MATH 244 Results

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December 2021

Contents

§1 Introduction

Definition 1.1. Given two sets X and Y, a **function** f from X to Y is a rule which assigns to every element $x \in X$, an element $f(x) \in Y$.

Definition 1.2. A **relation** between two sets X and Y is a subset of $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.

Definition 1.3. A function $f : X \to Y$ is a relation R_f between X and Y which also satisfies the property:

- Every $x \in X$ is related to a unique $y \in Y$.
- $\forall x \in X\ \exists! y \in Y$ s.t. $(x, y) \in R_f$.

Definition 1.4. A function f is **injective** (i.e. one-to-one) if $\forall x_1, x_2 \in X$, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

Definition 1.5. A function f is **surjective** (i.e. onto) if $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$.

Definition 1.6. A function is **bijective** if it is both injective and surjective.

Definition 1.7. If $f : X \to Y$ and $g : Y \to Z$ then $g \circ f$ is the function $g \circ f = X \to Z$ defined by $(g \circ f)(x) = g(f(x))$.

Definition 1.8. If $(x, y) \in R$, we write xRy.

Definition 1.9. A relation R between a set X and itself is an **equivalence relation** if it satisfies the following 3 properties:

i) $\forall x \in X, xRx$ (reflexive)

i) $\forall x, y \in X, xRy \implies yRx$ (symmetric)

i) $\forall x, y, z \in X, xRy$ and $yRz \implies xRz$ (transitive)

Definition 1.10. A relation R between X and itself is an **ordering** if it satisfies

i) $\forall x \in X, xRx$ (reflexive)

i) $\forall x, y \in X, xRy$ and $yRx \implies x = y$ (anti-symmetric)

i) $\forall x, y, z \in X, xRy$ and $yRz \implies xRz$ (transitive)

Definition 1.11. An ordering R is a **linear order** (or **total order**) if $\forall x, y \in X$, either xRy or yRx.

Definition 1.12. If $\sim$ is an equivalence relation on X and $x \in X$, we define $[x] = \{y \in X \mid y \sim x\}$. This is the **equivalence class** of x.

Proposition 1.13
If $\sim$ is an equivalence relation on X, then

i) $\forall x \in X$, $[x]$ is nonempty.

ii) $\forall x, y \in X$, $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

iii) $\sim$ is uniquely determined by the set of its equivalence classes.

Definition 1.14. A **finite partition** of a set X is a collection of nonempty subsets $X_1, X_2, \dots, X_k \subseteq X$ such that

1. $\bigcup_{i=1}^{k} X_i = X$
2. $\forall 1 \leq i, j \leq k$, $X_i \cap X_j = \emptyset$

§2 Big O Notation

Definition 2.1. If $f(n), g(n)$ are 2 nonnegative functions on $\mathbb{N}$, we say $f(n) = O(g(n))$ if $\exists n_0 > 0$ and $C > 0$ such that $\forall n \geq n_0$, $f(n) \leq Cg(n)$.

Proposition 2.2
If $\forall n_1,\ f_1(n) \le f_2(n)$ and $f_2(n) = O(g(n))$, then $f_1(n) = O(g(n))$.

Proposition 2.3
We have the following properties of Big-O:

- If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then
$$f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$$
$$f_1(n)f_2(n) = O(g_1(n)g_2(n))$$
- $n^\alpha = O(n^\beta)$ if $0 \le \alpha \le \beta$
- $n^\alpha = O(a^n)$ if $a > 1$
- $a^n = O(n!)$ if $a > 1$
- $n! = O(n^n)$

Proposition 2.4
Hierarchy of growth:

1. Bounded
2. $(\log n)^\alpha,\ \alpha > 0$
3. n^β, $\beta > 0$
4. e^γ, $\gamma > 0$
5. $n!$
6. n^n

Theorem 2.5 (Stirling's Approximation)
$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

Theorem 2.6

$$e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n \quad \forall n \geq 1$$

Theorem 2.7

Assume $k \leq \frac{n}{2}$. Then

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!}.$$

§3 Counting

Definition 3.1. A bijection $[n] \to [n]$ is also called a **permutation**.

Definition 3.2. $\binom{n}{j} = \frac{n!}{j!(n-j)!}$

Definition 3.3. If X is a set, $\binom{X}{j}$ is the set of all j-element subsets of X.

Proposition 3.4

$\left|\binom{X}{j}\right| = \binom{n}{j}$ if $|X| = n$.

Theorem 3.5

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Proposition 3.6

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Definition 3.7. If $n_1 + \ldots + n_k = n$ where $n_i \geq 0$ $\forall 1 \leq i \leq k$, then

$$\binom{n}{n_1, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}.$$

Proposition 3.8

$\binom{n}{i,j,k}$ is the number of ways to partition a set of size n into 3 subsets of sizes i, j, k respectively.

