Introduction to Circles

BCA Math Team AMC Lecture

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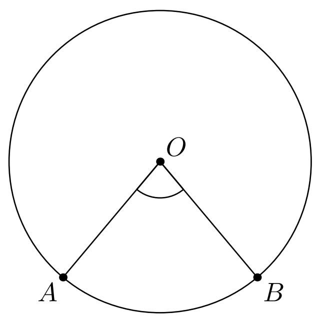
§1 Inscribed Angles

Consider the figure below, where O is the center of a circle that contains points A and B.



Definition 1.1. In the figure above, $\angle AOB$ is a **central angle** that is **subtended** by the **arc** $\widehat{\mathrm{AB}}$.

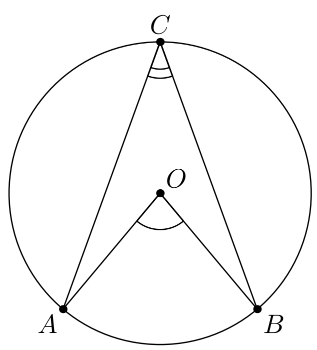
How do we measure how wide or narrow an arc spans? In fact, this "distance" is entirely determined by the central angle.

Definition 1.2. The central angle $\angle AOB$ is the **angular distance** of $\widehat{\mathrm{AB}}$.

Henceforth, we will let $\widehat{\mathrm{AB}}$ also refer to the angular distance of the arc, and thus we can write

$$m\angle AOB = \widehat{\mathrm{AB}}.$$

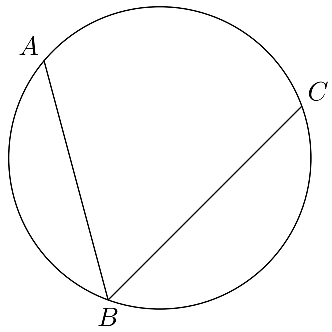
But what if we considered an angle formed by three points that all lie on the circle?



Definition 1.3. $\angle ACB$ is **inscribed** in arc $\widehat{\mathrm{AB}}$, and $\angle ACB$ is called an **inscribed angle**.

Inscribed angles, while simple, are incredibly important and useful in geometry. One of the main reasons for this is due to the following significant result:

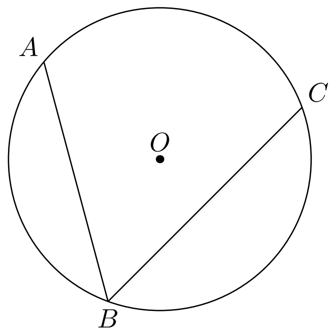
Theorem 1.4 (Inscribed Angle Theorem)
The measure of an inscribed angle is one-half the measure of the arc it intercepts.



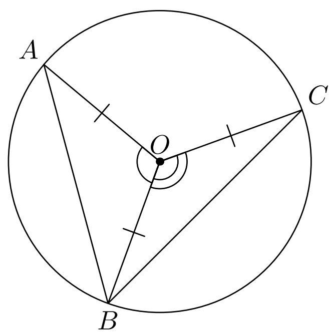
In other words, based on the figure above, $m\angle ABC = \frac{1}{2}\overset{\frown}{\text{AC}}$.

Proof. Let O be the center of the circle, with points A, B, C on the circle. There are many cases we have to consider.

1. Case 1:



How do we start? Looking at the theorem statement, we see that the measure of $\overset{\frown}{\text{AC}}$ is mentioned. So far, the only fact we know that is related to the measure of the arc is that it is equal to the subtended central angle, which must be $\angle AOC$ in this context. Thus, the center of the circle must be closely related to the three points. Then, draw lines OA, OB, and OC so we can get a clearer picture of what is going on.



Now, we start by noticing a basic property of the circle. Since OA, OB, and OC are radii of the circle, they are all equal. Then, $\triangle AOB$ and $\triangle COB$ are isosceles triangles. By the Isosceles Triangle Theorem, $\angle OAB \cong \angle OBA$ and $\angle OCB \cong \angle OBC$.

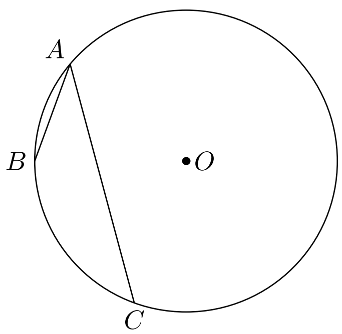
Now we introduce an important technique: **angle chasing**. We initially assign unknown variables to some angles and then try to determine the values of the rest of the angles in the diagram in terms of those unknown variables using our knowledge of geometry.

Let $m\angle AOB = \alpha$ and $m\angle COB = \beta$. Then $m\angle OAB = m\angle OBA = \frac{180-\alpha}{2}$ and $m\angle OCB = m\angle OBC = \frac{180-\beta}{2}$. Therefore,

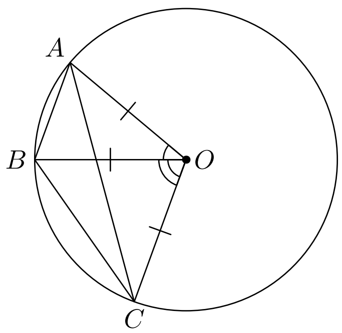
$$m\angle ABC = m\angle ABO + m\angle OBC = \frac{180-\alpha}{2} + \frac{180-\beta}{2} = \frac{360-\alpha-\beta}{2}.$$

How do we relate this to the measure of $\overset{\frown}{\mathrm{AC}}$? Note that $\angle AOC$ is a central angle, so $\overset{\frown}{\mathrm{AC}} = m\angle AOC$. But $m\angle AOC = 360 - m\angle AOB - m\angle COB = 360 - \alpha - \beta$. Therefore, $m\angle ABC = \frac{1}{2}\overset{\frown}{\mathrm{AC}}$.

