

Commutative Algebra

Notes on MATH 7830

Daniel Koizumi

January 24, 2023

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Let R be a commutative Noetherian ring, and let M be an R -module. What does it mean for an element $r \in R$ to be a **zero-divisor**? It simply means that for some $m \neq 0$, $r \cdot m = 0$.

$$zdr_R(M) = \{r \in R \mid r \text{ is a zero divisor on } M\} = \bigcup_{\mathfrak{p} \in \text{Ann}_R M} \mathfrak{p}.$$

We can say $r \in R$ is a non-zero divisor if it is not a zero divisor (abbrev. nzd). Fix a sequence $\mathbf{x} = x_1, \dots, x_n \in R$.

Definition 1. We say that \mathbf{x} is a **weakly M -regular** sequence on M if x_{i+1} is not a zero divisor on $\frac{M}{(x_1, \dots, x_i)M}$ for all applicable i . It becomes a **regular sequence** if in addition $\frac{M}{\mathbf{x}M} \neq 0$.

Example. If $R = \mathbb{k}[x_1, \dots, x_n]$, and note $\mathbf{x} = x_1, \dots, x_n$ is a regular sequence on R .

We now introduce Koszul complexes. Given $r \in R$, we can write $K(r, R)$ to be the complex

$$0 \rightarrow R \rightarrow R \rightarrow 0.$$

there $R \rightarrow R$ is the homothetic map multiplication by r . The left first copy of R is labeled degree 1. Here, taking the homology functor of the sequence provides 0 on the left R if and only if r is a nzd. We have

$$K(\mathbf{x}, R) = \bigotimes_{i=1}^n K(x_i, R).$$

We will get

$$0 \rightarrow R \rightarrow R^n \rightarrow R^{n \text{choose } 2} \rightarrow \dots \rightarrow R^{n \text{choose } 2} \rightarrow R^n \rightarrow R \rightarrow 0.$$

(exercise calculate the first and last maps). Given $M \in \mathcal{C}(R)$,

$$K(\mathbf{x}, M) = K(\mathbf{x}, R) \otimes_R M.$$

If M is just an R -module, it is merely replacing copies of R with copies of M . We denote $H_i(\mathbf{x}, M) = H_i(K(\mathbf{x}, M))$. Note

$$H_0(\mathbf{x}, M) = \frac{M}{\mathbf{x}M}.$$

$$H_1(\mathbf{x}, M) = \{m \in M \mid x_i \cdot m = 0 \forall i\} = (0 :_M (\mathbf{x})).$$

Remark: Note

$$K(\mathbf{x}, M) = K(x_1, R) \otimes K(x_2, R) \otimes \dots \otimes K(x_n, R) \otimes_R M \\ K(x_1, R) \otimes K(\mathbf{x}_{\geq 2}, M).$$

So we have

$$K(\mathbf{x}, M) = K(x_1, K(\mathbf{x}_{\geq 2}, M)).$$

Remark:

We have $X, Y \in \mathcal{C}(R)$, we get the isomorphism

$$X \otimes_R Y \rightarrow Y \otimes_R X.$$

via $x \otimes_R y \mapsto (-1)^{(x)(y)} y \otimes_R x$ For any $\sigma \in S_n$,

$$K(x_1, \dots, x_n) \cong K(x_{\sigma(1)}, \dots, x_{\sigma(n)}, R).$$

Also, we have a second perspective on Koszul complexes: that they are the iterated mapping cones. Given a morphism of complexes

$$f : X \rightarrow Y.$$

recall the **cone** is defined

$$\text{cone}(f) = (Y \oplus \Sigma X, \begin{pmatrix} \partial^Y & f \\ 0 & -\partial^X \end{pmatrix}).$$

We get that

$$0 \rightarrow Y \rightarrow \text{cone}(f) \rightarrow \Sigma X \rightarrow 0.$$

$y \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ \Sigma x \end{pmatrix} \mapsto \Sigma X$. The long exact sequence in homology yields

$$\dots \rightarrow H_i(X) \rightarrow H_i(Y) \rightarrow H_i(\text{cone}(f)) \rightarrow H_i(\Sigma X) \cong H_{i-1}(X) \rightarrow \dots$$

Where the connecting map $H_i(X) \rightarrow H_i(Y)$ is just $H_i(f)$.

Now consider $x \in R$, and the homothetic map $f : R \rightarrow R$.

