conversely, if  $N \triangleleft G$  and  $\frac{G}{N}$  is abelian, then

$$G' \subset N$$

The reason is that  $aNbNa^{-1}Nb^{-1}N = eN$  implies

$$aba^{-1}b^{-1}N = eN$$

or  $[a, b] \in N$ , so  $G' \subset N$ .

**Remark.** Any homomorphism  $G \rightarrow H$  with  $H$  abelian factors as  $G \rightarrow \frac{G}{G'} \rightarrow H$  where the composition is the original map.

**Proposition 6.** Suppose  $H \triangleleft G$ . Then  $G$  is solvable if and only if  $H$  and  $\frac{G}{H}$  are solvable.

*Proof.* Lets prove the if direction. If we have abelian tower

$$H = H_0 \triangleright H_1 \triangleright \dots \triangleright H_m = \{e\}$$

and abelian tower of  $\frac{G}{H}$ , we can get

$$\frac{G}{H} \triangleright \frac{G_1}{H} \triangleright \frac{G_2}{H} \triangleright \dots \triangleright \frac{G_n}{H} \frac{H}{H}$$

where  $G_1$  is a normal subgroup of  $G$ ,  $G_i$  is a normal subgroup of  $G_{i-1}$ , and

$$\frac{G_i/H}{G_{i+1}/H}$$

Then  $G$  is solvable since we have abelian tower

$$G \triangleright G_1 \triangleright G_2 \triangleright \dots \triangleright G_n = H \triangleright H_1 \triangleright H_2 \triangleright \dots \triangleright H_m = \{e\}$$

Now we prove the only if direction. Suppose there exists Abelian tower

$$G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_n = \{e\}$$

and define  $H_i = G_i \cap H$ . We have a natural inclusion

$$H_i \hookrightarrow G_i$$

and map

$$H_i \rightarrow \frac{G_i}{G_{i+1}} = H_i \cap G_{i+1} = (G_i \cap H \cap G_{i+1}) = H_{i+1}$$

Now we have an induced map

$$\frac{H_i}{H_{i+1}} \hookrightarrow \frac{G_i}{G_{i+1}}$$

To see  $\frac{G}{H}$  is solvable, use

$$\frac{G}{H} = \frac{G_0}{H} \triangleright \frac{G_1}{G_1 \cap H} \triangleright \frac{G_2}{G_2 \cap H} \triangleright \dots \triangleright \frac{G_n}{G_n \cap H} = \{e\}$$

and we have

$$\frac{G_i/G_i \cap H}{G_{i+1}/G_{i+1} \cap H} \cong G_i/G_{i+1}$$

which is abelian by assumption. The isomorphism comes from the first isomorphism theorem (exercise for later maybe).  $\square$

We now provide a more formal discussion on symmetric groups. We look at  $S_n = \text{Perm}\{1, \dots, n\}$ . Let  $e_1, \dots, e_n$  be standard basis vectors for  $\mathbb{R}^n$ . We can view  $S_n$  as  $\text{Perm}\{e_1, \dots, e_n\}$ , and it provides an action on  $\mathbb{R}^n$ . Each element of  $S_n$  acts on  $\mathbb{R}^n$  as a permutation matrix. For  $\sigma \in S_n$ , define  $\text{sgn}(\sigma) = \det(\text{permutation matrix of } \sigma) = \pm 1$ . An element  $\pi \in \text{Perm}\{1, \dots, n\}$  can be written as

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$$

Fix  $\pi$  as above. Let  $\langle \pi \rangle$  act on  $\{1, \dots, n\}$ , and like any group acting on any set it partitions it into disjoint orbits. We can use this to write  $\pi$  in **cyclic notation**. The action of  $\pi$  on each orbit can be represented as a **cyclic** permutation.

**Example.** A **cyclic permutation** can for instance be written  $(1, \pi(1), \pi^2(1), \pi^3(1), \dots, \pi^m(1))$  where  $m$  is the smallest integer so that  $\pi^{m+1}(1) = 1$ .

## 5 January 27

Last time, we defined  $\text{sgn}\pi$ , for  $\pi$  a permutation, as the determinant of the corresponding matrix. Notation:

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$$

Alternatively, we can write  $\pi$  with cyclic notation. Consider the orbits of  $\langle \pi \rangle$  acting on  $\{1, \dots, n\}$ . We can write each cycle as

$$(1 \ \pi(1) \ \pi^2(1) \ \dots \ \pi^{k-1}(1))$$

and likewise for other orbits. Since orbits partition the entire set into disjoint subsets,  $\pi$  can be expressed as a product of disjoint cycles.

**Example.** Consider

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix} \\ = (1)(2\ 5)(3\ 4) = (2\ 5)(3\ 4)$$

**Example.** We can compose cycles:  $(1\ 2\ 3)(3\ 4\ 5) = (1\ 2\ 3\ 4\ 5)$ . Also note that the order of a cycle is discernible from its length. The above cycle has order 5. Also, for example,  $(1\ 2\ 3)(4\ 5)$  has order 6.

We have

$$\pi(x_1\ x_2\ \dots\ x_k)\pi^{-1} = (\pi(x_1)\ \dots\ \pi(x_k))$$

Check that:

$$(\pi(x_1\ x_2\ \dots\ x_k)\pi^{-1})(\pi(x_i)) = \pi(x_{i+1})$$

(maybe except when  $i = k$ , in which case the resulting element is  $\pi(x_1)$ ). We have

$$(\pi(x_1\ x_2\ \dots\ x_k)\pi^{-1})(\pi(y)) = \pi(y)$$

for  $y \neq x_i$  for all  $i$ . By the cycle structure of  $\sigma \in S_n$ , we mean the number of 2 cycles, number of 3 cycles, etc when  $\sigma$  is written as disjoint cycles. Disjoint cycles commute, so the order doesn't matter. By what we have proved above, conjugation preserves the cyclic structure. For example,

$$\pi(1\ 2)(3\ 4\ 5)\pi^{-1} = (a\ b)(c\ d\ e)$$

We have

- Each element of  $S_n$  can be written as a product of disjoint cycles.
- It can be written as a product of 2-cycles (ie we can write it as a product of transpositions).
- Every element can be written as a product of 2-cycles involving 1. For example,

$$(2\ 3) = (1\ 2)(1\ 3)(1\ 2)$$

- $S_n = \langle (1\ 2), (1\ 3), \dots, (1\ n) \rangle$ . We can also write

$$S_n = \langle (1\ 2), (2\ 3), \dots, (n-1\ n) \rangle$$

- $S_n = \langle (1\ 2), (1\ 2\ \dots\ n) \rangle$ . If we call the latter generator  $\pi$ , we have this because  $\pi(1\ 2)\pi^{-1} = (2\ 3)$ , and so on, giving the generators for the previous item.

We have a caveat:

$$S_4 \neq \langle (1\ 3), (1\ 2\ 3\ 4) \rangle$$

Call the former  $\sigma = (1\ 3)$  and  $\tau = (1\ 2\ 3\ 4)$ . We have

$$\sigma\tau\sigma^{-1} = (3\ 2\ 1\ 4) = (4\ 3\ 2\ 1) = \tau^{-1}$$

so  $\langle \tau \rangle \triangleleft \langle \sigma, \tau \rangle = \langle \sigma \rangle \langle \tau \rangle$ . We will later come to the conclusion that this is a **dihedral group**. On the other hand,  $\tau$  is not a normal subgroup of  $S_4$ , because

$$(1\ 2)\tau(1\ 2) = (2\ 1\ 3\ 4) \notin \langle \tau \rangle$$

Let  $p$  be prime. Then  $S_p = \langle (1\ 2), \tau \rangle$  for any  $p$ -cycle  $\tau$ . because some power of  $\tau$  has the form  $(1\ 2\ 3\ \dots\ p)$ . In other news, we have

$$\text{sgn} : S_n \rightarrow \{\pm 1\}$$

is a group homomorphism, since determinants are multiplicative. We call  $A_n = \ker \text{sgn}$ , called the **alternating group**. We automatically have  $A_n \triangleleft S_n$  because it is a kernel. If  $n \geq 2$ ,  $(S_n : A_n) = 2$ . Also  $\text{sgn}(i\ j) = -1$ .