2. Case 2: Here, we want to prove that $\angle BAC = \frac{1}{2}\overset{\frown}{\mathrm{BC}}$.



Again, we draw the three radii OA, OB, and OC. Noticing that $\triangle AOB$ is now an isosceles triangle, it would also help to draw in BC since $\triangle BOC$ would also be isosceles:



Again, we do some angle chasing: let $m\angle AOB = \alpha$ and $m\angle BOC = \beta$. Then the Isosceles Triangle Theorem gives us that $m\angle OAB = m\angle OBA = \frac{180-\alpha}{2}$ and $m\angle OBC = m\angle OCB = \frac{180-\beta}{2}$.

Keep our eye on the prize: showing the relation between $\angle BAC$ and $\overset{\frown}{\mathrm{BC}}$. How do we find the measure of $\angle BAC$?

Notice that line AC divides $\angle OAB$ (whose value we already know) into angles $\angle BAC$ (which is what we're looking for) and $\angle OAC$. Then, we just have to find what $m\angle OAC$ is to solve for $m\angle BAC$.

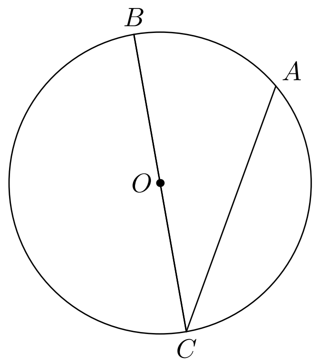
It is important to realize that $\angle OAC$ is part of a larger triangle, namely $\triangle OAC$. Particularly, since $OA = OC$, $\triangle OAC$ is also an isosceles triangle with $m\angle AOC = m\angle AOB + m\angle BOC = \alpha + \beta$. Then, $m\angle OAC = m\angle OCA = \frac{180-(\alpha+\beta)}{2}$.

We now have the information necessary to finish the problem: note that $m\angle OAB = m\angle OAC + m\angle BAC$, i.e.

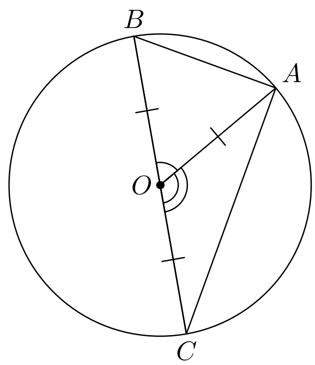
$$m\angle BAC = m\angle OAB - m\angle OAC = \frac{180-\alpha}{2} - \frac{180-(\alpha+\beta)}{2} = \frac{\beta}{2}.$$

The central angle $\angle BOC$ tells us that $\widehat{\mathrm{BC}} = \beta$, and therefore $m\angle BAC = \frac{1}{2}\widehat{\mathrm{BC}}$.

3. Case 3: Now consider the case when BC is a diameter of the circle.



Again, we fill in the rest of the lines so we can use isosceles triangles and angle chasing.



Let $m\angle AOB = \alpha$ and $m\angle AOC = \beta$, so that $m\angle BCA = \frac{180-\beta}{2}$. Note that $\widehat{\mathrm{BA}} = \alpha$ due to central angle $\angle AOB$.

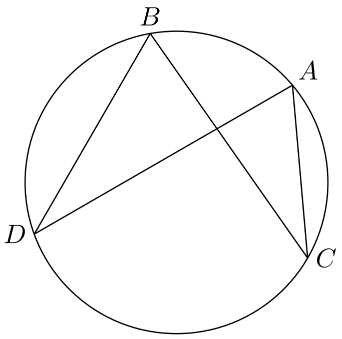
How do we show that these two quantities are equal? We need to find a relation between α and β. Observe that BOC is a straight line, so $m\angle AOB + m\angle AOC = \alpha + \beta = 180$. Thus, $\alpha = 180 - \beta$. Then, $m\angle BCA = \frac{\alpha}{2}$, proving that $m\angle BCA = \frac{1}{2}\widehat{\mathrm{BA}}$.

Furthermore, we can compute $m\angle BAC = m\angle BAO + m\angle CAO = \frac{180-\alpha}{2} + \frac{180-\beta}{2} = \frac{360-\alpha-\beta}{2}$.

But $\alpha + \beta = 180$, so $m\angle BAC = \frac{360-180}{2} = 90$, so $\angle BAC$ is actually a right angle! In fact, as long as BC is a diameter of the circle, $m\angle BAC$ is always 90. This is a special result called **Thales's Theorem**.

