"Open Source Macroeconomics Laboratory Boot Camp Perturbation Methods"

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Outline

Types of Variables

- Z_t is a vector of n_Z exogenous state variables.
- X_{t-1} is a vector of n_X endogenous state variables.
- Y_t is a vector of n_Y implicity-defined non-state or “jump” variables.
- D_t is a vector of n_D explicity-defined non-state or “jump” variables.

Note we can lump Y_t and D_t into X_t if we like. This may increase computational cost, but is otherwise sound logically. We are searching for a polucy function $X_t = \Phi(X_{t-1}, Z_t)$, and perhaps a jump function $Y_t = \Psi(X_{t-1}, Z_t)$.

Dynamic Behavior

We will take the characterizing equations for the model and write them as a vector of two sets of functions stacked in the following form:

$$\Gamma_Y(X_t, X_{t-1}, Y_t, Z_{t+1}, Z_t)\} = 0 \tag{1}$$

$$E_t\{\Gamma_X(X_{t+1}, X_t, X_{t-1}, Y_{t+1}, Y_t, Z_{t+1}, Z_t)\} = 0 \tag{2}$$

X_{t+1}, X_t and X_{t-1} are $n_X \times 1$ vectors. Y_{t+1} and Y_t are $n_Y \times 1$ vectors. Z_{t+1} and Z_t are $n_Z \times 1$ vectors and Γ outputs a $n_X \times 1$ vector.

Dynamic Behavior

We can use a first-order Taylor-series approximation of these equations to get a linear approximation of the characterizing equations.

$$A\tilde{X}_t + B\tilde{X}_{t-1} + C\tilde{Y}_t + D\tilde{Z}_t = 0 \tag{3}$$

$$E_t\left\{F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + J\tilde{Y}_{t+1} + K\tilde{Y}_t + L\tilde{Z}_{t+1} + M\tilde{Z}_t\right\} = 0 \tag{4}$$

where $\tilde{X}_t$ denotes $X_t - \bar{X}$

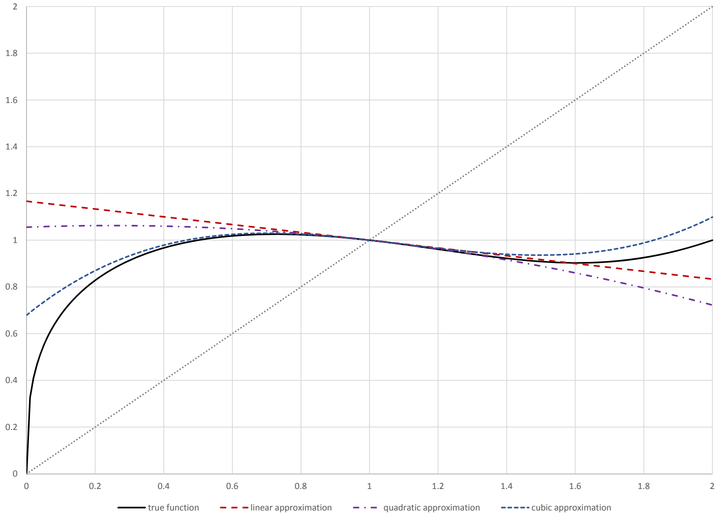
Linear approximations of policy and jump functions are:

$$\tilde{X}_t = U + P\tilde{X}_{t-1} + Q\tilde{Z}_t \tag{5}$$

$$\tilde{Y}_t = V + R\tilde{X}_{t-1} + S\tilde{Z}_t \tag{6}$$

Outline

Illustration of Polynomial Approximations



Linear approximation is:

$$\tilde{X}_t = P\tilde{X}_{t-1} + Q\tilde{Z}_t$$

Quadratic approximation is:

$$\tilde{X}_t = H_X\tilde{X}_{t-1} + H_Z\tilde{Z}_t + \frac{1}{2}\left[H_{XX}\tilde{X}_{t-1}^2 + H_{ZZ}\tilde{Z}_t^2 + 2H_{XZ}\tilde{X}_{t-1}\tilde{Z}_t\right]$$

Outline

First-Order Peturbations

Suppose we have a condition on a potentially nonlinear bivariate function:

$$F(x, u) = 0 \tag{7}$$

Assume u is an exogenously given variable, and x will be choson to satisfy (7). Denote the solution to this condition as $x(u)$ and assume that the value of $x(u_0)$ is known.