Example. $\text{cone}(f) = (R \oplus R, \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}) = K(x, R)$. Ditto for the homothetic map on modules.

$$\text{cone}(M \rightarrow M) = K(x, M).$$

Thus, $K(\mathbf{x}, M) = K(x_1, K(\mathbf{x}_{\geq 2}, M))$ is $\text{cone}(K(\mathbf{x}_{\geq 2}, M) \rightarrow K(\mathbf{x}_{\geq 2}, M))$. This gives

$$H_i(\mathbf{x}_{\geq 2}, M) \rightarrow H_i(\mathbf{x}_{\geq 2}, M) \rightarrow H_i(\mathbf{x}, M) \rightarrow H_{i-1}(\mathbf{x}_{\geq 2}, M) \rightarrow \dots$$

where the connecting morphism is multiplication by x_1 up to sign. By looking at the images/cokernels/kernels of one segment in this sequence, we get induced SES

$$0 \rightarrow H_i(\mathbf{x}_{\geq 2}, M)/x_1 H_i(\mathbf{x}_{\geq 2}, M) \rightarrow H_i(\mathbf{x}, M) \rightarrow (0 :_{H_{i-1}(\mathbf{x}_{\geq 2}, M)} x_1) \rightarrow 0.$$

If M is an R -module, $\mathbf{x} = x_1, \dots, x_n \subset R$,

$$K(\mathbf{x}, M) \twoheadrightarrow H_0(\mathbf{x}, M) = \frac{M}{\mathbf{x}M}.$$

So,

$$K(\mathbf{x}, M) \rightarrow \frac{M}{\mathbf{x}M}.$$

is a weak equivalence if and only if

$$H_i(\mathbf{x}, M) = 0 \forall i \geq 1.$$

Lemma 1. When \mathbf{x} is a weakly M -regular,

$$K(\mathbf{x}, M) \rightarrow \frac{M}{\mathbf{x}M}$$

which is also a weak equivalence.

Proof. When $n = 1$,

$$0 \rightarrow M \rightarrow M \rightarrow 0$$

has zero homology at degree 1 if and only if x is a nonzero divisor on M .

Now say when $n \geq 2$, we know that $K(\mathbf{x}, M) = K(x_n, K(\mathbf{x}_{\leq n-1}, M))$. By our induction hypothesis,

$$K(\mathbf{x}_{\leq n-1}, M) \rightarrow \frac{M}{(\mathbf{x}_{\leq n-1})M}.$$

We have

$$K(x, R) = (0 \rightarrow R \rightarrow R \rightarrow 0).$$

is semi-free.

$$K(\mathbf{x}, M) = K(x_n, R) \otimes_R K(\mathbf{x}_{\leq n-1}, M) \rightarrow K(x_n, \frac{M}{\mathbf{x}_{\leq n-1}M}).$$

Exercise 1

Prove this using the Koszul homology long exact sequence.

□

Definition 2. \mathbf{x} is **Koszi-regular** on M if

$$K(\mathbf{x}, M) \rightarrow \sim \frac{M}{\mathbf{x}M}.$$

. Note that x_1, \dots, x_n is Koszi-regular on M if and only if any permutation

$$x_{\sigma(1)}, \dots, x_{\sigma(n)}$$

is Koszi-regular on M for any $\sigma \in S_n$.

Exercise 2

(Weakly) regular sequences are sensitive to permutations.

Theorem 1. Say $\mathbf{x} \subset J(R)$ and $M \neq 0$ is finitely generated as an R -module. Then the following are equivalent:

1. \mathbf{x} is regular (\equiv weakly regular).
2. $H_i(\mathbf{x}, M) = 0$ for all $i \geq 1$.
3. $H_1(\mathbf{x}, M) = 0$.

Our main application is when R is a local ring and $\mathbf{x} \subset \mathfrak{m}_R$. We use Nakayama's lemma: $J(R) \neq M$, so regularity is equivalent to weak regularity.

Proof. We know $1 \Rightarrow 2 \Rightarrow 3$. It remains to show $3 \Rightarrow 1$. We want to examine $H_*(x_1, \dots, x_{n-1}, x_n, M)$. The module

$$K(\mathbf{x}, M) = K(x_n, K(\mathbf{x}_{\leq n-1}, M))$$

provides long exact sequence containing

$$0 \rightarrow H_i(\mathbf{x}_{\leq n-1}, M)/(x_n)H_i(\mathbf{x}_{\leq n-1}, M) \rightarrow H_i(\mathbf{x}, M) \rightarrow (0 :_{H_{i-1}(\mathbf{x}_{\leq n-1}, M)} x_n).$$

