Misc. Problems on Sequences, Series, Limits, and Induction

Daniel Kim

Problems:

1. Compute $\sum_{i=1}^{4} \left(\sum_{j=1}^{3} i + j \right)$.

2. The sum of an infinite geometric series is 10. The sum of the same series but with each of its terms squared is 12. What is its fifth term?

3. Prove $\lim_{n\to\infty} \frac{2n+5}{3n-7} = \frac{2}{3}$ using the definition of a limit.

4. Prove $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n}$ by induction.

5. Let $f(n) = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)}$.

   i) Compute $f(n)$ for $n = 1, 2, 3, 4, 5$.
   ii) Hypothesize a formula.
   iii) Prove your formula by induction.
   iv) Compute $\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)}$.

6. Let $f(n) = \prod_{k=1}^{n} \left(1 - \frac{1}{k+1}\right)$.

   i) Compute $f(n)$ for $n = 1, 2, 3, 4, 5$.
   ii) Hypothesize a formula.
   iii) Prove your formula by induction.
   iv) Compute $\prod_{k=1}^{\infty} \left(1 - \frac{1}{k+1}\right)$.

7. When expanded, what is the full term of the form $kx^{1776}y^r$ in $(4x - 5y)^{2017}$?

8. If $\lim_{n\to\infty} a_n = L$, prove $\lim_{n\to\infty} 2a_n = 2L$ using the definition of a limit.

9. Find all sequences a, b, c such that a, b, c and $a + 1, b + 1, c + 1$ are both geometric.

10. Compute $\sum_{k=10}^{20} k^2$ in two ways by:

a) Subtracting two sums.

b) Reindexing the summation starting from 1 and breaking the sum into parts.

11. In any finite arithmetic sequence, prove that the median is equal to the average of the first term and last term of the sequence.

Solutions:

1. $$\begin{aligned}\sum_{i=1}^{4}\left(\sum_{j=1}^{3} i+j\right) &= \sum_{j=1}^{3} 1+j+\sum_{j=1}^{3} 2+j+\sum_{j=1}^{3} 3+j+\sum_{j=1}^{3} 4+j \\ &= (2+3+4)+(3+4+5)+(4+5+6)+(5+6+7) \\ &= 9+12+15+18 \\ &= \boxed{54}\end{aligned}$$

2. $$\begin{aligned} a+ar+ar^2+\ldots = \frac{a}{1-r} &= 10 \\ a^2+a^2r^2+a^2r^4+\ldots = \frac{a^2}{1-r^2} &= 12\end{aligned}$$

   The first equation gives $a = 10-10r$, and the second gives $a^2 = 12-12r^2$. Therefore, we can solve for r with the equation:

   $$12-12r^2 = (10-10r)^2$$

   We end up with the quadratic $14r^2-25r+11=0$, which can be factored into $(r-1)(14r-11)=0$, giving $r=1,\ \frac{11}{14}$. Since it is an infinite geometric sequence, $|r|<1$, so the common ratio must be $\frac{11}{14}$.

   Therefore $a = 10-10\cdot\frac{11}{14} = \frac{15}{7}$, so the fifth term, which is ar^4, would be $\boxed{\frac{15}{7}\cdot\left(\frac{11}{14}\right)^4}$.

3. Let $N = \frac{29+21\varepsilon}{9\varepsilon}$. Note that $n>N$ suggests:

   $$\begin{aligned} 3n &> 3N \\ 3n-7 &> 3N-7 \\ \frac{1}{3n-7} &< \frac{1}{3N-7} \\ \therefore \frac{\frac{29}{3}}{3n-7} &< \frac{\frac{29}{3}}{3N-7}\end{aligned}$$

   Note that

   $$\left|\frac{2n+5}{3n-7}-\frac{2}{3}\right| = \left|\frac{2n+5}{3n-7}-\frac{2n-\frac{14}{3}}{3n-7}\right| = \left|\frac{2n+5-\left(2n-\frac{14}{3}\right)}{3n-7}\right| = \left|\frac{\frac{29}{3}}{3n-7}\right| = \frac{\frac{29}{3}}{3n-7}$$

   Using the results above, and based on the definition of a limit, if $n>N$, then

   $$\left|\frac{2n+5}{3n-7}-\frac{2}{3}\right| = \frac{\frac{29}{3}}{3n-7} < \frac{\frac{29}{3}}{3N-7} = \frac{\frac{29}{3}}{3\left(\frac{29+21\varepsilon}{9\varepsilon}\right)-7} = \varepsilon$$

   In other words, for any ε, we have found an N such that for all $n>N$, $\left|\frac{2n+5}{3n-7}-\frac{2}{3}\right|<\varepsilon$. Therefore, $\lim\limits_{n\to\infty}\frac{2n+5}{3n-7} = \frac{2}{3}$.