First-Order Peturbations

Taking the derivative of (7) with respect to u gives:

$$F_x\{x(u), u\}x_u(u) + F_u\{x(u), u\} = 0 \tag{8}$$

If we evaluate this at $u = u_0$ and solve for the first derivative of $x(u)$, we have:

$$x_u(u_0) = -\frac{F_u\{x(u_0), u_0\}}{F_x\{x(u_0), u_0\}}$$

First-Order Peturbations

Since $x(u_0)$ is known, as long as $F_x\{x(u_0), u_0\} \neq 0$ we can find the value for the first derivative. The first-order (linear) Taylor-series approximation of $x(u)$ will be:

$$x(u) = x(u_0) + x_u(u_0)(u - u_0)$$

Second-Order Peturbations

To find the second-order terms we differentiate (8) again with respect to *u*.

$$
\begin{aligned}
&F_{xx}\{x(u),u\}x_u(u)x_u(u) + F_{xu}\{x(u),u\}x_u(u) \\
&+F_x\{x(u),u\}x_{uu}(u) + F_{xu}\{x(u),u\}x_u(u) \\
&+F_{uu}\{x(u),u\} = 0
\end{aligned}
$$

Second-Order Peturbations

Again evaluating at $u = u_0$ and solving this time for the second derivative of $x(u)$, we have:

$$x_{uu}(u_0) = -\frac{F_{xx}\{x(u_0), u_0\}[x_u(u_o)]^2 + 2F_{xu}\{x(u_0), u_0\}x_u(u_o) + F_{uu}}{F_x\{x(u_0), u_0\}}$$

Hence, the second-order (quadratic) Taylor-series approximation of $x(u)$ will be:

$$x(u) = x(u_0) + x_u(u_0)(u - u_0) + \frac{1}{2}x_{uu}(u_0)(u - u_0)^2$$

Highr-Order Peturbations

Higher order terms can be obtained by successive differentiation, setting $u = u_0$ and solving for the appropriate derivative.

Each will be a function of the various derivatives of $F(x, u)$ and the lower-order derivatives of $x(u)$ obtained from previous iterations.

Outline

Brock and Mirman

Recall the Euler equation for the non-stochastic version of the model is:

$$\frac{1}{K_t^\alpha - K_{t+1}} - \beta \frac{\alpha K_{t+1}^{\alpha-1}}{K_{t+1}^\alpha - K_{t+2}} = 0$$

In terms of our notatation from the previous section we have:

$$\begin{aligned} u &= K_t \\ x &= x(u) = K_{t+1} \\ y &= x(x) = K_{t+2} \\ F(y,x,u) &= F(y(x(u)), x(u), u) \\ &= \frac{1}{K_t^\alpha - K_{t+1}} - \beta \frac{\alpha K_{t+1}^{\alpha-1}}{K_{t+1}^\alpha - K_{t+2}} = 0 \end{aligned} \tag{9}$$

Differentiate

Take the derivative of (9) with respect to $u = K_t$:

$$F_y(y(x(u)), x(u), \bar{u})x_u(x(u))x_u(u) \\ + F_x(y(x(u)), x(u), u)x_u(u) + F_u(y(x(u)), x(u), u) = 0 \tag{10}$$

Differentiate

Evaluating (10) at $u = \bar{u} = \bar{K}$ and noting that $x(\bar{u}) = \bar{u}$:

$$F_y(y(x(\bar{u})), x(\bar{u}), \bar{u})x_u(x(\bar{u}))x_u(\bar{u})$$
$$+ F_x(y(x(\bar{u})), x(\bar{u}), \bar{u})x_u(\bar{u}) + F_u(y(x(\bar{u})), x(\bar{u}), \bar{u}) = 0$$
$$F_y(\bar{u}, \bar{u}, \bar{u})\ x_u(\bar{u})^2 + F_x(\bar{u}, \bar{u}, \bar{u})\ x_u(\bar{u}) + F_u(\bar{u}, \bar{u}, \bar{u}) = 0$$

Note that $F_y(\bar{u}, \bar{u}, \bar{u})$ is the same as F from the linearization notes. Similarly, $F_x(\bar{u}, \bar{u}, \bar{u})$ is G, and $F_u(\bar{u}, \bar{u}, \bar{u})$ is H. Also note that $x_u(\bar{u})$ is P. As in those notes the value of $x_u(\bar{u})$ comes from a solving a quadratic.

