Personal Math Contest Problem Inventory

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These are a collection of math problems that I've written over the years to be proposed for various math contests. These are not ordered by difficulty.

Contents

§1 Algebra

1. Simplify $\sqrt{31 \cdot 33 \cdot 35 \cdot 37 + 16}$.

2. Find all real solutions to $x^4 + (4-x)^4 = 466$.

3. Simplify
$$\left(\sqrt{2\sqrt{3\sqrt{4\sqrt{2\sqrt{3\sqrt{4\cdots}}}}}}\right)^7.$$

4. How many ways are there to write 39 as a sum of more than two consecutive integers?

5. Given an arithmetic sequence $\{a_n\}$ with $a_1+a_2+\ldots+a_{10} = 10$ and $a_{2011}+a_{2012}+\ldots+a_{2020} = 412$, find the last three digits of the sum of the first 2020 terms of the sequence.

6. Find all real solutions to $(3x-1)^4 + (x+2)^4 = (4x+1)^4$.

7. Find all integer values of x such that $x^2 + 17x + 70$ is a perfect square.

8. The first two terms of a sequence $\{a_n\}$ are $a_1 = 1$ and $a_2 = \frac{\sqrt{3}}{3}$. For $n \geq 1$,
$$\frac{1}{a_n a_{n+1} a_{n+2}} + \frac{1}{a_n} + \frac{1}{a_{n+1}} = \frac{1}{a_{n+2}}.$$
Compute a_{2020}.

9. Find the area of the region bounded by $|x| + |y| + |x-y| = 10$.

10. Evaluate
$$\sum_{i=0}^{\infty} \sum_{j=1}^{i} \sum_{k=j}^{i} \frac{1}{2^{i+j+k}}.$$

11. How many positive integer values of r are there such that for all real numbers x and y, $x^2 + y^2 \leq r^2$ and $\left|x^2 + 2xy - y^2\right| \leq 2020$?

12. Find the shortest distance from the origin to the graph of
$$4x^2 + 4xy - 12x + y^2 - 6y + 9 = 0.$$

13. Find the number of solutions $x \in (0, 10)$ to
$$x^{(x/2)^{\tan x}} = x^{(x/2)^{\sin(2x)}}.$$

14. Let N be the real solution to
$$x^3 - 3\sqrt{3}x^2 + 9x = 3\sqrt{3} + 5.$$
The value of N can be expressed in the form $a^b + c^d$, where a and c are positive integers and b and d are rational numbers in the simplest form. Find $a + b + c + d$.

15. Find the area of the largest square contained inside the region $\left|x^2+y\right| \leq 1$ whose sides are parallel to the coordinate axes.

16. Let $n!$ denote the factorial function. A clerk performs the following computation

$$1\% \text{ of } 2\% \text{ of } 3\% \text{ of } \ldots 100\% \text{ of } n$$

and gets the number $99!$. How many digits in n are zero?

17. Find the positive solution to the equation

$$\left(\sqrt[x]{x\sqrt[x]{x\sqrt[x]{x\cdots}}}\right)^2 = 3.$$

18. Find the minimum value of $\cos^2 x + \sec^2 x$ over the real numbers.

19. Find the maximum value of $\sqrt{\frac{25-x^2}{x^2+1}}$ on $x \in [-5, 5]$.

20. A circle centered at the origin is tangent to the line $x + ay = 1$, where $a > 0$. Let region R be the portion of the circle in the first quadrant, and define the region $S = \{(x, y) \mid x \geq 0, y \geq 0, x + ay \leq 1\}$. Find the maximum of the ratio of the area of R to the area of S.

21. A cubic polynomial $x^3 + ax^2 + bx + 2020$ has roots r, s, and t. If a and b are positive integers less than or equal to 2020, find the number of ordered pairs (a, b) such that $r^2 + s^2 + t^2 = 0$.

22. Find the area of the region bounded by $y = |2x|$ and $y = |x| + 2$.

23. The function

$$f(x) = \begin{cases} \frac{1}{1+\tan^2(2x)} & x \neq 45^\circ \\ 0 & x = 45^\circ \end{cases}$$

is defined on the interval $[1^\circ, 89^\circ]$. Compute

$$\sum_{k=1^\circ}^{89^\circ} f(k).$$

24. Let $f(\theta) = \cos\theta + i\sin\theta$, where $i = \sqrt{-1}$ and θ is in radians. Compute the imaginary part of

$$\prod_{k=1}^{\infty} f\left(\frac{(2k-1)\pi}{9 \cdot 2^k}\right).$$

25. The area of the smallest rectangle whose sides are parallel to the coordinate axes that contains the graph of $f(x) = \cos^{-1}(\cos^{-1} x)$ is of the form $A - B\cos C$, where A, B, C are real numbers and C is in radians. Find $A + B + C$.

26. If $\sin x + \cos x = \dfrac{1}{2}$ and $\dfrac{\pi}{2} < x < \pi$ where x is in radians, compute $\tan x$.

27. Compute
$$\prod_{n=2}^{2020}\prod_{k=0}^{\infty}\frac{1}{1+n^{-(2^k)}}.$$

28. Given $\sin\theta+\cos\theta=\frac{1}{3}$, evaluate $\cos^4\theta-2\sin^2\theta+3\sin^4\theta$.

29. Let
$$S=\left\{\sqrt{2+\sqrt{3}},\sqrt{3+\sqrt{5}},\sqrt{4+\sqrt{7}},\sqrt{5+\sqrt{9}},\cdots,\sqrt{2020+\sqrt{4039}}\right\}.$$
Find the last two digits of the sum
$$\sum_{x\in S}\left(2x-\sqrt{2}\right)^2.$$

30. Derek starts at the origin $O=(0,0)$ and walks 1 unit due east. He turns 90° counterclockwise and walks half the previous distance. He repeats this step indefinitely as he arbitrary approaches a point P. Compute the distance of P from the origin.

31. A monic degree 4 real-valued polynomial $P(x)$ satisfies $P(1+\sqrt{2})=0$, $P(3)=3$, and $P(4)=14$. Find $P(6)$.

32. Real numbers x and y satisfy $x=3-y=\frac{1}{y}$. Compute $x^4+x^3y+x^2y^2+xy^3+y^4$.

33. Let $S=\{1,2,\ldots,n\}$. The product of all elements in S divided by the sum of all elements in S is equal to 180. Find n.

34. Let a,b, and c be positive integers such that $a^2+b^2+c^2+4^2+37^2=2020$. Find $a+b+c$.

35. Find the real solution to $x(x(x+3)+2)+1=11^2$.

36. If $3^x=5$, compute
$$\frac{\log_3(\log_{5^{1/10}}(\sqrt{9^x}))}{\log_{25^{1/x}}(10)}.$$

37. A function f satisfies $f(2-\sqrt{x})=x^2-9x$. Find the sum of all distinct roots of f.

38. Let $f(x)=x^3-2x^2-2x-3$ have roots r,s, and t. Compute
$$\left(\frac{1}{r-1}\right)^2+\left(\frac{1}{s-1}\right)^2+\left(\frac{1}{t-1}\right)^2.$$

39. Find the minimum value of $|x-1|+|2-x|+|x+3|$ over all real numbers x.

40. A sequence $\{a_n\}$ is defined by $a_1=0$ and for any two positive integers m and n,
$$a_{mn}=a_ma_n+na_m+ma_n.$$
Then a_{2020} can be written as a multivariate polynomial in a_p where p is a prime number. Find the sum of the coefficients of this polynomial.

41. Two points A and B are chosen on the parabola $y = x^2$ such that the line containing A and B has slope 4. What is the x-coordinate of the midpoint of A and B?

42. Evaluate $5.5^3 + 2.5^3$.

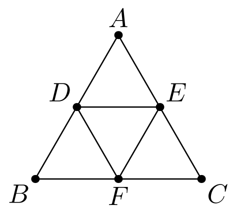
43. Simplify

$$\sqrt{\frac{4}{27+27\sqrt{\frac{4}{27+27\sqrt{\frac{4}{27+\cdots}}}}}}.$$

44. Find the number of solutions to $x^2 = |\lfloor x \rfloor + x|$.

§2 Combinatorics

1. In the diagram below, $\triangle ABC$ is equilateral, and $\triangle DEF$ is an equilateral triangle formed by the midpoints of the sides of $\triangle ABC$.



A bunny walks along the line segments of this figure, starting at point A. At every step of the process, the bunny walks from one labeled point to an adjacent labeled point. Find the number of paths with 7 steps that begin at point A and end at point B.

2. Let
$$S_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{\binom{n}{2i}}{2i+1}.$$
Compute
$$\sum_{k=1}^{\infty} \frac{1}{S_k}.$$

3. How many ordered tuples (a, b, c, d) exist such that $a + b + c + d = 20$ where a, b, c and d are integers, $a \geq 0$, $b \geq 1$, $c \geq 2$, and $d \geq 3$?

4. A real number x is chosen such that $0 < x < 100$. If $\lfloor x \rfloor$ is the greatest integer less than or equal to x, what is the probability that $\lfloor \sqrt[3]{x} \rfloor$ is odd?

§3 Geometry

1. Let $ABCD$ be a square with side length 2. Let points E, F, G, and H be the midpoints of AB, BC, CD, and DA respectively. Find the intersection of the areas of $\triangle BHG$ and $\triangle DEF$.

2. Consider $\triangle ABC$ with $AB = 3$ and $BC = 5$. Let the incircle of $\triangle ABC$, with incenter I, meet side BC at point D. Extend AB and DI to meet at point E. Find the length of AE.

3. Let $\triangle ABC$ be a right triangle with $\angle C = 90^\circ$ and $\angle A = 60^\circ$. Let D be the point where the incircle of $\triangle ABC$ meets $\overline{AC}$. Let the incircle of $\triangle ABC$ and line segment $\overline{BD}$ intersect at point E. The ratio of $\overline{BE}$ to $\overline{BD}$ can be represented as $\frac{p+q\sqrt{r}}{s}$, where p, q, r, and s are positive integers, r is not divisible by the square of any prime, and s is minimized. Find $p + q + r + s$.

4. Circles A and B are externally tangent at point C. Circle A has radius 9 and circle B has diameter 9. Let a common external tangent to both circles meet circle A at point D and meet circle B at point E. Find the area of $\triangle CDE$.

5. Suppose two semicircles have the same center, with one semicircle having twice the radius as the other. Let the larger semicircle have endpoints A and B. Randomly pick a point C on the arc AB. What is the probability that $\triangle ABC$ does not completely contain the smaller semicircle?

6. In $\triangle ABC$, let points D and E be on side BC such that $BD : DE : EC = 5 : 2 : 1$. Let point F be on side AB such that $AF : FB = 2 : 3$. Cevian CF meets AD at point G and AE at point H. The ratio of the areas of triangles $\triangle AGF$ and $\triangle AHG$ can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

7. Connect the midpoints of a unit square to form four right triangles. Compute the area of the quadrilateral with vertices as the incenters of these four right triangles.

8. Let $P_0P_3P_6P_9$ be a unit square. Let points P_1, P_2 lie on $\overline{P_0P_3}$, P_4, P_5 lie on $\overline{P_3P_6}$, P_7, P_8 lie on $\overline{P_6P_9}$, and P_{10}, P_{11} lie on $\overline{P_9P_0}$, such that for $k = 0, \ldots, 10$, $P_kP_{k+1} = P_{k+1}P_{k+2}$ where $P_{12} = P_0$. The area of the intersection of polygons $P_1P_4P_7P_{10}$ and $P_2P_5P_8P_{11}$ can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

9. Suppose $\triangle ABC$ has side lengths $AB = 7$, $AC = 8$, and $BC = 9$. Construct $\triangle DEF$ such that $DF \parallel AC$, $EF \parallel BC$, $DE \parallel AB$, and the circumcircle of $\triangle DEF$ is the incircle of $\triangle ABC$. The area of $\triangle DEF$ can be expressed in the form $\frac{p\sqrt{q}}{r}$, where p, q, r are positive integers, q is not divisible by the square of any prime, and $\gcd(p, r) = 1$. Find $p + q + r$.

10. Let T be the number you will receive. Circle ω_1 has radius $4T$ and circle ω_2 has radius T. Suppose a third circle ω_3 is tangent to ω_1, ω_2, and one of the common external tangents to the two circles. Compute the radius of ω_3.

