Firm Entry and Exit (Clementi and Palazzo AEJ Macro 2016)

The goal of this problem set is to introduce you to numerically solving heterogeneous firm models. We will analyze a simple version of Clementi and Palazzo (2016)'s model of the firm lifecycle (which itself builds on Hopenhyan (1992) and Hopenhayn and Rogerson (1993)). A secondary goal of the homework is to introduce you to this class of models since we do not have time to discuss them in lecture.

This homework will only study the steady state of the model without aggregate shocks. I suspect that you will solve a model with aggregate shocks next week with Tony.

The model is as follows. There are two groups of firms. The first group of *incumbent firms* behaves similarly to the Khan and Thomas (2008) model, except they face convex capital adjustment costs rather than fixed costs. Specifically, each of these incumbent firms has access to a decreasing returns to scale production function $y_{jt} = e^{\varepsilon_{jt}} k_{jt}^{\theta} n_{jt}^{\nu}$, where y_{jt} is output, ε_{jt} is idiosyncratic productivity, k_{jt} is the firm's capital stock, n_{jt} is the firm's labor input, and $\theta + \nu < 1$. Idiosyncratic productivity ε_{jt} follows a Markov chain described below. Firms accumulate capital according to the accumulation equation $k_{jt+1} = (1-\delta)k_{jt} + i_{jt}$. Capital accumulation incurs the convex adjustment cost $-\frac{\varphi}{2}\left(\frac{i_{jt}}{k_{jt}}\right)^2 k_{jt}$, in units of output. At the beginning of each period, incumbent firms must pay a fixed cost c_f units of output to remain in operation. A firm that does not pay this fixed cost does not produce, sells its entire capital stock with value $(1-\delta)k$, and permanently exits the economy.

There is a continuum of the second group of firms, the *potential entrants*. These firms are ex-ante identical. At the beginning of each period, each firms decides whether to pay a fixed entry cost c_e and enter the economy. If a potential entrant enters the economy, it draws a value for idiosyncratic productivity ε_{jt} from some distribution ν and begins as an incumbent firm with $k_{jt} = 0$. Assume that there are no adjustment costs at $k_{jt} = 0$. We assume there is free entry among potential entrants, which implies that the expected value from entering is less than or equal to the entry cost c_e, with equality if entry actually takes place. In equations, this condition is $c_e \leq \int v(\varepsilon, 0)\nu(d\varepsilon)$, with equality if $m^* > 0$ (where $v(\varepsilon, k)$ is the value function of an incumbent firm and m^* is the mass of entrants in equilibrium).

Finally, there is a representative household with preferences over consumption C_t and labor supply N_t represented by the expected utility function $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\log C_t - aN_t)$, where β and a are parameters. Output can be used for consumption, investment, capital adjustment costs, entry costs, or operating costs.

A steady state recursive competitive equilibrium of this economy is a set of incumbent value functions $v(\varepsilon, k)$, policy rules $k'(\varepsilon, k)$ and $n(\varepsilon, k)$, a mass of entrants per period m^*, a measure of active firms at the beginning of the period $g^*(\varepsilon, k)$, and real wage w^* such that (i) incumbent firms maximize their firm value; (ii) the free entry condition holds; (iii) the labor market clears; and (iv) the measure of active firms $g^*(\varepsilon, k)$ is stationary.

We will calibrate the fixed costs c_f and c_e in this problem set. Throughout, the following parameter values are fixed: $\theta = 0.21$, $\nu = 0.64$, $\delta = 0.1$, $\beta = 0.96$, and $\varphi = 0.5$. Assume that the distribution of idiosyncratic shocks ε_{jt} follows the AR(1) process $\varepsilon_{jt+1} = \rho\varepsilon_{jt} + \omega_{jt+1}$ where $\omega_{jt+1} \sim N(0, \sigma^2)$ with parameters $\rho = 0.9$ and $\sigma = 0.02$. Numerically, approximate this process with a Markov chain with $n_\varepsilon = 10$ points using Tauchen's method (if you have not seen this, google some notes on how to do it – it is straightforward). Finally, assume the distribution of new entrants' productivity $\nu(\varepsilon)$ is the stationary distribution associated with this process.

1. **Define the recursive competitive equilibrium** Please define the recursive competitive steady state equilibrium of this model precisely (similar to my definition of the Khan and Thomas (2008) model in class). Use the short description above as a template.

2. **Solve for representative agent steady state** Before computing for the steady state of this model, it will be convenient to compute the steady state equilibrium of the model in which there is a representative firm with the same production function as the heterogeneous firms and productivity equal to the mean value of ε according to its stationary distribution. Assume the representative firm rents capital and labor in competitive input markets as in the RBC model from lecture. Compute the steady state capital stock K^*_{rep} and steady state wage w^*_{rep} of the representative agent model. Assume that steady state labor supply is $N^*_{\text{rep}} = 0.6$. [Hint: use the first order conditions for the firm's input decision and plug in $N^*_{\text{rep}} = 0.6$. You do not need to know the value of the parameter a. My `Dynare` codes for the RBC model use the same trick.]