We have that

$$H_1(\mathbf{x}, M) = 0 \Rightarrow H_1(\mathbf{x}_{\leq n-1}, M) = (x_n)H_1(\mathbf{x}_{\leq n-1}, M).$$

so apply Nakayama's. We are doing the proof of equivalence by induction on n (it is already proven for $n = 1$), so we have

$$x_1, \dots, x_{n-1}.$$

is M -regular. This implies further that

$$H_i(\mathbf{x}_{\leq n-1}, M) = 0.$$

for all $i \geq 1$. Moreover, applying this to our exact sequence above, $(0 : x_n) = 0$, so $H_0(\mathbf{x}_{\leq n}, M) = \ker\left(\frac{M}{\mathbf{x}_{\leq n-1}M} \rightarrow \frac{M}{\mathbf{x}_{\leq n}M}\right)$. \square

Corollary 1. $\mathbf{x} \subset J(R)$, M finitely generated. The property that \mathbf{x} is M -regular does not depend on the ordering of \mathbf{x} .

Lemma 2. Suppose we have a sequence $x_1, \dots, x_n \subset R$ (now we drop the assumption regarding the Jacobson radical). Let M be an R -module. The following are equivalent:

1. \mathbf{x} is Koszi-regular on M .
2. $\{x_1^{a_1}, \dots, x_n^{a_n}\}$ is Koszi-regular on M for any choice $a_i \geq 1$.
3. $\mathbf{x}^{\mathbf{a}}$ is Koszi-regular on M for some $\mathbf{a} \geq (1, \dots, 1)$.

Proof. It suffices to prove x_1, \dots, x_n is Koszi-regular on M if and only if x_1^a, \dots, x_n^a for some $a \geq 1$. Recall that Koszi-regularity means

$$K(x_1^a, x_2, \dots, x_n, M) \rightarrow^{\sim} K(x_1^a, \frac{M}{(x_{\geq 2})M}).$$

Replacing M with $\frac{M}{(\mathbf{x}_{\geq 2})M}$, we are reduced to proving x is weakly M -regular if and only if x^a is weakly M -regular for some $a \geq 1$. x is not a zero divisor on M if and only if x^a is not a zero divisor on M for some or all $a \geq 1$. \square

Exercise 3

(this is also a theorem, called the rigidity of Koszul homology). If we take $\mathbf{x} \subset J(R)$ and M a finitely generated R -module, then $H_i(\mathbf{x}, M) = 0$ for some $i \geq 0$ implies that $H_j(\mathbf{x}, M) = 0$ for all $j \geq i$.

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Let R be a commutative and Noetherian ring, and $M, N \in \mathcal{C}(R)$. Note

$$\mathrm{RHom}_R(M, N) = \mathrm{Hom}_R(pM, N).$$

where $pM \xrightarrow{\sim} M$ is a K -projective resolution. Recall

$$\mathrm{Ext}_R^*(M, N) = H^*(\mathrm{RHom}_R(M, N)).$$

For any $M, N, P \in \mathcal{C}(R)$, there exists

$$\theta : \mathrm{RHom}_R(M, N) \otimes_R^L P \rightarrow \mathrm{RHom}_R(M, N \otimes_R^L P).$$

Lemma 3. This is a weak equivalence when P is **perfect**. In particular

$$P \xrightarrow{\sim} (0 \rightarrow P_b \rightarrow \dots \rightarrow P_c \rightarrow 0).$$

Where P_i is finitely generated as a projective R -module. We get a morphism of complexes

$$\mathrm{Hom}_R(pM, N) \otimes_R p(P) \rightarrow \mathrm{Hom}_R(pM, N \otimes_R p(P)).$$

Defined by

$$f \otimes x \mapsto \left(m \mapsto (-1)^{|x||m|} f(m) \otimes x \right).$$

In the category of modules over R , if we look at

$$\mathrm{Hom}_R(M, N) \otimes_R P \rightarrow \mathrm{Hom}_R(M, N \otimes_R P)$$

to prove this when P is a finitely generated projective.

Lemma 4. Rees' Lemma. Let $\mathbf{x} \subset R$ be a finite subset. Let M, N be R -modules. Let N be an R -module such that $\mathbf{x}N = 0$. And let M be an R -module such that \mathbf{x} is Koszi-regular on M . This means that

$$K(\mathbf{x}, M) \xrightarrow{\sim} \frac{M}{\mathbf{x}M}.$$

Lemma 5.

$$\mathrm{RHom}_R(N, \frac{M}{\mathbf{x}M}) \xrightarrow{\sim} \mathrm{RHom}_R(N, M) \otimes_R \bigwedge^* (\Sigma R^c).$$

In particular,

$$\mathrm{Ext}_R^*(N, \frac{M}{\mathbf{x}M}) \cong \mathrm{Ext}_R^*(N, M) \otimes_R \bigwedge^* (\Sigma R^c).$$

Where c denotes the rank of the free module.