11. Let $A_1B_1C_1D_1$ be a square of side length 4. Let points E_1, F_1 trisect A_1B_1 such that $A_1E_1 = E_1F_1 = F_1B_1$. Inscribe a square $A_2B_2C_2D_2$ in quadrilateral $E_1F_1C_1D_1$ such that one of its sides rests on C_1D_1. Repeat the process for $A_2B_2C_2D_2$ to create a new square $A_3B_3C_3D_3$, and so on. If $[ABCD]$ denotes the area of quadrilateral $ABCD$, compute $\sum\limits_{k=1}^{\infty} [A_kB_kC_kD_k]$.

12. Let $\triangle ABC$ be a triangle with side lengths $AB = 7$, $AC = 8$, and $BC = 10$. Let point D be on $\overline{BC}$ such that $\angle BAD = \angle CAD$. Let the altitudes from D meet $\overline{AB}$ and $\overline{AC}$ at points E and F respectively. The area of quadrilateral $AEDF$ can be expressed in the form $\frac{p\sqrt{q}}{r}$, where p, q, r are positive integers, q is not divisible by the square of any prime, and $\gcd(p, r) = 1$. Find $p + q + r$.

13. Triangle $\triangle ABC$ has side lengths $AB = 4$, $BC = 3$, and $AC = 5$. The angle bisector of $\angle ACB$ intersects the perpendicular at point A to $\overline{AC}$ at point D. Find the area of quadrilateral $ACBD$.

14. Six towns are arranged in the form of a regular hexagon with side length 2. Two towns are *adjacent* if they are consecutive vertices on the hexagon. All roads between two towns exist if and only if those two towns are not adjacent. Sally starts at one town and visits all other towns. What is the length of the shortest path Sally can take?

15. Let T be the number that you will receive. Let $ABCD$ be a square of side length T. Point E is the midpoint of $\overline{AB}$. Inscribe a circle ω in the square, and let points F and G be the intersections of $\overline{ED}$ and $\overline{EC}$ with circle ω (other than point E) respectively. Find the area of $\triangle EFG$.

16. Points D and E are located on side $\overline{BC}$ of $\triangle ABC$ such that $BD = CE$, $\angle BAD = \angle CAE = 30^\circ$, and $\angle DAE = 60^\circ$. Side $\overline{BC}$ has length 7. Find the length of $\overline{AB}$.

17. Let T be the number you will receive. $ABCDEFGH$ is a regular hexagon of side length T. Points U, V, W, X, Y, and Z are the midpoints of sides $\overline{BC}, \overline{CD}, \overline{DE}, \overline{FG}, \overline{GH}$, and $\overline{HA}$ respectively. The area of hexagon $UVWXYZ$ can be represented as $\frac{p+q\sqrt{r}}{s}$, where p, q, r, and s are positive integers, r is not divisible by the square of any prime, and s is minimized. Find $p + q + r + s$.

18. Point D is chosen on minor arc AC of the circumcircle of equilateral triangle $\triangle ABC$ with side length 2 such that the perimeter of quadrilateral $ABCD$ is 7. Find the length of BD.

19. A triangle with side lengths a, b, and c has circumradius 5 and inradius 3. Find
$$\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}.$$

20. A triangle has side lengths 2020, 2022, and 2018. Let S be the sum of the squares of the three medians of the triangle. Find the last three digits of S.

21. A semicircle of radius 1 has diameter AB. Point C is on the arc AB to form $\triangle ABC$. Compute the maximum value of the product of the three side lengths of $\triangle ABC$.

22. A square $ABCD$ has midpoints E, F, G and H on sides AB, BC, CD, and DA respectively. Let the square bounded by line segments AG, CE, BH, and DF be $\mathcal{P}$. Compute the ratio of the area of $\mathcal{P}$ to the area of $ABCD$.

23. Let S be the quadrilateral whose vertices are the intersections of the curves $y = x^2$, $y = 8x^2$, $xy = 1$, and $xy = 8$. Find the area of S.

24. A circle of radius 2 rolls around a triangle $\triangle ABC$ with $AC = 6$, $AB = 3$ and $\angle ABC = 90°$. Find the total area swept by the rolling circle.

25. A triangle with side lengths $5, 6$, and 7 is inscribed in a circle. Find the area of the region consisting of points inside the circle but outside the triangle.

§4 Number Theory

1. A positive integer is called *picky* if it is not divisible by any single digit prime. How many *picky* positive integers are there less than 2020?

2. How many zeroes does the product $123! \cdot 987!$ end in?

3. Let T be the number you will receive. Define $f(n) = \left\lceil (\sqrt{n+1} + \sqrt{n})^4 \right\rceil$ for $n > 0$, where $\lceil x \rceil$ is the least integer greater than or equal to x. Find the remainder when $f(T)$ is divided by 1000.

4. $\triangle ABC$ has side lengths $AC = 7$, $AB = 8$, and $BC = a$. A cevian from point A meets side $\overline{BC}$ at point D that is not point B or C, and let $CD = m$. If a and m are positive integers, find the number of ordered pairs (a, m) which satisfy these conditions.

5. Calculate the remainder when
$$\frac{1 + 2020 + 2020^2 + 2020^3 + \ldots + 2020^{2020}}{1 + \frac{1}{2020} + \frac{1}{2020^2} + \frac{1}{2020^3} + \ldots + \frac{1}{2020^{2020}}}$$
is divided by 7.

6. The number $64 \cdot 11^4 + 81 \cdot 7^4$ has 6 positive divisors. Find the sum of its prime factors.

7. Compute
$$\sum_{k=0}^{2020} \sin\left(\frac{\pi}{2} \cdot k^3 + \frac{\pi}{4}\right).$$

8. A set of three consecutive integers, all between 2 and 100 inclusive, is chosen. What is the probability that the product of the three numbers is divisible by 7?

9. Find the sum of the prime factors of 159999.

10. Find the sum of the prime factors of $3^{12} + 3^9 + 3^5 + 1$.

11. What is the largest power of 2020 that divides 2020!?

12. A *special* polygon is eight-sided and has interior angles which form an arithmetic sequence. Two polygons are called *distinct* if they have different sets of interior angles. How many *distinct special* polygons are there?

§5 Solutions

§5.1 Algebra

1. Considering the general expression $\sqrt{(x-3)(x-1)(x+1)(x+3)+16}$, we see that this rearranges to $\sqrt{(x^2-9)(x^2-1)+16}$, or $\sqrt{x^4-10x^2+25}$. This then simplifies to x^2-5. Thus our answer is $34^2-5=\boxed{1151}$.

2. We let $y=2-x$ to exploit symmetry so that $x^4+(4-x)^4$ rewrites as $(2-y)^4+(2+y)^4$, which simplifies to $2(y^4+24y^2+16)$. This equals 466, so we end up with $y^4+24y^2-217=0$, which factors to $(y^2+31)(y^2-7)=0$. We are looking for real solutions, so we have $y=\pm\sqrt{7}$. This gives us $x=\boxed{2+\sqrt{7},2-\sqrt{7}}$ as our real solutions.

3. Let N equal the expression inside the parentheses. Then we have $N=\sqrt{2\sqrt{3\sqrt{4N}}}$. This simplifies to $N^7=2^4\cdot 3^2\cdot 4=\boxed{576}$.

4. As the problem statement suggests, 39 can be represented as $m+(m+1)+(m+2)+\ldots+(n-1)+n$ for arbitrary values of m and n. Now note that $m+(m+1)+(m+2)+\ldots+(n-1)+n=\frac{(n-m+1)(n+m)}{2}$. Thus, we should investigate

$$(n-m+1)(n+m)=78.$$

We can take advantage of m and n being integers, and the different parities of $(n-m+1)$ and $(n+m)$ (i.e. one is odd and one is even). We can enumerate all possibilities below:

$(n-m+1)$	$(n+m)$	Factorization of 78 into odd and even 'pairs'	Solution (n,m)
$n-m=1$	$n+m=39$	$2\cdot 39$	$(20,19)$
$n-m=5$	$n+m=13$	$6\cdot 13$	$(9,4)$
$n-m=25$	$n+m=3$	$26\cdot 3$	$(14,-11)$
$n-m=77$	$n+m=1$	$78\cdot 1$	$(39,-38)$
$n-m=38$	$n+m=2$	$39\cdot 2$	$(20,-18)$
$n-m=12$	$n+m=6$	$13\cdot 6$	$(9,-3)$
$n-m=2$	$n+m=26$	$3\cdot 26$	$(14,12)$
$n-m=0$	$n+m=78$	$1\cdot 78$	$(39,39)$

From here, we see that there are $\boxed{6}$ ways to express 39 as a sum of more than 2 consecutive integers.

5. Note that $(a_1+a_2+\ldots+a_{10})$, $(a_{11}+a_{12}+\ldots+a_{20})$, …, $(a_{2011}+a_{2012}+\ldots+a_{2020})$ is also an arithmetic sequence. We are given the first term of this new sequence to be 10, and the 202nd term to be 412. Then the common difference of this sequence is 2. Thus, the sum of the first 2020 terms is simply $10+12+14+\ldots+412$. Note that

$$\begin{aligned}10+12+14+\ldots+412 &= 202\cdot 10+(2+4+\ldots+402)\\ &= 2020+2(1+2+\ldots+201)\\ &= 2020+201\cdot 202.\end{aligned}$$

Now note that $201 \cdot 202 = (200 + 1)(200 + 2) = 200^2 + 3 \cdot 200 + 2 \equiv 602 \mod 1000$, and $2020 \equiv 20 \mod 1000$. Thus, our answer is $\boxed{622}$.

6. Noticing that $(3x - 1) + (x + 2) = 4x + 1$, let $a = 3x - 1$ and $b = x + 2$. Then we have the equation $a^4 + b^4 = (a + b)^4$, which reduces to $2ab(2a^2 + 3ab + 2b^2) = 0$. Thus, we have

$$2(3x - 1)(x + 2)(2(3x - 1)^2 + 3(3x - 1)(x + 2) + 2(x + 2)^2) = 0.$$

Now, $2(3x-1)^2+3(3x-1)(x+2)+2(x+2)^2$ simplifies to $29x^2+11x+4$, whose discriminant $121 - 4 \cdot 4 \cdot 29 < 0$. Hence there are no real solutions from that quadratic. Thus, our only real solutions are $x = \boxed{-2,\ \frac{1}{3}}$.

7. Let $x^2 + 17x + 70 = k^2$. Complete the square:

$$\left(x + \frac{17}{2}\right)^2 + 70 - \frac{289}{4} = k^2.$$

Multiply both sides by 4 and simplify:

$$(2x + 17)^2 - 9 = 4k^2.$$

We rearrange this as $(2x+17)^2 - 4k^2 = 9$, and the left hand side can be factored as a difference of squares:

$$(2x + 2k + 17)(2x - 2k + 17) = 9.$$

Now we proceed by casework: $1 \cdot 9, 3 \cdot 3, 9 \cdot 1, -1 \cdot -9, -3 \cdot -3$, and $-9 \cdot 1$. We can solve for x in each of these cases, giving us four unique values: $x = \boxed{-11,\ -10,\ -7,\ -6}$.

8. The relation rearranges to

$$a_{n+2} = \frac{a_n a_{n+1} - 1}{a_n + a_{n+1}}.$$

Note that this resembles the angle addition formula for cotangent. If $a_n = \cot(\alpha)$ and $a_{n+1} = \cot(\beta)$, then $a_{n+2} = \frac{\cot(\alpha)\cot(\beta)-1}{\cot(\alpha)+\cot(\beta)} = \cot(\alpha + \beta)$.

The initial values $a_1 = 1$ and $a_2 = \frac{\sqrt{3}}{3}$ suggest $a_1 = \cot(45°)$ and $a_2 = \cot(60°)$. Noting that cotangent has a period of 180°, the angles corresponding to the next few terms of the sequence can be quickly computed in modulo 180:

$$105°, 165°, 90°, 75°, 165°, 60°, 45°, 105°, 150°, 75°, 45°, 120°,$$
$$165°, 105°, 90°, 15°, 105°, 120°, 45°, 165°, 30°, 15°, 45°, 60°, \ldots$$

We can see that the sequence repeats every 24 terms (also note that no angle that is a multiple of 180° has appeared, so our redefinition of cotangent is well-defined). Thus, $a_{2020} = a_4 = \cot(165°)$. This is $\cot(120° + 45°)$, so we can use the angle addition formula to get $\cot(165°) = \boxed{-2 - \sqrt{3}}$.