4. Proceed by induction.

   Let $P(n) : \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n}$.

   <u>Base Case</u>: $P(1) : \sum_{k=1}^{1} \frac{1}{\sqrt{k}} = 1$.

   1 is less than $2\sqrt{1} = 2$, so the base case is true.

   <u>Inductive Step</u>: Suppose $P(n) : \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n}$ is true.

   We want to prove: $P(n+1) : \sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} < 2\sqrt{n+1}$.

   By assumption we have:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

   Add $\frac{1}{\sqrt{n+1}}$ to both sides.

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}}$$

   We need to use the fact that $2\sqrt{n} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1}$, which can be proved as follows:

   Proof. Note that $n \in \mathbb{Z}^{+}$. Then,

$$\begin{aligned}
\left(\sqrt{n} - \sqrt{n+1}\right)^2 &> 0 \\
n - 2\sqrt{n(n+1)} + n + 1 &> 0 \\
2\sqrt{n(n+1)} &< 2n + 1 \\
2\sqrt{n(n+1)} + 1 &< 2n + 2 \\
2\sqrt{n(n+1)} + 1 &< 2(n+1) \\
2\sqrt{n}\sqrt{n+1}\left(\frac{1}{\sqrt{n+1}}\right) + 1\left(\frac{1}{\sqrt{n+1}}\right) &< 2(n+1)\left(\frac{1}{\sqrt{n+1}}\right) \\
2\sqrt{n} + \frac{1}{\sqrt{n+1}} &< 2(n+1)\left(\frac{\sqrt{n+1}}{n+1}\right) \\
\therefore 2\sqrt{n} + \frac{1}{\sqrt{n+1}} &< 2\sqrt{n+1}
\end{aligned}$$

$\square$

   Therefore, using this fact,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1}$$

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}<2\sqrt{n+1}$$

which concludes the inductive step, so our proof by induction is complete.

5. i)
   - $f(1)=\frac{1}{1\cdot 3}=\boxed{\frac{1}{3}}$
   - $f(2)=\frac{1}{3}+\frac{1}{3\cdot 5}=\boxed{\frac{2}{5}}$
   - $f(3)=\frac{2}{5}+\frac{1}{5\cdot 7}=\boxed{\frac{3}{7}}$
   - $f(4)=\frac{3}{7}+\frac{1}{7\cdot 9}=\boxed{\frac{4}{9}}$
   - $f(5)=\frac{4}{9}+\frac{1}{9\cdot 11}=\boxed{\frac{5}{11}}$

   ii) Noticing the pattern, a possible formula is: $f(n)=\boxed{\frac{n}{2n+1}}$.

   iii) We must prove the statement: $\sum\limits_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\frac{n}{2n+1}$.

   We proceed by induction. Let $P(n):\sum\limits_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\frac{n}{2n+1}$.

   <u>Base Case</u>: $P(1):\sum\limits_{k=1}^{1}\frac{1}{(2k-1)(2k+1)}=\frac{1}{(2\cdot 1-1)(2\cdot 1+1)}=\frac{1}{3}$

   $\frac{n}{2n+1}$ where $n=1$ is $\frac{1}{3}$, so the base case is true.

   <u>Inductive Step</u>: Assume $P(n):\sum\limits_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\frac{n}{2n+1}$ is true.