Elements of  $A_n$  are precisely those that are a product of an even number of 2-cycles. We can make a conclusion about the commutator subgroup  $S'_n$ . We have  $S'_n \subset A_n$ . Are they always equal? Let's look at examples:

$$S'_1 = A_1$$

$$S'_2 = A_2$$

After some work,  $S'_3 = A_3$ .

**Lemma 4.**  $A_n$  is generated by 3-cycles.

*Proof.* If  $n = 1, 2$  then this is vacuously true. For  $n = 3$ ,

$$A_3 = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$$

so it is true for  $n = 3$ . Otherwise, any element can be written an even product of 2-cycles. Each pair is a product of 3-cycles:

$$\begin{aligned} (1\ 2)(2\ 3) &= (1\ 2\ 3) \\ (1\ 2)(3\ 4) &= [(1\ 2\ 4), (1\ 2\ 3)] \\ &= (1\ 2\ 4)(1\ 2\ 3)(4\ 2\ 1)(3\ 2\ 1) \\ &= (1\ 2\ 4)(4\ 3\ 2) = (1\ 2)(3\ 4) \end{aligned}$$

□

**Proposition 7.**  $S'_n = A_n$ .

*Proof.*  $\subset$  is true as remarked before the lemma. For  $\supset$ , it suffices to do it for  $n \geq 3$  (we already noted  $n = 1, 2$ ). Note

$$[(1\ 2\ 3), (1\ 2)] = (1\ 2\ 3)(1\ 2)(3\ 2\ 1)(1\ 2) = (1\ 2\ 3)(3\ 1\ 2) = (1\ 3\ 2)$$

So  $S'_n$  contains  $(1\ 3\ 2)$ , and hence every 3-cycle by a similar argument. □

**Proposition 8.**

$$A'_1 = A_1$$

$$A'_2 = A_2$$

$$A'_3 = \{e\}$$

$$A'_4 = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} = N$$

$$A'_n = A_n \quad \forall n \geq 5$$

*Proof.* By explicit calculation,  $N$  is a subgroup. Conjugation preserves cycle structure, and  $N$  contains all of the possible pairs of disjoint 2-cycles. So conjugation by any element gives back an element in  $N$ .

$$\left| \frac{A_4}{N} \right| = 3$$

So  $A_4/N$  is Abelian. Hence  $A'_4 \subset N$ . But

$$(1\ 2)(3\ 4) \in A'_4$$

which is equal to

$$[(1\ 2\ 4), (1\ 2\ 3)]$$

The typical element of  $N$  can be written as a commutator. Hence  $A'_4 = N$ . We now prove the conclusion for  $n \geq 5$ .

We saw  $[(1\ 2\ 3), (1\ 2)] = (1\ 3\ 2)$ . We do have, however

$$[(1\ 2\ 3), (1\ 2)(4\ 5)] = (1\ 3\ 2)$$

giving us all 3-cycles. □

**Corollary 3.** If  $n \geq 5$ , we have  $A_n^{(k)} = A_n \forall k \geq 1$ . Recall that  $G' = [G, G] = \langle [\sigma, \tau] : \sigma, \tau \in G \rangle$ . We now define

$$G^{(2)} = G'' = (G')'$$

and so on for  $G^{(n)}$ .

We also have  $A_4$  is solvable because  $A_4 \triangleright N \triangleright \{e\}$  is an Abelian tower.  $|A_4/N| = 3$  so its abelian. Also,  $|N| = 4$ , so it is also abelian (in particular, it is  $\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$ ). Our corollary implies that  $A_n$  is not solvable. Anything that contains the commutator which we would mod out by to make an Abelian group must be the entirety of  $A_n$ , which is no abelian alone.

**Proposition 9.**  $G$  is solvable if and only if  $G^{(n)} = \{e\}$  for some  $n$ .

*Proof.* The if direction comes from our previous discussion. Because then

$$G = G^{(0)} \triangleright G^{(1)} \triangleright \dots \triangleright G^{(n)} = \{e\}$$

is an Abelian tower. For the other direction, if  $G$  is solvable, we have

$$G = G_0 \triangleright G_2 \triangleright G_3 \triangleright \dots \triangleright G_m = \{e\}$$

where  $G_i/G_{i+1}$  is abelian for all  $i$ . This fact says  $G'_0 \subset G_1$ , and  $G'_i \subset G_{i+1}$  in general. We have that  $G_0^{(n)} \subset G_n$ , so  $G^{(m)} = \{e\}$ . □

We saw  $S'_n = A_n$  for all  $n$ . The shape of the tower for symmetric groups?

$$S'_2 = \{e\}$$

$$S_3 \triangleright S'_3 = A_3 \triangleright \{e\}$$

$$S_4 \triangleright S'_4 = A_4 \triangleright A'_4 = N \triangleright \{e\}$$

$$S_5 \triangleright S'_5 = A_5 \triangleright A'_5 = A_5 \triangleright \dots$$

so  $S_5$  not solvable. We would like to prove that  $A_5$  and higher are simple in some future class. For now, we talk about dihedral groups.

$$A_n < S_n$$

**Definition 3.**  $D_n < S_n$  is the group of rigid symmetries of a regular  $n$ -gon. Meaning:

We have two kinds of elements in  $D_n$  :

- Rotations  $a = (1\ 2\ \dots\ n)$  and
- Reflections

$$b = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & n & n-1 & \dots & 3 & 2 \end{pmatrix}$$

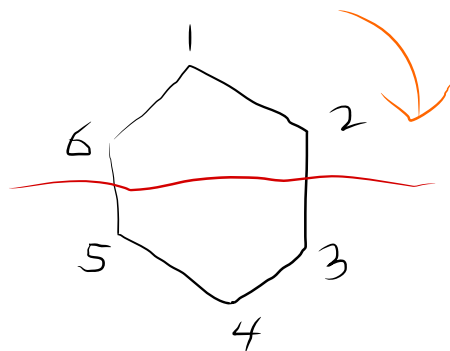


Figure 1: Symmetries include rotations and reflections

We have  $b^2 = e$ ,  $a^n = e$ . We also have

$$bab^{-1} = (1 \ n \ n-1 \ \dots \ 2) = a^{-1}$$

Said otherwise,

$$\langle a \rangle \triangleleft \langle a, b \rangle$$

and

$$\langle a, b \rangle = \langle b \rangle \langle a \rangle$$

is a group of order  $2n$ . One might ask if  $a$  and  $b$  together give us other reflections. Is  $D_n$  solvable? Yes!

$$D_n \triangleright \langle a \rangle \triangleright \{e\}$$

shows that  $D_n$  is solvable, since

$$\left| \frac{D_n}{\langle a \rangle} \right| = 2$$

What is  $D'_n$ ?

$$\begin{aligned} [a, b] &= aba^{-1}b^{-1} \\ &= (1 \ 2 \ \dots \ n)(2 \ 3 \ \dots \ n \ 1) = a^2 \end{aligned}$$

So  $a^2 \in D'_n$ . And  $\left| \frac{D_n}{\langle a^2 \rangle} \right| = 4$ .