We have exhausted all cases, proving our general result. $\square$

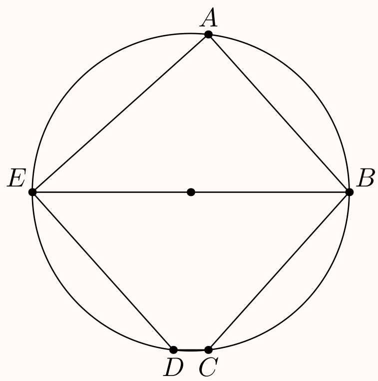
This result immediately tells us that two inscribed angles which intercept the same arc are equal:



In this figure, $\angle BDA \cong \angle ACB$ since they both intercept the same arc $\widehat{\mathrm{BA}}$.

Example 1.5 (2011 AMC 10B #17)

In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4:5$. What is the degree measure of angle BCD?



(A) 120 **(B)** 125 **(C)** 130 **(D)** 135 **(E)** 140

Solution. Since EB is a diameter of the circle, by Thales's Theorem, $m\angle EAB = 90$, so $m\angle AEB + m\angle ABE = 90$. But these angles are in the ratio $4:5$, therefore $m\angle AEB = 40$ and $m\angle ABE = 50$. Given that $AB \parallel ED$, $m\angle ABE = m\angle BED = 50$ by alternate interior angles.

We wish to find $m\angle BCD$, and currently we know $m\angle BED$. How do we relate these two quantities?

We can use inscribed angles. By the main theorem, $m\angle BED = \frac{1}{2}\widehat{\mathrm{BCD}}$, and $m\angle BCD = \frac{1}{2}\widehat{\mathrm{BEAD}}$. But notice that $\widehat{\mathrm{BCD}}$ and $\widehat{\mathrm{BEAD}}$ together constitute the entire circle! Therefore

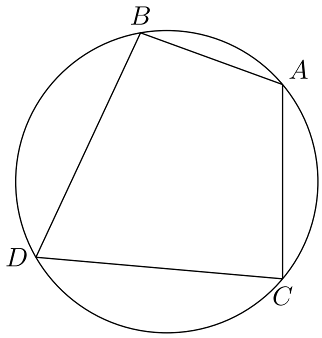
$\widehat{\text{BCD}} + \widehat{\text{BEAD}} = 360$, so

$$m\angle BCD = \frac{1}{2}\widehat{\text{BEAD}} = \frac{1}{2}(360 - \widehat{\text{BCD}}) = 180 - \frac{1}{2}\widehat{\text{BCD}} = 180 - m\angle BED = 130,$$

yielding $\boxed{C}$ as our answer. In fact, this solution was powerful enough that we did not even use the given fact that EB was parallel to DC.

In fact, $EBCD$ is a special kind of quadrilateral. Since of its vertices lie on the circle, we call $EBCD$ a **cyclic quadrilateral**. □

In general, given a cyclic quadrilateral $ABCD$ as shown,

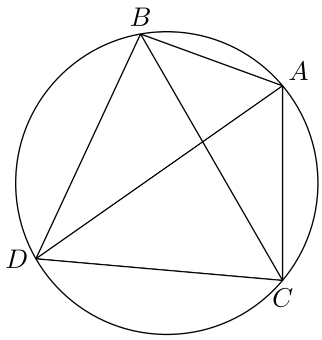


the opposite angles in a cyclic quadrilateral are supplementary (add up to 180), i.e.

$$m\angle BAC + m\angle CDB = m\angle DBA + m\angle ACD = 180.$$

Also, the backwards direction is also true: if opposite angles of a quadrilateral add up to 180, then the quadrilateral is cyclic.

Furthermore, if we draw the diagonals of the quadrilateral,

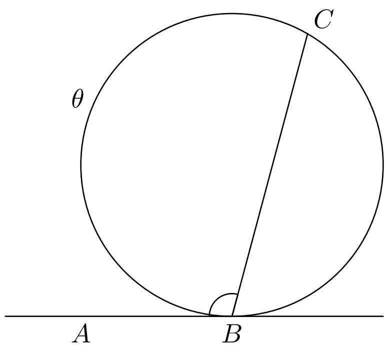


we obtain a plethora of angle congruences just using the Inscribed Angle Theorem. (Can you list them all?)

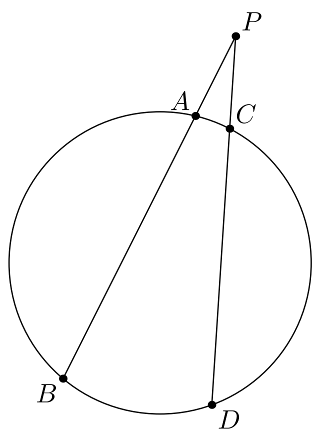
There are more properties to explore about cyclic quadrilaterals, but for now it remains important to just recognize these special quadrilaterals in circle geometry problems and to be able to angle chase effectively using inscribed angles.

Additionally, the Inscribed Angle Theorem brings about various important results:

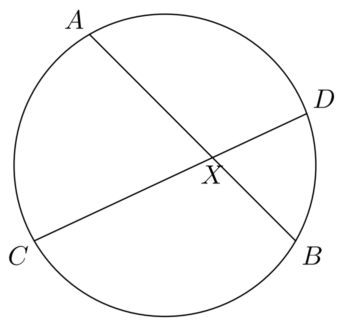
1. Given a tangent AB to the circle also containing point C, $m\angle ABC = \frac{\theta}{2}$.



2. Given a circle containing points A, B, C, D and a point P outside, $m\angle BPD = \frac{1}{2}\left(\widehat{\mathrm{BD}} - \widehat{\mathrm{AC}}\right)$.