9. By casework, we find that the bounds are $x = \pm 5$, $y = \pm 5$, $y = x - 5$, and $y = x + 5$. Then the area of the region is the area of a square with side length 10 minus two isosceles right triangles with leg 5. This is $10^2 - 2 \cdot \frac{1}{2} \cdot 5^2 = \boxed{75}$.

10. The finite limit of i on the two inner summations renders this expression rather cumbersome to solve directly. Instead, we can switch the order of summations to allow us to apply the infinite geometric series formula more conveniently:

$$\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\sum_{i=k}^{\infty}\frac{1}{2^{i+j+k}}.$$

Then,

$$\begin{aligned}
\sum_{j=1}^{\infty}\sum_{k=j}^{\infty}\sum_{i=k}^{\infty}\frac{1}{2^{i+j+k}} &= \sum_{j=1}^{\infty}\frac{1}{2^j}\left(\sum_{k=j}^{\infty}\frac{1}{2^k}\left(\sum_{i=k}^{\infty}\frac{1}{2^i}\right)\right)\\
&= \sum_{j=1}^{\infty}\frac{1}{2^j}\left(\sum_{k=j}^{\infty}\frac{1}{2^{2k-1}}\right)\\
&= \sum_{j=1}^{\infty}\frac{1}{2^j}\cdot\frac{2}{3}\cdot\frac{1}{4^{j-1}}\\
&= \frac{2}{3}\cdot\frac{4}{7}\\
&= \boxed{\frac{8}{21}}.
\end{aligned}$$

11. Fix $r > 0$. Let $x = r'\cos\theta$, $y = r'\sin\theta$, where $0 \le r' \le r$. Then,

$$\begin{aligned}
\left|x^2 + 2xy - y^2\right| &= r'^2 \cdot \left|\cos^2\theta - \sin^2\theta + 2\sin\theta\cos\theta\right|\\
&= r'^2 \cdot |\cos 2\theta - \sin 2\theta|\\
&= r'^2 \cdot \sqrt{2} \cdot \left|\frac{1}{\sqrt{2}}\cos 2\theta - \frac{1}{\sqrt{2}}\sin 2\theta\right|\\
&= \sqrt{2}r'^2\left|\sin(2\theta + \tfrac{\pi}{4})\right|\\
&\le r^2\sqrt{2},
\end{aligned}$$

with equality iff $\theta = \frac{\pi}{4}k - \frac{\pi}{8}$ or $\frac{3\pi}{4}k - \frac{\pi}{8}$ for $k \in \mathbb{Z}$.

Hence, it suffices to count the values of $r \in \mathbb{Z}$, $r > 0$ such that $r^2\sqrt{2} \le 2020$. This rearranges to $r^2 \le 1010\sqrt{2} \approx 1428$. Then $1 \le r \le 37$, so there are $\boxed{37}$ values of r.

12. Notice that the first term $4x^2$ is also $(2x)^2$, which motivates the observation that $4xy = (2x+y)^2 - (2x)^2 - y^2$. Then,

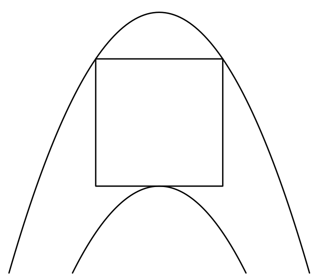
$$\begin{aligned}
4x^2 + 4xy - 12x + y^2 - 6y + 9 &= 4x^2 + ((2x+y)^2 - 4x^2 - y^2) - 12x + y^2 - 6y + 9\\
&= (2x+y)^2 - 12x - 6y + 9\\
&= (2x+y)^2 - 6(2x+y) + 9\\
&= (2x+y-3)^2.
\end{aligned}$$

Then $(2x+y-3)^2 = 0 \implies 2x+y-3=0$, which is the line $y = 2x-3$. Drop an altitude from the origin to the line and consider the triangle formed by the coordinate axes and the line. Calculating the area in two different ways (or using similar triangles) gives us the altitude length $\boxed{\frac{3\sqrt{5}}{5}}$.

13. Don't forget solutions $x = 1$ and 2. Now focus on $\tan x = \sin(2x)$. Carefully graphing $y = \tan x$ and $y = \sin(2x)$ from $x = 0$ to $x = \frac{7}{2}\pi$ reveals 9 solutions on $(0, 10)$. Hence, there are $9 + 2 = \boxed{11}$ solutions.

14. Rearrange to $x^3 - 3\sqrt{3}x^2 + 9x - 3\sqrt{3} = 5$, and notice that the left-hand side is actually $(x-\sqrt{3})^3$ by the Binomial Theorem. Hence, we have $x - \sqrt{3} = \sqrt[3]{5}$, so $x = \sqrt{3} + \sqrt[3]{5}$. Our answer is $3 + \frac{1}{2} + 5 + \frac{1}{3} = \boxed{\frac{53}{6}}$.

15. The region is bounded by the parabolas $y = -x^2 - 1$ and $y = -x^2 + 1$. Consider the following configuration:



Suppose the top side of this square lies on the line $y = k$, for $0 < k < 1$. This line intersects the top parabola $y = -x^2 + 1$ at points $(-\sqrt{1-k}, k)$ and $(\sqrt{1-k}, k)$. This gives a side length of $2\sqrt{1-k}$.

The bottom side of this square lies on the line $y = -1$. This means that either the right or left side of this square is the perpendicular distance between lines $y = -1$ and $y = k$, which is $k + 1$. This is equal to the side length of the square.

Hence, we solve $2\sqrt{1-k} = k + 1$ to obtain $k = 2\sqrt{3} - 3$. This gives us an area of $4(1-k) = \boxed{16 - 8\sqrt{3}}$.

16. In fact, the expression

$$1\% \text{ of } 2\% \text{ of } 3\% \text{ of } \ldots 100\% \text{ of } n$$

is equivalent to

$$\frac{1}{100} \cdot \frac{2}{100} \cdot \frac{3}{100} \cdots \frac{100}{100} \cdot n,$$

or $\frac{100!}{100^{100}}n$. We are given that this is equal to $99!$. Then, $n = 99! \cdot \frac{100^{100}}{100!} = \frac{100^{100}}{100} = 100^{99} = (10^2)^{99} = 10^{198}$. This number is just 1 followed by $\boxed{198}$ zeroes.

17. Let $y = \sqrt[x]{x\sqrt[x]{x\sqrt[x]{x\cdots}}}$. Then $y = \sqrt[x]{xy}$, which rearranges to $y = x^{\frac{1}{x-1}}$. We are given $y^2 = 3$, so $y = 3^{\frac{1}{2}}$. Hence, $x = \boxed{3}$ satisfies this equation.

18. The given equation rearranges to
$$\frac{\cos^4 x + 1}{\cos^2 x},$$
and let $y = \cos^2 x$. Then, we need to minimize $\frac{y^2+1}{y} = y + \frac{1}{y}$. By the AM-GM inequality, $y + \frac{1}{y} \geq 2\sqrt{y \cdot \frac{1}{y}} = 2$. Equality holds if and only if $y = \frac{1}{y}$. Since $y = \cos^2 x = 1$ is definitely obtainable, the minimum value is $\boxed{2}$.

19. Since $x \in [-5, 5]$, let $x = 5\sin\theta$. The given expression turns into
$$\frac{5\cos\theta}{\sqrt{25\sin^2\theta + 1}} = \frac{5\cos\theta}{\sqrt{26 - 25\cos^2\theta}} = \frac{5}{\sqrt{\frac{26}{\cos^2\theta} - 25}},$$
assuming $\cos\theta \neq 0$.

    We want to minimize the denominator, which is equivalent to minimizing $\frac{26}{\cos^2\theta}$. The maximum value of $\cos^2\theta$ is 1. Hence, our answer is $\frac{5}{\sqrt{26-25}} = \boxed{5}$.

20. Note that S is a right triangle with side lengths 1 and $\frac{1}{a}$. Then the altitude to the hypotenuse has length $\frac{1}{\sqrt{a^2+1}}$, which is also equal to the radius of the circle. The area of R is $\frac{\pi}{4(a^2+1)}$ and the area of S is $\frac{1}{2a}$, so the ratio is
$$\frac{\pi a}{2(a^2+1)}.$$
It suffices to maximize $\frac{a}{a^2+1}$ for $a > 0$. Let $a = \tan\theta$ for $\theta \in \left(0, \frac{\pi}{2}\right)$. Then, the expression rearranges to $\frac{\tan\theta}{\sec^2\theta} = \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$. As $\sin 2\theta$ achieves its maximum of 1 given $\theta \in \left(0, \frac{\pi}{2}\right)$, we know that $\frac{a}{a^2+1}$ achieves its maximum of $\frac{1}{2} \cdot 1 = \frac{1}{2}$ for some positive value of a.

    Hence, the maximal ratio is $\boxed{\frac{\pi}{4}}$.

21. By Vieta's Formulas, $r^2 + s^2 + t^2 = (r+s+t)^2 - 2(rs + st + rt) = a^2 - 2b = 0$. It suffices to find $a, b \in [1, 2020]$ such that $a^2 = 2b$. This is equivalent to finding the number of even perfect squares up to $2 \cdot 2020 = 4040$. Since $\lfloor\sqrt{4040}\rfloor = 63$, we have the ordered pairs $(2,2), (4,8), (6,18), \ldots, \left(62, \frac{62^2}{2}\right)$, giving $\boxed{31}$ pairs.

22. Note that the region is symmetric about the y-axis, so it suffices to find the area in the first quadrant and multiply by 2.

    Properly graphing these two equations reveals that the area of the region in the first quadrant is equal to the area of a 2×4 rectangle minus the area of two right triangles. This gives $2 \cdot 4 - \frac{1}{2} \cdot 2 \cdot 2 - \frac{1}{2} \cdot 2 \cdot 4 = 2$. Hence, our answer is $2 \cdot 2 = \boxed{4}$.

23. First, note that
$$\frac{1}{\tan^2(2k) + 1} = \frac{1}{\sec^2(2k)} = \cos^2(2k) = \frac{1}{2}(\cos(4k) + 1).$$
The piecewise definition of the function allows us to assume that it is well-behaved, so we can assume that $f(k) = \frac{1}{2}(\cos(4k) + 1)$ for $k = 1°, 2°, \ldots, 89°$.

Then,

$$\begin{aligned}\sum_{k=1^\circ}^{89^\circ} \frac{1}{\tan^2(2k)+1} &= \sum_{k=1^\circ}^{89^\circ} \frac{1}{2}(\cos(4k)+1) \\ &= \frac{89}{2} + \frac{1}{2}\sum_{k=1^\circ}^{89^\circ} \cos(4k) \\ &= \frac{89}{2} + \frac{1}{2}\sum_{k=1}^{89} \cos\left(\frac{\pi k}{45}\right).\end{aligned}$$

Note that $e^{\frac{\pi k i}{45}} = e^{\frac{2\pi k i}{90}}$ for $k = 0, 1, \ldots, 89$ are roots of the equation $z^{90} - 1 = 0$. By Vieta's Formulas, the sum of these roots is 0, i.e.

$$\sum_{k=0}^{89} e^{\frac{\pi k i}{45}} = 0.$$

Then $\sum\limits_{k=1}^{89} e^{\frac{\pi k i}{45}} = -1 \implies \operatorname{Re}\left(\sum\limits_{k=1}^{89} e^{\frac{\pi k i}{45}}\right) = \operatorname{Re}(-1) \implies \sum\limits_{k=1}^{89} \cos\left(\frac{\pi k}{45}\right) = -1$. Thus, our answer is

$$\frac{89}{2} + \frac{1}{2}\cdot(-1) = \boxed{44}.$$

24. Note that

$$\prod_{k=1}^{\infty} f\left(\frac{(2k-1)\pi}{9\cdot 2^k}\right) = f\left(\frac{\pi}{9}\sum_{k=1}^{\infty}\frac{2k-1}{2^k}\right),$$

by De Moivre's Theorem. Furthermore,

$$\sum_{k=1}^{\infty}\frac{2k-1}{2^k} = \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \cdots$$

which is an arithmetico-geometric series. Let this sum be S. Then

$$\frac{1}{2}S = \frac{1}{4} + \frac{3}{8} + \frac{5}{16} + \cdots,$$

so

$$S - \frac{1}{2}S = \frac{1}{2} + \left(\frac{3}{4} - \frac{1}{4}\right) + \left(\frac{5}{8} - \frac{3}{8}\right) + \left(\frac{7}{16} - \frac{5}{16}\right) + \cdots = \frac{1}{2} + \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{3}{2},$$

hence $S = 3$.