Differentiate Again

$$
\begin{aligned}
&F_{yy}(\bar{u},\bar{u},\bar{u})\ x_u(\bar{u})^4 + F_{yx}(\bar{u},\bar{u},\bar{u})\ x_u(\bar{u})^3 \\
&+ F_{yu}(\bar{u},\bar{u},\bar{u})\ x_u(\bar{u})^2 + F_{y}(\bar{u},\bar{u},\bar{u})\ x_{uu}(\bar{u})\ x_u(\bar{u})^2 \\
&+ F_{y}(\bar{u},\bar{u},\bar{u})\ x_u(\bar{u})\ x_{uu}(\bar{u}) + F_{yx}(\bar{u},\bar{u},\bar{u})\ x_u(\bar{u})^3 \\
&+ F_{xx}(\bar{u},\bar{u},\bar{u})\ x_u(\bar{u})^2 + F_{xu}(\bar{u},\bar{u},\bar{u})\ x_u(\bar{u}) \\
&+ F_{x}(\bar{u},\bar{u},\bar{u})\ x_{uu}(\bar{u}) + F_{yu}(\bar{u},\bar{u},\bar{u})\ x_u(\bar{u})^2 \\
&+ F_{xu}(\bar{u},\bar{u},\bar{u})\ x_u(\bar{u}) + F_{uu}(\bar{u},\bar{u},\bar{u}) = 0
\end{aligned}
$$

Differentiate Again

Supressing the function arguments for the sake of clarity we can rewrite (**??**) as below.

$$(F_{yy}\, x_u^4 + 2F_{yx}\, x_u^3 + 2F_{yu}\, x_u^2 + F_{xx}\, x_u^2 + 2F_{xu}\, x_u + F_{uu})$$
$$+ (F_y\, x_u^2 + F_y\, x_u + F_x)x_{uu} = 0$$

Note the F_{ij} are all second-derivatives evaluated at the steady state. Since x_u has already been solved we can solve this for x_{uu}.

Differentiate Again

$$x_{uu} = -\frac{F_{yy}\, x_u^4 + 2F_{yx}\, x_u^3 + 2F_{yu}\, x_u^2 + F_{xx}\, x_u^2 + 2F_{xu}\, x_u + F_{uu})}{(F_y\, x_u^2 + F_y\, x_u + F_x)}$$

The quadratic approximation to the policy function is given by:

$$\tilde{K}_{t+1} = x_u \tilde{K}_t + \frac{1}{2} x_{uu} \tilde{K}_t^2 \tag{11}$$

Outline

DSGE Perturbation

Write our stacked dynamic equations as:

$$E_t\{\Gamma(X_{t+1}, X_t, X_{t-1}, Y_{t+1}, Y_t, Z_{t+1}, Z_t)\} = 0 \tag{12}$$

Recall the exogenous law of motion:

$$Z_t = NZ_{t-1} + v\Omega\varepsilon_t; \varepsilon_t \sim (0, I_{n_Z}) \tag{13}$$

where v is a scalar, and Ω is a matrix that determines correlations of the elements in ε_t.

The policy function and jump functions are:

$$X_t = H(X_{t-1}, Z_t, v) \tag{14}$$

$$Y_t = G(X_{t-1}, Z_t, v) \tag{15}$$

DSGE Perturbation

For notational ease define the following.

$$
\begin{aligned}
A_t &\equiv \begin{bmatrix} X_{t+1} & X_t & X_{t-1} & Y_{t+1} & Y_t & Z_{t+1} & Z_t \end{bmatrix}^T \\
S_t &\equiv \begin{bmatrix} X_{t-1} & Z_t & v \end{bmatrix} \\
n_A &\equiv 3n_X + 2n_Y + 2n_Z \\
n_s &\equiv n_X + n_Z + 1
\end{aligned}
$$

DSGE Perturbation

The Taylor-series approximation of Γ with second-order terms for the variance is:

$$\Gamma(A_t) \doteq \Gamma(\bar{X}, ..., \bar{Z}) + \begin{bmatrix} \Gamma_1 & \cdots & \Gamma_7 \end{bmatrix} \begin{bmatrix} \tilde{X}_{t+1} \\ \vdots \\ \tilde{Z}_t \end{bmatrix}$$
$$+ \frac{1}{2} \left(I_{n_A} \otimes \begin{bmatrix} \tilde{X}_{t+1} & \vdots & \tilde{Z}_t \end{bmatrix} \right) \begin{bmatrix} \Gamma_{11} & \cdots & \Gamma_{17} \\ \vdots & \ddots & \vdots \\ \Gamma_{71} & \cdots & \Gamma_{77} \end{bmatrix} \begin{bmatrix} \tilde{X}_{t+1} \\ \vdots \\ \tilde{Z}_t \end{bmatrix} \quad (16)$$

Γ_1 through Γ_7 are combinations of the A through M matrices in Uhlig's notation. Γ_{11} through Γ_{77} are all Magnus and Neudecker matrices of second-order coefficients.