3. **Solve For Incumbent Firms' Decision Rules** We will begin solving the heterogeneous firm model with the incumbent firms' decision rules. The Bellman equation of incumbent firms' optimization problem is

$$v(\varepsilon, k) = \max\{(1-\delta)k, v^1(\varepsilon, k) - c_f\} \tag{1}$$

$$v^1(\varepsilon, k) = \max_{k', n} \varepsilon k^\theta n^\nu - w^* n - (k' - (1-\delta)k) - \frac{\varphi}{2}\left(\frac{k'}{k} - (1-\delta)\right)^2 k + \beta\mathbb{E}[v(\varepsilon', k')] \tag{2}$$

where w^* is the steady state real wage. The first Bellman equation corresponds to the exit decision and the second corresponds to the optimal decisions of continuing firms. For now, set $w^* = w^*_{\text{rep}}$; in Part 4 we will solve for the true steady state wage. We will also need a parameter value for c_f. For now, set $c_f = 0.01$; in Part 6 we will calibrate this parameter to match the average exit rate in the economy.

Solve this dynamic programming problem using value function iteration, as you learned in week 1. Please use whichever method you see fit. Plot the value function, capital accumulation policy function, and an indicator for whether the firm will continue. Note that the labor decision n can be solved in closed form from the first order condition. [Hint: since we assume there is no capital adjustment costs at $k = 0$, you should solve that gridpoint separately from the rest. Once you have solved for the value function for all points $k > 0$, you can compute the object $v(\varepsilon, 0) \equiv \widehat{v}(\varepsilon) = \max_{k'} -k' + \beta\mathbb{E}[v(\varepsilon', k')]$.

4. **Compute the market-clearing wage** We will study equilibria of this model in which there is entry and exit, i.e., $m^* > 0$. In this case, the free-entry condition must hold with equality, i.e. $c_e = \int \widehat{v}(\varepsilon)\nu(d\varepsilon)$. The real wage w^* will adjust in order to ensure this condition holds. Please solve for the value of the real wage w^* such that the free entry condition holds. I suggest casting this problem as a root-finding problem $F(w^*) = 0$, where the function F performs the following computations:

   - Solve for the incumbent's firm decision rules $v(\varepsilon, k)$ given the value w^* (as discussed in Part 3).
   - Compute the value function of new entrants, $\widehat{v}(\varepsilon)$ (again, as discussed in Part 3).
   - Return the value $c_e - \int \widehat{v}(\varepsilon)\nu(d\varepsilon)$.

   It is possible that there is no solution to the equation $F(w^*) = 0$. This is an indication that the entry costs are so high that firms will never find it profitable to enter the economy. If you find this to be the case, decrease c_e until you can find a solution with entry. Note that you are not solving for the amount of entry, m^*, yet.

5. **Compute the stationary measure of firms and mass of entrants** Now we will solve for the mass of entrants m^* and the stationary measure of incumbent firms $g^*(\varepsilon, k)$. As discussed by Hopenhayn and Rogerson (1993), the stationary measure is linearly homogenous in the amount of entry m^*. That is, if we double the amount of entrants m^*, then you will double the total mass of firms in the economy but the fraction of firms with any particular pair (ε, k) is constant. Therefore, we proceed in two steps: first, solve for the stationary measure with $m^* = 1$; and second, solve for the mass of entrants m^* itself.

   (a) *Step 1: solve for stationary measure with* $m^* = 1$ In principle, the stationary distribution must satisfy the law of motion

   $$g^*(\varepsilon', k') = \int X(\varepsilon, k)Pr(\varepsilon'|\varepsilon)\mathbb{1}\{k'(\varepsilon, k) = k'\}dg^*(\varepsilon, k) + m^* \Pr(\nu = \varepsilon') \qquad (3)$$

   where $X(\varepsilon, k)$ is an indicator variable for whether a firm (ε, k) will survive (i.e., choose not to exit the economy).

   You will iterate on a numerical approximation of this equation with $m^* = 1$. In order to do so, assume that the capital stock k takes values in a discrete grid $\mathbf{k} = (k_1, ..., k_N)$ where $k_i > k_{i-1}$ and n_k is the total number of grid points. Under this assumption, the measure $m(\varepsilon, k)$ can be represented by a $n_\varepsilon \times n_k$ matrix or, in more compact form, a $n_\varepsilon n_k \times 1$ vector. [If you approximated the decision rules with a continuous function, it would be helpful to interpolate it over this discrete grid.] We will approximate the stationary distribution with a probability mass function over the discretized state space $g(\varepsilon_i, k_j)$. This function must satisfy the discretized law of motion

$$g(\varepsilon_k, k_l) = \sum_{i=1}^{n_\varepsilon} \sum_{j=1}^{N} \Pr(\varepsilon' = \varepsilon_k | \varepsilon = \varepsilon_i) \mathbb{1}\left\{k'_{ij} \in k_l\right\} X(\varepsilon_i, k_j) g(\varepsilon_i, k_j) + m^* \Pr(\nu(\varepsilon) = \varepsilon_k) \tag{4}$$

where the notation $\mathbb{1}\left\{k'_{ij} \in k_l\right\}$ means that k_l is the closest gridpoint to $k'(\varepsilon_i, k_j)$.