Finally, we have $f\left(\frac{\pi}{9}\cdot 3\right) = f\left(\frac{\pi}{3}\right)$, so the imaginary part is $\boxed{\dfrac{\sqrt{3}}{2}}$.

25. The smallest such rectangle is formed by the domain and range of the function.

For $\cos^{-1}(\cos^{-1} x)$ to be defined, the domain of the outer $\cos^{-1}$ must be $[-1, 1]$. Note that the range of the inner $\cos^{-1} x$ is $[0, \pi]$; this does not fit inside the domain of the outer $\cos^{-1}$, so we must restrict the range of the inner $\cos^{-1} x$ to $[0, 1]$.

If the range of $\cos^{-1} x$ is $[0,1]$, then $0 \leq \cos^{-1} x \leq 1$. Taking the cosine of this inequality yields $1 \geq x \geq \cos 1$. We flip the inequality signs because $\cos 1$ is clearly less than 1.

Therefore the domain of $\cos^{-1}(\cos^{-1} x)$ is $[\cos 1, 1]$.

Because the range of $\cos^{-1} x$ is $[0,1]$, the range of $\cos^{-1}(\cos^{-1} x)$ would be $[\cos^{-1} 1, \cos^{-1} 0]$ (because $\cos^{-1} 1 < \cos^{-1} 0$), or $\left[0, \frac{\pi}{2}\right]$.

Thus the area of the region is $\frac{\pi}{2}(1-\cos 1) = \frac{\pi}{2} - \frac{\pi}{2}\cos 1$, and the answer is $\frac{\pi}{2} + \frac{\pi}{2} + 1 = \boxed{\pi + 1}$.

26. First, note that

$$(\sin x + \cos x)^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x = 1 + \sin 2x = \left(\frac{1}{2}\right)^2,$$

from which we get $\sin 2x = -\dfrac{3}{4}$.

If we know $\sin x + \cos x$, it may be helpful to also find out $\sin x - \cos x$. Observe that

$$(\sin x - \cos x)^2 = \sin^2 x - 2\sin x \cos x + \cos^2 x = 1 - \sin 2x = \frac{7}{4}.$$

Thus, we can take the square root of both sides of $\sin x - \cos x = \pm\dfrac{\sqrt{7}}{2}$. Now, we use the fact that $\dfrac{\pi}{2} < x < \pi$. On this interval, $\sin x > \cos x$ (this can be seen from looking at the unit circle or comparing graphs of sine and cosine). Thus, since $\sin x - \cos x > 0$, we choose the positive sign: $\sin x - \cos x = \dfrac{\sqrt{7}}{2}$.

Now we have the equations $\sin x + \cos x = \dfrac{1}{2}$ and $\sin x - \cos x = \dfrac{\sqrt{7}}{2}$. If we add them together, we get $2\sin x = \dfrac{1+\sqrt{7}}{2}$. If we subtract either equation from the other, we get $2\cos x = \dfrac{1-\sqrt{7}}{2}$.

Finally, we can compute

$$\tan x = \frac{2\sin x}{2\cos x} = \frac{1+\sqrt{7}}{1-\sqrt{7}} = \frac{(1+\sqrt{7})^2}{-6} = \frac{8+2\sqrt{7}}{-6} = \boxed{-\frac{4+\sqrt{7}}{3}}.$$

27. Let $x = \frac{1}{n}$, and consider $\prod_{k=0}^{m}\left(1 + x^{(2^k)}\right) = (1+x)(1+x^2)(1+x^4)\cdots\left(1+x^{(2^m)}\right)$. Multiplying this product by $1 - x$ gives

$$\begin{aligned}(1-x)(1+x)(1+x^2)(1+x^4)\cdots\left(1+x^{(2^m)}\right) &= (1-x^2)(1+x^2)(1+x^4)\cdots\left(1+x^{(2^m)}\right) \\ &= (1-x^2)(1+x^2)(1+x^4)\cdots\left(1+x^{(2^m)}\right) \\ &= (1-x^4)(1+x^4)\cdots\left(1+x^{(2^m)}\right) \\ &\vdots\end{aligned}$$

$$= 1 - x^{(2^{m+1})}.$$

Hence,

$$\prod_{k=0}^{m} \left(1 + x^{(2^k)}\right) = \frac{1 - x^{(2^{m+1})}}{1 - x}.$$

Since $0 < x < 1$ where $2 \le n \le 2020$, it follows that $x^{(2^{m+1})}$ goes to 0 as m goes to ∞. This yields

$$\prod_{k=0}^{\infty} \left(1 + x^{(2^k)}\right) = \frac{1}{1 - x}.$$

Thus, our answer is

$$\prod_{n=2}^{2020} \left(1 - \frac{1}{n}\right) = \boxed{\frac{1}{2020}}$$

by classic telescoping.

28. We can convert the even powers of sine into cosine (i.e. substitute $\sin^2\theta = 1 - \cos^2\theta$), so we obtain the expression $4\cos^4\theta - 4\cos^2\theta + 1 = (2\cos^2\theta - 1)^2 = \cos^2(2\theta) = 1 - \sin^2(2\theta)$. We can find the value of $\sin 2\theta$ by squaring both sides of the given equation $\sin\theta + \cos\theta = 1$ (elaborated below).

    Alternatively, we utilize the Sophie-Germain identity, which states that

    $$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab) = ((a + b)^2 + b^2)((a - b)^2 + b^2).$$

    Substituting in $a = \cos\theta$ and $b = \sin\theta$, we have

    $$\cos^4\theta + 4\sin^4\theta = ((\sin\theta + \cos\theta)^2 + \sin^2\theta)((\cos\theta - \sin\theta)^2 + \sin^2\theta).$$

    Given $\sin\theta + \cos\theta = \frac{1}{3}$, we square both sides and rearrange to obtain $\sin 2\theta = -\frac{8}{9}$. Then $(\cos\theta - \sin\theta)^2 = 1 - \sin 2\theta = \frac{17}{9}$. Thus, our expression simplifies to

    $$\cos^4\theta + 4\sin^4\theta = \left(\frac{1}{9} + \sin^2\theta\right)\left(\frac{17}{9} + \sin^2\theta\right) = \frac{17}{81} + 2\sin^2\theta + \sin^4\theta.$$

    We rearrange to get

    $$\cos^4\theta - 2\sin^2\theta + 3\sin^4\theta = \boxed{\frac{17}{81}}.$$

29. Observe that S consists of radicals of the form $\sqrt{n + \sqrt{2n - 1}}$ for $n = 2$ to 2020. Set $(a + b\sqrt{2n - 1})^2 = n + \sqrt{2n - 1}$ to get a system of equations

    $$\begin{aligned} a^2 + (2n - 1)b^2 &= n, \\ 2ab\sqrt{2n - 1} &= \sqrt{2n - 1}. \end{aligned}$$

One can notice that $a = b = 1$ would work for $2n + 2\sqrt{2n-1}$, therefore $a = b = \frac{1}{\sqrt{2}}$ satisfies the relation for $n + \sqrt{2n-1}$. Hence,

$$\sqrt{n+\sqrt{2n-1}} = \frac{1+\sqrt{2n-1}}{\sqrt{2}} = \frac{1}{2}\left(\sqrt{2}+\sqrt{4n-2}\right).$$

Now consider the sum $\sum\limits_{x\in S}\left(2x-\sqrt{2}\right)^2$. Indeed, for $x = \frac{1}{2}\left(\sqrt{2}+\sqrt{4n-2}\right)$, the expression $\left(2x-\sqrt{2}\right)^2$ simplifies considerably to $4n-2$. Thus, this sum is equal to

$$\sum_{n=2}^{2020}(4n-2) = 4\cdot\sum_{n=2}^{2020} n - 2\cdot 2019 = 4(1010\cdot 2021 - 1) - 2\cdot 2019 \equiv \boxed{98} \pmod{100}.$$

30. Observe his first few steps. His first stop is $(1,0)$. Then he adds $\frac{1}{2}$ to his y-coordinate to arrive at the point $\left(1,\frac{1}{2}\right)$. Then, he subtracts $\frac{1}{4}$ from his x-coordinate to stop at the point $\left(1-\frac{1}{2},\frac{1}{2}\right)$. Afterward, he subtracts $\frac{1}{8}$ from his y-coordinate to stop at the point $\left(1-\frac{1}{2},\frac{1}{2}-\frac{1}{8}\right)$.

    Continuing this process indefinitely, we obtain infinite series

    $$\left(1-\frac{1}{4}+\frac{1}{16}-\frac{1}{64}+\cdots,\frac{1}{2}-\frac{1}{8}+\frac{1}{32}-\frac{1}{128}+\cdots\right) = \left(\frac{1}{1+\frac{1}{4}},\frac{\frac{1}{2}}{1+\frac{1}{4}}\right) = \left(\frac{4}{5},\frac{2}{5}\right),$$

    so the distance is $\sqrt{\left(\frac{4}{5}\right)^2+\left(\frac{2}{5}\right)^2} = \boxed{\frac{2\sqrt{5}}{5}}$.

31. Note that if $1+\sqrt{2}$ is a root of $P(x)$, then its radical conjugate $1-\sqrt{2}$ must also be a root. The monic quadratic having that conjugate pair as its roots is x^2-2x-1, obtained by Vieta's Formulas.

    Thus, $P(x) = (x^2-2x-1)Q(x)$ for another quadratic $Q(x)$, since P is monic and has degree 4. We're given that

    $$\begin{aligned}P(3) &= 2Q(3) = 3,\\ P(4) &= 7Q(4) = 14.\end{aligned}$$

    Thus, $Q(3) = \frac{3}{2}$ and $Q(4) = 2$, so $Q(x) - \frac{x}{2}$ has roots 3 and 4. Since Q must have degree 2 and also be monic, $Q(x) = (x-3)(x-4)+\frac{x}{2}$. Thus, $P(6) = 23\cdot(3\cdot 2+3) = \boxed{207}$.

32. It would be unwise to solve for y and substitute, since computation with quadratic surds can get quite intensive. Instead, we rearrange the given equations to $x+y=3$ and $xy=1$, which are elementary symmetric polynomials. The given expression is symmetric, so it can be reduced in terms of the elementary symmetric polynomials.

    Notice that $x^4+x^3y+x^2y^2+xy^3+y^4$ resembles $(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$, but the latter has an additional $3x^3y+5x^2y^2+3xy^3$. One can almost factor out a $3xy$ from $3x^3y+5x^2y^2+3xy^3$, but the middle term has a coefficient of 5 instead of 6. Thus, simply add and subtract a term of x^2y^2, so $3x^3y+5x^2y^2+3xy^3 = 3xy(x^2+2xy+y^2)-x^2y^2 = 3xy(x+y)^2-(xy)^2$. Hence,

    $$x^4+x^3y+x^2y^2+xy^3+y^4 = (x+y)^4-3xy(x+y)^2+(xy)^2,$$

    which is equal to $3^4-3\cdot 3^2+1 = \boxed{55}$.

33. The product is $n!$, and the sum is $\frac{n(n+1)}{2}$. Their quotient is $\frac{2\cdot(n-1)!}{n+1}$, which is equal to 180. Note that the factorial function grows rapidly, and therefore a small value of n is necessary ($7! = 5040$ which is way too large). Brief guess-and-check reveals $n = \boxed{7}$ as the answer.

34. Note that $2020 - 37^2 - 4^2 = 635$, and a perfect square close to this number is $625 = 25^2$. Fortunately, $635 - 25^2 = 10$ which is simply $1^2 + 3^2$. Thus, our answer is $1 + 3 + 25 = \boxed{29}$.

35. The given equation expands to $x^3 + 3x^2 + 2x + 1 = 121$, and we "complete the cube" to get $(x+1)^3 - x = 121$. Note that 121 is 4 less than $125 = 5^3$, and indeed, $(4+1)^3 - 4 = 121$, so $x = \boxed{4}$ is the real solution.

36. If $3^x = 5$, then $x = \log_3 5$. Furthermore, if $\log_{a^n} b = c$, then $(a^n)^c = b \implies a^c = b^{1/n} \implies \log_a b^{1/n} = c \implies \frac{1}{n}\log_a b = c$, so $\log_{a^n} b = \frac{1}{n}\log_a b$. Using this fact, we have

$$\begin{aligned}\log_3(\log_{5^{1/10}}(\sqrt{9^x})) &= \log_3(\log_{5^{1/10}} 5)\\ &= \log_3(10\log_5 5)\\ &= \log_3 10,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\log_{25^{1/x}} 10} &= \log_{10}(25^{1/x})\\ &= \log_{10}(25^{\log_5 3})\\ &= \log_{10}((5^{\log_5 3})^2)\\ &= \log_{10} 9,\end{aligned}$$

so our answer is $\log_3 10 \cdot \log_{10} 9 = \boxed{2}$.

