As always, let R be a ring. We noted

$$D(R) \cong \mathrm{KProj}(R).$$

To recap, a complex $P \in \mathcal{C}(\mathrm{Proj}(R))$ is K projective we have the existing map in the following diagram:

$$\begin{tikzcd} & X \arrow[d, two heads, "\wr"'] \\ P \arrow[ur, dashed, "\exists"] \arrow[r] & Y \end{tikzcd}$$

For example,

Example. $P \in \mathcal{C}(\mathrm{Proj}(R))$ with $P_i = 0$ for all $i \ll 0$. To sketch a proof of this claim, construct liftings one step at a time.

$$\begin{tikzcd} & & & & & & & & X \arrow[d, "\wr", "\varepsilon"'] \\ (\quad \dots \arrow[r] & P_{n+1} \arrow[r] & P_n \arrow[r] & \dots \arrow[r] & P_{a+1} \arrow[r] \arrow[urrrr, dashed, "\tilde{\alpha}"] & P_a \arrow[r] \arrow[urrr] & 0 \arrow[r] & \dots \quad) \arrow[r, "\alpha"] & Y \end{tikzcd}$$

Here, $\tilde{\alpha}(S)$ should satisfy

- $\varepsilon\tilde{\alpha} = \alpha(S)$
- $\partial\tilde{\alpha}(S) = \tilde{\alpha}(\partial S)$.
- ε is surjective implies that ε is surjective on boundaries.
- $H(\varepsilon)$ is isomorphic (H is the homology functor) implies that $\varepsilon : Z(X) \twoheadrightarrow Z(Y)$
- $\ker(\varepsilon)$ is acyclic.

Once you construct these maps, the claim holds.

Note that every module has a K-projective resolution. Every module M admits a surjection $F_0 \twoheadrightarrow M$ from a free module F_0, call this map ε. The kernel of ε then admits a surjection $F_1 \to \ker\varepsilon$ from a free module F_1. Continuing the process one more step yields the following diagram:

$$\begin{tikzcd} F_2 \arrow[rr, "\partial_2"] \arrow[dr] & & F_1 \arrow[rr, "\partial_1"] \arrow[dr] & & F_0 \arrow[r, two heads, "\varepsilon"] & M \\ & \ker\partial_1 \arrow[ur] & & \ker\varepsilon \arrow[ur] & & \end{tikzcd}$$

If we continue this process, we get the K-projective resolution. To make the choice of resolution canonical, we choose the free modules to be freely generated by the elements of the kernels that they surject onto. This yields a functor from modules to K-projective resolutions.

Definition 1. A K-projective resolution of $M \in \mathcal{C}(R)$ is a morphism $\varepsilon : P \to M$ such that

- ε is a quasi-isomorphism, and
- P is K-projective.

We can make this functorial by taking $R \times M \twoheadrightarrow M$,

$$\varepsilon(r, m) = rm.$$

We then set

$$\bigoplus_{m \in M} R \times m = R^{\oplus M} \twoheadrightarrow M.$$

The last surjection above is the ε map.

Theorem 1. For all $M \in \mathcal{C}(R)$, there exists a surjective K-projective resolution

$$P \twoheadrightarrow M$$

which is a weak equivalence.

Definition 2. An R-complex F is semi-free if it has a filtration

$$(0) \subset F_0 \subset F_1 \subset \ldots \subset \bigcup_{n \geq 0} F_n = F$$

such that

1. $F(n) \subset F$ is a subcomplex,
2. $F(n+1)/F(n)$ is a graded free module with trivial boundary maps ($\partial = 0$). In particular, $\partial(F_{n+1}) \subset F_n$.

Exercise 1

Show that semi-free implies K-projective. Also show that unions are K-projective.

Example. Take a complex of free modules

$$\ldots \to F_{n+1} \to F_n \to 0.$$

Let $F(n)$ be the truncation consisting of degrees n and lower. Then $\frac{F(n+1)}{F(n)}$ is precisely $\Sigma^{n+1} F_{n+1}$.

Theorem 2. Each $M \in \mathcal{C}(R)$ has a semi-free resolution

$$F \twoheadrightarrow M.$$

(The map above is a weak equivalence.)

Corollary 1. Every K-projective is a retract of a semi-free.

Proof. (Corollary proof): Let P be K-projective. The following diagram completes the proof:

$$\begin{tikzcd}
 & F \arrow[d, two heads, "\wr"'] \\
P \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, equal] & P
\end{tikzcd}$$

□

Proof. (Theorem proof): We use the small object argument. We want to construct

$$\begin{tikzcd}
F(1) \arrow[r, hook] \arrow[d, "\varepsilon(1)"'] & F(2) \arrow[r, hook] \arrow[dl, "\varepsilon(2)"] & \ldots \arrow[r, hook] & F(n) \arrow[dlll, "\varepsilon(n)"] \arrow[r, hook] & \ldots \arrow[r, hook] & \bigcup_{n \geq 0} F(n) \\
M
\end{tikzcd}$$

such that $\frac{F(n+1)}{F(n)}$ is a graded complex of free modules with zero differential which is surjective on cycles. We need that $\varepsilon(1)$ is surjective on cycles. We have

$$\begin{tikzcd} F(1) \arrow[r, equal] & H(F(1)) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\Longrightarrow"] & H(F(n)) \arrow[r, "\phi"] \arrow[dl, two heads, "H(\varepsilon(n))"] & H(F(n+1)) \\ & H(M) & & \end{tikzcd}$$

for all n where the kernel of ϕ contains the kernel of $H(\varepsilon(n))$, so that $H(\varepsilon(n))$ is surjective for all n.

Next, we note $\ker(H(\varepsilon(n))) \subset H(F(n))$ goes to 0 under $H(i(n))$. We have $H(F) \xrightarrow{\sim} H(M)$.

