Trigonometry and a Sequence of Polynomials

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Abstract

The **Chebyshev polynomials of the first kind** are a sequence of polynomials satisfying a specific set of trigonometric properties. Specifically, for any nonnegative integer n, the Chebyshev polynomial $T_n(x)$ satisfies $T_n(\cos(\theta)) = \cos(n\theta)$. We demonstrate the existence of such a polynomial for all nonnegative, integral n and implement a recurrence relation using trigonometric identities to generate these polynomials. Additionally, we develop a closed form utilizing a combinatorial bijection that provides the coefficients of all such Chebyshev polynomials. We extrapolate these results to develop powerful results in various moduli and compare to well-known approximations such as the Taylor series. By considering the general roots of a Chebyshev polynomial of the first kind, we deduce cases in which one Chebyshev polynomial of the first kind divides another.

Contents

1 Introduction **2**

2 Initial Results and Patterns **3**

3 Coefficients **7**
3.1 Patterns in Coefficients 7
3.2 Other Observations 9
3.3 Triangle A 10
3.4 Triangle B 12
3.5 Closed Form 14

4 Modulo p **16**

5 Roots **16**

6 Approximation **17**

§1 Introduction

As part of the PROMYS (Program in Mathematics for Young Scientists) First-Year Exploration Labs, the aim of this investigation was to discover various properties and results about Chebyshev polynomials. This was conducted over a period of four to five weeks, through a series of group meetings twice a week. We analyzed patterns in numerical experiments to develop conjectures, which were then proved or disproved.

First of all, we demonstrate the existence of such Chebyshev polynomials of the first kind.

Theorem 1.1

For each positive integer n, there exists a unique polynomial $T_n(x)$ such that $T_n(\cos(x)) = \cos(nx)$.

Proof. By DeMoivre's Theorem,

$$\operatorname{Re}((\cos\theta + i\sin\theta)^n) = \operatorname{Re}(\cos(n\theta) + i\sin(n\theta)) = \cos(n\theta).$$

Hence, we apply the Binomial Theorem to get

$$\begin{aligned}\cos(n\theta) &= \operatorname{Re}((\cos\theta + i\sin\theta)^n)\\ &= \operatorname{Re}\left(\sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\theta(i\sin\theta)^k\right)\\ &= \sum_{k \text{ even}}^{n}\binom{n}{k}\cos^{n-k}\theta\cdot(i\sin\theta)^k.\end{aligned}$$

For even k, $\frac{k}{2}$ is an integer. Then, note that

$$\sin^k\theta = (\sin^2\theta)^{k/2} = (1-\cos^2\theta)^{k/2},$$

so

$$\cos(n\theta) = \sum_{k \text{ even}}^{n}\binom{n}{k}\cos^{n-k}\theta\cdot(1-\cos^2\theta)^{k/2}\cdot i^k,$$

and note that $i^k = \pm 1$ for even k. Thus, we have shown that $\cos(n\theta)$ can be expressed in terms of powers of $\cos\theta$ with real coefficients, meaning $T_n(x)$ exists.

Moreover, this polynomial is unique, since any other polynomial $\widetilde{T}_n(x)$ would have to be equal to $T_n(x)$ at infinitely many points (i.e. all reals from 0 to 1). But two polynomials that are equal at infinitely many points must be equal everywhere, so $\widetilde{T}_n = T_n$. $\square$

Theorem 1.2

For $n \in \mathbb{N}$,

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} x^{n-2k}(x^2-1)^k \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} x^{n-2k} \sum_{j=k}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j}\binom{j}{k}(-1)^k. \end{aligned}$$

Proof. Recall from the proof of Theorem 1.1 that

$$\cos(n\theta) = \sum_{k \text{ even}}^{n} \binom{n}{k} \cos^{n-k}\theta \cdot (1-\cos^2\theta)^{k/2} \cdot i^k.$$

Then, we reindex the summation to get

$$\begin{aligned} \cos(n\theta) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \cos^{n-2k}\theta \cdot (1-\cos^2\theta)^k \cdot (-1)^k \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \cos^{n-2k}\theta \cdot (\cos^2\theta - 1)^k \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \cos^{n-2k}\theta \left(\sum_{j=0}^{k} \binom{k}{j} \cos^{2(k-j)}\theta \cdot (-1)^j \right). \end{aligned}$$

After some algebraic manipulation, we obtain

$$\cos(n\theta) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \cos^{n-2k}\theta \sum_{j=k}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j}\binom{j}{k}(-1)^k.$$

□

§2 Initial Results and Patterns

Furthermore, we prove an important recursive relation among Chebyshev polynomials.

Lemma 2.1

$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$.

Proof. The angle addition formulas state that $\cos(x+y) = \cos x \cos y - \sin x \sin y$ and $\cos(x-y) = \cos x \cos y + \sin x \sin y$. Hence, $\cos(x+y) + \cos(x-y) = 2\cos x \cos y$. Let $a = x+y$ and $b = x-y$, such that $x = \frac{a+b}{2}$ and $y = \frac{a-b}{2}$. Then, $\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$, as desired. □

Theorem 2.2
For $n \in \mathbb{N}$, $T_n(\cos(x)) = 2\cos(x) \cdot T_{n-1}(\cos(x)) - T_{n-2}(\cos(x))$.