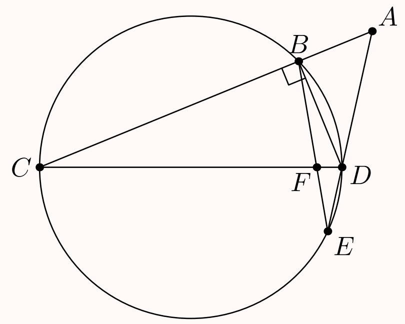
3. Given A, B, C, D on the circle and X inside, $m\angle AXC = m\angle BXD = \frac{1}{2}\left(\widehat{\mathrm{AC}} + \widehat{\mathrm{BD}}\right)$.



Exercise 1.6. Prove these results. (Hints: for 1, draw radii to B and C; for 2, draw either BC or AD; for 3, draw AD and BC).

Example 1.7

Consider the following diagram:



If $\angle CBD$ is a right angle, $m\angle CAE = 40$, $m\angle ACD = 30$, then find $m\angle CFE$ and $m\angle DBE$.

Solution. If $m\angle ACD = 30$, then $\overset{\frown}{\mathrm{BD}} = 60$. Furthermore, $m\angle CAE = \frac{1}{2}\left(\overset{\frown}{\mathrm{CE}} - \overset{\frown}{\mathrm{BD}}\right) = \frac{1}{2}\left(\overset{\frown}{\mathrm{CE}} - 60\right) = 40$. Then, $\overset{\frown}{\mathrm{CE}} = 140$. Therefore,

$$m\angle CFE = \frac{1}{2}\left(\overset{\frown}{\mathrm{CE}} + \overset{\frown}{\mathrm{BD}}\right) = \frac{1}{2}(140+60) = \boxed{100}.$$

To find $m\angle DBE$, it helps to find $\overset{\frown}{\mathrm{DE}}$. Now notice that if $\angle CBD$ is a right angle, then CD is a diameter of the circle. Then $\overset{\frown}{\mathrm{DE}} = 180 - \overset{\frown}{\mathrm{CE}} = 40$, so $m\angle DBE = \frac{1}{2} \cdot 40 = \boxed{20}$. □

Example 1.8 (Mandelbrot)

We are given points A, B, C, and D in the plane such that $AD = 13$ while $AB = BC = AC = CD = 10$. Find $\angle ADB$.

Solution. The key insight is that since $CA = CB = CD = 10$, points A, B, and D lie on the circle of radius 10 with center at point C. Then, using inscribed angles, to find $m\angle ADB$, we just have to find $\overset{\frown}{\mathrm{AB}}$. But notice that this subtends central angle $\angle ACB$. Thus, $m\angle ACB = \overset{\frown}{\mathrm{AB}}$. Now since $CA = CB = AB = 10$, $\triangle ABC$ is equilateral, so $m\angle ACB = 60$. Thus, $m\angle ADB = \frac{1}{2} \cdot 60 = \boxed{30}$. In fact, we didn't even use $AD = 13$ given in the problem statement. Just by recognizing a circle in the diagram, inscribed angles allowed us to arrive at the answer relatively quickly. □

§2 Power of a Point

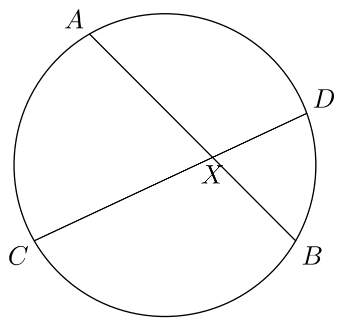
However, not all geometry problems only involve angles. Many problems ask for lengths of segments inside or outside circles, and using inscribed angles and similar triangles, we can derive another set of formulas to assist us in computing lengths.

Theorem 2.1 (Power of a Point Theorem)

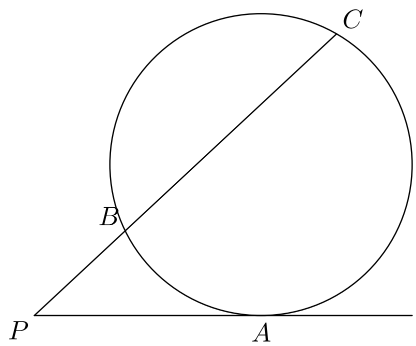
Suppose a line containing point P intersects a circle in two points, A and B. Then, $(PA)(PB)$ is constant for any line we choose.

The generality of this statement obscures its immediate impact, and can be better explained via three scenarios:

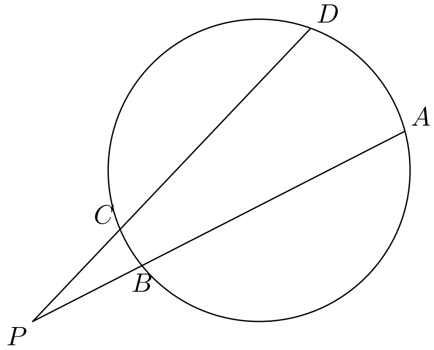
Theorem (Power of a Point, part 1)
In the diagram below, $AX \cdot BX = CX \cdot DX$.



Theorem (Power of a Point, part 2)
In the diagram below, $PA^2 = PB \cdot PC$.