37. Suppose $y = 2 - \sqrt{x}$. Then $x = (2-y)^2$, so

$$f(2-\sqrt{x}) = f(y) = (2-y)^2((2-y)^2 - 9) = (2-y)^2(y^2 - 4y - 5) = (2-y)^2(y+1)(y-5),$$

so the sum of distinct roots is $2 - 1 + 5 = \boxed{6}$.

38. We transform the polynomial. If r satisfies $f(r) = 0$, then $r - 1$ is a root of $f(x+1)$ since $f(r-1+1) = f(r) = 0$. Then, the polynomial with roots $r-1, s-1$, and $t-1$ is

$$\begin{aligned}f(x+1) &= (x+1)^3 - 2(x+1)^2 - 2(x+1) - 3\\ &= x^3 + x^2 - 3x - 6.\end{aligned}$$

Furthermore, if r satisfies $f(r) = 0$, then $\frac{1}{r}$ is a root of $x^3 f\left(\frac{1}{x}\right)$ (we multiply by x^3 to keep the degree of the polynomial - it doesn't affect the roots). Thus, our polynomial with roots $\frac{1}{r-1}$, $\frac{1}{s-1}$, and $\frac{1}{t-1}$ is

$$x^3\left(\frac{1}{x^3} + \frac{1}{x^2} - \frac{3}{x} - 6\right) = 1 + x - 3x^2 - 6x^3.$$

Lastly, note that $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+ac+bc)$, so our answer, reading from Vieta's Formulas, is $\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{6}\right) = \boxed{\frac{7}{12}}$.

39. Let $f(x) = |x-1| + |2-x| + |x+3|$. The key values of x for which the value of $f(x)$ changes are $x = -3, 1$, and 2.

- For $x < -3$, we have $x-1$ negative, $2-x$ positive, and $x+3$ negative, so in this interval, $f(x)$ is equal to
$$-(x-1) + (2-x) - (x+3) = -3x.$$
- For $-3 < x < 1$, we have $x-1$ negative, $2-x$ positive, and $x+3$ positive, so in this interval, $f(x)$ is equal to
$$-(x-1) + (2-x) + (x+3) = -x+6.$$
- For $1 < x < 2$, all parts are positive, so in this interval, $f(x)$ is equal to
$$(x-1) + (2-x) + (x+3) = x+4.$$
- For $x > 2$, we have $x-1$ positive, $2-x$ negative, and $x+3$ positive, so in this interval, $f(x)$ is equal to
$$(x-1) - (2-x) + (x+3) = 3x.$$

Hence,
$$f(x) = \begin{cases} -3x & x \le -3 \\ -x+6 & -3 \le x \le 1 \\ x+4 & 1 \le x \le 2 \\ 3x & x \ge 2 \end{cases}.$$

Plotting this on a graph, we see that the minimum is achieved at the intersection of lines $y = -x+6$ and $y = x+4$, which is the point $(1, 5)$, so the answer is $\boxed{5}$.

40. First, we observe
$$\begin{aligned} a_{mn} &= a_m a_n + n a_m + m a_n \\ a_{mn} + mn &= a_m a_n + n a_m + m a_n + mn \\ &= (a_m + m)(a_n + n), \end{aligned}$$
so define a new sequence $b_k = a_k + k$ such that $b_{mn} = b_m b_n$. Since $a_1 = 0$, we have $b_1 = 1$, so b_n is a completely multiplicative function. Hence,
$$b_{2020} = b_2^2 b_5 b_{101} = (a_2+2)^2(a_5+5)(a_{101}+101).$$

Since $b_{2020} = a_{2020} + 2020$, we have
$$a_{2020} = (a_2+2)^2(a_5+5)(a_{101}+101) - 2020.$$

To find the sum of the coefficients, "plug in" 1 for a_2, a_5, and a_{101} to obtain $3^2 \cdot 6 \cdot 102 - 2020 = \boxed{3488}$.

41. Let $A = (a, a^2)$ and $B = (b, b^2)$. Then the slope through those two points is $\frac{b^2-a^2}{b-a} = \frac{(b-a)(b+a)}{b-a} = b + a = 4$. The midpoint of A and B is $\left(\frac{a+b}{2}, \frac{a^2+b^2}{2}\right)$, so the x-coordinate is $\frac{4}{2} = \boxed{2}$.

42. We rewrite this as $\frac{11^3+5^3}{2^3}$. Factor $11^3 + 5^3$ as $(11+5)(11^2 - 11 \cdot 5 + 5^2) = 16 \cdot 91$, so our answer is $2 \cdot 91 = \boxed{182}$.

43. Set the given expression equal to x, and let $k = \frac{4}{27}$. Then we have $x = \sqrt{\frac{k}{1+x}}$, which we solve to get $x^2(x+1) = \frac{4}{27}$. Note that $\frac{4}{27}$ can be broken up into $\left(\frac{2}{3}\right)^2 \cdot \frac{1}{3}$ or $\left(\frac{1}{3}\right)^2 \cdot \frac{4}{3}$, and x cannot be negative. Only the latter makes sense, so $x = \boxed{\frac{1}{3}}$.

44. Since the LHS is quadratic in x and the RHS is approximately linear, we expect that x cannot be too big. Indeed, we show that $-3 \leq \lfloor x \rfloor \leq 2$. This is because $x^2 = |x + \lfloor x \rfloor| \leq 2x \implies \lfloor x \rfloor \leq 2$ for $x > 0$ and $x^2 = |x + \lfloor x \rfloor| \leq |x + x - 1| = -2x + 1 \implies \lfloor x \rfloor \geq -3$ for $x < 0$. We now do casework on $\lfloor x \rfloor$:

    - Case 1: $\lfloor x \rfloor = -3$. Then $x \in [-3, -2)$ in which x^2 decreases from $9 \to 4$ and $|x + \lfloor x \rfloor|$ decreases less steeply from $5 \to 6$, yielding exactly one solution.
    - Case 2: $\lfloor x \rfloor = -2$. Then $x \in [-2, -1)$ in which x^2 decreases from $4 \to 1$ and $|x + \lfloor x \rfloor|$ decreases less steeply from $3 \to 4$, yielding exactly one solution.
    - Case 3: $\lfloor x \rfloor = -1$. Then $x \in [-1, 0)$ in which x^2 decreases from $1 \to 0$ and $|x + \lfloor x \rfloor|$ decreases from $2 \to 1$, yielding no solutions (1 is not hit simultaneously).
    - Case 4: $\lfloor x \rfloor = 0$. Then $x \in [0, 1)$ in which x^2 increases from $0 \to 1$ and $|x + \lfloor x \rfloor|$ increases more steeply from $0 \to 1$, yielding exactly one solution since 1 is not achieved when $x \in [0, 1)$.
    - Case 5: $\lfloor x \rfloor = 1$. Then $x \in [1, 2)$ in which x^2 increases from $1 \to 4$ and $|x + \lfloor x \rfloor|$ increases less steeply from $2 \to 3$, yielding exactly one solution.
    - Case 6: $\lfloor x \rfloor = 2$. Then $x \in [2, 3)$ in which x^2 increases from $4 \to 9$ and $|x + \lfloor x \rfloor|$ increases more steeply from $4 \to 5$, yielding exactly one solution.

    Thus in total there are $\boxed{5}$ solutions.

§5.2 Combinatorics

1. We proceed by recursion. Let a_n be the number of paths that end at point A, and define b_n, d_n, and f_n for points B, D, F similarly. Note that the number of paths that end at points B or C are equal by symmetry (likewise for D and E). Thus, we have the recurrence relations

$$\begin{aligned} a_n &= 2d_{n-1}, \\ b_n &= d_{n-1} + f_{n-1}, \\ d_n &= a_{n-1} + b_{n-1} + d_{n-1} + f_{n-1}, \\ f_n &= 2b_{n-1} + 2d_{n-1}. \end{aligned}$$

The bunny starts at point A, so $a_0 = 1$, $b_0 = 0$, $d_0 = 0$, and $f_0 = 0$. We compute to eventually obtain $b_7 = \boxed{344}$.

2. Essentially, $S_n = \sum\limits_{k \text{ even}}^{n} \frac{1}{k+1}\binom{n}{k}$. Note that

$$\begin{aligned} \frac{1}{k+1}\binom{n}{k} &= \frac{1}{k+1} \cdot \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(k+1)!(n-k)!} \\ &= \frac{(n+1)!}{(k+1)!(n-k)!} \cdot \frac{1}{n+1} \\ &= \frac{1}{n+1}\binom{n+1}{k+1}. \end{aligned}$$

We also utilize Pascal's Identity to obtain

$$\begin{aligned} S_n &= \sum_{k \text{ even}}^{n} \frac{1}{k+1}\binom{n}{k} \\ &= \frac{1}{n+1} \sum_{k \text{ even}}^{n} \binom{n+1}{k+1} \\ &= \frac{1}{n+1} \sum_{k \text{ even}}^{n} \left[\binom{n}{k} + \binom{n}{k+1}\right] \\ &= \frac{2^n}{n+1}. \end{aligned}$$

Hence, our answer is

$$\sum_{k=1}^{\infty} \frac{n+1}{2^n} = \boxed{3},$$

by classic arithmetico-geometric techniques.

3. We substitute $b = b' + 1$, $c = c' + 2$, and $d = d' + 3$ where b', c', and $d' \geq 0$. Then we have

$$a + b' + c' + d' = 14,$$

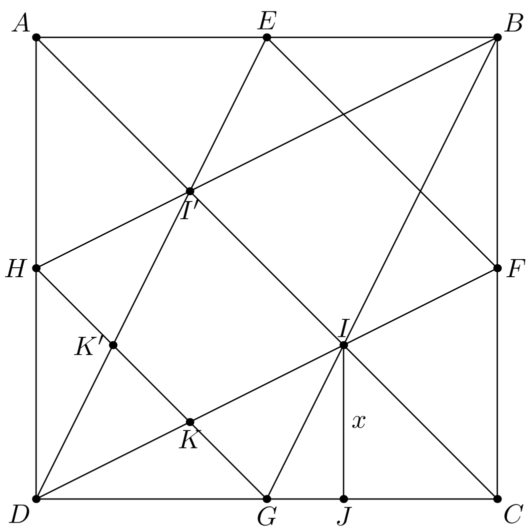
from which we use stars and bars to obtain $\binom{14}{3} = \boxed{364}$.

4. Since $5^3 = 125 > 100$, we only need to consider $0^3, 1^3, \ldots, 4^3$. The value of $\lfloor \sqrt[3]{x} \rfloor$ will be odd when $\lfloor \sqrt[3]{x} \rfloor = 1$ or 3, so if x is chosen such that $1 \leq x < 8$ or $27 \leq x < 64$, this will be satisfied. Since x is chosen from an interval of length 100, the probability is

$$\frac{(8-1)+(64-27)}{100} = \boxed{\frac{11}{25}}.$$

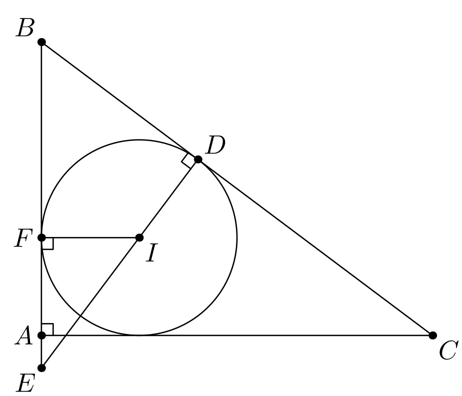
§5.3 Geometry

1. Let DF and BG meet at point I, DF meet GH at point K, DE meet AC at point I', and DE meet GH at point K'. Let the foot of the altitude from point I to CD be point J, and set $IJ = x$.



