2020 HMMT Lectures

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February 16, 2020

These are a collection of notes (typed live) of all 2020 HMMT lectures.

Contents

§1 Complex Analysis: Contour Integrals (Lecturer: Andrew Lin)

Our goal in this lecture is to explain the following:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{e}.$$

§1.1 Prerequisites

1. What is a derivative? Understand the meaning of $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.
2. What is an integral?

$$\int_a^b f(x)\,dx = \text{“adding up the little parts of } f(x) \text{ from } a \text{ to } b''.$$

3. Understand the Fundamental Theorem of Calculus.

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ if } F \text{ is the antiderivative (integral) of } f.$$

§1.2 Complex Numbers

Consider the mapping from $x \in \mathbb{C} \to f(x) \in \mathbb{C}$. Note that x has two dimensions: the real part and the imaginary part.

We define the derivative (in complex numbers) in the same way as with real numbers:

$$f'(x) = \frac{f(x+h)-f(x)}{h}.$$

But consider $f(x) = \Re(x)$. Considering the function along the real and imaginary axes, the function's derivative assumes different values, so it does not have a well-defined derivative in the complex numbers.

However, we are perfectly fine with functions like $f(x) = x^2, e^x, \sin x$.

Note two important facts:

1. If a function is differentiable, then it is infinitely differentiable.
2. If so, we can express $f(x)$ as a power series:

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

Because we are working with the real and imaginary axes, we actually care about the exact path we take from one complex number to another.

Definition 1.1. A **contour** is a path in the complex plane.

Consider a curve C. Then we represent the integral along the curve as

$$\int_C f(x)\,dx.$$

If the "curve" is simply a line from 0 to 5, then we would simply have

$$\int_0^5 f(x)\,dx.$$

Now consider the curve that is the unit circle, starting at 1 (counterclockwise) and ending at 1. Suppose we want to evaluate

$$\int_C \frac{1}{z}\,dz.$$

The key is to *parameterize the curve.* On the complex plane, the curve is e^{it} where $0 \le t \le 2\pi$. Let $x = e^{it}$, such that $dx = ie^{it}\,dt$. Then we have

$$\int_C \frac{1}{x}\,dx = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}\,dt = \int_0^{2\pi} 1\,dt = \boxed{2\pi i}.$$

What is special about this particular integral is that we started and ended at the same point and we evaluated an integral that is not zero. Integrating "easier" functions like x^n gives us zero.

This particular integral was not zero because of the pole (or "bad point") at 0, which the "circle" curve contains (however, this is beyond the scope of this lecture).

If a contour contains a "bad point," we can write the function as a power series

$$f(x) = \frac{C_{-2}}{x^2} + \frac{C_{-1}}{x} + C_0 + C_1 x + C_2 x^2 + \cdots$$

If we integrate each part of this power series, we get that the integration of every term is 0, EXCEPT the term $\frac{C_{-1}}{x}$, which we have already evaluated to be $C_{-1} 2\pi i$.

If we shift the pole around, such as the function $g(x) = \frac{1}{\sin(x-1)}$, we can write a power series

$$g(x) = \frac{d_{-1}}{x-1} + d_0 + d_1(x-1) + \cdots$$

When we integrate this over any contour containing the pole at $x = 1$, the answer would be $d_{-1}2\pi i$.

Now let's return to the integral:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx.$$

Consider the contour that is a "clockwise semicircle" with the base on the real axis. Consider $f(x) = \frac{\cos x}{x^2+1}$. We will instead consider

$$I = \int_C \frac{e^{ix}}{x^2+1}\,dx = \int_{\text{real}} \frac{e^{ix}}{x^2+1}\,dx + \int_{\text{arc}} \frac{e^{ix}}{x^2+1}\,dx.$$

As the radius of our semicircle gets bigger, note that

$$\int_{\text{arc}} \frac{e^{ix}}{x^2+1}\,dx$$

goes to 0. Then we just need to worry about the *base* of the semicircle, i.e.

$$\int_{\text{real}} \frac{e^{ix}}{x^2+1}\,dx,$$

which contributes to the value of the overall integral.

Note that $\frac{e^{ix}}{x^2+1}$ has a pole at $x = i$, so consider the power series

$$f(x) = \frac{a_{-1}}{x-i} + a_0 + a_1(x-i) + a_2(x-i)^2.$$

Since $f(x) = \frac{e^{ix}}{x^2+1} = \frac{e^{ix}}{(x-i)(x+i)}$, the degree of $x - i$ in the denominator is 1, so we just need to start at the term with a_{-1}.

Now multiply the power series by $x - i$ to get

$$(x-i)f(x) = a_{-1} + a_0(x-i) + a_1(x-i)^2 + \cdots,$$

i.e.

$$\frac{e^{ix}}{x+i} = a_{-1} + a_0(x-i) + a_1(x-i)^2 + \cdots$$

Now plug in $x = i$, so we get

$$\frac{1}{e \cdot 2i} = a_{-1}.$$

Finally, we obtain

$$I = a_{-1} \cdot 2\pi i = \boxed{\frac{\pi}{e}},$$

as desired.