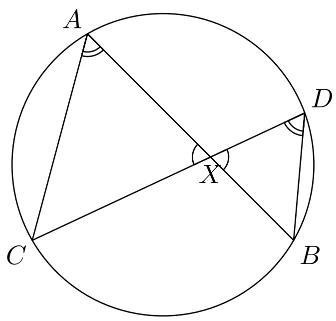


Theorem (Power of a Point, part 3)
In the diagram below, $PB \cdot PA = PC \cdot PD$.



Proof. We prove each case separately.

1. First draw AC and BD.

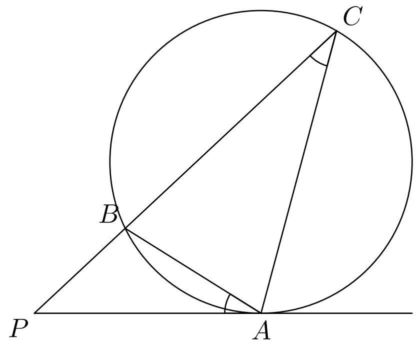


By vertical angles, $\angle AXC \cong \angle DXB$. Since $\angle XAC$ and $\angle XDB$ intercept the same arc $\widehat{\mathrm{BC}}$, $\angle XAC \cong \angle XDB$. Therefore, by AA similarity, $\triangle AXC \sim \triangle DXB$. Then,

$$\frac{AX}{DX} = \frac{CX}{BX},$$

which rearranges to $AX \cdot BX = CX \cdot DX$.

2. Draw AB and AC.



By reflexivity, $\angle BPA \cong \angle APC$. By inscribed angle, $m\angle PCA = \frac{1}{2}\widehat{\mathrm{AB}}$. Furthermore, $m\angle PAB = \frac{1}{2}\widehat{\mathrm{AB}}$, so $\angle PAB \cong \angle PCA$. Therefore, by SS similarity, $\triangle PBA \sim \triangle PAC$. Then,

$$\frac{PA}{PB} = \frac{PC}{PA},$$

so $PA^2 = PB \cdot PC$.

3. Draw a tangent to the circle (suppose the tanget was at point E) and apply part 2 of Power of a Point to both PA and PD. It follows that $PE^2 = PB \cdot PA = PC \cdot PD$ as desired. □

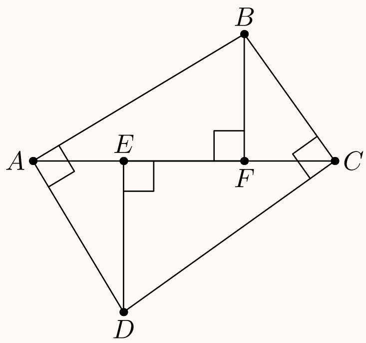
Example 2.2 (Source: HMMT)

Let A, B, C, and D be points on a circle such that $AB = 11$ and $CD = 19$. Point P is on segment AB with $AP = 6$, and Q is on segment CD with $CQ = 7$. The line through P and Q intersects the circle at X and Y. If $PQ = 27$, find XY.

Solution. Suppose X, P, Q, Y lie in that order. Let $PX = x$ and $QY = y$. By Power of a Point from P, $x \cdot (27 + y) = 30$, and by Power of a Point from Q, $y \cdot (27 + x) = 84$. Subtracting the first from the second, $27 \cdot (y - x) = 54$, so $y = x + 2$. Now, $x \cdot (29 + x) = 30$, and we find $x = 1, -30$. Since -30 makes no sense, we take $x = 1$ and obtain $XY = 1 + 27 + 3 = \boxed{31}$. □

Example 2.3 (1990 AHSME #20)

In the figure $ABCD$ is a quadrilateral with right angles at A and C.



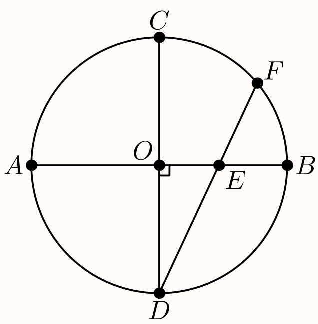
Points E and F are on $\overline{AC}$, and $\overline{DE}$ and $\overline{BF}$ are perpendicular to $\overline{AC}$. If $AE = 3, DE = 5$, and $CE = 7$, then the length of BF is

(A) 3.6 **(B)** 4 **(C)** 4.2 **(D)** 4.5 **(E)** 5

Solution. Since opposite angles $\angle A$ and $\angle C$ add to 180, $ABCD$ is a cyclic quadrilateral. Now extend DE to meet at point G. Since $\angle BAD$ and $\angle BGD$ intercept the same arc $\overset{\frown}{\mathrm{BCD}}$, $m\angle BGD = 90$. Therefore, $GB \parallel EF$, and $GE = BF$ because they are perpendiculars of parallel lines. By Power of a Point, $AE \cdot CE = DE \cdot GE$. Therefore, $3 \cdot 7 = 5 \cdot GE \Rightarrow GE = BF = \boxed{4.2}$. □

Example 2.4 (1995 AHSME #26)

In the figure, $\overline{AB}$ and $\overline{CD}$ are diameters of the circle with center O, $\overline{AB} \perp \overline{CD}$, and chord $\overline{DF}$ intersects $\overline{AB}$ at E. If $DE = 6$ and $EF = 2$, then the area of the circle is



(A) 23π **(B)** $\frac{47}{2}\pi$ **(C)** 24π **(D)** $\frac{49}{2}\pi$ **(E)** 25π

Solution. Let the radius of the circle be r, such that $AO = OB = r$. Since we are given the lengths of segments DE and EF, and lots of intersecting chords inside this circle, applying Power of a Point (particularly using our given segments) may be helpful.

