Untangling Illiquidity: Optimal Asset Allocation with Private Assets

Daniel Dimitrov

University of Amsterdam, De Nederlandsche Bank

.

This paper does not represent the views of DNB or the Eurosystem.

Introduction to Private Assets

- Institutional investors increasingly allocate to private assets.
- Expected benefits:
  - Higher returns due to e.g. illiquidity premia
  - Diversification
- Private asset classes:
  - Private equity
  - Private real estate
  - Infrastructure
  - Hedge funds

Challenges of Private Asset Allocation

- **Illiquidity risk**:
  - Funds locked up for uncertain periods.
  - Limits ability to rebalance
  - Limits ability to withdraw (consume)
- **Illiquidity risk premia**: Creates potential for unintended overallocation.
- **Increased consumption volatility**: Affects long-term investor welfare.

Modeling Strategic Asset Allocation (SAA) with Illiquidity

- Extend *dynamic portfolio choice model* to include illiquidity effects. Merton [1971] $\rightarrow$ Ang, e.A. [2014]:
- **Liquidity arrival modeled as Poisson shocks** $\rightarrow$ uncertainty in liquidity timing.
- Use *capital market assumptions (CMAs)* (JP Morgan 2025) to determine optimal allocation.
- Set up a **numerical algorithm** to solve the dynamic optimization problem.

Related Literature

- The applied SAA literature Terhaar et al. (2003); Ilmanen et al. (2020) emphasizes unsmoothening of private equity
- Theoretical work on dynamic portfolio choice with liquidity frictions: Zabel (1973); Magill and Constantinides (1976); Gennotte and Jung (1994); Boyle and Lin (1997); Dai et al. (2011); Longstaff (2001); Miklós and Ádám (2002); Dimmock et al. (2023); Ang et al. (2014); Jansen and Werker (2022).
- Incorporates stochastic liquidity events into dynamic programming Ang et al. (2014) and numerical solutions from Cai et al. (2013).
- Contributes to private equity allocation literature Korteweg and Westerfield (2022); Gourier et al. (2024); Giommetti and Sorensen (2021); Luxenberg et al. (2022).

Key Findings on Portfolio Impact

- Investors **preemptively reduce withdrawals** and tilt their portfolios away from illiquid assets.
- When liquid substitutes exist, investors prefer them over costly private assets.
- If substitution is not possible, investors **increase holdings in liquid low-risk assets** to buffer against illiquid asset fluctuations.
- Higher returns and diversification benefits **interact negatively with illiquidity risk**.

Welfare Implications of Illiquidity

- **Private assets improve Certainty Equivalent Consumption (CEC)** for long-term investors.
- Ignoring illiquidity risk leads to **substantial welfare losses**.
- Welfare improvements (in CEC terms) by asset class after illiquidity risk is factored in:
  - **Private equity**: 5.5%–9%
  - **Infrastructure**: 2%–4%
  - **Real estate**: 16%
  - **Diversified hedge funds**: 7%
  - **Macro hedge funds**: 0.2%
- Merton (Mean-variance) optimization **over-allocates private assets** without liquidity considerations.
  - This results in significant welfare losses (Misallocation Risk)

Model: The Market

- Asset returns move as a multivariate Geom. Brownian Motion

- The n-th asset is illiquid:
  - It's liquidity is exogenous and random...
  - ...can be accessed when a Poisson Shock N_t with intensity η hits.
  - $1/\eta$ avarage waiting time to trade
- The expected returns and risk sensitivities are

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_w \\ \mu_x \end{bmatrix}, \boldsymbol{\sigma} = \begin{bmatrix} \boldsymbol{\sigma}_w \\ \boldsymbol{\sigma}_x \end{bmatrix}$$

Model: Wealth Dynamics

$$
\begin{aligned}
dW_t/W_t &= (r + \boldsymbol{\theta}_t'(\boldsymbol{\mu}_w - r\mathbb{1}) - c_t)dt + \boldsymbol{\theta}_t'\boldsymbol{\sigma}_w \boldsymbol{dZ}_t - dI_t/W_t \\
dX_t/X_t &= \mu_x dt + \boldsymbol{\sigma}_x \boldsymbol{dZ}_t + dI_t/X_t \\
dQ_t &= dW_t + dX_t
\end{aligned} \tag{1}
$$