Using 3-Dimensional Tensors

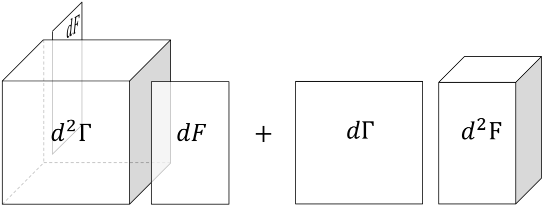
We need to get a matrices of first and second derivatives for the functions Γ, H, and G. All of these are vector-valued functions of vectors.

- The outputs are 1-dimensional
- The Jacobians are 2-dimensional
- The Hessians are 3-dimensional

We can either work with matrix algebra extended beyond two dimensions - tensors. Or we can restack 3-dimensional matrices into two dimensions. The latter is what Magnus and Neudecker (1999) discuss.

Using Tensors

$$\Gamma(A) = 0; \quad A = F(S)$$
$$\Gamma_F(F(S))F_S(S) = 0$$
$$F_S(S)^T \Gamma_{FF} F_S(S) + \Gamma_F(F(S))F_{SS}(S)$$



Let's NOT do this.

Magnus and Neudecker Matrices

If $Y = F(X)$ is a vector-valued function of a vector input.
Then the Magnus and Neudecker Jacobian is:

$$\begin{bmatrix} \frac{\partial Y_1}{\partial X_1} & \cdots & \frac{\partial Y_1}{\partial X_{n_X}} \\ \vdots & \ddots & \vdots \\ \frac{\partial Y_{n_Y}}{\partial X_1} & \cdots & \frac{\partial Y_{n_Y}}{\partial X_{n_X}} \end{bmatrix}$$

The Hessian is...

Magnus and Neudecker Matrices

$$
\begin{bmatrix}
\frac{\partial^2 Y_1}{\partial X_1 \partial X_1} & \cdots & \frac{\partial^2 Y_1}{\partial X_{n_X} \partial X_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 Y_{n_Y}}{\partial X_1 \partial X_1} & \cdots & \frac{\partial^2 Y_{n_Y}}{\partial X_{n_X} \partial X_1} \\
\frac{\partial^2 Y_1}{\partial X_1 \partial X_2} & \cdots & \frac{\partial^2 Y_1}{\partial X_{n_X} \partial X_2} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 Y_{n_Y}}{\partial X_1 \partial X_2} & \cdots & \frac{\partial^2 Y_{n_Y}}{\partial X_{n_X} \partial X_2} \\
 & \vdots & \\
\frac{\partial^2 Y_1}{\partial X_1 \partial X_{n_X}} & \cdots & \frac{\partial^2 Y_1}{\partial X_{n_X} \partial X_{n_X}} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 Y_{n_Y}}{\partial X_1 \partial X_{n_X}} & \cdots & \frac{\partial^2 Y_{n_Y}}{\partial X_{n_X} \partial X_{n_X}}
\end{bmatrix}
$$

DSGE Perturbation

The Taylor-series approximation of H with second-order terms for the variance is:

$$H(X_{t-1}, Z_t, v) \doteq H(\bar{X}, \bar{Z}, \bar{v}) + \begin{bmatrix} H_X & H_Z & H_v \end{bmatrix} \begin{bmatrix} \tilde{X}_{t-1} \\ \tilde{Z}_t \\ \tilde{v} \end{bmatrix}$$

$$+\frac{1}{2}\left(I_{n_Y+n_X} \otimes \begin{bmatrix} \tilde{X}_{t-1}^T & \tilde{Z}_t^T & \tilde{v} \end{bmatrix}\right) \begin{bmatrix} H_{XX} & H_{XZ} & 0 \\ H_{ZX} & H_{ZZ} & 0 \\ 0 & 0 & H_{vv} \end{bmatrix} \begin{bmatrix} \tilde{X}_{t-1} \\ \tilde{Z}_t \\ \tilde{v} \end{bmatrix} \tag{17}$$

H_X and H_Z terms are the P and Q matrices in Uhlig's notation. H_{XX}, H_{ZZ}, H_{ZX}, H_{XZ}^T and H_{vv} are all Magnus and Neudecker matrices of second-order coefficients.