Indeed, let $OE = x$ and therefore $BE = r - x$. Then Power of a Point yields $(r + x)(r - x) = 2 \cdot 6 = 12$. This gives us $r^2 - x^2 = 12$. Conveniently, the Pythagorean Theorem on $\triangle ODE$ gives us $x^2 + r^2 = 6^2 = 36$. Therefore, we solve

$$\begin{aligned} r^2 - x^2 &= 12, \\ r^2 + x^2 &= 36, \end{aligned}$$

to get solutions $r = 2\sqrt{6}$ and $x = 2\sqrt{3}$. Then our answer is $r^2\pi = 24\pi \Rightarrow \boxed{C}$. $\square$

§3 Going Beyond

We have barely scratched the surface of circle problems, and there are many ideas related to circles that are left to explore (and may cover in future lectures):

- internal and external tangents
- common internal tangents
- more properties of cyclic quadrilaterals
- circumscribed quadrilaterals
- incircle
- circumcircle

To practice with more problems or review the material covered, check out chapters 12 and 13 in *Introduction to Geometry* by Richard Rusczyk.

As a brief sidenote, the Power of a Point theorem can be generalized by introducing the following definitions:

Definition 3.1. If P is a point, and $k(O, r)$ is a circle with center O and radius r in the plane, then

$$\operatorname{pow}(P, k) = OP^2 - r^2$$

is called the **power** of P with respect to k.

Then, the various cases (two chords, secant-secant, tangent-chord) directly follow writing the power of a point in two different ways.

This generalized definition of the power yields interesting objects like the **radical axis** which have far-reaching applications in olympiad geometry.

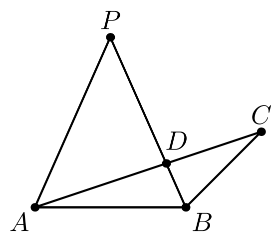
§4 Problem Set

§4.1 Easy Problems

1. Chords AB and CD of a given circle are perpendicular to each other and intersect at a right angle. Given that $BE = 16, DE = 4$, and $AD = 5$, find CE.

   Source: ARML

2. Triangle $\triangle ABC$ and point P in the same plane are given below.

   Point P is equidistant from A and B, angle APB is twice angle ACB, and $\overline{AC}$ intersects $\overline{BP}$ at point D. If $PB = 3$ and $PD = 2$, then $AD \cdot CD =$

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

   Source: 1997 AHSME #26

3. Two circles with centers P and Q intersect at points X and Y. Point A is located on $\overleftrightarrow{XY}$ such that $AP = 10$ and $AQ = 12$. If the radius of the circle centered at Q is 7, find the radius of the circle centered at P.

4. Square ABCD of side length 10 has a circle inscribed in it. Let M be the midpoint of $\overline{AB}$. Find the length of that portion of the segment $\overline{MC}$ that lies outside of the circle.

5. In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

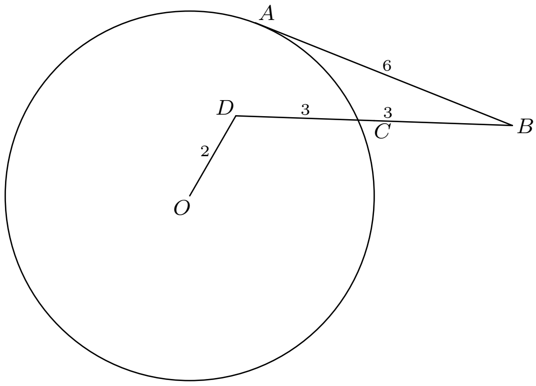
   (A) 11 **(B)** 28 **(C)** 33 **(D)** 61 **(E)** 72

   Source: 2013 AMC 10A #23

§4.2 Medium Problems

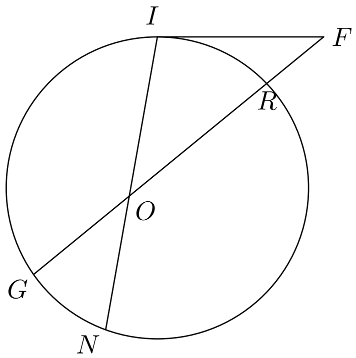
1. In the adjoining figure, AB is tangent at A to the circle with center O; point D is interior to the circle; and DB intersects the circle at C. If $BC = DC = 3$, $OD = 2$, and $AB = 6$, then the radius of the circle is

   (A) $3 + \sqrt{3}$ **(B)** $15/\pi$ **(C)** $9/2$ **(D)** $2\sqrt{6}$ **(E)** $\sqrt{22}$



Source: 1976 AHSME #18

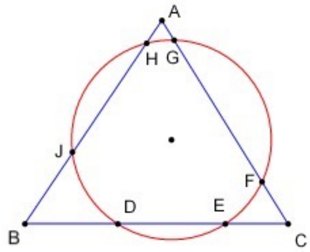
2. Segment $\overline{IF}$ is tangent to the circle at point I as shown in the diagram below. We are given that $IF = 21\sqrt{2}$, $IO = 20$, $ON = 12$, $RF = 18$, and $OR > GO$. Find OR.