- Investment choice:
  - Liquid Wealth W_t: liquid risky assets ($\boldsymbol{\theta}_t$), and a risk-free asset $(1 - \boldsymbol{\theta}_t'\mathbf{1})$
  - Illiquid wealth X_t: of a risky asset
  - Rebalancing between liquid and illiquid wealth through cash withdrawals dI_t
- Wealth dynamics

Model: The Investor's objective

- consumes continuously fraction c_t out of liquid wealth only
- decides in advance on portfolio composition when liquidity is possible (SAA)
- decides on rebalancing strategy when liquidity in one asset is not available (TAA)
- Investor optimizes lifetime utility of consumption

$$V(W_t, X_t) = \sup_{\theta_s, dI_s, c_s} E_t \int_t^{\infty} \mathrm{e}^{-\beta(s-t)} u(c_s W_s) ds$$

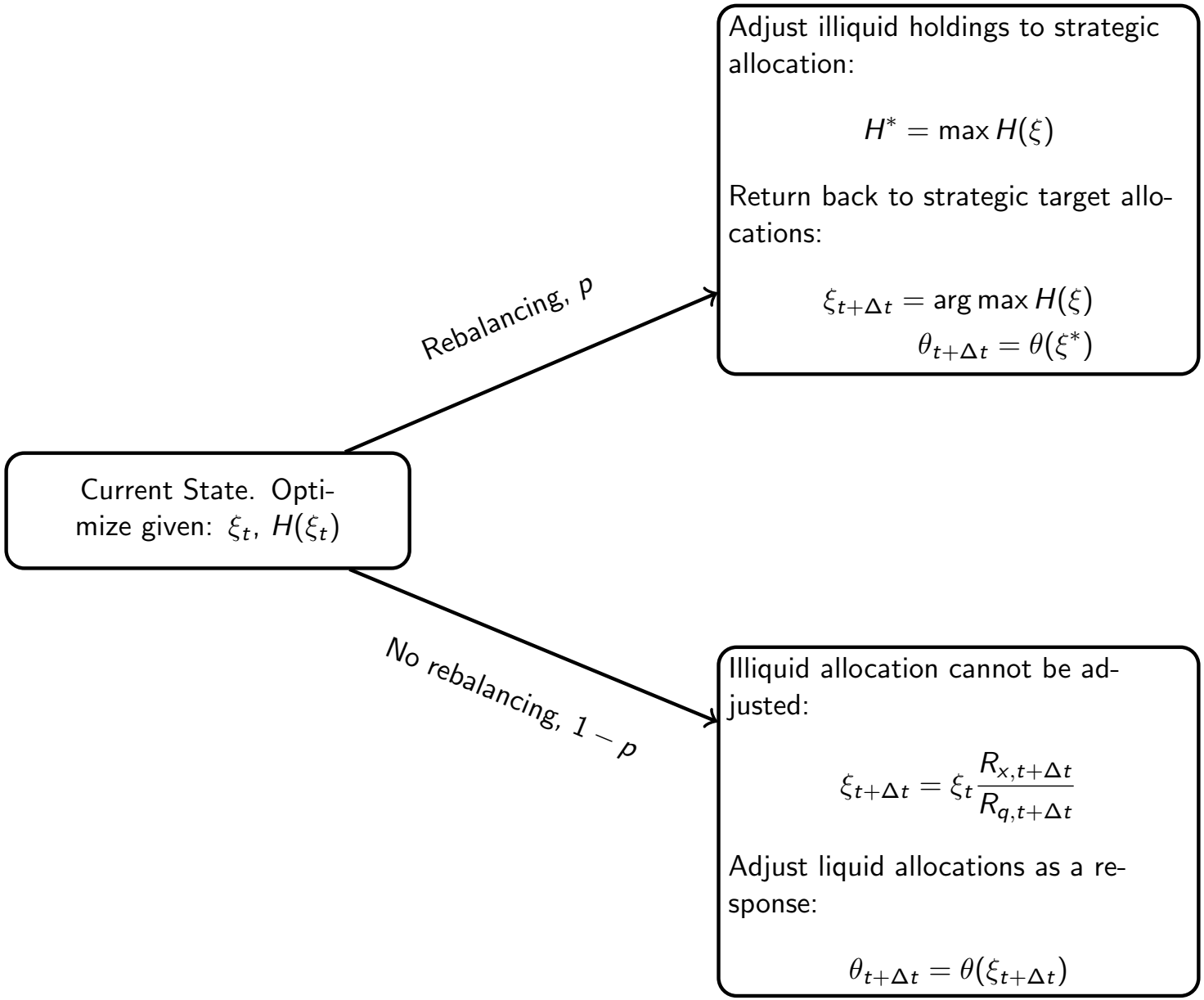
- Define $\xi_t = \frac{X_t}{X_t + W_t}$; Assume CRRA Utility $u(C_s) = \frac{C_s^{1-\gamma}}{1-\gamma}$