Theorem 3.9

$$(x+y+z)^n = \sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \binom{n}{i,j,k} x^i y^j z^k.$$

Theorem 3.10 (PIE)

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{I\in\binom{[n]}{k}} \left|\bigcap_{i\in I} A_i\right|.$$

Definition 3.11. Positive integers m, n are relatively prime if they have no common factors, i.e. $\gcd(m,n)=1$.

Theorem 3.12 (Euler)

If $n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$, then $\phi(n) = n\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right)$.

§4 Graph Theory

Definition 4.1. A **graph** $G=(V,E)$ consists of the following data:

- A finite set V (vertices)
- A finite set $E \subseteq \binom{V}{2}$ (edges)

Proposition 4.2

There exists precisely $2^{\binom{n}{2}}$ graphs $G=(V,E)$ with $V(G)=\{1,\ldots,n\}$.

Definition 4.3. Two graphs $G=(V,E)$ and $G'=(V',E')$ are called **isomorphic** if $\exists$ bijective function $f: V\to V'$ such that $(x,y)\in E \longleftrightarrow (f(x),f(y))\in E'$.

Proposition 4.4

Let $T(n)$ be the number of pairwise non-isomorphic graphs $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$. Then $T(n) \geq \frac{2^{\binom{n}{2}}}{n!}$.

Definition 4.5. Given a simple graph $G = (V, E)$, the **degree** of a vertex $x \in V$ is $|\{y \in V : (x, y) \in E\}|$. In other words, $\deg_G(x)$ is the number of edges containing vertex x as one of the endpoints.

Definition 4.6. When $\deg(x)$ is the same for all $x \in V$, we say that the graph is **regular**.

Proposition 4.7

Given a graph $G = (V, E)$, $\sum_{x \in V} \deg_G(x) = 2\,|E|$.

Corollary 4.8 (Handshake Lemma)

The number of vertices of odd degree in G is always even.

Definition 4.9. Given a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$, the sequence

$$(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$$

is called the **score**, or **degree sequence**, of G.

Theorem 4.10 (Score Theorem)

Let $D = (d_1, \ldots, d_n)$ be a sequence of natural numbers where $d_1 \leq d_2 \leq \cdots \leq d_n$. Define $D' = (d'_1, d'_2, \ldots, d'_{n-1})$ to be the sequence

$$d'_i = \begin{cases} d_i & \text{if } i < n - d_n \\ d_i - 1 & \text{if } i \geq n - d_n \end{cases}.$$

Then D is a graph score iff D' is a graph score.

Definition 4.11. Given graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$, the **adjacency matrix** of G is the $n \times n$ matrix $A = (a_{i,j})$ where rows and columns are indexed by V and

$$a_{i,j} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0\text{if } (v_i, v_j) \notin E \end{cases}.$$

Definition 4.12. Given a graph $G = (V, E)$, a **walk** in G is a sequence of vertices and edges $W = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$, where

- v_i is common endpoint of e_{i-1} and e_{i+1}
- vertices in this list may possibly repeat

Also, length(W) =number of edges $= k$.

Theorem 4.13
Let $G = (V, E)$ be a simple graph and A its adjacency matrix where $V = \{v_1, \ldots, v_n\}$. For each $k \geq n$, denote $A^k = (a_{i,j}^{(k)})_{1 \leq i,j \leq n}$. Then $a_{i,j}$ is the total number of walks of length k between vertices i and j.

Definition 4.14. K_n is the complete graph on n vertices

Definition 4.15. C_n is the cycle of length n

Definition 4.16. P_n is the path with n vertices

Definition 4.17. $K_{m,n}$ is the complete bipartite graph with m, n, i.e. $V = M \cup N$, $M \cap N = \emptyset$, and edges run only between M and N where $|M| = m$ and $|N| = n$

Definition 4.18. If $G = (V, E)$ is a graph, a **subgraph** is another graph $H = (V', E')$ where $V' \subseteq V$, $E' \subseteq E$, where E' connects vertices in V'.

Definition 4.19. If $G = (V, E)$ and $V' \subseteq V$, then the **induced subgraph** is the subgraph $H = (V', E')$ where

$$E' = \{\text{edge } e \in E \mid \text{endpoints of } e \text{ in } V'\}.$$

Definition 4.20. A **path** in a graph G is a subgraph isomorphic to P_n for some $n \geq 1$.

Definition 4.21. A **walk** in a graph is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that $v_i \sim v_{i+1}$, $\forall i = 1, \ldots, k-1$.