Source: Mandelbrot

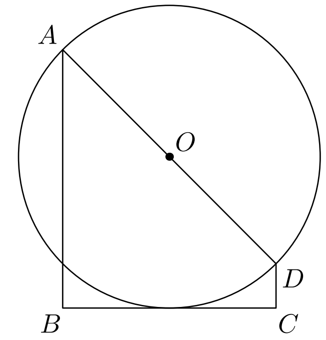
3. In the adjoining figure, the circle meets the sides of an equilateral triangle at six points. If $AG = 2, GF = 13, FC = 1$ and $HJ = 7$, then DE equals

(A) $2\sqrt{22}$ **(B)** $7\sqrt{3}$ **(C)** 9 **(D)** 10 **(E)** 13



Source: 1982 AHSME #24

4. In the figure, $AB \perp BC, BC \perp CD$, and BC is tangent to the circle with center O and diameter AD. In which one of the following cases is the area of $ABCD$ an integer?



(A) $AB = 3, CD = 1$ **(B)** $AB = 5, CD = 2$ **(C)** $AB = 7, CD = 3$
(D) $AB = 9, CD = 4$ **(E)** $AB = 11, CD = 5$

Source: 1988 AHSME #27

5. Let $\triangle ABC$ be a right triangle with $\angle C = 90°$ and $\angle A = 60°$. Let D be the point where the incircle of $\triangle ABC$ meets $\overline{AC}$. Let the incircle of $\triangle ABC$ and line segment $\overline{BD}$ intersect at point E. Find the ratio of $\overline{BE}$ to $\overline{BD}$.

   Source: Own

§4.3 Hard Problems

1. Chords $\overline{AB}$ and $\overline{CD}$ of a circle centered at point O are perpendicular at point P. Given $CP = 2$, $AP = 3$, and $PD = 6$, find
   a) OP.
   b) the radius of the circle.

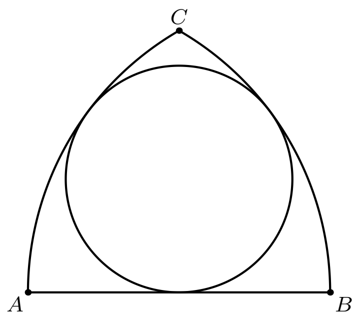
2. A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side?

   (A) 200 **(B)** $200\sqrt{2}$ **(C)** $200\sqrt{3}$ **(D)** $300\sqrt{2}$ **(E)** 500

   Source: 2016 AMC 10A #24

3. If circular arcs AC and BC have centers at B and A, respectively, then there exists a circle tangent to both $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$, and to $\overline{AB}$. If the length of $\overset{\frown}{BC}$ is 12, then the circumference of the circle is

   (A) 24 **(B)** 25 **(C)** 26 **(D)** 27 **(E)** 28



   Source: 2000 AMC 12 #24

4. The diameter $\overline{AB}$ of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment $\overline{AE}$ intersects the circle at a point C between A and E. What is the area of $\triangle ABC$?

   (A) $\frac{120}{37}$ **(B)** $\frac{140}{39}$ **(C)** $\frac{145}{39}$ **(D)** $\frac{140}{37}$ **(E)** $\frac{120}{31}$

   Source: 2017 AMC 10B #22

5. In acute $\triangle ABC, AB = 4$. Let D be the point on BC such that$\angle BAD = \angle CAD$. Let AD intersect the circumcircle of $\triangle ABC$ at X. Let Γ be the circle through D and X that is tangent to AB at P. If $AP = 6$, compute AC.

   Source: Duke Math Meet 2014 Team Round #6

Introduction to Circles: Problem Set Solutions

BCA Math Team AMC Lecture

Daniel Kim

October 12, 2019

§1 Solutions to Easy Problems

1. Since AB and CD are perpendicular to each other, $\triangle DEA$ is a right triangle. Then, by Pythagorean Theorem, $AE^2 + DE^2 = AD^2 \Rightarrow AE^2 = AD^2 - DE^2 = 5^2 - 4^2 = 9 \Rightarrow AE = 3$. Then, by Power of a Point, $AE \cdot BE = CE \cdot DE \Rightarrow 3 \cdot 16 = CE \cdot 4 \Rightarrow CE = \boxed{12}$.

2. Answer: $\boxed{A}$. Solution: https://artofproblemsolving.com/wiki/index.php/1997_AHSME_Problems/Problem_26

3. Let the circle centered at point P be ω_1 and the circle centered at point Q be ω_2. Let B be the intersection of AP and ω_1, and C be the intersection of AQ and ω_2. Let the radius of ω_1 be r, so $BP = r$ and $AP = 10 - r$. Since $AQ = 12$ and the radius of ω_2 is 7, we have $AC = 5$ and $QC = 7$.

   Let $\overrightarrow{BP}$ meet ω_1 at point B', and $\overrightarrow{CQ}$ meet ω_2 at point C'. By Power of a Point on A and ω_1, $AB \cdot AB' = AX \cdot AY$. By Power of a Point on A and ω_2, $AC \cdot AC' = AX \cdot XY$. Thus, $AB \cdot AB' = AC \cdot AC'$, i.e. $(10 - r)(10 + r) = 5 \cdot 19$. We solve this to get $r = \boxed{\sqrt{5}}$.