$$\Longrightarrow \; V(W_t, X_t) = (X_t + W_t)^{1-\gamma} H(\xi_t)$$

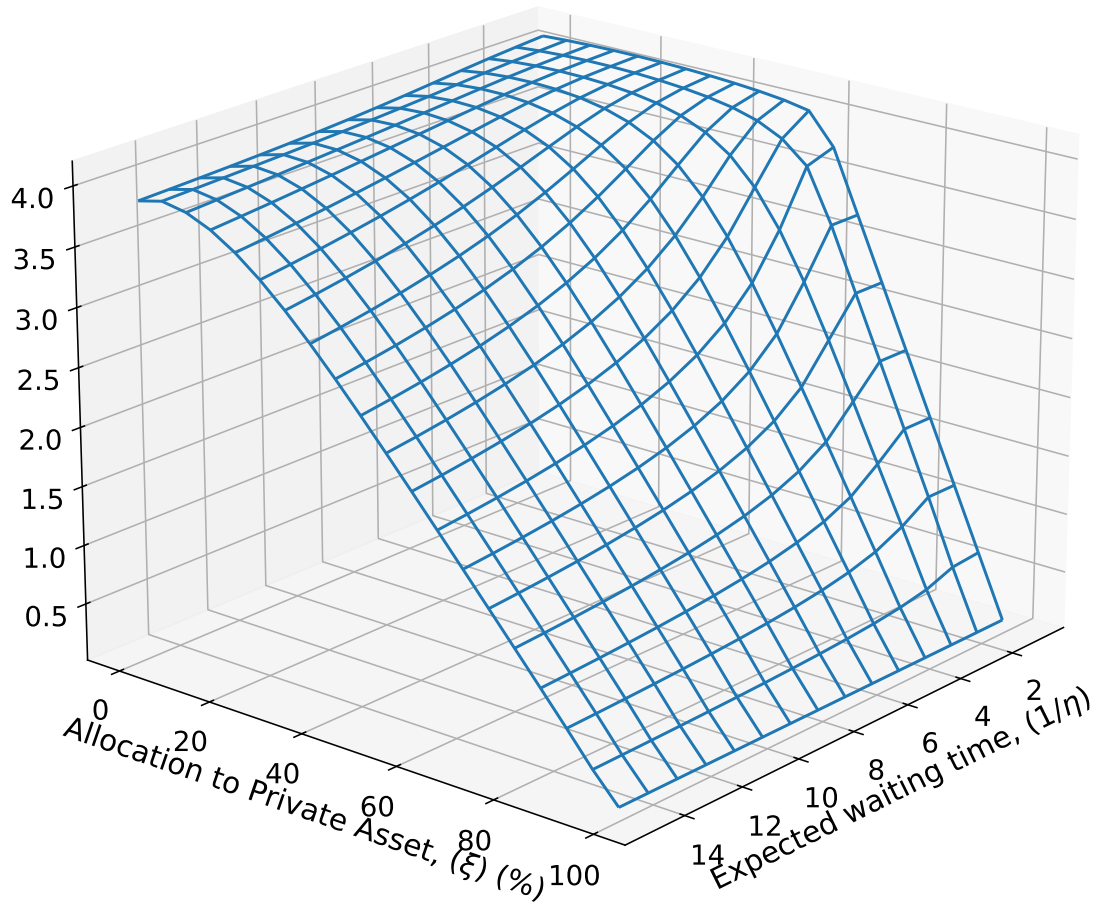
- Whenever liquidity is available, the investor reshuffles the portfolio such that $\xi^* = arg\max_{\xi} H(\xi)$.
- ξ_t then has a dual nature
  - **decision variable** when liquidity arrives, ξ^*
  - **state variable** in illiquid periods
- Investor's optimal policy functions: $\{\xi^*, \theta_t = \theta(\xi_t, H(\xi_t)), c_t = c(\xi_t, H(\xi_t))\}$;
- Solve *numerically* for the unknown function $H(\xi_t)$
- Discretize and solve numerically through *value function iteration* and *multivariate quadrature techniques*.