Corollary 2.

$$\inf \mathrm{Ext}_R^*(N, M) = \inf \mathrm{Ext}_R^*(N, \frac{M}{\mathbf{x}M}) + c.$$

We also have

$$\mathrm{Ext}_R^*(N, M) \cong \mathrm{Ext}_R^{*+c}(N, \frac{M}{\mathbf{x}M}).$$

Recalling the alternating product complex will have zero differentials.

Proof. We want to compute

$$\begin{aligned} \mathrm{RHom}_R(N, \frac{M}{\mathbf{x}M}) &\xrightarrow{\sim} \mathrm{RHom}_R(N, K(\mathbf{x}, M)) \xrightarrow{\sim} \mathrm{RHom}_R(N, M \otimes_R^L K(\mathbf{x}, R)). \\ &\xrightarrow{\sim} \mathrm{RHom}_R(N, M) \otimes_R^L K(\mathbf{x}, R). \end{aligned}$$

since $K(\mathbf{x}, R)$ is perfect. Since $\mathbf{x} \cdot N = 0$, $\mathbf{x} \cdot \mathrm{Ext}_R^*(N, M) = 0$ (Exercise, show this is true). Using this and long exact sequence associated to Koszul complexes, one can calculate the isomorphism at the level of Ext .

Alternatively,

$$\mathrm{RHom}_R(N, M) \cong \mathrm{Hom}_R(N, I).$$

where $M \cong I$ is an injective resolution (\cong denotes weak equivalence in $M \cong I$). Now

$$\mathbf{x} \cdot \mathrm{Hom}_R(N, I) = 0.$$

$$\begin{aligned} &\mathrm{RHom}_R(N, M) \otimes_R K(\mathbf{x}, R) \\ &\cong \mathrm{Hom}_R(N, I) \otimes_R K(\mathbf{x}, R) \\ &\cong \mathrm{Hom}_R(N, I) \otimes_R K(\mathbf{0}, R). \end{aligned}$$

where $\mathbf{0}$ is a zero sequence of length c . To get the in particular part of lemma 3, take homology. The details are an exercise. \square

If we want to compute $\mathrm{Ext}_R^n(N, \frac{M}{\mathbf{x}M})$ it would be

$$\begin{aligned} &\left(\mathrm{Ext}_R(N, M) \otimes \bigwedge (\Sigma R^c) \right)^n. \\ &= \bigoplus_i \mathrm{Ext}_R^i(N, M) \otimes_R \left(\bigwedge (\Sigma R^c) \right)^{n-i}. \end{aligned}$$

So

$$\mathrm{Ext}_R^n(N, \frac{M}{\mathbf{x}M}) \cong \bigoplus_i \mathrm{Ext}_R^i(N, M) \otimes_R R^{choose i - n}.$$

If we had a \mathbb{Z} -graded object V , we think of it having upper and lower gradings via

$$V^i = V_{-i}.$$

Notation-wise, the supremum of the graded object V ,

$$\begin{aligned} \sup V^* &= \sup\{i \mid V^i \neq 0\}. \\ \inf V^* &= \inf\{i \mid V^i \neq 0\}. \end{aligned}$$

We brought all of this up to discuss **depth**. Now fix $I \subset R$ an ideal. We can define for any $M \in \mathcal{C}(R)$,

$$\mathrm{depth}_R(I, M) = \inf \mathrm{Ext}_R^* \left(\frac{R}{I}, M \right).$$

This is called the I -depth of M . We could get a few important properties.

Remark. We have the following.

1. Given an exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ of complexes, we get a long exact sequence in Ext . If the Ext groups for L and N vanish, then so too must those of M . Hence we get

$$\mathrm{depth}_R(I, M) \geq \min\{\mathrm{depth}_R(I, L), \mathrm{depth}_R(I, N)\}.$$

This is all from that exact sequence

$$\mathrm{Ext}_R^i \left(\frac{R}{I}, L \right) \rightarrow \mathrm{Ext}_R^i \left(\frac{R}{I}, M \right) \rightarrow \mathrm{Ext}_R^i \left(\frac{R}{I}, N \right) \rightarrow \mathrm{Ext}_R^{i+1} \left(\frac{R}{I}, L \right) \rightarrow \dots$$

Theorem 2. Let $\mathbf{x} = x_1, \dots, x_c$ be a generating set for the ideal I . Then we can compute

$$\text{depth}_R(I, M) = c - \sup H_*(\mathbf{x}, M).$$

This is true for any $M \in \mathcal{C}(R)$.