DSGE Perturbation

A similar setup is used for the approximation of the G function.

$$G^k(X_{t-1}, Z_t, v) \doteq G(\bar{X}, \bar{Z}, \bar{v}) + \begin{bmatrix} G_X & G_Z & G_v \end{bmatrix} \begin{bmatrix} \tilde{X}_{t-1} \\ \tilde{Z}_t \\ \tilde{v} \end{bmatrix}$$

$$+\frac{1}{2}\left(I_{n_Y+n_X} \otimes \begin{bmatrix} \tilde{X}_{t-1}^T & \tilde{Z}_t^T & \tilde{v} \end{bmatrix}\right) \begin{bmatrix} G_{XX} & G_{XZ} & 0 \\ G_{ZX} & G_{ZZ} & 0 \\ 0 & 0 & G_{vv} \end{bmatrix} \begin{bmatrix} \tilde{X}_{t-1} \\ \tilde{Z}_t \\ \tilde{v} \end{bmatrix} \tag{18}$$

G_X and G_Z terms are the R and S matrices in Uhlig's notation.

DSGE Perturbation

We can substitute (14), (15) and (13) into our definition of A_t to get the following function:

$$A_t = F(S_t) = \begin{bmatrix} H\left(H(X_{t-1}, Z_t, v), NZ_t + v\Omega\varepsilon_{t+1}, v\right) \\ H(X_{t-1}, Z_t, v) \\ X_{t-1} \\ G\left(H(X_{t-1}, Z_t, v), NZ_t + v\Omega\varepsilon_{t+1}, v\right) \\ G(X_{t-1}, Z_t, v) \\ NZ_{t-1} + v\Omega\varepsilon_t \\ Z_t \end{bmatrix} \tag{19}$$

See the chapter handout for the Jacobian and Hessian matrices, $F_S(S_t)$ and $F_{SS}(S_t)$.

DSGE Perturbation

Using (19) in (12) we get $\Delta(S_t) \equiv \Gamma(F(S_t)) = 0$. Magnus and Neudecker (1999) show that the chain-rule for this function with our setup for the organization of the Jacobian and Hessian matrices is as follows.

$$\Delta_{SS} = (F_S \otimes I_{n_X+n_Y})^T \Gamma_{AA} F_S + (I_{n_S} \otimes \Gamma_A) F_{SS} \qquad (20)$$

Solving for Linear Terms

We can solve for the linear terms as we did in the linearization chapter. This will generate the coefficient matrices H_X, H_Z, G_X and G_Z. (Uhlig's P, Q, R and S.) Discussion in the chapter shows that H_v and G_v are zero.

Solving for Quadratic Terms

With the first-order coeffiecients for the *H* and *G* functions known, we can use the expectation of (20) to solve for the second-order coefficients. We note that F_S is a function of the first-order coeffiecients as shown in the chapter handout. Similarly, we know that F_{SS} a function of both the first and second-order coefficients.

Solving for Quadratic Terms

Before taking expectations, we need to multiply out the term $\Lambda \equiv (F_S \otimes I_{n_X+n_Y})^T \Gamma_{AA} F_S$. Examine the F_S matrix and note that terms with ε_{t+1} appear only in the thrid column.

$$F_S = \begin{bmatrix} H_X H_X & H_X H_Z + H_Z N & H_X H_v + H_Z \Omega \varepsilon_{t+1} + H_v \\ H_X & H_Z & H_v \\ 1 & 0 & 0 \\ G_X H_X & G_X H_Z + G_Z N & G_X H_v + G_Z \Omega \varepsilon_{t+1} + G_v \\ G_X & G_Z & G_v \\ 0 & N & \Omega \varepsilon_{t+1} \\ 0 & 1 & 0 \end{bmatrix}$$

Solving for Quadratic Terms

If we take the expectation of F_S the ε_{t+1} terms disappear.

$$E\{F_S\} = \begin{bmatrix} H_X H_X & H_X H_Z + H_Z N & H_X H_v + H_v \\ H_X & H_Z & H_v \\ 1 & 0 & 0 \\ G_X H_X & G_X H_Z + G_Z N & G_X H_v + G_v \\ G_X & G_Z & G_v \\ 0 & N & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Solving for Quadratic Terms

Let's look at the $(3,3)$ block in Λ. Recall that H_v and G_v are zeros.

$$\Lambda(3,3) = \varepsilon_{t+1}^T(\Omega H_Z^T \Gamma_{11} H_Z \Omega + \Omega G_Z^T \Gamma_{44} G_Z \Omega)\varepsilon_{t+1}$$

This is the only term where the quadratic form of ε_{t+1} appears. In every other term it is either absent or appears as a linear term. Hence when expectations are taken the terms with ε_{t+1} disappear. This means we can use $E\{\Lambda\} = (E\{F_S\} \otimes I_{n_X+n_Y})^T \Gamma_{AA} E\{F_S\}$ and then replace the $(3,3)$ term.