## 6 February 1

Last time, we defined the Dihedral group  $D_n = \langle a, b \rangle \subset S_n$  where  $a = (1 \ 2 \ 3 \ \dots \ n)$  (rotation) and

$$b = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & n & n-1 & \dots & 3 & 2 \end{pmatrix}$$

Note that

$$bab^{-1} = (1 \ n \ n-1 \ \dots \ 3 \ 2) = a^{-1}$$

and  $|D_n| = 2n$ . We also defined  $[b, a] = bab^{-1}a^{-1} = a^{-2}$ , so  $a^{-2} \in D'_n$ .

What is  $D'_n$ ? So far, we know that  $a^2 \in D'_n$ . What happens if we conjugate  $a^2$  by some other element? For instance,

$$ba^2b^{-1} = a^{-2}$$

This shows  $\langle a^2 \rangle \triangleleft D_n$ . We discuss

$$\left| \frac{D_n}{\langle a^2 \rangle} \right| = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even.} \end{cases}$$

So  $D_n / \langle a^2 \rangle$  is abelian. Hence  $D'_n = \langle a^2 \rangle$ . Hence  $D'_n = \langle a^2 \rangle$ .  $|a| = n$ ,  $|b| = 2$ . We have

$$bab^{-1} = a^{-1}$$

We have an action

$$\langle b \rangle \rightarrow \text{Aut} \langle a \rangle$$

conjugation by  $b$  or not. **Semi-direct Products:**

**Definition 4.** Suppose that  $N$  and  $H$  are groups, and we have a homomorphism  $\alpha : H \rightarrow \text{Aut}(N)$ . We define a group structure on  $\{(n, h) : n \in N, h \in H\}$  by

$$(n, h) \cdot (n', h') = (n\alpha(h)(n'), hh')$$

It is left to check the following properties:

1. Associativity (see exercise)
2.  $(n, h) \cdot (e, e) = (n\alpha(h)(e), h) = (n, h)$  and

$$(e, e) \cdot (n, h) = (e\alpha(e)(n), h) = (n, h)$$

3. Inverses.  $(n, h)^{-1} = (\alpha(h^{-1})(n^{-1}), h^{-1})$  (see exercise).

We write the defined group  $N \rtimes H$ , or  $N \rtimes_{\alpha} H$ .

### Exercise 2

Check the associative property of this group. Also check that the inverse in property 3 is actually an inverse.

Note that  $N \times \{e\} \triangleleft N \rtimes H$ . In other words, we can reverse the operation of multiplying by  $N$  on the left by semi-direct product. We check that  $N \times \{e\}$  is a normal subgroup indeed. We check

$$\begin{aligned} (n, h) \cdot (n', e) \cdot (n, h)^{-1} &= (n\alpha(h)(n'), h) \cdot (\alpha(h^{-1})(n^{-1}), h^{-1}) \\ &= (n\alpha(h)(n')\alpha(h)(\alpha(h^{-1})(n^{-1})), hh^{-1}) \\ &= (n\alpha(h)(n')n^{-1}, e) \end{aligned}$$

We also have

$$\{e\} \times H < N \rtimes H$$

but not necessarily normality.



**Proposition 10.** Suppose  $G$  is a group,  $N \triangleleft G$ ,  $H < G$  such that

1.  $NH = G$ .
2.  $N \cap H = \{e\}$ .

Then  $G \cong N \rtimes_{\alpha} H$  where  $\alpha : H \rightarrow \text{Aut}(N)$  is conjugation  $\alpha(h)(n) = hnh^{-1}$ .

*Proof.* Define  $N \rtimes_{\alpha} H \rightarrow G$  by

$$(n, h) \mapsto nh$$

so it is at least a map. We check it's a homomorphism:

$$\begin{aligned} ((n, h) \cdot_{\alpha} (n', h')) &= (nhn'h^{-1}, hh') \\ &\mapsto nhn'h^{-1}hh' = nhn'h' \end{aligned}$$

It's surjective by property 1, and injective by 2. □

$$D_n \cong \langle a \rangle \rtimes \langle b \rangle$$

where  $\langle a \rangle \cong \frac{\mathbb{Z}}{n}$ ,  $\langle b \rangle \cong \frac{\mathbb{Z}}{2}$ . So we can define

$$D_{\infty} = \langle a \rangle \rtimes_{\alpha} \langle b \rangle$$

where  $\langle a \rangle \cong \mathbb{Z}$  and  $\langle b \rangle \cong \frac{\mathbb{Z}}{2}$ . It has generators  $a, b$  by  $|a| = \infty$  and  $|b| = 2$ . What is  $\text{Aut}\mathbb{Z}$ ? It's  $\frac{\mathbb{Z}}{2}$ . Let's look at the  $\alpha$  we define for the product:

$$\begin{aligned} (\alpha(b))(a) &= a^{-1} \\ bab^{-1} &= a^{-1} \end{aligned}$$

We now look to the following question: What are **all** groups of order 6? There must be a 3-Sylow  $P \cong \frac{\mathbb{Z}}{3}$  which is a normal subgroup. Cauchy's theorem says there is a subgroup of order 2. A group of order 6 must hence be

$$\frac{\mathbb{Z}}{3} \rtimes_{\alpha} \frac{\mathbb{Z}}{2}$$

for some choice of  $\alpha$ . We have

$$\alpha : \frac{\mathbb{Z}}{2} \rightarrow \text{Aut}\frac{\mathbb{Z}}{3} \cong \frac{\mathbb{Z}}{2}$$

So if  $\alpha$  is trivial, the semi-direct product is the direct product.

$$\frac{\mathbb{Z}}{3} \times \frac{\mathbb{Z}}{2} \cong \frac{\mathbb{Z}}{6}$$

If  $\alpha$  is nontrivial, there's only one choice for it:  $\alpha(1) = 2$

$$bab^{-1} = a^{-1}$$

$a = (1 \ 2 \ 3), b = (1 \ 2)$ .

$$bab^{-1} = a^{-1}$$

When can we say different (nontrivial)  $\alpha$ 's give us isomorphic groups?

**Proposition 11.** Suppose  $H$  is cyclic, and  $\alpha, \beta : H \rightarrow \text{Aut}(N)$  with the property that  $\alpha(h)$  and  $\beta(h)$  are conjugate subgroups of  $\text{Aut}(N)$ . Then  $N \rtimes_{\alpha} H \cong N \rtimes_{\beta} H$

*Proof.* Let  $H = \langle h \rangle = \langle h^k \rangle$  and  $\beta(h) = \varphi \circ \alpha(h^k) \circ \varphi^{-1}$  (ie  $\beta(h)$  is the conjugate of some generator of  $\alpha(H)$ ).