Discretized Bellman equation of the form: $H(\xi_t) = \max_{(\theta_t, c_t)} \left\{ u(c_t(1-\xi_t))\Delta t + \delta \left(pH^* E_{\xi_t} \left[R_{q,t+\Delta t}^{1-\gamma} \right] + (1-p) E_{\xi_t} \left[R_{q,t+\Delta t}^{1-\gamma} H(\xi \right. \right. \right.$
where

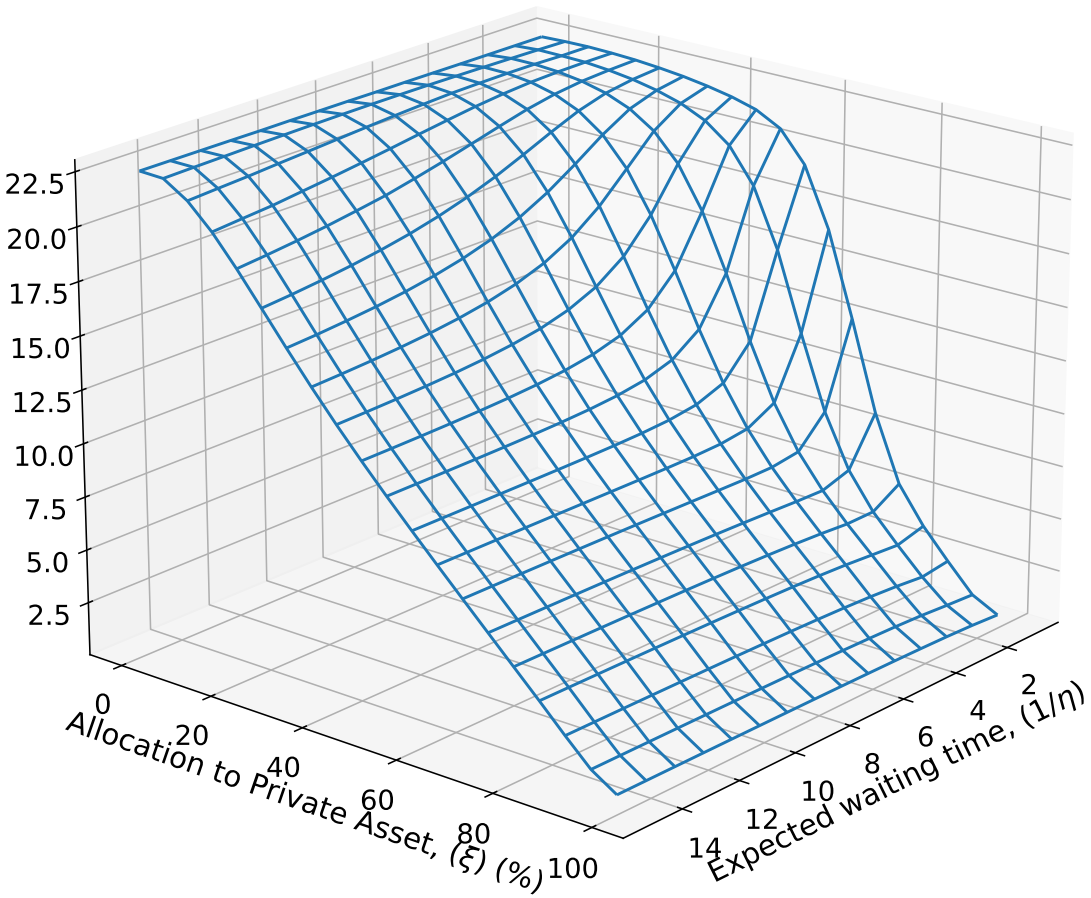
$$
\begin{aligned}
R_{w,t+\Delta t} &= 1 + (r + \boldsymbol{\theta}_t'(\boldsymbol{\mu}_w - r\not{1}) - c_t)\Delta t + \boldsymbol{\theta}_t' \boldsymbol{\sigma}_w \sqrt{\Delta t} \boldsymbol{\Delta Z}_t \\
R_{x,t+\Delta t} &= 1 + \mu_x \Delta t + \boldsymbol{\sigma}_x \sqrt{\Delta t} \boldsymbol{\Delta Z}_t \\
R_{q,t+\Delta t} &= (1-\xi_t) R_{w,t+\Delta t} + \xi R_{x,t+\Delta t} \\
\xi_{t+\Delta t} &= \xi_t \frac{R_{x,t+\Delta t}}{R_{q,t+\Delta t}}
\end{aligned}
\tag{2}
$$