Then $\triangle IJG \sim \triangle BCG$, so $\frac{x}{GJ} = \frac{BC}{CG} = 2$. But $GJ = 1 - CJ = 1 - x$, so we can solve to get $x = \frac{2}{3}$. Then $IC = \frac{2\sqrt{2}}{3}$, implying that IC is a third of diagonal AC. Similarly, $I'A$ is also a third of the diagonal. Hence, triangles $\triangle AI'D$, $\triangle I'ID$, and $\triangle ICD$ have equal bases with equal height, so they each have area $\frac{2}{3}$. Since $\triangle DK'K$ is similar to $\triangle DI'I$ by ratio of similitude $\frac{1}{2}$, the area of $\triangle DK'K$ is $\frac{2}{3} \cdot \left(\frac{1}{2}\right)^2$. Hence the area of quadrilateral is $I'IKK'$ $\frac{2}{3} - \frac{2}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$. By symmetry, the area of the entire intersection is $\frac{1}{2} \cdot 2 = \boxed{1}$.

2. First notice that $\triangle ABC$ is a $3, 4, 5$ right triangle with a right angle at A. Using $[ABC] = rs$ to find the inradius, we have $3 \cdot 4 \cdot \frac{1}{2} = r \cdot \left(\frac{3+4+5}{2}\right) \implies r = 1$.

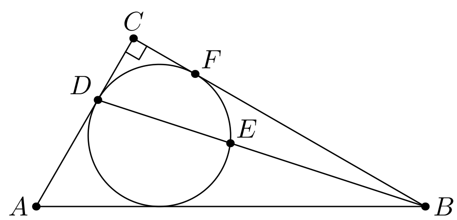


Let $AE = x$. Then by the Pythagorean Theorem on $\triangle BDE$, $DE^2 + DB^2 = BE^2 \implies DE^2 = (3 + x)^2 - 2^2 = x^2 + 6x + 5$. Letting the incircle meet $\overline{AB}$ at point F, by AA, $\triangle IFE \sim \triangle BDE \implies \frac{IF}{FE} = \frac{BD}{DE} \implies \frac{IF^2}{FE^2} = \frac{BD^2}{DE^2}$. We know that $IF = r = 1$, and it is clear that $FA = IF = 1$ as well, implying $FE^2 = (FA + AE)^2 = (1 + x)^2$. Finally $BD = BF = BA - FA = 2$, and substituting everything gives

$$\frac{1^2}{(1+x)^2} = \frac{2^2}{x^2+6x+5}.$$

This rearranges to $3x^2 + 2x - 1 = 0$, whose only positive solution is $AE = x = \boxed{\frac{1}{3}}$.

3. Let $AC = s$, such that $BC = s\sqrt{3}$ and $AB = 2s$.



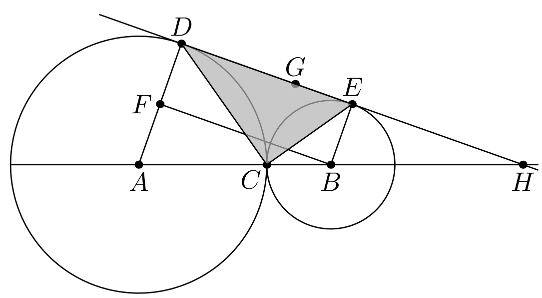
Then, using the formula $[ABC]$ = inradius times the semiperimeter, we obtain $r = \frac{s\sqrt{3}}{3+\sqrt{3}}$. Let the incircle of $\triangle ABC$ meet BC at point F. Since the length of CF must be equal to the inradius, we get $BF = s\sqrt{3} - \frac{s\sqrt{3}}{3+\sqrt{3}} = s\left(\frac{\sqrt{3}+1}{2}\right)$. By Power of a Point, $BF^2 = BE \cdot BD$. Then $BE = \frac{BF^2}{BD}$, so the ratio $\frac{BE}{BD}$ can be rewritten as $\frac{BF^2}{BD^2}$. By the Pythagorean Theorem, $BD^2 = BC^2 + CD^2 = (s\sqrt{3})^2 + \left(\frac{s\sqrt{3}}{3+\sqrt{3}}\right)^2 = s^2\left(4 - \frac{\sqrt{3}}{2}\right)$. Then the ratio is

$$\frac{BE}{BD} = \frac{BF^2}{BD^2} = \frac{s^2\left(\frac{\sqrt{3}+1}{2}\right)^2}{s^2\left(4-\frac{\sqrt{3}}{2}\right)} = \frac{19+10\sqrt{3}}{61}.$$

Thus the answer is $19 + 10 + 3 + 61 = \boxed{93}$.

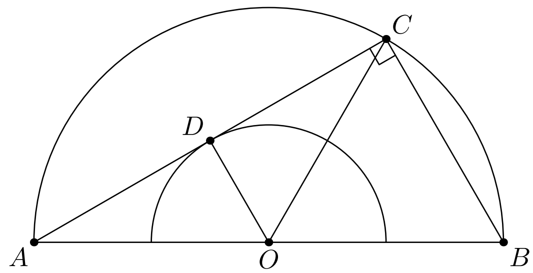
Note that the work could be simplified by letting $s = 1$, as the ratio stays the same regardless of the value of s.

4. Let point F be the midpoint of $\overline{AD}$, and let point G be the foot of the altitude from C to $\overline{DE}$.



Since $AF = BE = \frac{9}{2}$ and $AB = AC+CB = \frac{3r}{2}$, we apply the Pythagorean Theorem to $\triangle AFB$ to obtain $FB = DE = 9\sqrt{2}$. Now extend AB and DE to meet at point H. By similarity of $\triangle ADH$ and $\triangle BEH$, we get $BH = \frac{27}{2}$. Then $CH = CB + BH = 18$. By similarity of $\triangle BEH$ and $\triangle CGH$, we obtain $CG = 6$. Now, the area of $\triangle CDE$ is $6 \cdot 9\sqrt{2} \cdot \frac{1}{2} = \boxed{27\sqrt{2}}$.

5. Let point O be the common center, and suppose the smaller semicircle has radius R.



If we let the external tangent from point A to the smaller semicircle meet the larger semicircle at point C, we can see that choosing any point C' between arc BC will result in $\triangle AC'B$ not completely containing the smaller semicircle. Thus we want to avoid choosing any point on the arc BC. The same conclusion can be drawn from reflecting the diagram horizontally, so the probability is simply $\frac{180-2\angle COB}{180^\circ}$.

To find $\angle COB$, we use similar triangles. Let point D be the foot of the perpendicular from point O to external tangent AC. Then the conditions $OD = R$, $AO = 2R$, and $\angle ADO = 90^\circ$ imply that $\triangle ADO$ is a 30-60-90 right triangle. Hence, $\angle DAO = 30^\circ$. Furthermore, by Thales's theorem, $\angle ACB = 90^\circ$. Thus, $\angle ABC = 60^\circ$ from which we conclude $\triangle COB$ is equilateral (so $\angle COB = 60^\circ$). Our answer is $\frac{180-2\cdot 60}{180} = \boxed{\frac{1}{3}}$.

6. Apply mass point geometry on two systems: one excluding segment AD, and the other one excluding segment AE. This yields $FG : GC = 2 : 3$ and $FH : HC = 14 : 5$. Letting $FG = x$, $GH = y$, and $HC = z$ gives us $\frac{x}{y+z} = \frac{2}{3}$ and $\frac{x+y}{z} = \frac{14}{5}$. From this, we can conclude that $\frac{x}{y} = \frac{19}{16}$. Since $\triangle AGF$ and $\triangle AHG$ have the same height, the ratio of their areas is the ratio of their bases, which we already found to be $\frac{19}{16}$. The answer is $19 + 16 = \boxed{35}$.

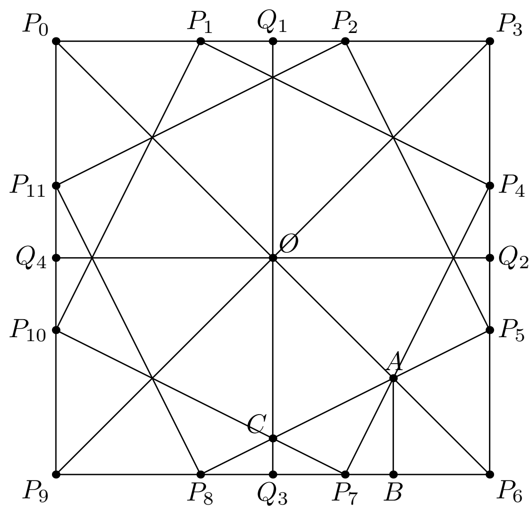
7. Consider one such right triangle with leg $\frac{1}{2}$. By the formula $A = rs$, we obtain

$$r = \frac{A}{s} = \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}+\frac{\sqrt{2}}{2}\right)} = \frac{2-\sqrt{2}}{4}.$$

The area of the resulting quadrilateral (which is a square) is

$$(1-2r)^2 = \left(1 - \frac{2-\sqrt{2}}{2}\right)^2 = \boxed{\frac{1}{2}}.$$

8. Let points Q_1, Q_2, Q_3, and Q_4 be the midpoints of P_0P_3, P_3P_6, P_6P_9, and P_9P_0. Let point O be the center of the square. Also, let point A be the intersection of P_4P_7 and P_5P_8. Let point B be the foot of the altitude from A to P_6P_7. Lastly, let point C be the intersection of P_7P_{10} and P_5P_8. Refer to the diagram below.



It suffices to find the area of $\triangle ACO$ and multiply it by 8. Note that the length of the altitude from A to CO is equal to the length of Q_3B, so we just have to find Q_3B and OC.

First, let's find Q_3B. Since $\triangle ABP_7 \sim \triangle P_4P_6P_7$, we have the ratio $\frac{AB}{BP_7} = \frac{P_4P_6}{P_6P_7} = 2$. Noticing that $BP_7 = 1 - AB$, we solve to get $AB = \frac{2}{9}$. Then, $Q_3B = Q_3P_6 - BP_6 = \frac{1}{2} - \frac{2}{9} = \frac{5}{18}$.

Now, we need to find CO. Since $\triangle CQ_3P_8 \sim \triangle P_5P_6P_8$, we have $\frac{CQ_3}{Q_3P_8} = \frac{P_5P_6}{P_6P_8} = \frac{1}{2}$. Also, $Q_3P_8 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, we solve to get $CQ_3 = \frac{1}{12}$. Then, $CO = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$.

Finally, the total area is

$$8 \cdot [ACO] = 8 \cdot \frac{1}{2} \cdot \frac{5}{12} \cdot \frac{5}{18} = \frac{25}{54},$$

so our answer is $25 + 54 = \boxed{79}$.

9. By Heron's formula, the area of $\triangle ABC$ is $12\sqrt{5}$. We use the formulas $A = rs$ and $A = \frac{abc}{4R}$ to obtain that the inradius is $\sqrt{5}$ and the circumradius is $\frac{21\sqrt{5}}{10}$.

   Then, the ratio of similitude between triangles $\triangle DEF$ and $\triangle ABC$ is the ratio of the inradius and the circumradius. Hence, the area of $\triangle DEF$ is

$$12\sqrt{5} \cdot \left(\frac{\sqrt{5}}{\frac{21\sqrt{5}}{10}} \right)^2 = \frac{400}{147}\sqrt{5},$$

   so the answer is $400 + 147 + 5 = \boxed{552}$.

10. Let O_1, O_2, and O_3 be the centers of ω_1, ω_2, and ω_3 respectively. Let r_1, r_2, r_3 be the radii of ω_1, ω_2, and ω_3 respectively.

    Draw a segment beginning at O_2 parallel to the common external tangent and meeting the radius of O_1 perpendicular to the common external tangent. This forms a right triangle with hypotenuse $r_1 + r_2$ and a leg $r_1 - r_2$. Then the length of the common external tangent is the other leg, which we compute to be $2\sqrt{r_1 r_2}$ using the Pythagorean Theorem.

    Draw a segment through O_3 parallel to the common external tangent that meet the radii of ω_1 and ω_2 perpendicular to the common external tangent. Considering $\overline{O_1O_3}$ and $\overline{O_2O_3}$, we find two right triangles: one with hypotenuse $r_1 + r_3$ and a leg $r_1 - r_3$, and the other with hypotenuse $r_2 + r_3$ and a leg $r_2 - r_3$. Then, the sum of the lengths of the other legs of these two right triangles is equal to the length of the common external tangent. The Pythagorean Theorem gives us the equation

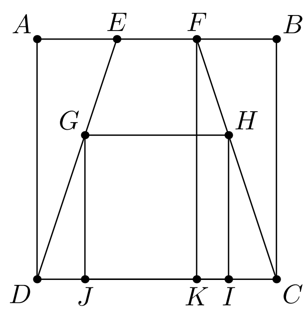
$$2\sqrt{r_1 r_3} + 2\sqrt{r_2 r_3} = 2\sqrt{r_1 r_2},$$

and solving this for r_3 gives

$$r_3 = \frac{r_1 r_2}{r_1 + r_2 + 2\sqrt{r_1 r_2}}.$$

Plugging in $r_1 = 4T$ and $r_2 = T$ yields $r_3 = \boxed{\frac{4}{9}T}$.