Define

$$N \rtimes_{\beta} H \rightarrow N \rtimes_{\alpha} H$$

by

$$\begin{aligned} (n, h) &\mapsto (\varphi(n), h^k) \\ (n, h) \cdot_{\beta} (n', h') &= (n\beta(h)(n'), hh') \end{aligned}$$

the product of the two elements' images on the left is

$$\begin{aligned} &(\varphi(n), h^k) \cdot_{\alpha} (\varphi(n'), (h')^k) \\ &= (\varphi(n)\alpha(h^k)(\varphi(n')), h^k(h')^k) \end{aligned}$$

and the latter maps to

$$\begin{aligned} &(\varphi(n)\varphi\varphi^{-1}\alpha(h^k)\varphi(n'), (hh')^k) \\ &= (\varphi(n)\alpha(h^k)\varphi(n'), h^k(h')^k) \end{aligned}$$

□

**Example.** On different  $\rtimes$  structures.

$$\frac{\mathbb{Z}}{8} \rtimes \frac{\mathbb{Z}}{2}$$

We have  $\text{Aut}(\frac{\mathbb{Z}}{8}) = (\frac{\mathbb{Z}}{8})^{\times} = \{1, 3, 5, 7\}$ . Write  $\frac{\mathbb{Z}}{8}$  as  $\langle x \rangle$ ,  $x^8 = 1$  and  $\frac{\mathbb{Z}}{2}$  as  $\langle y \rangle$  where  $y^2 = 1$ .

$$xyx^{-1} = \begin{cases} x \\ x^3 \\ x^5 \\ x^7 \end{cases}$$

Let's examine:

$$xyx^{-1} = x^3$$

In any of the above cases,

$$G = \{x^i, yx^i : 0 \leq i \leq 7\}$$

$x^4$  is an element of order 2, but we also have

$$(yx^i)^2 = yx^i yx^i = yx^i y^{-1} x^i = x^{3i} x^i = x^{4i}$$

This has order 2 if  $i = 2, 4, 6, 8$ . In this group, all the elements of order 2 are  $\{x^4, yx^2, yx^4, yx^6, y\}$ , so there are 5 elements. Similarly, when

$$xyx^{-1} = x^5,$$

we have

$$(yx^i)^2 = x^{6i}$$

which is 1 when  $i = 0, 4$ . So elements of order 2 are

$$\{x^4, y, yx^4\}.$$

We also have for

$$xyx^{-1} = x^7$$

(which is the actual dihedral group)

$$(yx^i)^2 = x^{8i} = 1$$

for all  $i$ . That is, the set of order 2 elements are

$$\{x^4, y, yx, yx^2, yx^3, yx^4, yx^5, yx^6, yx^7\}$$

However, when

$$yxy^{-1} = x$$

we have

$$(yx^i)^2 = x^{2i} = 1$$

when  $i = 0, 4$  (but in this case we are working with  $\frac{\mathbb{Z}}{8} \rtimes \frac{\mathbb{Z}}{2}$ ). Hence the order 2 elements are

$$\{x^4, y, yx^4\}.$$

We have that  $\frac{\mathbb{Z}}{8} \rtimes \frac{\mathbb{Z}}{2}$  has three nonisomorphic nonabelian  $\rtimes$  structures.

Let  $\mathbb{F}$  be a field, and consider  $\mathrm{GL}_n(\mathbb{F})$ . We can consider

$$\mathrm{SL}_n(\mathbb{F}) \triangleleft \mathrm{GL}_n(\mathbb{F})$$

Denote  $G$  to be the bigger group and  $N = \mathrm{SL}_n(\mathbb{F})$ . Take

$$H = \left\{ \begin{pmatrix} a & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \right\}$$

We prove we can write  $G = N \rtimes H$ . We have

$$\mathrm{GL}_n(\mathbb{F}) \xrightarrow{\det} \mathbb{F}^\times \xrightarrow{\sim} H$$

the former map has kernel  $\mathrm{SL}_n(\mathbb{F})$ . By a form of the first isomorphism theorem we may do later, the desired semi-direct product follows.

## 7 February 3

Suppose  $R$  is a commutative ring, and  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  is an exact sequence of  $R$ -modules. This sequence splits if we can split the surjection and injection. In other words, there exists a map  $N \rightarrow M$  such that

$$L \rightarrow M \rightarrow N$$

is the identity. This is the same thing as saying there exists  $M \rightarrow L$  such that

$$L \rightarrow M \rightarrow L$$

is the identity. The same thing holds for groups, except we usually write 1 for multiplicative notation for groups. For an exact sequence of groups,

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

we say this sequence splits if there exists a map  $Q \rightarrow G$  such that

$$Q \rightarrow G \rightarrow Q$$

is the identity.

**Example.**

$$1 \rightarrow A_3 \rightarrow S_3 \rightarrow \frac{S_3}{A_3} \rightarrow 1$$

Suppose we take a nontrivial element in  $\frac{S_3}{A_3}$  to a 2-cycle. Now  $\frac{S_3}{A_3} \cong \frac{\mathbb{Z}}{2}$ , so this forms a group homomorphism. But note that we cannot have a map  $S_3 \rightarrow A_3$ , because there is no normal subgroup of order 2. The takeaway here is a difference between the commutative and noncommutative case. The latter map is required to split, but the former map does not necessarily split if the latter one does.

**Example.** Consider

$$0 \rightarrow \frac{\mathbb{Z}}{p} \xrightarrow{\cdot p} \frac{\mathbb{Z}}{p^2} \rightarrow \frac{\mathbb{Z}}{p} \rightarrow 0$$

which does not split. (after reading the ahead discussion, note that there is no possible way to write  $\frac{\mathbb{Z}}{p} \rtimes \frac{\mathbb{Z}}{p}$  other than  $\frac{\mathbb{Z}}{p} \times \frac{\mathbb{Z}}{p}$ .)

*Proof.* We would like to ultimately show that  $G$  is a semidirect product of  $Q$  and  $N$  if it is in a split exact sequence. We saw

$$N \cong \{(n, 1) : n \in N\} \triangleleft N \rtimes H$$

$$H \cong \{(1, h) : h \in H\} < N \rtimes H$$

We get

$$1 \rightarrow N \rightarrow N \rtimes H \rightarrow H \rightarrow 1$$

is split exact. The split map is  $h \mapsto (1, h)$ . Suppose  $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$  is split exact. Then there exists  $Q' < G$  with  $Q' \cong Q$  and  $\varphi : Q \rightarrow G$ . Suppose the sequence is split. We claim  $G = NQ'$ . We have a proposition  $G \cong N \rtimes Q' \cong N \rtimes Q$ .

Suppose  $x \in N \cap Q'$ .

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

$$x \mapsto x \mapsto 1$$

by exactness. Since  $x \in Q'$ , there exists  $y \in Q$  such that  $\varphi(y) = x$ . But  $y \mapsto \varphi(y) \mapsto 1 = y$ . So  $x = 1$ .

$$\frac{G}{N} \cong Q \cong Q'$$

$NQ' < G$ , and

$$\frac{NQ'}{N} \cong \frac{Q'}{Q' \cap N}$$

$$NQ' < G$$

and  $\frac{NQ'}{N} \cong \frac{G}{N}$ . Using that subgroups of  $\frac{G}{N}$  correspond to subgroups of  $G$  containing  $N$ , we get  $NQ' = G$ .  $\square$

Suppose  $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$  is exact. We say  $G$  is an extension of  $Q$  by  $N$ . For example,  $\frac{\mathbb{Z}}{p^2}$  is an extension of  $\frac{\mathbb{Z}}{p}$  by  $\frac{\mathbb{Z}}{p}$ .  $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ . We have the rules

$$i \cdot j = k = -j \cdot i$$

$$j \cdot k = i = -k \cdot j$$

$$k \cdot i = j = -i \cdot k$$

$$1, -1 \in Z(Q)$$

Those elements have order 2, and  $i, j, k$  have order 4, their squares being  $-1$ . We have

$$\langle i \rangle \triangleleft Q$$

via

$$1 \rightarrow \langle i \rangle \rightarrow Q \rightarrow \frac{Q}{\langle i \rangle} \rightarrow 1$$

so  $Q$  is an extension of  $\frac{\mathbb{Z}}{2}$  by  $\frac{\mathbb{Z}}{4}$ . This sequence does not split. The only element of order 2 is  $-1$ , which cannot be mapped to, since  $-1 \in \langle i \rangle$ .

$$Q \neq \frac{\mathbb{Z}}{4} \rtimes \frac{\mathbb{Z}}{2}$$

for any choice of  $\alpha : \frac{\mathbb{Z}}{2} \rightarrow \text{Aut}(\frac{\mathbb{Z}}{4}) = \frac{\mathbb{Z}}{2}$  which is either trivial or nontrivial. In the nontrivial case, we are looking at  $D_4$ . In this group, there are 5 elements of order 2. But  $Q$  has 1 element of order 2.

