Trigonometry and a Sequence of Polynomials

Daniel Kim, Sameer Pai, Simon Sun, Jesse Yang

PROMYS

July 2019

Notation

Notation

- $T_n(x)$ is the unique polynomial satisfying $T_n(\cos(x)) = \cos(nx)$ for all $x \in \mathbb{R}, n \in \mathbb{N}$.

Notation

- $T_n(x)$ is the unique polynomial satisfying $T_n(\cos(x)) = \cos(nx)$ for all $x \in \mathbb{R}, n \in \mathbb{N}$.
- $\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$.

Notation

- $T_n(x)$ is the unique polynomial satisfying $T_n(\cos(x)) = \cos(nx)$ for all $x \in \mathbb{R}, n \in \mathbb{N}$.
- $\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$.
- Let $\alpha(n, k)$ denote the coefficient of x^k in $T_n(x)$.

Closed Form for $T_n(x)$

Theorem 1

For $n \in \mathbb{N}$,

$$T_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} x^{n-2k}(x^2-1)^k$$
$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} x^{n-2k} \sum_{j=k}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j}\binom{j}{k}(-1)^k.$$

Closed Form for $T_n(x)$

Theorem 1

For $n \in \mathbb{N}$,

$$T_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} x^{n-2k}(x^2-1)^k$$
$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} x^{n-2k} \sum_{j=k}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \binom{j}{k} (-1)^k.$$

Sketch of Proof.

$$\text{Consider } \cos(n\theta) = \operatorname{Re}((\cos\theta + i\sin\theta)^n)$$
$$= \operatorname{Re}\left(\sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta (i\sin\theta)^k \right).$$

Table of Coefficients

n	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0	0
2	-1	0	2	0	0	0	0	0	0	0	0	0	0
3	0	-3	0	4	0	0	0	0	0	0	0	0	0
4	1	0	-8	0	8	0	0	0	0	0	0	0	0
5	0	5	0	-20	0	16	0	0	0	0	0	0	0
6	-1	0	18	0	-48	0	32	0	0	0	0	0	0
7	0	-7	0	56	0	-112	0	64	0	0	0	0	0
8	1	0	-32	0	160	0	-256	0	128	0	0	0	0
9	0	9	0	-120	0	432	0	-576	0	256	0	0	0
10	-1	0	50	0	-400	0	1120	0	-1280	0	512	0	0
11	0	-11	0	220	0	-1232	0	2816	0	-2816	0	1024	0
12	1	0	-72	0	840	0	-3584	0	6912	0	-6144	0	2048

First, we compiled a table of numerical data in hopes of finding a pattern, particularly a closed form, for any coefficient.

Patterns in Coefficients

Consider the first few polynomials:

$$\begin{array}{lll} 1, & x, & 2x^2-1, \\ 4x^3-3x, & 8x^4-8x^2+1, & 16x^5-20x^3+5x \end{array}$$

The coefficients alternate in sign:

Lemma 2

$\forall k \leq n$,

$$\operatorname{sgn}(\alpha(n,k)) = \begin{cases} 0 & k \not\equiv n \pmod 2 \\ -1 & k \equiv n+2 \pmod 4 \\ 1 & k \equiv n \pmod 4 \end{cases}.$$

Coefficients

Let $T_n(x) = \sum_{i=0}^{n} a_i x^i$. We made the following observation:

n	$T_n(x)$	$\sum \lvert a_i \rvert$
0	1	1
1	x	1
2	$2x^2 - 1$	$3 = 2 \cdot 1 + 1$
3	$4x^3 - 3x$	$7 = 2 \cdot 3 + 1$
4	$8x^4 - 8x^2 + 1$	$17 = 2 \cdot 7 + 3$
5	$16x^5 - 20x^3 + 5x$	$41 = 2 \cdot 17 + 7$
6	$32x^6 - 48x^4 + 18x^2 - 1$	$99 = 2 \cdot 41 + 17$
7	$64x^7 - 112x^5 + 56x^3 - 7x$	$239 = 2 \cdot 99 + 41$
8	$128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$	$577 = 2 \cdot 239 + 99$

The sum of the absolute value of the coefficients follow a recursive pattern.

Coefficients

Lemma 3

$|\alpha(n,k)| = 2\,|\alpha(n-1,k-1)| + |\alpha(n-2,k)|.$

This follows from the recursion $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$.

