Newton's Forward Difference Formula

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Newton's Forward Difference Formula is a very powerful tool when you need to figure out the equation that models to a given sequence or arithmetic series with a pattern.

The general form of this formula is:

$$x_0 + x_1 \cdot n + \frac{x_2(n)(n-1)}{2!} + \frac{x_3(n)(n-1)(n-2)}{3!} + \frac{x_4(n)(n-1)(n-2)(n-3)}{4!} + \cdots$$

Given:

- n is the dummy variable for the resulting equation
- $x_0, x_1, x_2, \ldots$ are the 0th terms of the 1st, 2nd, 3rd, $\ldots$ differences of the original sequence or series

Example 0.1

$7, 15, 27, 43, 63, \ldots$ What is the formula of the sequence? What is the 100th term?

Solution. First, take the difference between each pair of terms (1st and 2nd, 2nd and 3rd, 3rd and 4th, etc.).

Original:	7		15		27		43		63
1st differences:		8		12		16		20	

Do the same thing, but with the "1st differences" row:

Original:	7		15		27		43		63
1st differences:		8		12		16		20	
2nd differences:			4		4		4		

Eventually, the terms become constant, and you stop. Now, let's go backwards; find the "0th terms" of each row by subtracting the difference from each 1st term:

Original:	$\boxed{3}$		7		15		27		43		63		
1st differences:		$\boxed{4}$		8		12		16		20			
2nd differences:			$\boxed{4}$		4		4		4				

The 0th terms are boxed. The formula of the sequence, using Newton's Forward Difference Formula, would be

$$\left.\begin{aligned} x_0 &= 3 \\ x_1 &= \text{the first } 4 \\ x_2 &= \text{the second } 4 \end{aligned}\right\} 3 + 4n + \frac{4n(n-1)}{2} \implies \boxed{2n^2 + 2n + 3}.$$

The 100th term is $2 \cdot 10000 + 2 \cdot 100 + 3 = \boxed{20203}$. □

We can also derive famous results using this theorem.

Example 0.2

Find the closed form of $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$.

Solution. This is a series, but we can turn it into a sequence by letting the first term be the sum when $n = 1$, second term when $n = 2$, and so on:

Original: $1, 5, 14, 30, 55, 91, 140, \ldots$
1st differences: $4, 9, 16, 25, 36, 49, \ldots$
2nd differences: $5, 7, 9, 11, 13, \ldots$
3rd differences: $2, 2, 2, 2, \ldots$

Now we find the 0^{th} terms:

Original: $\boxed{0}, 1, 5, 14, 30, 55, 91, 140, \ldots$
1st differences: $\boxed{1}, 4, 9, 16, 25, 36, 49, \ldots$
2nd differences: $\boxed{3}, 5, 7, 9, 11, 13, \ldots$
3rd differences: $\boxed{2}, 2, 2, 2, 2, \ldots$

Newton's Forward Difference Formula yields $0 + n + \frac{3n(n-1)}{2!} + \frac{2n(n-1)(n-2)}{3!}$.

Simplifying it gives the famous formula $\boxed{\dfrac{n(n+1)(2n+1)}{6}}$. □

Example 0.3

Find the closed form of $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3$.

Solution. The process is the same: turn it into a sequence, find the differences, and work backwards to find the 0th terms.

Original: $\boxed{0}, 1, 9, 36, 100, 225, 441, \ldots$
1st differences: $\boxed{1}, 8, 27, 64, 125, 216, \ldots$
2nd differences: $\boxed{7}, 19, 37, 61, 91, \ldots$
3rd differences: $\boxed{12}, 18, 24, 30, \ldots$
4th differences: $\boxed{6}, 6, 6, \ldots$

$$0 + n + \frac{7n(n-1)}{2!} + \frac{12n(n-1)(n-2)}{3!} + \frac{6n(n-1)(n-2)(n-3)}{4!}$$

$$\Longrightarrow \frac{n^4 + 2n^3 + n^2}{4} \Longrightarrow \frac{n^2(n^2 + 2n + 1)}{4} \Longrightarrow \frac{n^2(n+1)^2}{2^2} \Longrightarrow \boxed{\left(\frac{n(n+1)}{2}\right)^2}$$

Therefore, $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = (1 + 2 + 3 + 4 + \cdots + n)^2$. $\square$