**Groups of order 12** (see homework). We know there is a 2-sylow and a 3-sylow. The 3-sylow is isomorphic to  $\frac{\mathbb{Z}}{3}$  and the 2-sylow (because it's of order 4) is isomorphic to either  $\frac{\mathbb{Z}}{4}$  or  $\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$ . Suppose  $P \triangleleft G$  and  $|G| = 12$ . Then  $G = PQ$ .

$$P \cap Q = 1$$

So  $G \cong P \rtimes Q$ . There are some cases:

1.  $\frac{\mathbb{Z}}{3} \rtimes_{\alpha} \frac{\mathbb{Z}}{4}$ . This means we should have a map  $\frac{\mathbb{Z}}{4} \rightarrow \text{Aut}(\frac{\mathbb{Z}}{3}) = (\frac{\mathbb{Z}}{3})^{\times} = \frac{\mathbb{Z}}{2}$ . Two options exist.  $\alpha$  is trivial, giving

$$\frac{\mathbb{Z}}{3} \times \frac{\mathbb{Z}}{4} \cong \frac{\mathbb{Z}}{12}$$

or it is nontrivial. This yields a nonabelian

$$\frac{\mathbb{Z}}{3} \rtimes \frac{\mathbb{Z}}{4}.$$

2.  $\frac{\mathbb{Z}}{3} \rtimes_{\alpha} (\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2})$ , where  $\alpha : \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2} \rightarrow \frac{\mathbb{Z}}{2}$ . You could kill  $(1, 0)$ ,  $(0, 1)$ , or  $(1, 1)$ . Some options: If  $\alpha$  is trivial, we have

$$\frac{\mathbb{Z}}{3} \times \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$$

If not,  $\frac{\mathbb{Z}}{2} \times (\frac{\mathbb{Z}}{3} \rtimes \frac{\mathbb{Z}}{2}) \cong \frac{\mathbb{Z}}{2} \times S_3$  since there is only one nonabelian group of order 6.

Suppose  $P$  is not normal. There are several 3-Sylow subgroups.  $s_3 = 4$  of them, to be exact. This gives exactly eight elements of order 3, and the remaining ones constitute  $Q$ . Hence  $Q \triangleleft G$ . So  $Q$  is normal. Suppose  $\frac{\mathbb{Z}}{4} \rtimes \frac{\mathbb{Z}}{3}$ .  $\alpha : \frac{\mathbb{Z}}{3} \rightarrow \text{Aut}(\frac{\mathbb{Z}}{4}) = \frac{\mathbb{Z}}{2}$  must be trivial. And the other group of order 4 is  $\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$ , and so we have

$$\left( \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2} \right) \rtimes \frac{\mathbb{Z}}{3}$$

Now  $\alpha : \frac{\mathbb{Z}}{3} \rightarrow \text{Aut}(\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2})$ . Any automorphisms permute the three possible generators. In particular, we work with  $\text{GL}_2(\mathbb{F}_2)$ . What have we of

$$|\text{GL}_n(\mathbb{F}_p)| = (p^n - 1)(p^n - p)(p^n - p^2) \cdots (p^n - p^{n-1}).$$

Hence  $\text{GL}_2(\mathbb{F}_2) = S_3$  since its of order 6 and nonabelian.  $\alpha$  is either trivial (a case we already covered) or nontrivial, in which case it hits some order 3 element. The new group is

$$\left( \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2} \right) \rtimes \frac{\mathbb{Z}}{3}$$

Now we would like to distinguish groups that we have already found. Note that

$$\frac{\mathbb{Z}}{2} \times \left( \frac{\mathbb{Z}}{3} \rtimes \frac{\mathbb{Z}}{2} \right) \cong \frac{\mathbb{Z}}{6} \rtimes \frac{\mathbb{Z}}{2} = D_6$$

The distinguishing factor between  $(\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}) \rtimes \frac{\mathbb{Z}}{3}$  and  $D_6$  is that one has a normal 2 sylow and the other has a normal 3 sylow, but neither has both or else they would be abelian. In  $A_4$ , notice that we had a normal 2-sylow. It should accordingly be

$$A_4 = \left( \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2} \right) \rtimes \frac{\mathbb{Z}}{3}$$

We now classify groups of order 8. If there exists elements of order 8,  $G \cong \frac{\mathbb{Z}}{8}$ .

1. If there is an element of order 8,  $G \cong \frac{\mathbb{Z}}{8}$ .
2. If  $x^2 = 1$  for all  $x \in G$ , then we have commutativity, so  $G \cong \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$ .
3. If there exists  $x \in G$  of order 4, it generates a normal subgroup (since it's of index 2). So  $\langle x \rangle \triangleleft G$ . Suppose there exists  $y \in G \setminus \langle x \rangle$  of order 2. Then  $\langle x \rangle \langle y \rangle$ . Hence

$$G \cong \frac{\mathbb{Z}}{4} \rtimes_{\alpha} \frac{\mathbb{Z}}{2}$$

So  $\alpha$  is trivial or not. In the trivial case we have  $\frac{\mathbb{Z}}{4} \times \frac{\mathbb{Z}}{2}$ . In the nontrivial case, we have

$$\frac{\mathbb{Z}}{4} \rtimes \frac{\mathbb{Z}}{2}$$

which is nonabelian so it is  $D_4$ .

4. Last case: every element of  $G \setminus \langle x \rangle$  has order 4. Pick  $y \in G \setminus \langle x \rangle$ . Then  $y^2 \in \langle x \rangle$ , so in particular  $y^2 = x^2$ . Our full list of group elements:  $1, x, x^2, x^3, y, yx, yx^2, yx^3$ . We call  $x^2 = -1$ , and  $x = i$ ,  $x^3 = -i$ . Call  $y = j$ ,  $yx = -k$ ,  $yx^2 = -j$ ,  $yx^3 = k$ . This group is the quaternion group  $Q$ .

## 8 February 8

We classify groups of order  $p^3$  where  $p$  is an odd prime. When we did the groups of order 8, they were built up from cyclic groups by using direct products and semidirect products, with the exception of the quaternion group. It had elements of order 4, call it  $i$ ,  $\langle i \rangle$  complement also only had elements of order 4.

- Suppose there exists  $x$  of order  $p^3$ , then  $G$  is cyclic.
- Suppose  $x^p = 1$  for all  $x \in G$ . Noting a result,  $|G| = p^k$   $k \geq 1$  implies  $Z \neq \{1\}$ . Pick  $x$  of order  $p$  in  $Z$ , and  $y \in G \setminus \langle x \rangle$ . Then  $\langle x, y \rangle$  is abelian, and

$$\langle x, y \rangle \cong \frac{\mathbb{Z}}{p} \times \frac{\mathbb{Z}}{p}$$

Also pick  $w \in G \setminus \langle x, y \rangle$  be an element. Then

$$G \cong \left( \frac{\mathbb{Z}}{p} \times \frac{\mathbb{Z}}{p} \right) \rtimes \frac{\mathbb{Z}}{p}$$

where the last copy is  $\langle w \rangle$ . Also pick

$$\alpha : \frac{\mathbb{Z}}{p} \rightarrow \text{Aut} \left( \frac{\mathbb{Z}}{p} \times \frac{\mathbb{Z}}{p} \right) = \text{GL}_2(\mathbb{F}_p)$$

Last time we calculated

$$|\text{GL}_2(\mathbb{F}_p)| = (p^2 - 1)(p^2 - p)$$

The highest power of  $p$  dividing it is  $p$ . If  $\alpha$  is nontrivial, then its image is a  $p$ -Sylow of  $\text{GL}_2(\mathbb{F}_p)$ , and all such are conjugate. We have two options.