Coefficients

Lemma 3

$|\alpha(n,k)| = 2\,|\alpha(n-1,k-1)| + |\alpha(n-2,k)|.$

This follows from the recursion $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$.

Proposition 4

Let $s(n) = \sum_{i=0}^{n} |a_i|$ *where* $T_n(x) = \sum_{i=0}^{n} a_i x^i$. *Then* $s(n) = 2s(n-1) + s(n-2)\ \forall\ n \geq 2$, *and* $s(0) = s(1) = 1$.

Table of Coefficients

n	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0	0
2	-1	0	2	0	0	0	0	0	0	0	0	0	0
3	0	-3	0	4	0	0	0	0	0	0	0	0	0
4	1	0	-8	0	8	0	0	0	0	0	0	0	0
5	0	5	0	-20	0	16	0	0	0	0	0	0	0
6	-1	0	18	0	-48	0	32	0	0	0	0	0	0
7	0	-7	0	56	0	-112	0	64	0	0	0	0	0
8	1	0	-32	0	160	0	-256	0	128	0	0	0	0
9	0	9	0	-120	0	432	0	-576	0	256	0	0	0
10	-1	0	50	0	-400	0	1120	0	-1280	0	512	0	0
11	0	-11	0	220	0	-1232	0	2816	0	-2816	0	1024	0
12	1	0	-72	0	840	0	-3584	0	6912	0	-6144	0	2048

Recall our table of coefficients. We seek a closed formula to calculate the coefficient of any power of x in any polynomial $T_n(x)$.

Table of Coefficients

n	1	x	x^2	x^3	x^4	x^5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	−1	0	2	0	0	0
3	0	−3	0	4	0	0
4	1	0	−8	0	8	0
5	0	5	0	−20	0	16
6	−1	0	18	0	−48	0
7	0	−7	0	56	0	−112
8	1	0	−32	0	160	0
9	0	9	0	−120	0	432
10	−1	0	50	0	−400	0
11	0	−11	0	220	0	−1232
12	1	0	−72	0	840	0

Triangle A

$m = 0$:						1						
$m = 1$:					1		1					
$m = 2$:				1		3		2				
$m = 3$:			1		5		8		4			
$m = 4$:		1		7		18		20		8		
$m = 5$:	1		9		32		56		48		16	

Figure: Triangle A

Triangle A

$m=0$:						1						
$m=1$:					1		1					
$m=2$:				1		3		2				
$m=3$:			1		5		8		4			
$m=4$:		1		7		18		20		8		
$m=5$:	1		9		32		56		48		16	

Figure: Triangle A

For every nonzero value a in the Table of Coefficients, which corresponds to some value of n and some x power, x^k, place $|a|$ in row m of Triangle A where $n+k=2m$. Let a be ordered as the k^{th} index in row m (where indices are ordered left to right from 0 to m).

Triangle A

$m = 0$:						1						
$m = 1$:					1		1					
$m = 2$:				1		3		2				
$m = 3$:			1		5		8		4			
$m = 4$:		1		7		18		20		8		
$m = 5$:	1		9		32		56		48		16	

Definition 5

Let $\beta(m, k)$ denote the k^{th} index in row m of Triangle A. Then, by our construction, $\beta(m, k) = |\alpha(2m - k, k)|$.

Triangle A

Lemma 6

When both quantities are defined,
$\beta(m, k) = 2\beta(m-1, k-1) + \beta(m-1, k)$.

Triangle A

Lemma 6

When both quantities are defined, $\beta(m, k) = 2\beta(m-1, k-1) + \beta(m-1, k)$.

Proposition 7

For $m \geq 0$, $\sum_{i=0}^{m+1} \beta(m+1, i) = 3 \cdot \sum_{i=0}^{m} \beta(m, i)$. *Hence,*

$$\sum_{i=0}^{m+1} \beta(m+1, i) = 2 \cdot 3^m.$$

Triangle B

$m = 0$:						1						
$m = 1$:					$\frac{1}{2}$		$\frac{1}{2}$					
$m = 2$:				$\frac{1}{6}$		$\frac{1}{2}$		$\frac{1}{3}$				
$m = 3$:			$\frac{1}{18}$		$\frac{5}{18}$		$\frac{4}{9}$		$\frac{2}{9}$			
$m = 4$:		$\frac{1}{54}$		$\frac{7}{54}$		$\frac{1}{3}$		$\frac{10}{27}$		$\frac{4}{27}$		
$m = 5$:	$\frac{1}{162}$		$\frac{1}{18}$		$\frac{16}{81}$		$\frac{28}{81}$		$\frac{8}{27}$		$\frac{8}{81}$	

Figure: Triangle B

Now consider the bijection of Triangle A to Triangle B; namely, Triangle B represents each element as a probabilistic value.