Please compute the stationary measure $\mathbf{g}$ for m^* by iterating on (4) with $m* = 1$ (i.e., plug in the current iteration for $\mathbf{g}$ on the RHS and get the new iteration on th LHS. [Hint: it is computationally efficient to write the RHS of (4) as a matrix operation $\mathbf{g}^{n+1} = \mathbf{T}\mathbf{g}^n$, where $\mathbf{T}$ is a $n_{\varepsilon N} \times n_{\varepsilon N}$ transition matrix whose $(ij, kl)^{th}$ entry contains the corresponding entry to the RHS of (4).

Plot the stationary measure corresponding to $m^* = 1$, $g(\varepsilon, k)$. If there is a positive mass of firms at either end of the state space, this indicates the grid bounds on the capital stock k are too narrow. If there is empty space at either end of the state space, this indicates that the grid bounds are too wide. Adjust the grid bounds appropriately.

(b) *Step 2: solve for the mass of entrants* m^* As discussed above, the equilibrium measure of firms is $g^*(\varepsilon, k) = m^* g(\varepsilon, k)$, where m^* is the equilibrium mass of entrants. This mass of entrants m^* must be consistent with labor market clearing, i.e. aggregate labor demand $N^{\mathrm{d}}(m^*)$ equals aggregate labor supply $N^{\mathrm{s}}(m^*)$.

As discussed in Part 1, a common way to calibrate business cycle models is to choose the disutility of labor supply, a, so that steady state equilibrium labor supply is $N^{\mathrm{s}}(m^*) = 0.6$. This calibration strategy will simplify the computation of the equilibrium mass of entrants m^*. First, we will compute the mass of entrants m^* such that aggregate labor demand $N^{\mathrm{d}}(m^*) = 0.6$. Second, given this value for the wage, we will use the household's first order condition to back out the parameter a which ensures $N^{\mathrm{s}}(m^*) = 0.6$.

Write a function that takes as input a guess of mass of entrants m^* and outputs excess aggregate labor demand. This function $F(m^*)$ should perform the following steps.

- Compute the stationary mass of firms $g^*(\varepsilon, k) = m^* g(\varepsilon, k)$, where $g(\varepsilon, k)$ is the object computed in step 1.

- Compute aggregate labor demand $N^{\mathrm{d}}(m^*) = \int n(\varepsilon, k) dg^*(\varepsilon, k)$, where $n(\varepsilon, k)$ is the decision rule from Part 2.
- Return $N^{\mathrm{d}}(m^*) - 0.6$.

Similarly to Part 4, the steady state mass of entrants must satisfy the equation $F(m^*) = 0$. Solve this equation. What is the equilibrium mass of entrants m^*?

The household's first order condition for labor is $w^* C = a$, where C is aggregate consumption. Compute aggregate consumption using the resource constraint and using the firm's output, investment decisions, adjustment costs, and entry/exit costs, integrated against the stationary distribution. What is the implied value of a?

6. **Calibrate the entry and exit costs** So far, we have used ad-hoc values of the operating cost c_f and entry cost c_e. Let's loosely calibrate these parameters to target (i) the annual exit rate of 10% and (ii) the average size of new firms to be 40% the average size of all firms. To do so, consider parameter values on the coarse grids $c_e = \{0.01, 0.05, 0.1, 0.2, 0.5\}$ and $c_f = \{0.01, 0.05, 0.1, 0.2, 0.5\}$. Find the combination that gets closest to the empirical targets (i) and (ii). [NB: these grids may be very far from the correct parameter values. Feel free to choose values of the parameters off the grids if they perform much better.] [NB: I made up these targets based on a range of values I have seen for different datasets in the literature. If you are feeling adventurous, you can compute the targets yourself using the publicly-available BDS data I mentioned in the first lecture.]

7. **Analyze calibrated model** Now let's analyze the behavior of firms in this model.
   - Compute the dispersion of investment rates across firms, $\sigma(\frac{i_{jt}}{k_{jt}})$ (excluding new entrants). How does it compare to the dispersion in the Cooper and Haltiwanger (2006)?
   - Compute the dispersion in the log marginal product of capital across firms, $\theta\sigma(\log y_{jt} - k_{jt})$. How does it compare to Hsieh and Klenow (2009)?
   - Plot (i) the average exit rate and (ii) the average "Davis-Haltiwanger-Schur" employment growth rate $\frac{n_{jt} - n_{jt-1}}{0.5(n_{jt} + n_{jt-1})}$ as a function of firm age. How does it compare to Clementi and Palazzo (2016)?

8. **Congratulations!** You have now solved, calibrated, and analyzed the steady state of a heterogeneous firm lifecycle model. The skills you learned here can be applied to a wide range of heterogeneous agent models: firm dynamics, entrepreneurship, banks, households, etc. If you are interested in any of these models, I would like to help you get started in the relevant literature. I will save 10-15 minutes in the last class to do so. Please come prepared with a particular topic of interest (if applicable); I will provide references and an overview of the relevant literature to students who ask about a topic.