   We want to prove: $P(n+1):\sum\limits_{k=1}^{n+1}\frac{1}{(2k-1)(2k+1)}=\frac{n+1}{2n+3}$.

$$\begin{aligned}&\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)(2n+1)}+\frac{1}{(2n+1)(2n+3)}\\&=\frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)}\\&=\frac{n(2n+3)+1}{(2n+1)(2n+3)}\\&=\frac{2n^2+3n+1}{(2n+1)(2n+3)}\\&=\frac{(2n+1)(n+1)}{(2n+1)(2n+3)}\\&=\frac{n+1}{2n+3}\end{aligned}$$

We have proved $P(n+1) : \sum_{k=1}^{n+1} \frac{1}{(2k-1)(2k+1)} = \frac{n+1}{2n+3}$, so our proof by induction is complete.

iv) $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)}$$
$$= \lim_{n\to\infty} \frac{n}{2n+1}$$
$$= \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}}$$
$$= \boxed{\frac{1}{2}}$$

6. i)

- $f(1) = \left(1-\frac{1}{2}\right) = \boxed{\frac{1}{2}}$
- $f(2) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) = \frac{1}{2}\cdot\frac{2}{3} = \boxed{\frac{1}{3}}$
- $f(3) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4} = \boxed{\frac{1}{4}}$
- $f(4) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5} = \boxed{\frac{1}{5}}$
- $f(5) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{6}\right) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6} = \boxed{\frac{1}{6}}$

ii) The pattern of the first five terms suggests that the formula is: $f(n) = \boxed{\frac{1}{n+1}}$.

iii) The statement we must prove is: $\prod_{k=1}^{n} \left(1-\frac{1}{k+1}\right) = \frac{1}{n+1}$.

Let $P(n) : \prod_{k=1}^{n} \left(1-\frac{1}{k+1}\right) = \frac{1}{n+1}$.

Base Case: $P(1) : \prod_{k=1}^{1} \left(1-\frac{1}{k+1}\right) = 1-\frac{1}{2} = \frac{1}{2}$

$\frac{1}{n+1}$ where $n=1$ is $\frac{1}{2}$, so the base case is true.

Inductive Step: Assume $P(n) : \prod_{k=1}^{n} \left(1-\frac{1}{k+1}\right) = \frac{1}{n+1}$.

We want to prove: $P(n+1) : \prod_{k=1}^{n+1} \left(1-\frac{1}{k+1}\right) = \frac{1}{n+2}$.

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n+1}\right)\left(1-\frac{1}{n+2}\right) = \left(\frac{1}{n+1}\right)\left(1-\frac{1}{n+2}\right)$$
$$= \left(\frac{1}{n+1}\right)\left(\frac{n+1}{n+2}\right)$$

$$= \frac{1}{n+2}$$

So the statement for $P(n+1)$ is true, which concludes the proof by induction.

iv) $$\prod_{k=1}^{\infty}\left(1-\frac{1}{k+1}\right) = \lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\frac{1}{k+1}\right)$$
$$= \lim_{n\to\infty}\frac{1}{n+1}$$
$$= \boxed{0}$$

7. Recall the Binomial Theorem, which states that a general term in a binomial expansion to the n power is of the form
$$\binom{n}{k}x^{n-k}y^k$$
Clearly $n = 2017$ and $n - k = 1776$, so $k = 241$. Therefore the term we have to find is:
$$\binom{2017}{241}(4x)^{1776}(-5y)^{241}$$
$$\boxed{-\binom{2017}{241}\left(4^{1776}\right)\left(5^{241}\right)x^{1776}y^{241}}$$

8. Consider the definition of $\lim_{n\to\infty} a_n = L$:
$$\forall\widetilde{\varepsilon} > 0,\ \exists N \text{ s.t. } \forall n > N,\ |a_n - L| < \widetilde{\varepsilon}$$
Choose $\varepsilon = \frac{\widetilde{\varepsilon}}{2}$. Then,
$$\forall\varepsilon > 0,\ \exists N \text{ s.t. } \forall n > N,\ |a_n - L| < \varepsilon$$
$$\text{i.e. } |a_n - L| < \frac{\widetilde{\varepsilon}}{2}$$
$$\text{i.e. } 2|a_n - L| < \widetilde{\varepsilon}$$
$$\text{i.e. } |2||a_n - L| < \widetilde{\varepsilon}$$
$$\text{i.e. } |2(a_n - L)| < \widetilde{\varepsilon}$$
$$\text{i.e. } |2a_n - 2L| < \widetilde{\varepsilon}$$
which implies $\lim_{n\to\infty} 2a_n = 2L$.