Triangle B

$$
\begin{array}{lccccccccccc}
m=0: & & & & & & 1 & & & & & \\
m=1: & & & & & \frac{1}{2} & & \frac{1}{2} & & & & \\
m=2: & & & & \frac{1}{6} & & \frac{1}{2} & & \frac{1}{3} & & & \\
m=3: & & & \frac{1}{18} & & \frac{5}{18} & & \frac{4}{9} & & \frac{2}{9} & & \\
m=4: & & \frac{1}{54} & & \frac{7}{54} & & \frac{1}{3} & & \frac{10}{27} & & \frac{4}{27} & \\
m=5: & \frac{1}{162} & & \frac{1}{18} & & \frac{16}{81} & & \frac{28}{81} & & \frac{8}{27} & & \frac{8}{81}
\end{array}
$$

Definition 8

Let $\gamma(m,k)$ represent the probability associated with the k^{th} index in row m:

$$\gamma(m,k) = \frac{\beta(m,k)}{\sum_{i=0}^{m} \beta(m,i)}.$$

Triangle B

$m = 0$: 1

$m = 1$: $\frac{1}{2}$ $\frac{1}{2}$

$m = 2$: $\frac{1}{6}$ $\frac{1}{2}$ $\frac{1}{3}$

$m = 3$: $\frac{1}{18}$ $\frac{5}{18}$ $\frac{4}{9}$ $\frac{2}{9}$

$m = 4$: $\frac{1}{54}$ $\frac{7}{54}$ $\frac{1}{3}$ $\frac{10}{27}$ $\frac{4}{27}$

$m = 5$: $\frac{1}{162}$ $\frac{1}{18}$ $\frac{16}{81}$ $\frac{28}{81}$ $\frac{8}{27}$ $\frac{8}{81}$

Lemma 9

$$\gamma(m,k) = \frac{2}{3}\gamma(m-1,k+1) + \frac{1}{3}\gamma(m-1,k).$$

Triangle B

Theorem 10

We start a random walk down Triangle B. The following rules govern the walk:

- *We start at $m = 1$, with $k = 0$ or $k = 1$ with equal probability.*

Triangle B

Theorem 10

We start a random walk down Triangle B. The following rules govern the walk:

- *We start at* $m = 1$, *with* $k = 0$ *or* $k = 1$ *with equal probability.*
- *At each state, we step one row down and either left or right, moving left with probability* $\frac{1}{3}$ *and right with probability* $\frac{2}{3}$. *(Formally, we move from* (m, k) *to* $(m + 1, k)$ *one-thirds of the time and to* $(m + 1, k + 1)$ *two-thirds of the time.)*

Triangle B

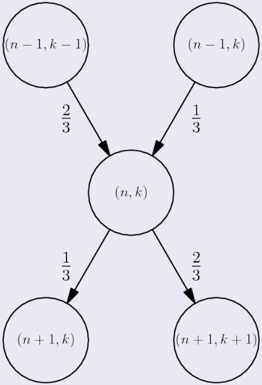
Theorem 10

We start a random walk down Triangle B. The following rules govern the walk:

- *We start at* $m = 1$*, with* $k = 0$ *or* $k = 1$ *with equal probability.*
- *At each state, we step one row down and either left or right, moving left with probability* $\frac{1}{3}$ *and right with probability* $\frac{2}{3}$*. (Formally, we move from* (m, k) *to* $(m+1, k)$ *one-thirds of the time and to* $(m+1, k+1)$ *two-thirds of the time.)*
- *Then* $\gamma(m, k)$ *is the probability that we reach* (m, k) *on this walk.*

Triangle B

Proof By Picture.



$$\gamma(n,k) = \frac{2}{3}\gamma(n-1,k-1) + \frac{1}{3}\gamma(n-1,k)$$

$\square$

Closed Form

Proposition 11

For all $m > 0$, $\gamma(m,k) = \frac{1}{3^{m-1}}\left(\binom{m-1}{k}\cdot 2^{k-1} + \binom{m-1}{k-1}\cdot 2^{k-2}\right)$.