If we look at $K(\mathbf{x}, R) \rightarrow \frac{R}{\mathbf{x}R} = \frac{R}{I}$, we $+-+$. We prove this theorem when M is a module. Koszul complexes revisited. We started by introducing it as a tensor product as short complexes. Instead, we could start with an exterior algebra, end up with the differential. It is the same as giving a map $f : F \rightarrow R$ where F is a finite free R -module and with fixed chosen basis of rank c . One can choose a Koszul complex attached to f . Look at

$$K(f) = \left(\bigwedge^* (\Sigma F), \partial \right).$$

The former module is an exterior algebra on F . Taking a differential of a typical element, it has form $\partial(e_{i_1} \wedge \dots \wedge e_{i_n}) = \sum_{j=1}^n (-1)^{j-1} f(e_{i_j}) e_{i_1} \wedge \dots \wedge \widehat{e_{i_j}} \wedge \dots \wedge e_{i_n}$.

For example,

$$e_1 \wedge e_2 \xrightarrow{\partial} f(e_1)e_2 - e_1f(e_2).$$

Lemma 6. Suppose we have $\mathbf{x} = x_1, \dots, x_c \subset R$. For any $y \in (\mathbf{x})$, then

$$K(\mathbf{x}, y; M) \cong K(\mathbf{x}, 0, M).$$

The above is isomorphism as R -complexes. The latter is just

$$K(\mathbf{x}, M) \otimes K(0, R).$$

Proof. We stare at the following picture: We get

$$\begin{array}{ccc} R^{n+1} & \xrightarrow{[x_1 x_2 \dots x_c y]} & R \\ \sim \uparrow & & \parallel \\ R^{n+1} & \xrightarrow{[x_1 x_2 \dots x_c 0]} & R \end{array}$$

In particular,

$$\sup H_*(\mathbf{x}, y; M) = 1 + \sup H_*(\mathbf{x}, M).$$

Thus,

$$c + 1 - \sup H_*(\mathbf{x}, y; M) = c - \sup H_*(\mathbf{x}, M).$$

□

Corollary 3. (Check this corollary.) The quantity

$$c - \sup H_*(\mathbf{x}, M).$$

is independent of the choice of generating set for the ideal I .

Theorem 3. We have

$$\text{depth}_R(I, M) = c - \sup H_*(\mathbf{x}, M)$$

where $\mathbf{x} = x_1, \dots, x_c$ generates the ideal I .

Proof. We prove this when M is a module. What does it mean for

$$\text{depth}_R(I, M) = 0?$$

It precisely means that

$$\text{Hom}_R\left(\frac{R}{I}, M\right) \neq 0.$$

This is because the zeroth Ext group is the homology. The depth zero is if and only if

$$I \subset \text{zdr}_R(M).$$

which holds if and only if $H_c(\mathbf{x}, M) \neq 0$. This is also if and only if

$$\sup H_*(\mathbf{x}, M) = c.$$

So we can assume that $\text{depth}_R(I, M) \geq 1$. In particular, there exists $y \in I$ which is nonzero divisor on M . Then in particular y is Koszi-regular on M . We would like to compute

$$\text{Ext}_R^*\left(\frac{R}{I}, \frac{M}{\mathbf{x}M}\right).$$

What is the supremum of the above complex? Rees's lemma (that corollary afterwards) says

$$\inf \text{Ext}_R^*\left(\frac{R}{I}, \frac{M}{yM}\right) = \inf \text{Ext}_R^*\left(\frac{R}{I}, M\right) - 1.$$

This applies because

$$y \cdot \left(\frac{R}{I}\right) = 0.$$

In terms of depth, it tells us that

$$\text{depth}_R(I, M) = 1 + \text{depth}_R\left(I, \frac{M}{\mathbf{x}M}\right) = 1 + c - \sup H_*(\mathbf{x}, M).$$

We also have

$$H_*(\mathbf{x}, \frac{M}{yM}) = H_*(\mathbf{x}, K(y, M)) = H_*(\mathbf{x}, y; M).$$

We just saw that this is exactly

$$H_*(\mathbf{x}, 0; M).$$

because $y \in (\mathbf{x})$. If we calculate the supremum, the supremums are the same. In particular,

$$\sup H_*(\mathbf{x}, \frac{M}{yM}) = \sup H_*(\mathbf{x}, 0; M) = 1 + \sup H_*(\mathbf{x}, M).$$

So this justifies the proof of the theorem by completing an induction step. \square

One huge takeaway from the story: we have that the depth is the longest Koszi-regular sequence in I . Next time, we discuss depth in the context of local.