11. Consider a general square $ABCD$ with side length s. Let points E, F be on $\overline{AB}$ such that $AE = EF = FB$. Let $GHIJ$ be the inscribed square with side length x, as shown in the diagram below.



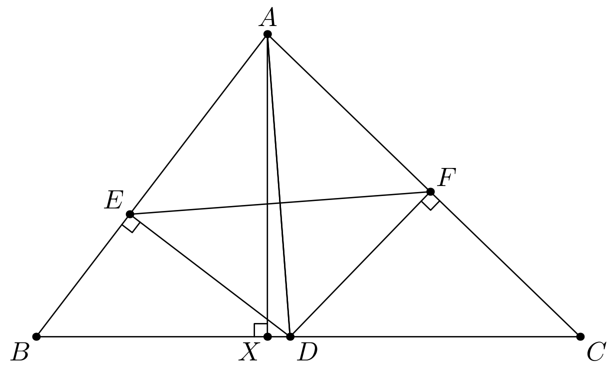
Let point K be the foot of the altitude from point F to segment CD. Then $\triangle FKC \sim \triangle HIC$, giving us $\frac{FK}{KC} = \frac{HI}{IC}$. Then $IC = \frac{KC \cdot HI}{FK} = \frac{\frac{s}{3} \cdot x}{s} = \frac{x}{3}$. Then we see that $x = IJ = s - 2 \cdot \frac{x}{3}$, which gives us $x = \frac{3}{5}s$.

Thus, if we start with a square of side length s, the process gives us a smaller square with side length $\frac{3}{5}s$. Then, the infinite sum of the areas of these squares is

$$\begin{aligned} s^2 + \left(\frac{3}{5}s\right)^2 + \left(\left(\frac{3}{5}\right)^2 s\right)^2 + \left(\left(\frac{3}{5}\right)^3 s\right)^2 + \cdots &= s^2\left(1 + \frac{9}{25} + \left(\frac{9}{25}\right)^2 + \left(\frac{9}{25}\right)^3 + \cdots\right) \\ &= s^2 \cdot \frac{1}{1 - \frac{9}{25}} \\ &= s^2 \cdot \frac{25}{16}. \end{aligned}$$

Plugging in $s = 4$ gives us the answer $\boxed{25}$.

12. First, notice that $\triangle AED \cong \triangle AFD$, so the area of $AEDF$ is 2 times the area of $\triangle AED$.

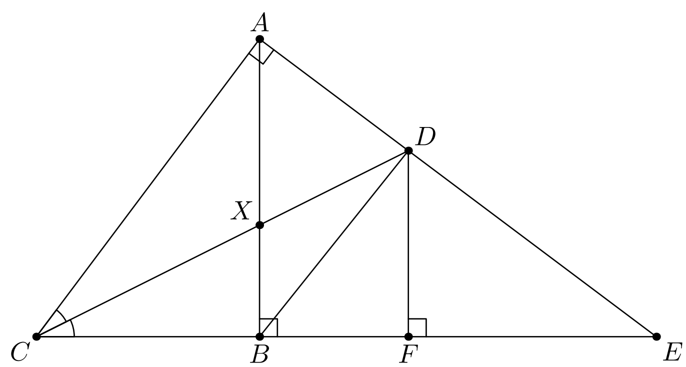


Drop an altitude from point A to $\overline{BC}$ and call the intersection point X. Then $\triangle AXB \sim \triangle DEB$. Heron's formula gives the area of $\triangle ABC$ as $\frac{15\sqrt{55}}{4}$, so $AX = \frac{3\sqrt{55}}{4}$. The Angle Bisector Theorem tells us that $BD = \frac{14}{3}$. Then, applying similarity gives us that $DE = \frac{\sqrt{55}}{2}$. By the Pythagorean Theorem, $BE = \frac{17}{6}$, so $AE = \frac{25}{6}$. Our area is

$$\frac{1}{2} \cdot \frac{\sqrt{55}}{2} \cdot \frac{25}{6} = \frac{25}{12}\sqrt{55},$$

giving the answer $25 + 12 + 55 = \boxed{92}$.

13. Extend $\overline{AD}$ to meet line BC at point E. Let point F be the foot of the altitude from point D to $\overline{BE}$. Let the angle bisector of $\angle ACB$ meet $\overline{AB}$ at point X. Refer to the diagram below.

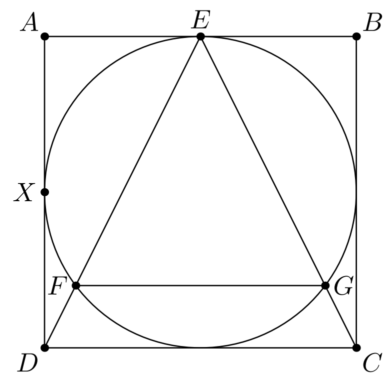


Now, we exploit similar right triangles. By the Angle Bisector Theorem, $BX = \frac{3}{2}$. Since $\triangle CXB \sim \triangle CDA$, we obtain $AD = \frac{5}{2}$. Because $\triangle ABE \sim \triangle CBA$, we get $BE = \frac{16}{3}$ and $AE = \frac{20}{3}$, so $DE = \frac{20}{3} - \frac{5}{2} = \frac{25}{6}$. Since $\triangle DFE \sim ABE$, $DF = \frac{5}{2}$. Thus, the area of $ACBD$ is equal to the area of $\triangle ACE$ minus the area of $\triangle DBE$, which is

$$\frac{1}{2} \cdot AC \cdot AE - \frac{1}{2} \cdot DF \cdot BE = \frac{1}{2} \cdot 5 \cdot \frac{20}{3} - \frac{1}{2} \cdot \frac{5}{2} \cdot \frac{16}{3} = \boxed{10}.$$

14. Let $ABCDEF$ be a regular hexagon with side length 2, and suppose Sally starts at town A. Then a minimum path would be $ACEBDF$, which has length $4 \cdot 2\sqrt{3} + 4 = \boxed{4 + 8\sqrt{3}}$.

15. Let ω meet $\overline{AD}$ at point X, as shown in the diagram below.



By the Pythagorean Theorem, $ED = \sqrt{T^2 + \frac{T^2}{4}} = \frac{T\sqrt{5}}{2}$. Let $EF = l$. By Power of a Point, $DX^2 = DF \cdot DE$, so

$$\frac{T^2}{4} = \left(\frac{T\sqrt{5}}{2} - l\right) \cdot \frac{T\sqrt{5}}{2}.$$

Solve to get $l = \frac{2T}{\sqrt{5}}$. Now note that $\triangle EFG \sim \triangle EDC$, so $\frac{EF}{FG} = \frac{ED}{DC}$. Substituting in previously obtained values and solving yields $FG = \frac{4}{5}T$. Draw an altitude from E to $\overline{FG}$ and apply Pythagorean Theorem to obtain that the height of $\triangle EFG$ is $\frac{4}{5}T$. Hence, the area of $\triangle EFG$ is $\frac{1}{2} \cdot \frac{4}{5}T \cdot \frac{4}{5}T = \boxed{\frac{8}{25}T^2}$. [Highly preferable if given T was divisible by 5 for clean integer answer.]

16. We claim that $AB = AC$. To prove this, consider a translation that maps point B to E, and point A to a point A'. Then $AA'CE$ is cyclic with $AA' \parallel EC$, so $AA'CE$ is an isosceles trapezoid. Hence, their diagonals are equal. As $A'E = AB$, we have $AB = AC$.

    Hence, $\angle ABC = \angle ACB = 30°$. We apply Law of Sines to get

$$\frac{7}{\sin 120°} = \frac{AB}{\sin 30°},$$

    which yields $AB = \boxed{\frac{7\sqrt{3}}{3}}$.

17. Let the octagon have side length $s = T$. Look at trapezoid $UVYZ$. We compute $UZ = s\left(1 + \frac{\sqrt{2}}{2}\right)$, and $VY = CH = s\left(1 + \sqrt{2}\right)$.

    Now it suffices to find the height of trapezoid $UVYZ$. This is equal to the distance from $\overline{VY}$ to point B minus the distance from $\overline{UZ}$ to point B. This gives us the height $\frac{s}{2}\left(1 + \frac{\sqrt{2}}{2}\right)$.

    The area of trapezoid $UVYZ$ is

$$\frac{1}{2} \cdot \frac{s}{2}\left(1 + \frac{\sqrt{2}}{2}\right) \cdot \left(s\left(1 + \sqrt{2}\right) + s\left(1 + \frac{\sqrt{2}}{2}\right)\right) = \frac{s^2}{4}\left(1 + \frac{\sqrt{2}}{2}\right)\left(2 + \frac{3\sqrt{2}}{2}\right),$$

so the area of $UVWXYZ$ is twice that area:

$$\frac{s^2}{2}\left(1+\frac{\sqrt{2}}{2}\right)\left(2+\frac{3\sqrt{2}}{2}\right)=\frac{s^2}{4}(7+5\sqrt{2}).$$

Given T, the area is $\frac{7T^2+5T^2\sqrt{2}}{4}$. If T is odd, then the answer is $12T^2+6$. Otherwise, T^2 will cancel out the 4 in the denominator which would mean we would have to slightly modify the problem statement. [The value of T is TBD...]

18. Note that $ABCD$ is a cyclic quadrilateral, so by Ptolemy's Theorem, $BD\cdot AC = AB\cdot CD + AD\cdot BC$. However, $AC = AB = BC$ since $\triangle ABC$ is equilateral, so the equation reduces to $BD = CD + AD$ (this is known as Van Schooten's theorem). The perimeter is $CD + AD + BC + AB = BD + 2\cdot 2 = 7$, so $BD = \boxed{3}$.

19. Note the formulas $A = rs = \frac{abc}{4R}$, where A is the area, r is the inradius, s is the semiperimeter, and R is the circumradius. Then $4Rr = \frac{abc}{s} \implies 2Rr = \frac{abc}{a+b+c}$. Take the reciprocal of both sides to get
$$\frac{1}{2Rr}=\frac{a+b+c}{abc}=\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc},$$
so our answer is simply $\frac{1}{2\cdot5\cdot3}=\boxed{\frac{1}{30}}$.

20. Given sides a, b and c, Stewart's Theorem gives that the median to side c has length $\sqrt{\frac{2a^2+2b^2-c^2}{4}}$ (also known as Apollonius's theorem). Then the sums of the squares of the three medians is $\frac{3}{4}(a^2+b^2+c^2)$. Let $n = 2020$. Then $a^2+b^2+c^2=(n-2)^2+n^2+(n+2)^2=3n^2+8$. Thus, we compute $\frac{3}{4}(3\cdot 2020^2+8)=\frac{9}{4}\cdot 2020^2+6=9\cdot 505\cdot 2020+6\equiv 9\cdot 505\cdot 20+6\equiv 9\cdot 100+6\equiv \boxed{906}$ (mod 1000).

21. By Thales's Theorem, $\triangle ABC$ is a right triangle with hypotenuse 2. Suppose it has legs a and b, such that $a^2+b^2=4$ by the Pythagorean Theorem.

    We want to maximize $2ab$. By AM-GM, $a+b\geq 2\sqrt{ab} \implies a^2+2ab+b^2\geq 4ab \implies 2ab\leq 4$, hence our maximum value is $\boxed{4}$.

22. Without loss of generality, let the side length of $ABCD$ be 1 (so the ratio is equal to the area of $\mathcal{P}$). Let AG and BH intersect at point X. Then $\triangle AXB \sim \triangle HAB$. The Pythagorean Theorem reveals $BH = \frac{\sqrt{5}}{2}$, so similarity yields $AX = \frac{1}{\sqrt{5}}$ and $BX = \frac{2}{\sqrt{5}}$. The side length of $\mathcal{P}$ is equal to $BX - AX = \frac{1}{\sqrt{5}}$ by rotational symmetry of the diagram, so the answer is $\boxed{\frac{1}{5}}$.