Current State. Optimize given: ξ_t, $H(\xi_t)$
Rebalancing, p
No rebalancing, $1-p$
Adjust illiquid holdings to strategic allocation:
$H^* = \max H(\xi)$
Return back to strategic target allocations:
$\xi_{t+\Delta t} = \arg\max H(\xi)$
$\theta_{t+\Delta t} = \theta(\xi^*)$
Illiquid allocation cannot be adjusted:
$\xi_{t+\Delta t} = \xi_t \frac{R_{x,t+\Delta t}}{R_{q,t+\Delta t}}$
Adjust liquid allocations as a response:
$\theta_{t+\Delta t} = \theta(\xi_{t+\Delta t})$

Consumption Rate

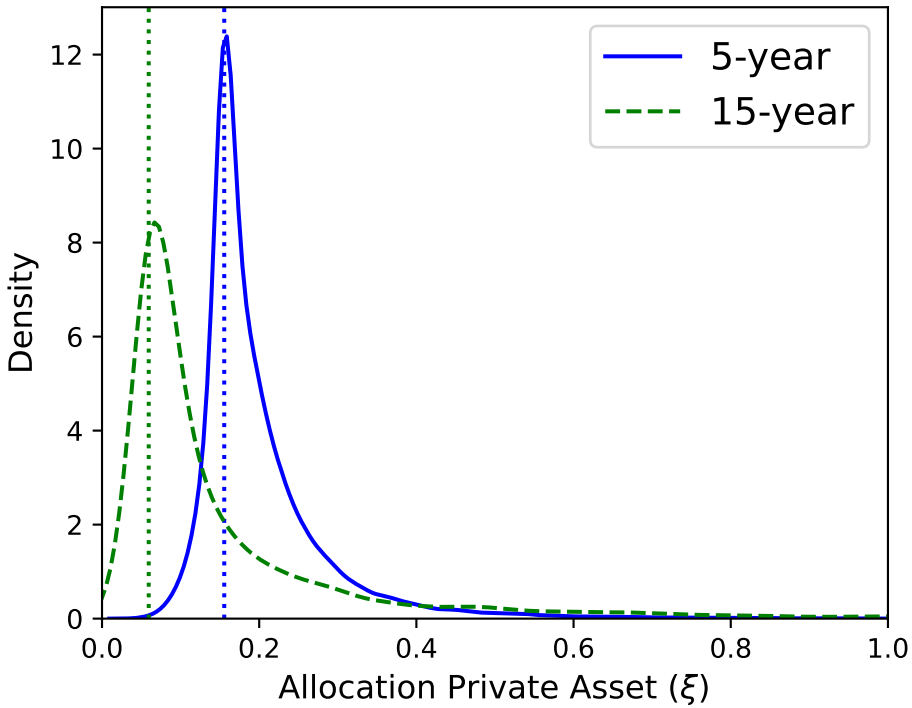


Liquid Risky Asset Allocation

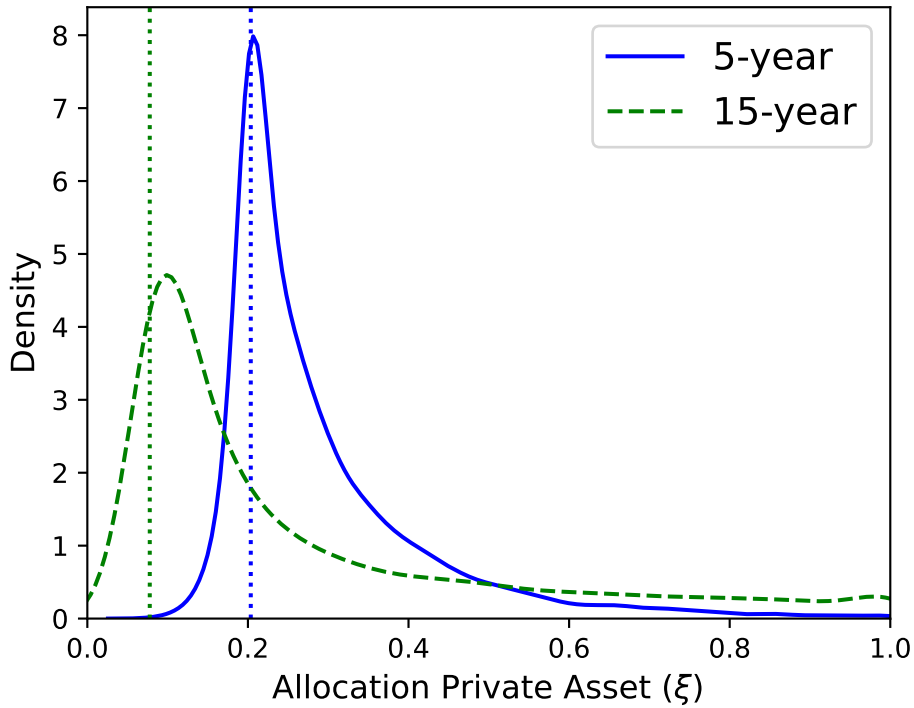


Realized Private Asset Allocation

(a) No Liquidity Premium

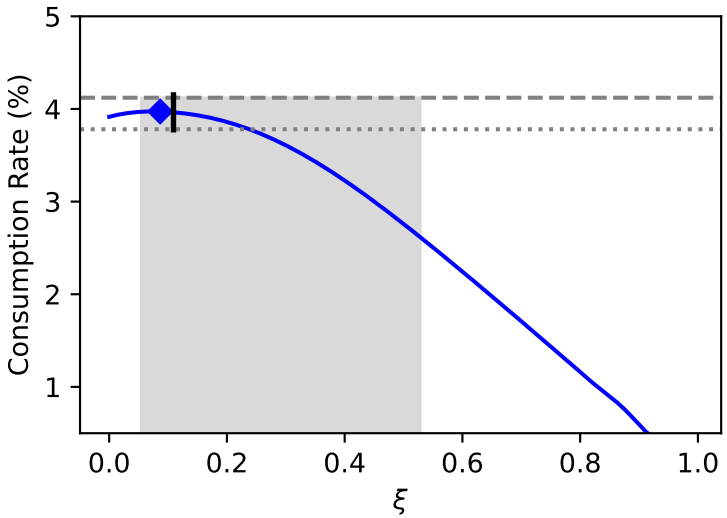


(b) With Liquidity Premium

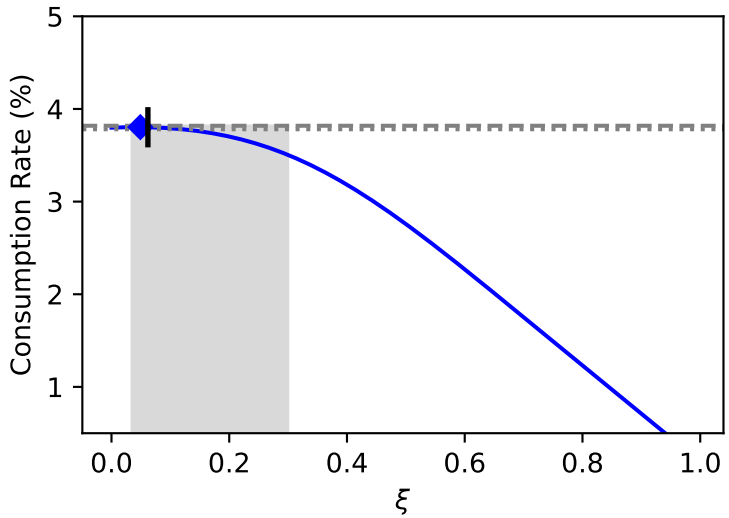


Consumption Variability

(a) $\rho = 0; lp = 0$

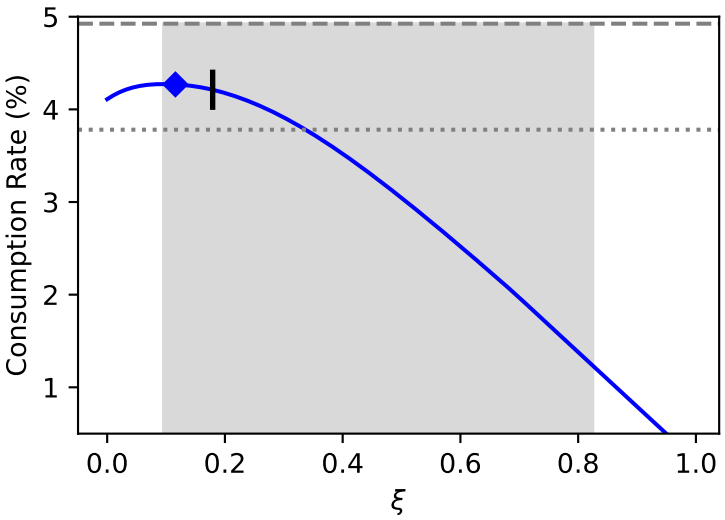


(b) $\rho = 0.8; lp = 0$