Remark. Given $\varepsilon : X \to Y$ where $Z(\varepsilon)$ and $H(\varepsilon)$ are surjective, we have that ε is surjective. See the four lemmas applied to the following diagrams:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & B(X) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & H(X) \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\ 0 \arrow[r] & B(Y) \arrow[r] & Z(Y) \arrow[r] & H(Y) \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & Z(X) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \varepsilon(B(X)) \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\ 0 \arrow[r] & Z(Y) \arrow[r] & Y \arrow[r] & \varepsilon(B(Y)) \arrow[r] & 0 \end{tikzcd}$$

Say we have calculated $\varepsilon(n) : F(n) \to M$. Choose cycles $\{z_\lambda\}$ in $F(n)$ that map to the generating set of $\ker(H(\Sigma(n)))$. Pick $w_\lambda \in M$ such that $\partial w_\lambda = \varepsilon(n)(z_\lambda)$. Set

$$F(n+1) = F(n) \oplus_\lambda R \cdot e_\lambda.$$

Above, we have $|e_\lambda| = |z_\lambda| + 1$ and $\partial|_{F(n)} = \partial^{F(n)}$. (Attached are some iPad notes, where the proof continues.) $\square$

$\varepsilon(n+1)(e_x) = w_x$.

$\varepsilon(n+1) : F(n+1) \to M$

- $\varepsilon(n+1)|_{F(n)} = \varepsilon(n)$

- $\varepsilon(n+1)(e_x) = w_x$

Can make a functorial realization of the semi-free resolution of any complex by making free module quite big. Given any cycle, also include info on how it becomes a bd.

There are other ways of constructing resolutions.

M

$H(M) = \{H_i(M)\}_{i \in \mathbb{Z}}$ graded module.

$P \xrightarrow{\sim} H(M)$ K-projective resolution.

One can perturb the differential on P to construct a K-projective resolution of M. (Adams's resolution)
(Cartan / Eilenberg resolution)

Example: P projective $\Rightarrow$ P_i projective modules.

Converse is false. (Dold)

$$\boxed{\begin{array}{l} R = \mathbb{Z}/4\mathbb{Z} \\ \Rightarrow R \xrightarrow{2} R \xrightarrow{2} R \to \cdots \\ \text{Not K-projective.} \end{array}}$$

Ex: Suppose P is K-projective and acyclic. Then P is contractible, i.e. $id \simeq 0$

$$\begin{tikzcd} K\mathrm{Proj}(R) \arrow[r, hook] & K(\mathrm{Proj}\, R) \\ D(R) \arrow[u, "\simeq"] & \end{tikzcd}$$

Fact: Say R is commutative Noetherian. Then
$K\mathrm{proj}(R) = K(\mathrm{Proj}\, R) \iff R$ is regular.

Ex: $\mathbb{Z}, \mathbb{Z}[x], k[x]$ are regular rings

Derived functors: Given an R-complex.
If $P \xrightarrow{\simeq} M$, $Q \xrightarrow{\simeq} M$ K-projective resolutions that $P \simeq Q$ in $K(\mathrm{Proj}\, R)$.

$$\begin{tikzcd} & F \arrow[d, "\simeq"] \\ P \arrow[ur, "\simeq"] \arrow[r, "\simeq"'] & M \end{tikzcd}$$

Given any N, set

$R\mathrm{Hom}_R(M,N) := \mathrm{Hom}_R(P,N)$ where P is a K-proj resoln.

$\mathrm{Ext}^i_R(M,N) := H^i(R\mathrm{Hom}_R(M,N)) = H_{-i}(\mathrm{Hom}_R(P,N))$.

$R\mathrm{Hom}_R(M,-) : \mathcal{C}(R) \longrightarrow K(\mathbb{Z})$
or $K(R)$ if R commutative.

$\mathrm{Ext}^0_R(M,N) = H^0(\mathrm{Hom}_R(P,N))$.

$\mathrm{Ext}^i_R(M,N) =$ morphisms $P \to \Sigma^i N$.

If $Q \xrightarrow{\simeq} N$ is K-projective resoln. then

$\mathrm{Hom}_R(P,Q) \xrightarrow{\simeq} \mathrm{Hom}_R(P,N)$.
is quasi-iso

$- \otimes_R -$ $\quad M \in \mathcal{C}(R^{op})$
cx. of right R-mods

$N \in \mathcal{C}(R)$

$M \otimes_R N$ $\quad$ cx (of $\mathbb{Z}$-modules)

$$\begin{array}{ccc} (M \otimes_R N)_n & = & \bigoplus_{i \in \mathbb{Z}} M_i \otimes N_{n-i} \\ \downarrow \partial & & \downarrow \partial^M \otimes 1 \pm 1 \otimes \partial^N \\ (M \otimes_R N)_{n-1} & & \end{array}$$

$$\partial(m\otimes n) = \partial m\otimes n + (-1)^{|m|} m\otimes \partial n.$$

Fact. $X \xrightarrow{\simeq} Y$ quasi-iso.

Then $P\otimes X \xrightarrow{\simeq} P\otimes Y$ for any K-Proj complex P

Tensor products behave nicely w/ colimits.

In absence of subscripts $\mathrm{Tor}^R(M,N)$ refers to $\mathrm{Tor}^R_*(M,N)$.

- Check for semi-free

Def^n: $M\otimes^L_R N = P\otimes_R N$

$P \xrightarrow{\simeq} M$

a K-proj resln.

$$\mathrm{Tor}^R_i(M,N) = H_i(P\otimes_R N).$$

$X \xrightarrow{\simeq} Y$ quasi-iso $\Rightarrow \mathrm{Tor}^R(M,X) \cong \mathrm{Tor}^R(M,Y)$