- $\frac{\mathbb{Z}}{p} \times \frac{\mathbb{Z}}{p} \times \frac{\mathbb{Z}}{p}$  which is abelian if  $\alpha$  is trivial.
- $\alpha$  is nontrivial  $\left( \frac{\mathbb{Z}}{p} \times \frac{\mathbb{Z}}{p} \right) \rtimes \frac{\mathbb{Z}}{p}$ .
- $\exists x \in G$  of order  $p^2$ . Suppose  $\exists y \in G \setminus \langle x \rangle$  of order  $p$ . Then  $G \cong \frac{\mathbb{Z}}{p^2} \rtimes \frac{\mathbb{Z}}{p}$ .

$$\alpha : \frac{\mathbb{Z}}{p} \rightarrow \text{Aut} \left( \frac{\mathbb{Z}}{p^2} \right) \cong \left( \frac{\mathbb{Z}}{p^2} \right)^{\times} \cong \frac{\mathbb{Z}}{p(p-1)}$$

(we will look next Thursday  $\frac{\mathbb{Z}}{p^n}$ ) <sup>$\times$</sup>  as a group and verify that it is cyclic). Now  $\frac{\mathbb{Z}}{p(p-1)}$  has a unique subgroup of order  $p$ . Two possibilities:

- $\alpha$  is trivial, so we have

$$\frac{\mathbb{Z}}{p^2} \times \frac{\mathbb{Z}}{p}$$

which is abelian, or

- $\alpha$  is nontrivial, so we have

$$\frac{\mathbb{Z}}{p^2} \rtimes \frac{\mathbb{Z}}{p}$$

which is nonabelian.

- Now suppose that we again have  $x \in G$  with order  $p^2$  and for all  $y \in G \setminus \langle x \rangle$   $y$  has order  $p^2$ . We will get a contradiction. So pick  $y \in G \setminus \langle x \rangle$ . Then  $y$  of course has order  $p^2$ . So  $y^p$  has order  $p$ . Hence  $y^p \in \langle x \rangle$ . Hence  $y^p = x^{pr}$  for some  $r \in \mathbb{Z}$ . We have

$$\langle x^r \rangle = \langle x \rangle$$

because  $r$  is relatively prime to  $p$ . We can replace  $x^r$  with  $x$ , so we may by abuse of notation assume  $r = 1$ . Hence  $x^p = y^p$ . Suppose  $G$  is abelian, then  $yx^{p^2-1}$  has order  $p$ . Separately note  $yx^{p^2-1} \in G \setminus \langle x \rangle$ . We get a contradiction however because of our claim about  $G \setminus \langle x \rangle$ . Hence  $G$  is not abelian. Then by a result from the homework,  $|Z| = p$  so  $|Z| = \langle x^p \rangle = \langle y^p \rangle$ . We have  $\frac{G}{Z}$  has order  $p^2$  and is abelian because of that. Hence  $G' \subset Z$ . In particular,  $[x, y] \in Z$ . We utilize exercise 10. We have

$$x^n y^n = (xy)^n [x, y]^{\binom{n}{2}}$$

So  $[x, y^{-1}] \in Z$  so

$$x^p y^{-p} = (xy^{-1})^p [x, y^{-1}]^{\binom{p}{2}}$$

now  $x^p = y^p$  implies the left is the identity. Now  $[x, y^{-1}] \in Z$ , and  $|Z| = p$ , so  $[x, y^{-1}]^p = 1$ . But  $\binom{p}{2}$  for odd primes. So we have  $xy^{-1}$  has order  $p$ . But then  $xy^{-1}$  is an element of order  $p$  in  $G \setminus \langle x \rangle$ , a contradiction.

The same argument above doesn't work when  $p = 2$ . Note the Quaternion group, as  $Q \setminus \langle i \rangle$  has only elements of order 4 =  $2^2$ . Recall that if

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

is an exact sequence of groups, then we say  $G$  is an extension of  $Q$  by  $N$ . This gets us to the following notion.

**Definition 5.**  $G$  is called **simple** if its only normal subgroups are  $\{1\}$  and  $G$ .

Every finite group is "built" from simple groups and extensions. What are finite simple groups?

- $\frac{\mathbb{Z}}{p}$
- $A_n$  for  $n \geq 5$ .
- Some Lie groups and matrix groups  $\text{PSL}_n(\mathbb{F}_q)$  or  $U_n(\mathbb{F}_q)$ . Symplectic and orthogonal groups yadda yadda
- 26 sporadic simple groups

We prove next that for  $n \geq 5$ ,  $A_n$  is simple.

**Theorem 5.**  $A_n$  is simple for  $n \geq 5$ .

As a first remark, note  $A_n$  is generated by 3-cycles.

**Lemma 5.** If  $n \geq 5$ , then all 3-cycles are conjugate in  $A_n$ .

*Proof.* Let  $(1\ 2\ 3)$  and  $(a\ b\ c)$  be 3-cycles. Consider

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ a & b & c & ? & \dots & ? \end{pmatrix}$$

In the way we constructed  $\pi$ ,

$$\pi(1\ 2\ 3)\pi^{-1} = (a\ b\ c)$$

If  $\pi \in A_n$  we are done. If  $\pi \notin A_n$ , we can compose it with a 2-cycle, say  $(4\ 5)$ . Then the resulting permutation does the same.  $\square$

*Proof.* (of the big theorem) Let  $N \neq \{e\}$  be a normal subgroup of  $A_n$ .

1. If  $N$  contains a three cycle, we are done. Since in that case  $N$  contains all 3-cycles, so  $N = A_n$ .
2. For the remaining cases, let  $\sigma \neq e$  be an element of  $N$ . Write  $\sigma$  as a product of disjoint cycles. Suppose  $\sigma$  includes a cycle of length  $\geq 4$ . So  $\sigma = (1\ 2\ \dots\ \tau)\zeta$  where  $\tau \geq 4, \zeta$  disjoint. Note

$$\begin{aligned} [(1\ 2\ 3), \sigma] &= (1\ 2\ 3)(1\ 2\ \dots\ \tau)(1\ 2\ 3)^{-1}(\tau\ \tau-1\ \dots\ 2\ 1) \\ &= (2\ 3\ 1\ 4\ \dots\ \tau)(\tau\ \tau-1\ \dots\ 3\ 2\ 1) \\ &= (1\ 2\ 4) \in N \end{aligned}$$

So  $N$  contains a 3-cycle, and we are done.

3. Now suppose  $\sigma$  consists of a three cycle and some disjoint 2 cycles. Then  $\sigma^2$  is a 3-cycle and we're done.
4. Now suppose  $\sigma$  consists of multiple three cycles and disjoint 2 cycles. By squaring it, we may assume there are no 2-cycles. So

$$\sigma = (1\ 2\ 3)(4\ 5\ 6)\zeta$$

where  $\zeta$  is disjoint.

$$\begin{aligned} [(1\ 2\ 4), \sigma] &= (1\ 2\ 4)(1\ 2\ 3)(4\ 5\ 6)(4\ 2\ 1)(3\ 2\ 1)(6\ 5\ 4) \\ &= (2\ 4\ 3)(1\ 5\ 6)(3\ 2\ 1)(6\ 5\ 4) = (1\ 2\ 5\ 3\ 4) \end{aligned}$$

so we are done by 2.