Definition 4.22. A graph is **connected** if for any two vertices $u, v \in V$, there is a path (or walk) starting at u and ending at v.

Definition 4.23. Given a graph G, define an equivalence relation $\approx$ on V where $u \approx v$ if there is a walk from u to v. The equivalence classes of $\approx$ are called components of G ("connected components").

Definition 4.24. The **distance** $d_G(x, y)$ between vertices x and y is the length (number of edges) of the shortest walk from x to y (or ∞ if $u \approx v$).

Proposition 4.25
$(A^k)_{u,v}$ is the number of walks of length exactly k from u to v

Definition 4.26. A graph is **Eulerian** if it has an Eulerian tour.

Definition 4.27. An **Eulerian tour** is a closed walk (start and end vertices are the same) which crosses each edge exactly once.

Theorem 4.28
G is Eulerian iff every vertex has even degree.

Definition 4.29. A **Hamiltonian tour** is a closed walk which visits every vertex once.

Definition 4.30. A graph G is **k-vertex connected** for $k \geq 1$ if it has $k+1$ vertices and if you remove any $k-1$ vertces and their incident edges, the remaining graph is connected.

Definition 4.31. A graph is **k-edge-connected** if it remains connected after removal of any $k-1$ edges.

Definition 4.32. The **vertex connectivity** of G is the maximum value of k where G is k-vertex-connected.

Theorem 4.33
G is 2-vertex-connected iff $\forall x, y \in V$, G contains a cycle containing x and y.

Definition 4.34. A graph is **minimally 2-vertex-connected** if it is 2 *vertex*-connected but removal of an edge leaves a graph with is not 2-vertex-connected.

Theorem 4.35
G is 2-vertex-connected iff it can be made, starting from a triangle, by a sequence of operations with are either (1) edge subdivision or (2) adding an edge joining existing vertices.

- Let $T(n)$ be the maximum number of edges in a graph on $[n]$ with no triangle subgraphs.

Claim — $T(n) = \left\lfloor \frac{n^2}{4} \right\rfloor$.

Definition 4.36. A **tree** is a connected graph with no cycles.

Definition 4.37. A **leaf** (or **end-vertex**) is a vertex of degree 1 in a tree.

Lemma 4.38

A tree with ≥ 2 vertices has at least 2 leaves.

Lemma 4.39

If $G = (V, E)$ is a tree and v is a leaf, then the induced subgraph on $V \subseteq V \in \{e\}$

Lemma 4.40

A **spanning tree** in G is a subgraph which contains all vertices and is a tree.

Proposition 4.41

Any connected grain G has a spanning tree. The construction is as follows:

1. Order edges
2. Add edges in order, but skip any edge if adding it would make a cycle

Definition 4.42. Let G be a connected graph. Let $c(e)$ be a cost associated to edge e. If T is a spanning tree, its cost is $c(T) = \sum_{e \in E} c(e)$.

Definition 4.43. A **minimal spanning tree** is a spanning tree of smallest $c(T) = \sum_{e \in E} c(e)$.

Theorem 4.44 (Kruskal's Algorithm)

You repeat the following steps:

1. Order edges in increasing cost:

$$c(e_1) \leq c(e_2) \leq \cdots$$

2. Build the tree as before, adding edges, skipping those which make cycles.

Definition 4.45. A graph is **planar** if it can be drawn in $\mathbb{R}^2$ so that no 2 edges intersect except possibly at their endpoints.

Definition 4.46. A graph has **genus** g if it can be drawn (without edge crossings) on a genus g surface but not on a genus $g-1$ surface.

Theorem 4.47
Every graph has a finite genus.

Definition 4.48. A **Jordan curve** is a continuous injective image of a circle in $\mathbb{R}^2$.

Theorem 4.49 (Jordan Curve Theorem)
A Jordan curve separates $\mathbb{R}^2$ into 2 components, an inside and an outside region.

Theorem 4.50
K_5 is not planar.

Theorem 4.51 (Kuratowski)
A graph G is planar iff G has no subgraph isomorphic to a subdivision of $K_{3,3}$ or K_5.

Definition 4.52. A **subdivision** of G is a graph obtained from G by adding vertices along its edge.

Theorem 4.53 (Euler's Formula)
For a connected planar graph, $|V| - |E| + |F| = 2$.

Theorem 4.54
A 2-vertex-connected planar graph has the property that all faces are bounded by cycles.

Definition 4.55. A convex **polyhedron** is a set in $\mathbb{R}^3$ which is the intersection of a finite number of half-spaces.

At each corner of a polyhedron, the **curvature** c is

$$c = 2\pi - \sum_{\text{adj. face } f} \theta_f.$$

Theorem 4.56 (Gauss-Bonnet)
THe sum of curvatures of vertices of a 3D polyhedron is 4π.