Closed Form

Proposition 11

For all $m > 0$, $\gamma(m,k) = \dfrac{1}{3^{m-1}}\left(\dbinom{m-1}{k}\cdot 2^{k-1} + \dbinom{m-1}{k-1}\cdot 2^{k-2}\right)$.

We finally have our closed form:

Closed Form

Proposition 11

For all $m > 0$, $\gamma(m,k) = \frac{1}{3^{m-1}}\left(\binom{m-1}{k}\cdot 2^{k-1} + \binom{m-1}{k-1}\cdot 2^{k-2}\right)$.

We finally have our closed form:

Theorem 12

For $n \neq k$, $\alpha(n,k) = \operatorname{sgn}(\alpha(n,k))\left[2^k \cdot \binom{\frac{n+k}{2}-1}{k} \cdot \frac{n}{n-k}\right]$. *Otherwise,* $n = k$ *and* $\alpha(n,n) = 2^{n-1}$.

Trigonometric Polynomials in $\mathbb{Z}_p[x]$

Trigonometric Polynomials in $\mathbb{Z}_p[x]$

The closed formula enables us to prove a powerful result regarding modulo p for prime p.

Trigonometric Polynomials in $\mathbb{Z}_p[x]$

The closed formula enables us to prove a powerful result regarding modulo p for prime p.

Lemma 13

For any $a, b \in \mathbb{N}$, $T_a(T_b(x)) = T_{ab}(x)$.

Trigonometric Polynomials in $\mathbb{Z}_p[x]$

The closed formula enables us to prove a powerful result regarding modulo p for prime p.

Lemma 13

For any $a, b \in \mathbb{N}$, $T_a(T_b(x)) = T_{ab}(x)$.

Theorem 14

For odd prime p and $a \in \mathbb{N}$, $T_{ap}(x) = T_a(x^p)$ in $\mathbb{Z}_p[x]$.

Trigonometric Polynomials in $\mathbb{Z}_p[x]$

Proof.

- By above, $T_{ap}(x) = T_a(T_p(x))$. Therefore, if we show that $T_p(x) \equiv x^p$ in $\mathbb{Z}_p[x]$, then we are done.

Trigonometric Polynomials in $\mathbb{Z}_p[x]$

Proof.

- By above, $T_{ap}(x) = T_a(T_p(x))$. Therefore, if we show that $T_p(x) \equiv x^p$ in $\mathbb{Z}_p[x]$, then we are done.
- First, we will verify that the coefficient of x^k is zero for $0 \leq k < p$. By our closed formula derived earlier, we know that

$$|\alpha(p,k)| = 2^k \binom{\frac{p+k}{2} - 1}{k} \cdot \frac{p}{p-k}.$$

Trigonometric Polynomials in $\mathbb{Z}_p[x]$

Proof.

- By above, $T_{ap}(x) = T_a(T_p(x))$. Therefore, if we show that $T_p(x) \equiv x^p$ in $\mathbb{Z}_p[x]$, then we are done.
- First, we will verify that the coefficient of x^k is zero for $0 \leq k < p$. By our closed formula derived earlier, we know that

$$|\alpha(p,k)| = 2^k \binom{\frac{p+k}{2} - 1}{k} \cdot \frac{p}{p-k}.$$

- Then taking modulo p, the $\frac{p}{p-k}$ term means that, since everything in the denominator (i.e. $p-k$ and $k!$) do not have any factors of p, $\alpha(p,k)$ is a multiple of p as desired. So, every coefficient of x^k with $k < p$ disappears in $\mathbb{Z}_p[x]$.

Trigonometric Polynomials in $\mathbb{Z}_p[x]$

Proof.