Proof. It suffices to prove the equivalent statement,

$$\cos(n\theta) = 2\cos\theta\cos((n-1)\theta) - \cos((n-2)\theta).$$

Since

$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right),$$

setting $a = n\theta$ and $b = (n-2)\theta$ gives us

$$\begin{aligned}\cos(n\theta) + \cos((n-2)\theta) &= 2\cos\left(\left(\frac{n+(n-2)}{2}\right)\theta\right)\cos\left(\left(\frac{n-(n-2)}{2}\right)\theta\right)\\ &= 2\cos((n-1)\theta)\cos\theta,\end{aligned}$$

from which rearranging gives us our result. $\square$

Corollary 2.3
For all $n \in \mathbb{N}$, $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$.

Proof. By Theorem 2.2, we see that the polynomials $T_n(x)$ and $2xT_{n-1}(x) - T_{n-2}(x)$ are equal at $\cos(\theta)$ for all θ. Therefore, they are equal at all $0 \leq x \leq 1$. Since this is infinitely many points, the polynomials must be equal. $\square$

Lemma 2.4
$\forall n \in \mathbb{N}$, $T_n(1) = 1$.

Proof. Since $T_n(\cos\theta) = \cos(n\theta)$, we substitute $\theta = 0$ to get $T_n(\cos 0) = \cos(n \cdot 0)$, or $T_n(1) = 1$. $\square$

It follows that the sum of the coefficients of $T_n(x)$ is always 1 for any $n \in \mathbb{N}$. Furthermore, we explored the derivatives of the first few $T_n(x)$, and noticed that the sum of their coefficients were perfect squares.

n	$T_n(x)$	$\frac{d}{dx}(T_n(x))$	Sum of coefficients of $\frac{d}{dx}(T_n(x))$
0	1	0	0
1	x	1	1
2	$2x^2 - 1$	$4x$	4
3	$4x^3 - 3x$	$12x^2 - 3$	9
4	$8x^4 - 8x^2 + 1$	$32x^3 - 16x$	16
5	$16x^5 - 20x^3 + 5x$	$80x^4 - 60x^2 + 5$	25
6	$32x^6 - 48x^4 + 18x^2 - 1$	$192x^5 - 192x^3 + 36x$	36

Proposition 2.5
$\forall n \in \mathbb{N} \cup \{0\}$, $\frac{d}{dx}(T_n(x))(1) = n^2$.

Proof. We proceed by induction on n. Note the base cases are $n = 0$ and $n = 1$. Then $\dfrac{d}{dx}(T_0(x)) = 0 = 0^2$, and $\frac{d}{dx}(T_1(1)) = 1 = 1^2$, so the base cases are true.

As for our inductive hypothesis, assume that the equation holds for $\frac{d}{dx}(T_n(1))$ and $\frac{d}{dx}(T_{n+1}(1))$. We wish to prove that the relationship holds for $\frac{d}{dx}(T_{n+2}(1))$. Recall by the recursion from Corollary 2.3 that $T_{n+2}(x) = 2x \cdot T_{n+1}(x) - T_n(x)$. Taking the derivative of both sides gives

$$\begin{aligned}
\frac{d}{dx}(T_{n+2}(x)) &= \frac{d}{dx}(2x \cdot T_{n+1}(x)) - \frac{d}{dx}(T_n(x)) \\
&= \frac{d}{dx}(2x) \cdot T_{n+1}(x) + 2x \cdot \frac{d}{dx}(T_{n+1}(x)) - \frac{d}{dx}(T_n(x)) \\
&= 2T_{n+1}(x) + 2x \cdot \frac{d}{dx}(T_{n+1}(x)) - \frac{d}{dx}(T_n(x)).
\end{aligned}$$

Then, plugging in $x = 1$ yields

$$\begin{aligned}
\frac{d}{dx}(T_{n+2}(x))(1) &= 2 \cdot 1 + 2 \cdot 1 \cdot \frac{d}{dx}(T_{n+1}(x))(1) - \frac{d}{dx}(T_n(x))(1) \\
&= 2 + 2 \cdot (n+1)^2 - (n)^2 \\
&= 2 + 2n^2 + 4n + 2 - n^2 \\
&= n^2 + 4n + 4 \\
&= (n+2)^2.
\end{aligned}$$

This concludes the induction. □

Proposition 2.6
$\forall n \in \mathbb{N}$, $\frac{d^2}{dx^2}(T_n(x))(1) = \frac{n^2(n^2-1)}{3}$.

Proof. Again, we induct on n. The base cases are $n = 0$ and $n = 1$. We have $\frac{d^2}{dx^2}(T_0(x)) = 0 = \frac{0^2(0^2-1)}{3}$, and $\frac{d^2}{dx^2}(T_1(x)) = 0 = \frac{1^2(1^2-1)}{3}$. Hence, the base cases are true.