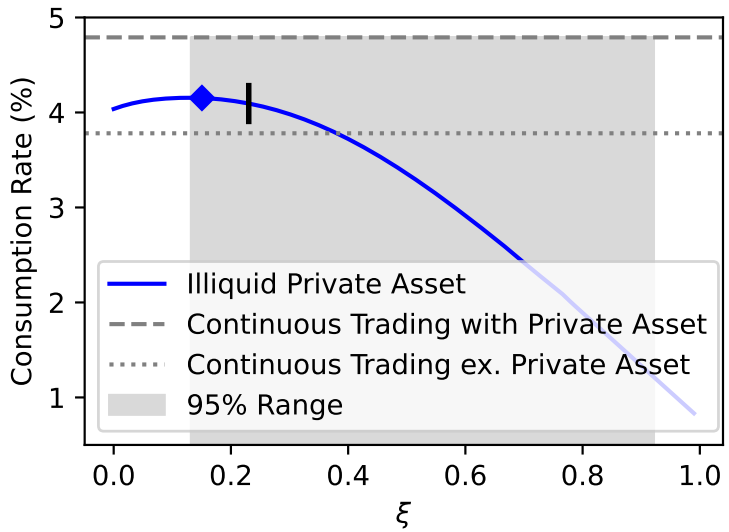


Consumption Variability

(a) $\rho = 0$; $lp = 0.03$



(b) $\rho = 0.8$; $lp = 0.03$



Definition: Allocation Benefit

Define the allocation benefit from investing in the private asset class as the improvement in CEC relative to investing only in liquid public assets. Formally:

$$\text{Allocation Benefit} = \frac{CEC(\xi^*)}{CEC_m(\xi = 0)} - 1, \tag{3}$$

where $CEC(\xi^*)$ based on allocation with the illiquid private asset, and $CEC_m(\xi = 0)$ based on optimal allocation without the private asset.

Definition: Misallocation Risk

Define the misallocation loss as the potential welfare loss from ignoring the illiquidity risk of the private asset. Formally:

$$\text{Misallocation Risk} = -\left(\frac{CEC(\xi^{liquid})}{CEC(\xi^{*})} - 1\right), \tag{4}$$

where $CEC(\xi^{liquid})$ based on the model with illiquid private asset but the mean-variance (Merton) is used; and the $CEC(\xi^{*})$ based on the optimal allocation with illiquidity properly considered.

Portfolio Metrics: Private Equity

	Liquid	Illiquid 5Y	Illiquid 10Y
Fixed Income	63.20	67.02	67.99
- Short Term	30.92	48.56	53.81
- Long Term	32.27	18.47	14.18
Equity	0.19	4.66	11.17
Liquid Alternatives	0.40	11.14	11.93
Private Assets	36.15	17.09	8.82
CEC	4.59	4.31	4.14
Median Realized ξ	36.15	20.34	13.41
Median Realized c	4.32	4.09	3.94
Allocation Benefit	16.65	9.51	5.11
Misallocation Risk	-	3.69	12.49

Portfolio Metrics: Infrastructure

	Liquid	Illiquid 5Y	Illiquid 10Y
Fixed Income	49.21	56.78	62.94
- Short Term	47.11	51.74	55.55
- Long Term	2.10	5.03	7.39
Equity	5.20	10.17	14.13
Liquid Alternatives	16.39	14.95	13.83
Private Assets	29.12	18.03	9.01
CEC	4.13	4.09	4.02
Median Realized ξ	29.12	20.44	12.13
Median Realized c	3.95	3.92	3.86
Allocation Benefit	4.78	3.87	2.20
Misallocation Risk	-	0.58	5.43