- By above, $T_{ap}(x) = T_a(T_p(x))$. Therefore, if we show that $T_p(x) \equiv x^p$ in $\mathbb{Z}_p[x]$, then we are done.
- First, we will verify that the coefficient of x^k is zero for $0 \leq k < p$. By our closed formula derived earlier, we know that

$$|\alpha(p,k)| = 2^k \binom{\frac{p+k}{2} - 1}{k} \cdot \frac{p}{p-k}.$$

- Then taking modulo p, the $\frac{p}{p-k}$ term means that, since everything in the denominator (i.e. $p-k$ and $k!$) do not have any factors of p, $\alpha(p,k)$ is a multiple of p as desired. So, every coefficient of x^k with $k < p$ disappears in $\mathbb{Z}_p[x]$.
- Now, the coefficient of x^p in $T_p(x)$ is 2^{p-1}, which is 1 in $\mathbb{Z}_p$ by Euler's Theorem. Thus, all coefficients less than x^p are 0, and the x^p coefficient is 1, so $T_p(x) = x^p$ in $\mathbb{Z}_p[x]$. $\square$

Roots

Roots

Lemma 15

The roots of $T_m(x)$ are of form $\cos\left(\frac{\pi+2\pi k}{2m}\right)$.

Roots

Lemma 15

The roots of $T_m(x)$ are of form $\cos\left(\frac{\pi+2\pi k}{2m}\right)$.

Theorem 16

$\forall k, m \in \mathbb{N},\ T_m(x) \mid T_{m(2k+1)}(x)$.

Roots

Proof.

- The roots of $T_m(x)$ are of form $\cos\left(\frac{\pi+2\pi a}{2m}\right)$ and the roots of $T_{m(2k+1)}(x)$ are of form $\cos\left(\frac{\pi+2\pi b}{2m(2k+1)}\right)$ for all natural numbers k, where a and b are integers.

Proof.

- The roots of $T_m(x)$ are of form $\cos\left(\frac{\pi+2\pi a}{2m}\right)$ and the roots of $T_{m(2k+1)}(x)$ are of form $\cos\left(\frac{\pi+2\pi b}{2m(2k+1)}\right)$ for all natural numbers k, where a and b are integers.
- For any a, we show that there exists b such that $\frac{\pi+2\pi b}{2m(2k+1)} = \frac{\pi+2\pi a}{2m}$. Indeed,

$$\frac{\pi + 2\pi b}{2m(2k+1)} = \frac{\pi + 2\pi a}{2m}$$

which simplifies to

$$b = 2ka + k + a.$$

Roots

Proof.

- The roots of $T_m(x)$ are of form $\cos\left(\frac{\pi+2\pi a}{2m}\right)$ and the roots of $T_{m(2k+1)}(x)$ are of form $\cos\left(\frac{\pi+2\pi b}{2m(2k+1)}\right)$ for all natural numbers k, where a and b are integers.
- For any a, we show that there exists b such that $\frac{\pi+2\pi b}{2m(2k+1)} = \frac{\pi+2\pi a}{2m}$. Indeed,

$$\frac{\pi + 2\pi b}{2m(2k+1)} = \frac{\pi + 2\pi a}{2m}$$

  which simplifies to

$$b = 2ka + k + a.$$

- Therefore, there exists an integer b such that $\frac{\pi+2\pi b}{2m(2k+1)} = \frac{\pi+2\pi a}{2m}$ for every a. This means that every root of $T_m(x)$ is a root of $T_{m(2k+1)}(x)$, which means that $T_m(x) \mid T_{m(2k+1)}(x)$. $\square$

Approximation

Proposition 17

As $n \to \infty$, for any even $k \in \mathbb{N}$, $\alpha(n, k)$ approaches the coefficient of x^k in the Taylor series for $\cos(nx)$.

Proposition 17

As $n \to \infty$, for any even $k \in \mathbb{N}$, $\alpha(n,k)$ approaches the coefficient of x^k in the Taylor series for $\cos(nx)$.

Proof.

It is well known that the coefficient of x^{2k} in the Taylor series for $\cos x$ is $\frac{(-1)^k}{(2k)!}$. So the coefficient in $\cos nx$ is $\frac{(-1)^k \cdot n^{2k}}{(2k)!}$. Now, the closed form for $|\alpha(n,k)|$ is

$$\left[2^k \cdot \binom{\frac{n+k}{2} - 1}{k} \cdot \frac{n}{n-k}\right].$$

Approximation

Proof.

Using Stirling's approximation for binomial coefficients, we see that this approaches

$$2^{2k} \cdot \frac{\left(\frac{n+2k}{2}\right)^{2k}}{(2k)!} \cdot \frac{n}{n-k} = \frac{(n+2k)^{2k}}{(2k)!} \cdot \frac{n}{n-k}.$$

As $n \to \infty$, both $n-k$ and $n+2k$ approach n, so this simplifies to $\frac{n^{2k}}{(2k)!}$ as desired. $\square$