5. Finally, suppose  $\sigma$  consists of just 2 cycles. Each element of  $N$  is a product of (an even number) disjoint 2-cycles. Let  $\sigma = (1\ 2)(3\ 4)\zeta \in N$ ,  $\zeta$  is disjoint. Playing the same game,

$$\begin{aligned} [(1\ 2\ 3), \sigma] &= (1\ 2\ 3)(1\ 2)(3\ 4)(1\ 2\ 3)^{-1}(2\ 1)(4\ 3) \\ &= (2\ 3)(1\ 4)(2\ 1)(4\ 3) = (1\ 3)(2\ 4) \in N \end{aligned}$$

Now we have

$$\begin{aligned} [(1\ 3\ 5), (1\ 3)(2\ 4)] &= (1\ 3\ 5)(1\ 3)(2\ 4)(1\ 3\ 5)^{-1}(3\ 1)(4\ 2) \\ &= (3\ 5)(2\ 4)(3\ 1)(4\ 2) \\ &= (1\ 5\ 3) \in N \end{aligned}$$

Which completes the proof!

$\square$



In a future point, we may discuss  $\text{PSL}_n(\mathbb{F}_q)$ . Also, we may discuss  $\text{Aut} S_n$ . As mentioned before, we will also discuss  $\text{Aut} \frac{\mathbb{Z}}{p^n}$ .

Automorphism groups of cyclic groups in general. An infinite cyclic group  $\mathbb{Z}$ , we have

$$\text{Aut} \mathbb{Z} = \frac{\mathbb{Z}}{2}$$

The elements are multiplication by  $\pm 1$ . So  $\text{Aut} \mathbb{Z} \cong \frac{\mathbb{Z}}{2}$ .

$$\text{Aut} \left( \frac{\mathbb{Z}}{n} \right) = \left( \frac{\mathbb{Z}}{n} \right)^\times$$

What is  $\left( \frac{\mathbb{Z}}{n} \right)^\times$ ? Suppose  $n = p_1^{e_1} \cdots p_k^{e_k}$  where  $p_i$  are distinct primes. The Chinese remainder theorem yields

$$\frac{\mathbb{Z}}{n} \cong \frac{\mathbb{Z}}{p_1^{e_1}} \times \cdots \times \frac{\mathbb{Z}}{p_k^{e_k}}.$$

A  $k$ -tuple is a unit precisely if each coordinate is a unit. So

$$\left( \frac{\mathbb{Z}}{n} \right)^\times \cong \left( \frac{\mathbb{Z}}{p_1^{e_1}} \right)^\times \times \cdots \times \left( \frac{\mathbb{Z}}{p_k^{e_k}} \right)^\times$$

This finally gets us to the question of  $\text{Aut} \left( \frac{\mathbb{Z}}{p^e} \right)$  when  $p$  is prime. This is the subject of the next lecture.

## 9 February 10

Recall a homework problem: Suppose  $G$  is a finite group in which  $x^d = 1$  has at most  $d$  solutions for each  $d$  dividing its order. Then  $G$  is cyclic.

**Corollary 4.** The multiplicative group  $\mathbb{F}^\times$  for a finite field  $\mathbb{F}$  is cyclic.

In particular, we have

**Corollary 5.** If  $p$  is a prime, then  $\left( \frac{\mathbb{Z}}{p} \right)^\times$  is cyclic. A generator is called a primitive root mod  $p$ .

Last time, what we mentioned was that if  $n$  is a positive integer, and  $n = \prod p_i^{e_i}$  where  $p_i$  are distinct, the Chinese remainder theorem yields

$$\frac{\mathbb{Z}}{n} \cong \prod_i \left( \frac{\mathbb{Z}}{p_i^{e_i}} \right)$$

Likewise, we get

$$\left( \frac{\mathbb{Z}}{n} \right)^\times \cong \prod_i \left( \frac{\mathbb{Z}}{p_i^{e_i}} \right)^\times$$

Now what is  $\left( \frac{\mathbb{Z}}{p^e} \right)^\times$ ? We do know

$$\left| \left( \frac{\mathbb{Z}}{p^e} \right)^\times \right| = \varphi(p^e) = p^e - p^{e-1}$$

the Euler phi function.

**Theorem 6.** Suppose  $p$  is an odd prime. Then  $\left(\frac{\mathbb{Z}}{p^e}\right)^\times$  is cyclic of order  $p^e - p^{e-1}$ .

*Proof.*  $e = 1$  is covered by an above corollary. Now about the general case. We have a canonical surjection

$$\frac{\mathbb{Z}}{p^e} \twoheadrightarrow \frac{\mathbb{Z}}{p}$$

If we have a ring element on the left that is a unit, it maps to a unit on the right. So we have

$$\frac{\mathbb{Z}^\times}{p^e} \twoheadrightarrow \frac{\mathbb{Z}^\times}{p}$$

So  $\left(\frac{\mathbb{Z}}{p}\right)^\times$  has an element of order  $p - 1$ , namely a primitive root. The inverse image of such an element has order  $(p - 1) \cdot p^k$ ,  $k \leq e - 1$ . Some power of the inverse image has order  $p - 1$ . In  $\left(\frac{\mathbb{Z}}{p^e}\right)^\times$ , we have proved the existence of an element  $\alpha$  of order  $p - 1$ . Next we want  $\beta \in \left(\frac{\mathbb{Z}}{p^e}\right)^\times$  of order  $p^{e-1}$ . If we do have such an element,  $\alpha\beta$  has order  $(p - 1)p^{e-1}$  since the group is commutative. Now a big claim.  $\beta = 1 + p$  has order  $p^{e-1}$ . So far, we haven't used the fact that  $p$  is odd. We have

$$\begin{aligned} \beta^{p^{e-1}} &\equiv 1 + \binom{p^{e-1}}{1}p + \binom{p^{e-1}}{2}p^2 + \dots + p^{p^{e-1}} \pmod{p^e} \\ &\equiv 1 \end{aligned}$$

So  $\beta$  has order  $p^{e-1}$  or some lower power of  $p$ . We have

$$\begin{aligned} \beta^{p^{e-2}} &\equiv 1 + \binom{p^{e-2}}{1}p + \binom{p^{e-2}}{2}p^2 + \dots + p^{p^{e-2}} \\ &\equiv 1 + p^{e-1} \not\equiv 1 \end{aligned}$$

□

**Theorem 7.** Suppose  $e \geq 3$ , then  $\left(\frac{\mathbb{Z}}{2^e}\right)^\times \cong \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2^{e-2}}$ .

*Proof.*  $\pm 1, \pm(2^{e-1} - 1)$ , are roots of  $x^2 = 1$  since

$$\begin{aligned} (2^{e-1} - 1)^2 &= 2^{2(e-1)} - 2 \cdot 2^{e-1} + 1 \\ &= 2^{2e-2} - 2^e + 1 \end{aligned}$$

So  $\left(\frac{\mathbb{Z}}{2^e}\right)^\times$  cannot be cyclic. Enough to find an element  $\beta$  of order  $2^{e-2}$ . Use the Abelian group structure theorem to prove the claim from there. We claim  $\beta = 5 = 1 + 2^2$  is up for the job.

$$\begin{aligned} (1 + 2^2)^{2^{e-2}} &= 1 + \binom{2^{e-2}}{1}2^2 + \binom{2^{e-2}}{2}2^4 + \dots + 2^{2(e-2)} \pmod{2^e} \\ &\equiv 1 \pmod{2^e} \end{aligned}$$

Likewise, as in the previous proof,  $(1 + 2^2)^{2^{e-3}} \not\equiv 1$ .

$$\begin{aligned} (1 + 2^2)^{2^{e-3}} &= 1 + \binom{2^{e-3}}{1}2^2 + \binom{2^{e-3}}{2}(2^2)^2 + \dots + (2^2)^{2^{e-3}} \\ &\equiv 1 + 2^{e-1} \end{aligned}$$

□

Consider  $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ . What is  $\text{Aut}(Q)$ ? As we will see on the homework, it will be  $S_4$ . What we will at least comment on is  $|\text{Aut}(Q)| = 4!$ . An automorphism does preserve order, so  $\varphi \in \text{Aut}(Q)$  implies that  $\varphi(-1) = -1$ .  $\varphi(i) = \{\pm i, \pm j, \pm k\}$ . Now note that  $\varphi(i)$  narrows down choices for  $\varphi(j)$ . From there,  $\varphi(i), \varphi(j)$  determine the whole automorphism.

$$\varphi(j) = \{\pm i, \pm j, \pm k\} \setminus \{\pm \varphi(i)\}$$

This gives us an upper bound on  $|\text{Aut}(Q)|$  by  $4!$ .