§2 Advanced Integration Techniques (Lecturer: Jeffrey Yu)

Note that

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a}.$$

By the Leibniz Integral Rule,

$$\frac{\partial}{\partial a}\int_0^\infty e^{-ax}\,dx = \int_0^\infty \frac{\partial}{\partial a}e^{-ax}\,dx = \int_0^\infty -xe^{-ax}\,dx = -\frac{1}{a^2}.$$

So, we have

$$\int_0^\infty xe^{-ax}\,dx = \frac{1}{a^2}.$$

We can apply this same technique to obtain

$$\int_0^\infty x^2e^{-ax}\,dx = \frac{2}{a^3}, \int_0^\infty x^3e^{-ax}\,dx = \frac{2\cdot 3}{a^4}, \cdots$$

and so on. Note that a standard method of integration by parts would be rather cumbersome compared to using the Leibniz Integral Rule.

We can inductively prove the following result:

Proposition 2.1
For $n \in \mathbb{N}$,

$$\int_0^\infty x^ne^{-ax}\,dx = \frac{n!}{a^{n+1}}.$$

Definition 2.2. The **Gamma Function** is

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt.$$

If $n \in \mathbb{N}$, then

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt = (n-1)!.$$

If we want to integrate the product of a polynomial and an exponential, we can perform a u-substitution with the expression in the exponent (not including the negative sign) to convert the integral into something that looks like the Gamma Function.

For example, consider

$$\int_0^\infty x^{2019}e^{-2020x}\,dx.$$

Let $u = 2020x$, such that $du = 2020dx$. Then, our u-substitution gives

$$\int_0^\infty \left(\frac{u}{2020}\right)^{2019} e^{-u}\,\frac{du}{2020} = \frac{1}{2020^{2020}}\int_0^\infty u^{2019}e^{-u}\,du = \frac{2019!}{2020^{2020}}.$$

It turns out that the Gamma Function can exist for all complex numbers, except the negative integers.

Recall $n! = n(n-1)!$. This suggests that $\Gamma(n+1) = n\Gamma(n)$.

Proposition 2.3

$$\Gamma(z+1) = z\Gamma(z).$$

Proof. We have
$$\Gamma(n+1) = \int_0^\infty t^n e^{-t}\, dt.$$
Perform a substitution by parts with $u = t^n$, $dv = e^{-t}\, dt$, and $du = nt^{n-1}\, dt$, $v = -e^{-t}$. We end up with
$$\int_0^\infty t^n e^{-t}\, dt = -t^n e^{-t} - \int_0^\infty (-e^{-t}) nt^{n-1}\, dt = n\int_0^\infty t^{n-1} e^{-t}\, dt = n\Gamma(n). \qquad \square$$

Theorem 2.4 (Reflection Formula)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Sketch of Proof. We use infinite products. Note that
$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right),$$
and
$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n. \qquad \square$$

Using this result, let $z = \frac{1}{2}$. Then $\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}) = \frac{\pi}{\sin(\frac{\pi}{2})} = \pi$. Thus,
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$
since $\Gamma(\frac{1}{2})$ is positive by the definition of the Gamma Function. This also gives the surprising fact,
$$\left(\frac{1}{2}\right)! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Now consider
$$\int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

We have
$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2}\, dx &= 2\int_0^\infty e^{-x^2}\, dx \\ &= 2\int_0^\infty e^{-u}\frac{1}{2}u^{-\frac{1}{2}}\, du,\end{aligned}$$
using the u-substitution $u = x^2 \implies dx = \frac{1}{2\sqrt{u}}\, du$.

Then, this equals
$$\int_0^\infty u^{-\frac{1}{2}} e^{-u}\, du = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Definition 2.5. The **beta function** is

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

Now let's segue into combinatorics. Suppose we have r red balls and g green balls, and suppose we choose a random ball at a random time $t \in [0,1]$. Then, the probability that all the red balls are chosen before a time t and all the green balls are chosen after a time t is

$$\int_0^1 t^r(1-t)^g\,dt.$$

However, we can also calculate the probability as

$$\frac{r!g!}{(r+g)!}.$$

These two probabilities are equal, giving us the identity:

Proposition 2.6

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

We have another formula that we won't prove in this lecture:

Proposition 2.7

$$\beta(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta.$$

Then, we have

$$\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta = \frac{1}{2}\beta(x,y) = \frac{(x-1)!(y-1)!}{2(x+y-1)!}.$$

Consider the example

$$\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta.$$

Normally we'd use the double angle formula for cosine, but we can apply the identities above that are related to the Beta Function.

We convert this to an expression involving the Beta Function:

$$\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \frac{1}{2}\beta\left(\frac{1}{2},\frac{3}{2}\right) = \frac{1}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)} = \frac{1}{2}\frac{\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{1} = \boxed{\frac{\pi}{4}}.$$

A slightly unrelated but still cool fact: we can quickly compute

$$\int_0^{2\pi} \cos^2 x\,dx = \int_0^{2\pi} \sin^2 x\,dx = \pi,$$

by noticing that from $x=0$ to 2π, the areas under the graphs of $\sin^2 x$ and $\cos^2 x$ are the same. Since

$$\int_0^{2\pi} \cos^2 x\,dx + \int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} 1\,dx = 2\pi,$$

we can conclude that each of $\int_0^{2\pi} \cos^2 x\,dx$ and $\int_0^{2\pi} \sin^2 x\,dx$ is π.