9. If a, b, c is geometric, then we know by the geometric mean that $b = \sqrt{ac}$, or $b^2 = ac$. Similarly, $(b+1)^2 = (a+1)(c+1)$.
$(b+1)^2 = (a+1)(c+1)$ simplifies to $b^2 + 2b = ac + a + c$. But we know that $b^2 = ac$, so $b^2 + 2b = b^2 + a + c$, yielding $b = \frac{a+b}{2}$.

This implies that a, b, c is an arithmetic sequence. The only possible solution for a sequence that is both geometric and arithmetic is $\boxed{a = b = c}$ (when all terms are equal to each other).
We can also get this result by letting $a = b - d$ and $c = b + d$, for some common difference d.
So $b^2 = ac$ becomes $b^2 = (b-d)(b+d)$, so $d = 0$, which implies the same result.

10. a) The sum of $10^2 + 11^2 + \ldots + 20^2$ is the same as subtracting the sum $1^2 + 2^2 + \ldots + 9^2$ from the total sum $1^2 + 2^2 + \ldots + 20^2$, making this more manageable with the formulas we already know:

$$\sum_{k=10}^{20} k^2 = \sum_{k=1}^{20} k^2 - \sum_{k=1}^{9} k^2 = \frac{20 \cdot 21 \cdot 41}{6} - \frac{9 \cdot 10 \cdot 19}{6} = 2870 - 285 = \boxed{2585}$$

b) Note that we can reindex the sum as:

$$\sum_{k=10}^{20} k^2 = \sum_{k=1}^{11} (k+9)^2 = \sum_{k=1}^{11} k^2 + 18k + 81$$

We can split $\sum_{k=1}^{11} k^2 + 18k + 81$ into a sum of partial sums, which we are able to compute individually using our formulas:

$$\begin{aligned}\sum_{k=1}^{11} k^2 + 18k + 81 &= \sum_{k=1}^{11} k^2 + 18 \sum_{k=1}^{11} k + \sum_{k=1}^{11} 81 \\ &= \frac{11 \cdot 12 \cdot 23}{6} + 18 \left(\frac{11 \cdot 12}{2} \right) + 11(81) \\ &= 506 + 1188 + 891 = \boxed{2585}.\end{aligned}$$

11. Consider an arithmetic sequence of n terms: $a, a+d, a+2d, \ldots, a+(n-1)d$.
Then the average of the first and last term is:

$$\frac{a + (a + (n-1)d)}{2} = \frac{2a + (n-1)d}{2} = a + \frac{n-1}{2} d$$

We must show that the median is equal to this. We can find the median by considering the parity of n:

a) If n is odd, then the median term is equally distant from the 1^{st} term and the n^{th} term. There would be $n-3$ terms to consider after the first, median, and last terms, so the distance between the median and the respective ends of the sequence is $\frac{n-3}{2}$ each. In other words, the sequence can be represented as follows:

$$1^{\text{st}}\text{term}, \underbrace{\ldots}_{\frac{n-3}{2}\text{ terms}}, \text{median term}, \underbrace{\ldots}_{\frac{n-3}{2}\text{ terms}}, n^{\text{th}}\text{ term}$$

Therefore the median is at the $a + \left(\frac{n-3}{2} + 1\right) d$ term, or $a + \frac{n-1}{2}d$, which is the average, so the median is equal to the average.

b) If n is even, then the median is the average of the pair of consecutive terms which is equidistant from the 1^{st} term and the n^{th} term. Similar to the odd case, we have a sequence in the form:

$$1^{\text{st}}\text{term}, \quad \underbrace{\ldots}_{\frac{n-4}{2}\text{ terms}}, \; x \text{ term}, \; y \text{ term}, \quad \underbrace{\ldots}_{\frac{n-4}{2}\text{ terms}}, \; n^{\text{th}} \text{ term}$$

The x and y terms are the pair of consecutive terms in discussion. The x term is $a + \left(\frac{n-4}{2} + 1\right) d$ and the y term is $a + \left(\frac{n-4}{2} + 2\right) d$, so the median is the average of them:

$$\frac{a + \left(\frac{n-4}{2} + 1\right) d + a + \left(\frac{n-4}{2} + 2\right) d}{2} = \frac{2a + (n - 4 + 3)\, d}{2} = a + \frac{n-1}{2} d$$

which is the average of the 1^{st} and n^{th} term.