### Exercise 3

Tedious exercise: prove that having chosen  $\varphi(i), \varphi(j)$  appropriately defines an automorphism.

The topic we want to understand is  $\text{Aut}(S_n)$ . Some remarks:

- If  $n \geq 3$ , then  $Z(S_n) = \{e\}$ .

*Proof.* Suppose  $\sigma \neq e$  in  $S_n$ . Let  $a \neq b$  be elements with  $\sigma(a) = b$ . There is a third letter  $c$  by  $n \geq 3$ . Note  $(b\ c)\sigma \neq \sigma(b\ c)$ .  $\square$

- If  $n \geq 5$ , then the only normal subgroups of  $S_n$  are  $\{e\}, A_n, S_n$ .

*Proof.* Suppose  $N \triangleleft S_n$ . Then  $A_n \cap N \triangleleft A_n$ . From last time though, we proved that  $A_n$  is simple for  $n \geq 5$ . So  $A_n \cap N = A_n$  or  $A_n \cap N = \{e\}$ . In the former,  $A_n$  has index 2 implies  $A_n = N$  or  $S_n = N$ . In the latter, if  $N \neq \{e\}$ , then it has an element that is not in  $A_n$ .

$$NA_n = S_n$$

$N$  and  $A_n$  are both normal implies then that we can write  $S_n$  as a direct product. This implies  $|N| = 2$ . In other words there is a nontrivial 2-cycle that is in the center, a contradiction to  $Z(S_n) = \{e\}$ .  $\square$

We note more things about  $\text{Aut}(S_n)$ .

1. Let  $\varphi \in \text{Aut}(S_n)$ . Then  $\varphi$  takes conjugate elements to conjugate elements. Given  $x, yxy^{-1}$ ,

$$\varphi(x)$$

is conjugate to

$$\varphi(y)\varphi(x)\varphi(y)^{-1}$$

2. More interesting is if  $\varphi \in \text{Aut}(S_n)$ , if  $\varphi$  takes 2-cycles to 2-cycles, then  $\varphi \in \text{Inn}(S_n)$ .
3. Say  $\varphi(1\ 2) = (a\ b)$ . If  $\varphi(1\ 3) = (c\ d)$  for  $c, d \notin \{a, b\}$ . Now  $(a\ b)$  and  $(c\ d)$  commute but  $\varphi(1\ 2)$  and  $\varphi(1\ 3)$  don't. We may write

$$\varphi(1\ 2) = (a\ b)$$

$$\varphi(1\ 3) = (a\ c)$$

We have

$$\varphi(1\ 4) = (a\ d) \text{ or } (b\ c)$$

In the latter case, then

$$(a\ b)(a\ c)(b\ c) = (a\ c)$$

Apply  $\varphi^{-1}$ .

$$(1\ 2)(1\ 3)(1\ 4) = (1\ 3)$$

a contradiction. Hence  $\varphi(1\ 4) = (a\ d)$ . We can keep labeling so that  $\varphi(1\ x) = (a\ y)$ . Define  $\pi \in S_n$  by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots \\ a & b & c & d & \dots \end{pmatrix}$$

We would like to show that  $\varphi$  is conjugation by  $\pi$ . Conjugation by  $\pi$  the cycles  $(1\ x)$  yields  $\varphi$ .

We can write

$$S_n/Z(S_n) \cong \text{Inn}(S_n) \triangleleft \text{Aut}(S_n)$$

Conjugacy classes of elements of order 2 in  $S_n$ . On the one hand we have the class of  $(1\ 2)$ ,  $(1\ 2)(3\ 4)$ ,  $(1\ 2)(3\ 4)(5\ 6)$ , etc etc. In the first form, we have  $\binom{n}{2}$  such elements. In the second we have  $\frac{1}{2!}\binom{n}{2}\binom{n-2}{2}$ . In the third form  $\frac{1}{3!}\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}$ . The conjugacy class of  $(1\ 2)(3\ 4)\dots(2k-1\ 2k)$  has size

$$\begin{aligned} & \frac{1}{k!}\binom{n}{2}\binom{n-2}{2}\dots\binom{n-2k+2}{2} \\ &= \frac{1}{k!}\frac{n(n-1)\dots(n-2k+2)(n-2k+1)}{2^k} \end{aligned}$$

So if  $\varphi$  takes the conjugacy classes of  $(1\ 2)$  to the conjugacy class of  $(1\ 2)(3\ 4)\dots(2k-1\ 2k)$ , we have

$$\begin{aligned} \binom{n}{2} &= \frac{1}{k!}\frac{n(n-1)\dots(n-2k+2)(n-2k+1)}{2^k} \\ &= \frac{n(n-1)}{2} \end{aligned}$$

So

$$\begin{aligned} & \frac{(n-2)(n-3)\dots(n-2k+1)}{k!2^k} = \frac{1}{2} \\ &= \binom{n-2}{n-2k} = \frac{k! \cdot 2^k}{(2k-2)!} \\ &= \frac{k}{(2k-3)(2k-5)\dots\cdot 3\cdot 1} \leq \frac{k}{2k-3} \\ &< 1 \text{ if } k > 3 \end{aligned}$$

so  $\varphi$  cannot change the conjugacy class of an element  $(1\ 2)$  to  $(1\ 2)(3\ 4)\dots(2k-1\ 2k)$  if  $k > 3$ .  $k = 1$ :  $\varphi$  takes 2-cycles to 2-cycles, so we have that the above formula that was contradicted is true. If  $k = 2$ , we have

$$\binom{n-2}{n-2k} = \frac{k! \cdot 2^k}{2(2k-2)!}$$

implies

$$\begin{aligned} \binom{n-2}{n-4} &= \frac{2 \cdot 2^2}{2 \cdot 2!} \\ &= \binom{n-2}{2} = 2 \end{aligned}$$

but  $\binom{*}{2}$  is never 2. In the case  $k = 3$ ,

$$\begin{aligned} \binom{n-2}{n-6} &= \frac{3! \cdot 2^3}{2(4!)} \\ \binom{n-2}{4} &= 1 \end{aligned}$$

so  $n = 6$ . So for  $\text{Aut}(S_n) = \text{Inn}(S_n)$  if  $n \neq 6$ . We could say

$$\text{Aut}(S_n) = \text{Inn}(S_n) \cong S_n \text{ if } n \neq 2, 6$$

We have  $\text{Aut}(S_2) = \{e\}$ . The only thing left to examine is  $\text{Aut}(S_6)$ . Next time, we will show

$$\left| \frac{\text{Aut}(S_6)}{\text{Inn}(S_6)} \right| = 2$$

so there is a

$$1 \rightarrow S_6 \rightarrow \text{Aut}(S_6) \rightarrow \frac{\mathbb{Z}}{2} \rightarrow 0$$

is exact.