4. By Pythagorean Theorem on $\triangle MBC$, $CM = 5\sqrt{5}$. Let MC intersect the inscribed circle at E, and let $CE = x$. Let AC intersect the circle at X and Y such that $CX < CY$. Then, by symmetry, $AY = CX$ and $XY = 10$ as it is a diameter of the circle. Furthermore, we compute $AC = 10\sqrt{2}$, so $AY = CX = \frac{10\sqrt{2}-10}{2} = 5\sqrt{2} - 5$. By Power of a Point,

$$CX \cdot CY = CE \cdot CM \Rightarrow (5\sqrt{2} - 5)(5\sqrt{2} - 5 + 10) = x \cdot 5\sqrt{5}.$$

   We solve to get $x = \boxed{\sqrt{5}}$.

5. Answer: $\boxed{D}$. Solution: https://artofproblemsolving.com/wiki/index.php/2013_AMC_10A_Problems/Problem_23

§2 Solutions to Medium Problems

1. Answer: $\boxed{\sqrt{22}}$. Solution:

   https://artofproblemsolving.com/community/c5t145f5h590003_find_the_radius_of_circle_o

2. Let $OR = x$ and $GO = y$. By Power of a Point, $FI^2 = FR \cdot FG \Rightarrow (21\sqrt{2})^2 = 18 \cdot (18 + x + y)$. This simplifies to $x + y = 31$. Power of a Point also tells us that $OI \cdot ON = OR \cdot OG \Rightarrow xy = 20 \cdot 12$. Combining these two equations, along with the given information that $x > y$, gives us $x = \boxed{16}$ and $y = 15$.

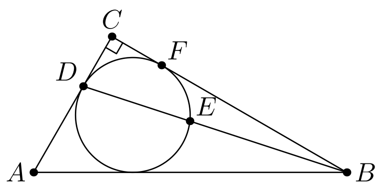
3. Answer: $\boxed{A}$. Solution:

   https://artofproblemsolving.com/community/c5t145f5h64939_nice_circle_and_equilateral_triangle_problem

4. Answer: $\boxed{D}$. Solution:

   https://artofproblemsolving.com/wiki/index.php/1988_AHSME_Problems/Problem_27

5. Let $AC = s$, such that $BC = s\sqrt{3}$ and $AB = 2s$.

   Then, using the formula $[ABC]$ = inradius times the semiperimeter, we obtain $r = \frac{s\sqrt{3}}{3+\sqrt{3}}$. Let the incircle of $\triangle ABC$ meet BC at point F. Since the length of CF must be equal to the inradius, we get $BF = s\sqrt{3} - \frac{s\sqrt{3}}{3+\sqrt{3}} = s\left(\frac{\sqrt{3}+1}{2}\right)$. By Power of a Point, $BF^2 = BE \cdot BD$. Then $BE = \frac{BF^2}{BD}$, so the ratio $\frac{BE}{BD}$ can be rewritten as $\frac{BF^2}{BD^2}$. By the Pythagorean Theorem, $BD^2 = BC^2 + CD^2 = (s\sqrt{3})^2 + \left(\frac{s\sqrt{3}}{3+\sqrt{3}}\right)^2 = s^2\left(4 - \frac{\sqrt{3}}{2}\right)$. Thus, our answer is

   $$\frac{BE}{BD} = \frac{BF^2}{BD^2} = \frac{s^2\left(\frac{\sqrt{3}+1}{2}\right)^2}{s^2\left(4 - \frac{\sqrt{3}}{2}\right)} = \boxed{\frac{19+10\sqrt{3}}{61}}.$$

   Note that the work could be simplified by letting $s = 1$, as the ratio stays the same regardless of the value of s.

§3 Solutions to Hard Problems

1. a) From the power of a point P we have $BP = \frac{CP \cdot DP}{AP} = \frac{2 \cdot 6}{3} = 4$. Let M be the midpoint of AB and N be the midpoint of CD. Then OM is part of a radius that bisects AB, so $OM \perp AB$, and similarly $ON \perp CD$. Thus, $MPNO$ is a rectangle.

Since $AM = \frac{AB}{2} = \frac{7}{2}$, we have $PM = MA - AP = \frac{1}{2}$. Similarly, we find $NP = CN - CP = \frac{CD}{2} - CP = 2$. Then, OP is the diagonal of rectangle $MPNO$ with side lengths 2 and $\frac{1}{2}$, so by Pythagorean Theorem, $OP = \boxed{\frac{\sqrt{17}}{2}}$.

b) Extend OP to meet the circle at X and Y, with X closer to P. Then the power of a point P gives us $XP \cdot PY = AP \cdot PB$. Let the radius be r. Then $\left(r - \frac{\sqrt{17}}{2}\right)\left(r + \frac{\sqrt{17}}{2}\right) = 12$, so $r = \boxed{\frac{\sqrt{65}}{2}}$.

2. Answer: $\boxed{E}$. Solution: Solution 5 in https://artofproblemsolving.com/wiki/index.php/2016_AMC_10A_Problems/Problem_24

3. Answer: $\boxed{D}$. Solution: https://artofproblemsolving.com/wiki/index.php/2000_AMC_12_Problems/Problem_24

4. Answer: $\boxed{D}$. Solution: https://artofproblemsolving.com/wiki/index.php/2017_AMC_10B_Problems/Problem_22

5. Answer: $\boxed{9}$. Solution:

https://artofproblemsolving.com/community/c4t145f4h613191_duke_math_meet_2014_team_round_6