23. In counter-clockwise order, the intersection points are $(2,4),(1,8),(\frac{1}{2},2)$, and $(1,1)$. Applying Shoelace Theorem on the coordinates gives
$$\frac{1}{2}\left|\left(2\cdot8+1\cdot2+\frac{1}{2}\cdot1+1\cdot4\right)-\left(4\cdot1+8\cdot\frac{1}{2}+2\cdot1+1\cdot2\right)\right|=\boxed{\frac{21}{4}}.$$

24. First note that $\triangle ABC$ is a $30-60-90$ triangle. When the circle pivots around point A, it covers a circular sector of radius 4 and angle $360^\circ-60^\circ-2\cdot90^\circ=120^\circ$, which has area

$\frac{1}{3}\pi \cdot 4^2$. When the circle pivots around point B, it covers a quarter circle of radius 4, which has area $\frac{1}{4}\pi \cdot 4^2$. When the circle pivots around point C, it covers a circular sector of radius 4 and angle $360° - 30° - 2 \cdot 90° = 150°$, which has area $\frac{15}{36}\pi \cdot 4^2$.

As the circle travels across a side of the triangle, it sweeps a rectangle of width the same as the side length and height equal to twice the radius, i.e. 4. To get the total area swept, we add up the areas of the rectangles and the circular sectors to get

$$6 \cdot 4 + 3\sqrt{3} \cdot 4 + 3 \cdot 4 + \frac{15}{36}\pi \cdot 4^2 + \frac{1}{3}\pi \cdot 4^2 + \frac{1}{4}\pi \cdot 4^2 = \boxed{36 + 12\sqrt{3} + 16\pi}.$$

For a faster solution, one can astutely notice that $120° + 90° + 150° = 360°$ which is a complete circle, so we can deduce that areas of the circular sectors add up to a complete circle.

25. By Heron's Formula, the area of the triangle is $6\sqrt{6}$. Since $A = \frac{5 \cdot 6 \cdot 7}{4R}$, the circumradius is $R = \frac{35}{4\sqrt{6}}$, so the area of the circle is $\frac{1225}{96}\pi$. Subtracting the two areas, we get $\boxed{\frac{1225}{96}\pi - 6\sqrt{6}}$.

§5.4 Number Theory

1. Note that $\phi(2 \cdot 3 \cdot 5 \cdot 7) = \phi(210) = 48$. This means that there are 48 positive integers less than 210 which are relatively prime to 210, i.e. are *picky*. If m is coprime with n, then $m + n$ is also coprime with n. Hence, there are $48 \cdot 10 = 480$ *picky* numbers less than $210 \cdot 10 = 2100$. Now it suffices to count the *picky* numbers from 2020 to 2100 and subtract from our current count of 480. Use a modified Sieve of Eratosthenes to count 19 extra *picky* numbers, so our answer is $480 - 19 = \boxed{461}$.

   Alternatively, apply Principle of Inclusion Exclusion to find the number of positive integers less than 2020 which are not *picky*. Then, subtract from 2020 to arrive at the same answer. However, this method is computationally cumbersome.

2. As the number of trailing zeroes is limited by the highest power of 5 dividing the product, Legendre's formula tells us that

$$\begin{aligned} \nu_5(123!) &= \left\lfloor \frac{123}{5} \right\rfloor + \left\lfloor \frac{123}{25} \right\rfloor \\ &= 24 + 4 = 28, \\ \nu_5(987!) &= \left\lfloor \frac{987}{5} \right\rfloor + \left\lfloor \frac{987}{25} \right\rfloor + \left\lfloor \frac{987}{125} \right\rfloor + \left\lfloor \frac{987}{625} \right\rfloor \\ &= 197 + 39 + 7 + 1 = 244. \end{aligned}$$

   If $123! = 10^{a_1} \cdot a_2$ and $987! = 10^{b_1} \cdot b_2$ where $a_2, b_2 \nmid 10$, then $123! \cdot 987! = 10^{a_1+b_1} \cdot a_2b_2$ where $a_2b_2 \nmid 10$. Hence, the number of trailing zeroes is the sum $28 + 244 = \boxed{272}$.

3. Note that for $n > 1$, $0 < \sqrt{n+1} - \sqrt{n} < 1$, so $0 < (\sqrt{n+1} - \sqrt{n})^4 < 1$. By the Binomial Theorem, we have

$$(\sqrt{n+1} + \sqrt{n})^4 + (\sqrt{n+1} - \sqrt{n})^4 = 16n^2 + 16n + 2,$$

   so $(\sqrt{n+1}+\sqrt{n})^4 = 16n^2+16n+2-(\sqrt{n+1}-\sqrt{n})^4$. Hence, $16n^2+16n+1 < (\sqrt{n+1}+\sqrt{n})^4 < 16n^2 + 16n + 2$, so for $n \in \mathbb{N}$, $f(n) = 16n^2 + 16n + 2$.

   The answer is computing $\boxed{16T^2 + 16T + 2 \pmod{1000}}$.

4. Let $AD = d$ and $BD = n$. Then, Stewart's Theorem gives

$$ad^2 + amn = n \cdot 7^2 + m \cdot 8^2.$$

   Note that $a \mid (ad^2 + amn)$, therefore $a \mid (49n + 64m)$ i.e. $a \mid (49(a - m) + 64m) \iff a \mid (49a - 49m + 64m)$. This implies $a \mid 15m$.

   By the Triangle Inequality, we have $7 + a > 8$ and $7 + 8 > a$. Combining these inequalities gives $1 < a < 15$, so $a = 2, 3, \ldots, 14$. For each value of a, the value of m goes from 1 to $a - 1$. We see that $a = 3$ gives 2 pairs, $a = 5$ gives 4 pairs, $a = 6$ gives 2 pairs, $a = 9$ gives 2 pairs, $a = 10$ gives 4 pairs, and $a = 12$ gives 2 pairs. Thus, there are $2 + 4 + 2 + 2 + 4 + 2 = \boxed{16}$ ordered pairs.

5. Multiply the numerator and denominator by 2020^{2020} to get
$$\frac{2020^{2020}(1+2020+2020^2+2020^3+\ldots+2020^{2020})}{2020^{2020}+2020^{2019}+\cdots+2020+1}=2020^{2020}.$$
Now it suffices to compute $2020^{2020} \pmod 7$. Note that $2020 \equiv 4 \pmod 7$, so $2020^{2020} \equiv 4^{2020} \equiv 2^{4040} \pmod 7$. Furthermore, $2^3 = 8 \equiv 1 \pmod 7$, so $2^{4040} \equiv 2^{4040 \pmod 3} \equiv 2^2 \equiv \boxed{4} \pmod 7$.

6. The given number is of the form $64 \cdot a^4 + 81 \cdot b^4$. Trying to "complete the square" gives
$$\begin{aligned}64 \cdot a^4 + 81 \cdot b^4 &= (8a^2+9b^2)^2 - 144a^2b^2\\ &= (8a^2+9b^2)^2 - (12ab)^2\\ &= (8a^2-12ab+9b^2)(8a^2+12ab+9b^2).\end{aligned}$$
Plugging in $a = 11$ and $b = 7$ gives $485 \cdot 2333$, and 485 is clearly divisible by 5, giving the factorization $5 \cdot 97 \cdot 2333$. We are given the fact that there are 6 positive divisors, so 2333 has to be prime. Thus, our sum is $5 + 97 + 2333 = \boxed{2435}$.

Alternatively, one may rewrite $64 \cdot 11^4 + 81 \cdot 7^4$ as $21^4 + 4 \cdot 22^4$ and apply Sophie-Germain Identity (which is essentially the same method).

7. First, note that
$$\sin\left(\frac{\pi}{2}\cdot k^3+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\left(\sin\left(\frac{\pi}{2}\cdot k^3\right)+\cos\left(\frac{\pi}{2}\cdot k^3\right)\right). \tag{$\star$}$$
Let's analyze the summation for $\sin\left(\frac{\pi}{2}\cdot k^3\right)$ and $\cos\left(\frac{\pi}{2}\cdot k^3\right)$. Note that
$$\sin\left(\frac{\pi}{2}\cdot n\right)=\begin{cases}0 & n\equiv 0,2 \pmod 4\\ 1 & n\equiv 1 \pmod 4\\ -1 & n \equiv 3 \pmod 4\end{cases},$$
$$\cos\left(\frac{\pi}{2}\cdot n\right)=\begin{cases}0 & n\equiv 1,3 \pmod 4\\ 1 & n\equiv 0 \pmod 4\\ -1 & n \equiv 2 \pmod 4\end{cases}.$$
For $n \equiv 0, 1, 2, 3 \pmod 4$, we have $n^3 \equiv 0, 1, 0, 3 \pmod 4$. Based on the cases above, observe that as k starts from 0 and increments, the value of $\sin\left(\frac{\pi}{2}\cdot k^3\right)$ cycles $0, 1, 0, -1$ and the value of $\cos\left(\frac{\pi}{2}\cdot k^3\right)$ cycles $1, 0, 1, 0$, so
$$\sin\left(\frac{\pi}{2}\cdot k^3\right)+\cos\left(\frac{\pi}{2}\cdot k^3\right)$$
cycles $1, 1, 1, -1$. By ($\star$),
$$\sum_{k=0}^{2020}\sin\left(\frac{\pi}{2}\cdot k^3+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\sum_{k=0}^{2020}\left[\sin\left(\frac{\pi}{2}\cdot k^3\right)+\cos\left(\frac{\pi}{2}\cdot k^3\right)\right],$$
so we can break the summation into 505 blocks of $1 + 1 + 1 - 1 = 2$ and the value for $k = 2020$ which is 1. Thus, our answer is $\frac{1}{\sqrt{2}} \cdot (505 \cdot 2 + 1) = \boxed{\frac{1011\sqrt{2}}{2}}$.

8. Suppose the chosen numbers are $n-1$, n, and $n+1$. Then their product is n^3-n, and testing $n \equiv 0, 1, \ldots, 6 \pmod 7$ reveals that $n \equiv 0, 1, 6 \pmod 7$ allows the quantity n^3-n to be divisible by 7. The value of n can go from 3 to 99, which contains the multiples $7, 14, \ldots, 7 \cdots 14 = 98$. We count 3 residues between each multiple of 7, so we initially count $13 \cdot 3 = 39$ such residues. However, $6, 7 \cdot 14 = 98$, and 99 also work, so our total number of valid sets is $39 + 3 = 42$. There are $99 - 3 + 1 = 97$ total sets, so the probability is $\boxed{\frac{42}{97}}$.

9. Note that $159999 = 160000 - 1 = 20^4 - 1$, so we use difference of squares:

$$\begin{aligned} 20^4 - 1 &= (20^2+1)(20^2-1) \\ &= 401 \cdot 399 \\ &= 401 \cdot 3 \cdot 133 \\ &= 401 \cdot 3 \cdot (140 - 7) \\ &= 401 \cdot 3 \cdot 7 \cdot 19, \end{aligned}$$

so the sum of the prime factors is $401 + 3 + 7 + 19 = \boxed{430}$.

10. Notice that $3^{12} + 3^9 + 3^5 + 1 = (3^4)^3 + 3 \cdot (3^4)^2 + 3 \cdot 3^4 + 1 = (3^4+1)^3 = 82^3 = 2^3 \cdot 41^3$, so our answer is $2 + 41 = \boxed{43}$.

11. Note that the largest prime factor of 2020 is 101, so it suffices to count the largest power of 101 which divides 2020!. This can be computed using Legendre's formula:

$$\nu_{101}(2020!) = \left\lfloor \frac{2020}{101} \right\rfloor = \boxed{20}.$$

12. The sum of the interior angles of an 8-sided polygon is $180 \cdot (8-2) = 180 \cdot 6$. Suppose the interior angles are $a, a+d, a+2d, \ldots, a+7d$ for a starting term a and common difference d. Then $a + (a+d) + (a+2d) + \cdots + (a+7d) = 8a + 28d = 180 \cdot 6 \implies 2a + 7d = 270$. Take this equation modulo 7 to get $2a \equiv 4 \pmod 7 \implies a \equiv 2 \pmod 7$ since 2 has a multiplicative inverse (namely 4) modulo 7. Then $a = 7n + 2$ for some integer n. We substitute this back into the equation and rearrange to obtain $d = 38 - 2n$. Thus, a special polygon's interior angles are determined by the ordered pair $(a, d) = (7n+2, 38-2n)$ for integers n, and it suffices to count the number of ordered pairs (a, d). Since angles must be positive, n can start from 0 and go up to 19 (at $n = 19$ the special polygon is actually an octagon), so there are $\boxed{20}$ distinct special polygons.