Definition 4.57. A **platonic solid** is a polyhedron with faces which are regular k-gons and vertices of constant degree d. For such a polyhedron the curvature at a vertex is equal to

$$2\pi - d\left(\pi - \frac{2\pi}{k}\right).$$

Theorem 4.58
For a planar graph of at least 3 vertices, $E \leq 3V - 6$.

Corollary 4.59
Every planar graph has a vertex of degree ≤ 5.

Corollary 4.60
K_5 is not planar.

Theorem 4.61
For a planar graph of at least 3 vertices and contains no K_3 as a subgraph, $E \leq 2V - 4$.

Corollary 4.62
$K_{3,3}$ is not planar.

Definition 4.63. A **proper coloring** is a function $c : V \to \{c_1, \ldots, c_k\}$ such that $c(u) \neq c(v)$ if $u \sim v$ i.e. u is adjacent to v.

Definition 4.64. The **chromatic number** $\mathcal{X}(G)$ of a graph G is the smallest m such that there is a proper coloring with m colors.

Definition 4.65. The **chromatic polynomial** $P_G(x)$ is the number of colorings with x colors.

Theorem 4.66
$P_G(x)$ is a polynomial.

Theorem 4.67 (Four-Color Theorem)
Any planar graph can be 4-colored.

Theorem 4.68
Any planar graph can be colored with ≤ 5 colors ($\mathcal{X}(G) \leq 5$).

Theorem 4.69 (Sperner's Lemma)
Let G be a triangulation of a triangle with vertices A_1, A_2, A_3. Assign vertices labels $1, 2$, or 3 with the following rules:

1. A_i has color i
2. A vertex on the side $\overline{A_iA_j}$ has color i or j
3. Remaining vertices (inside the triangle) are colored arbitrarily

Then, there exists a face with all 3 labels.

Theorem 4.70 (Brouwer Fixed Point Theorem)
Let $T = (x, y) \mid x \geq 0,\ y \geq 0,\ x + y \leq 1$. If $f : T \to T$ is continuous, then f has a fixed point, i.e. $\exists (x, y) \in T$ such that $f(x, y) = (x, y)$.

Theorem 4.71
If $f : [0, 1] \to [0, 1]$ is continuous, then f has a fixed point, i.e. $\exists x \in [0, 1]$ such that $f(x) = x$.

Theorem 4.72

The number of spanning trees of K_n with degree sequence $d_1, \dots, d_n$ is

$$\frac{(n-2)!}{(d_1-1)\cdots(d_n-1)!}.$$

Corollary 4.73

$$\sum_{\text{spanning trees of } K_n} x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} = x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}$$

Definition 4.74. Let the **Prüfer code** of a tree be an encoding of a labeled tree on $[n]$ with a sequence $(k_1, k_2, \dots, k_{n-2})$ where each $k_i \in [n]$. The procedure to obtain this sequence is a series of steps: at each step, remove the leaf with the smallest label and record its neighbor, until the K_2 graph is left (i.e. one edge and two vertices).

Definition 4.75. Given a graph $G = (V, E)$, the **incidence matrix** of G is the $|V| \times |E|$ matrix A such that

- Rows of A correspond to vertices of G
- Columns of A correspond to edges of G
- $a_{x,e} = \begin{cases} 1 & \text{if } x \text{ is endpoint of } e \\ 0 & \text{otherwise} \end{cases}$

Definition 4.76. The **Laplacian matrix** of a graph G is $\Delta = DD^\intercal$ where D is the incidence matrix of G.

Corollary 4.77

$\Delta_{i,i} = \deg_G(v_i)$ and $\Delta_{i,j} = \begin{cases} -1 & \text{if } v_i \sim v_j \\ 0 & \text{else} \end{cases}$ for $i \neq j$.

Theorem 4.78 (Matrix-Tree Theorem)

If G is connected, the number of spanning trees is $\det(\Delta_{11})$, where Δ_{11} is obtained from Δ by removing row 1 and column 1. In other words,

$$\det(\Delta_{11}) = \frac{1}{n} \prod \lambda_i$$

where λ_i are the non-zero eigenvalues of Δ.

Theorem 4.79 (Binet-Cauchy Formula)
If A is an $m \times n$ matrix, B is $n \times m$ matrix, then

$$\det(AB) = \sum_{\substack{S \subseteq [n] \\ |S|=m}} \det(A_s)\det(B_s) = \sum \det(A_s B_s),$$

where A_s is the submatrix of A using columns in S, and B_s is the submatrix of B using rows in S.

Corollary 4.80
K_n has n^{n-2} spanning trees.