Portfolio Metrics: Real Estate

	Liquid	Illiquid
Fixed Income	30.57	54.87
- Short Term	9.13	39.89
- Long Term	21.44	14.99
Equity	18.02	20.43
Liquid Alternatives	0.05	4.07
Private Assets	51.31	20.55
CEC	5.25	4.60
Median Realized ξ	51.31	24.60
Median Realized c	4.83	4.31
Allocation Benefit	33.46	16.78
Misallocation Risk	-	16.68

Portfolio Metrics: Diversified Hedge Funds

	Liquid	Illiquid
Fixed Income	23.07	29.13
- Short Term	8.99	15.71
- Long Term	14.08	13.41
Equity	3.34	5.29
Liquid Alternatives	14.42	14.08
Private Assets	59.13	51.46
CEC	4.24	4.21
Median Realized ξ	59.13	51.46
Median Realized c	3.98	4.02
Allocation Benefit	7.70	6.98
Misallocation Risk	-	0.19

Portfolio Metrics: Macro Hedge Funds

	Liquid	Illiquid
Fixed Income	57.97	58.25
- Short Term	47.90	48.18
- Long Term	10.07	10.07
Equity	16.47	16.50
Liquid Alternatives	13.34	13.33
Private Assets	12.13	11.84
CEC	3.95	3.95
Median Realized ξ	12.13	11.84
Median Realized c	3.81	3.81
Allocation Benefit	0.27	0.27
Misallocation Risk	-	0.00

Conclusion: The Need for Liquidity-Aware Allocation

- Institutional investors often **underestimate illiquidity risks**.
- Portfolio construction must **account for liquidity constraints** to avoid overexposure.
- Future research: Refining models for **strategic vs. tactical asset allocation (SAA vs. TAA)**.

Thank You!

Ang, A., Papanikolaou, D., and Westerfield, M. M. (2014). Portfolio choice with illiquid assets. *Management Science*, 60(11):2737–2761.

Boyle, P. P. and Lin, X. (1997). Optimal portfolio selection with transaction costs. *North American Actuarial Journal*, 1(2):27–39.

Cai, Y., Judd, K. L., and Xu, R. (2013). Numerical Solution of Dynamic Portfolio Optimization with Transaction Costs. *NBER Working Paper No. w18709*.

Dai, M., Jin, H., and Liu, H. (2011). Illiquidity, position limits, and optimal investment for mutual funds. *Journal of Economic Theory*, 146(4):1598–1630.

Dimmock, S. G., Wang, N., and Yang, J. (2023). The endowment model and modern portfolio theory. *Management Science*, 70(3):1554–1579.

Gennotte, G. and Jung, A. (1994). Investment Strategies under Transaction Costs: The Finite Horizon Case. *Management Science*, 40(3):385–404.

Giommetti, N. and Sorensen, M. (2021). Optimal allocation to private equity. *Tuck School of Business Working Paper*, (3761243).

Gourier, E., Phalippou, L., and Westerfield, M. M. (2024). Capital commitment. *The Journal of Finance*, 79(5):3407–3457.

Ilmanen, A., Chandra, S., and McQuinn, N. (2020). Demystifying illiquid assets: Expected returns for private equity. *The Journal of Alternative Investments*, 22(3):8–22.

Jansen, K. A. and Werker, B. J. (2022). The shadow costs of illiquidity. *Journal of Financial and Quantitative Analysis*, 57(7):2693–2723.

Korteweg, A. and Westerfield, M. M. (2022). Asset allocation with private equity. *Foundations and Trends® in Finance*, 13(2):95–204.

Longstaff, F. A. (2001). Optimal Portfolio Choice and the Valuation of Illiquid Securities. *Review of Financial Studies*, 14(2):407–31.

Luxenberg, E., Boyd, S., van Beek, M., Cao, W., and Kochenderfer, M. (2022). Strategic asset allocation with illiquid alternatives. In *Proceedings of the third ACM international conference on AI in finance*, pages 249–256.

Magill, M. J. P. and Constantinides, G. M. (1976). Portfolio selection with transactions costs. *Journal of Economic Theory*, 13(2):245–263.

Miklós, K. and Ádám, S. (2002). Portfolio Choice with Illiquid Assets. Rajk László Szakkollégium Working Papers 6, Rajk László College.

Terhaar, K., Staub, R., and Singer, B. D. (2003). Appropriate policy allocation for alternative investments. *The Journal of Portfolio Management*, 29(3):101–110.

Zabel, E. (1973). Consumer Choice, Portfolio Decisions, and Transaction Costs. *Econometrica*, 41(2):321–35.