Solving for Quadratic Terms

To take expectations of $\Lambda(3,3)$ it is useful to know that if the elements of a column vector of random variables $\varepsilon \sim iid(0, I)$, then $E\{\varepsilon^T A\varepsilon\} = \text{tr}(A)$.
So we replace the zero in the $(3,3)$ block with
$\text{tr}(\Omega[H_Z^T\Gamma_{11}H_Z] + [G_Z^T\Gamma_{44}G_Z]\Omega)$.

Solving for Quadratic Terms

Recall equation (20):

$$\Delta_{SS} = (F_S \otimes I_{n_X+n_Y})^T \Gamma_{AA} F_S + (I_{n_S} \otimes \Gamma_A) F_{SS}$$

Unfortunately, $I_{n_A} \otimes \Gamma_A$ is not a square matrix and therefore not invertable. However, we can solve for the second-order coefficients numerically.

Solving for Quadratic Terms

The coefficients we need to find are $\Theta = \{H_{XX}, H_{XZ}, H_{ZZ}, H_{vv}, G_{XX}, G_{XZ}, G_{ZZ}, G_{vv}\}$. $E\{F_S\}$, Γ_A and Γ_{AA} are known. We can therefore write a Δ_{SS} function as shown below and numerically solve for the values of Θ that set it equal to zero. We note that Δ_{SS} will return a matrix of size $n_S(n_X + n_Y) \times n_S$. This will be $n_X + n_Y$ blocks of symmetric $n_S \times n_S$ matrices.

$$\Delta_{SS}(\Theta) = (E\{\Lambda\} + (I_{n_S} \otimes \Gamma_A)E\{F_{SS}(\Theta)\} = 0$$

$$\Lambda = (F_S \otimes I_{n_X+n_Y})^T \Gamma_{AA} F_S$$

Solving for Quadratic Terms

The symmetric blocks in the Δ_{SS} matrix will be denoted Δ^i_{SS} for $i \in \{1, 2, \ldots, n_X + n_Y\}$ and can be decomposed into nine parts.

$$\Delta^i_{SS} = \begin{bmatrix} \Delta^i_{XX} & \Delta^i_{XZ} & 0 \\ (\Delta^i_{XZ})^T & \Delta^i_{ZZ} & 0 \\ 0 & 0 & \Delta^i_{vv} \end{bmatrix} \tag{21}$$

Hence we have $n_X^2 + n_Z^2 + n_X n_Z + 1$ unique values for each i, for a total of $(n_X^2 + n_Z^2 + n_X n_Z + 1)(n_X + n_Y)$. We have $(n_X^2 + n_X nZ + n_Z + 1)n_X$ terms in the H_{SS} coefficients and $(n_X^2 + n_X nZ + n_Z + 1)n_Y$ terms in the G_{SS} coefficients. Hence the $\Delta_{SS} = 0$ condition will exactly identify Θ.

To summarize. We get the quadratic terms by:

- Taking first and second deriviatives of the Γ function at the steady state: Γ_A and Γ_{AA}.
- Finding the first order terms: H_X and H_Z.
- These allow us to get $E\{F_S\}$.
- We then use a numerical equation solver to solve for $\Theta = \{H_{XX}, H_{XZ}, H_{ZZ}, H_{vv}, G_{XX}, G_{XZ}, G_{ZZ}, G_{vv}\}$, which are inputs into the F_{SS} function in the equation below.
- The (3.3) element of Λ is assigned as discussed.

$$\begin{aligned}
\Delta_{SS}(\Theta) &= 0 \\
E\{\Lambda\} + (I_{n_S} \otimes \Gamma_A)E\{F_{SS}(\Theta)\} &= 0 \\
E\{(F_S \otimes I_{n_X+n_Y})^T \Gamma_{AA} F_S\} + (I_{n_S} \otimes \Gamma_A)E\{F_{SS}(\Theta)\} &= 0
\end{aligned}$$