As our inductive hypothesis, suppose the relation holds for $n = k$ and $n = k+1$. Recall from the previous proof that

$$\frac{d}{dx}(T_{k+2}(x)) = 2T_{k+1}(x) + 2x \cdot \frac{d}{dx}(T_{k+1}(x)) - \frac{d}{dx}(T_k(x)).$$

Then, we differentiate both sides:

$$\begin{aligned}\frac{d^2}{dx^2}(T_{k+2}(x)) &= 2\frac{d}{dx}(T_{k+1}(x)) + \frac{d}{dx}\left(2x \cdot \frac{d}{dx}(T_{k+1}(x))\right) - \frac{d^2}{dx^2}(T_k(x)) \\ &= 4\frac{d}{dx}(T_{k+1}(x)) + 2x \cdot \frac{d^2}{dx^2}(T_{k+1}(x)) - \frac{d^2}{dx^2}(T_k(x)).\end{aligned}$$

Now evaluate $\frac{d^2}{dx^2}(T_{k+2}(x))$ at $x = 1$. We refer to Proposition 2.5 and our inductive hypothesis to conclude:

$$\begin{aligned}\frac{d^2}{dx^2}(T_{k+2}(x))(1) &= 4\frac{d}{dx}(T_{k+1}(x))(1) + 2 \cdot 1 \cdot \frac{d^2}{dx^2}(T_{k+1}(x))(1) - \frac{d^2}{dx^2}(T_k(x))(1) \\ &= 4(k+1)^2 + 2 \cdot \frac{(k+1)^2((k+1)^2-1)}{3} - \frac{k^2(k^2-1)}{3} \\ &= \frac{(k+1)(k+2)^2(k+3)}{3} \\ &= \frac{(k+2)^2((k+2)^2-1)}{3},\end{aligned}$$

concluding our inductive step. □

Proposition 2.7

For any $a, b \in \mathbb{N}$, $T_a(T_b(x)) = T_{ab}(x)$.

Proof. Note that for all x, $T_a(T_b(\cos(x)) = T_a(\cos(bx)) = \cos(a(bx)) = T_{ab}(\cos(x))$. Then since the polynomials $T_a(T_b(x))$ and $T_{ab}(x)$ are equal at infinitely many points, the polynomials are equal. □

Proposition 2.8

For any $a, b \in \mathbb{N}$, $T_a(x) \cdot T_b(x) = \frac{1}{2}\left(T_{a+b}(x) + T_{a-b}(x)\right)$.

Proof. First, consider

$$\cos((a+b)x) + \cos((a-b)x).$$

Note that

$$\begin{aligned}
\cos((a+b)x) &= \cos(ax+bx) \\
&= \cos(ax)\cos(bx) - \sin(ax)\sin(bx) \\
\cos((a-b)x) &= \cos(ax-bx) \\
&= \cos(ax)\cos(bx) + \sin(ax)\sin(bx).
\end{aligned}$$

Thus,

$$\cos((a+b)x) + \cos((a-b)x) = 2\cos(ax)\cos(bx).$$

Hence,

$$\cos(ax)\cos(bx) = \frac{1}{2}\left(\cos((a+b)x) + \cos((a-b)x)\right).$$

This directly translates to $T_a(x) \cdot T_b(x) = \frac{1}{2}\left(T_{a+b}(x) + T_{a-b}(x)\right)$. □

§3 Coefficients

§3.1 Patterns in Coefficients

Let $T_n(x) = \sum\limits_{i=0}^{n} a_i x^i$. We made the following observation:

n	$T_n(x)$	$\sum \lvert a_i \rvert$
0	1	1
1	x	1
2	$2x^2 - 1$	$3 = 2 \cdot 1 + 1$
3	$4x^3 - 3x$	$7 = 2 \cdot 3 + 1$
4	$8x^4 - 8x^2 + 1$	$17 = 2 \cdot 7 + 3$
5	$16x^5 - 20x^3 + 5x$	$41 = 2 \cdot 17 + 7$
6	$32x^6 - 48x^4 + 18x^2 - 1$	$99 = 2 \cdot 41 + 17$
7	$64x^7 - 112x^5 + 56x^3 - 7x$	$239 = 2 \cdot 99 + 41$
8	$128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$	$577 = 2 \cdot 239 + 99$

The sum of the absolute value of the coefficients follow a recursive pattern. To fully capture this idea in mathematical language, we introduce a few more definitions.

Definition 3.1. Define $\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}.$

Definition 3.2. Let $\alpha(n,k)$ denote the coefficient of x^k in $T_n(x)$.

Proposition 3.3
$\forall k \leq n$,

$$\operatorname{sgn}(\alpha(n,k)) = \begin{cases} 0 & k \not\equiv n \pmod 2 \\ -1 & k \equiv n+2 \pmod 4 \\ 1 & k \equiv n \pmod 4 \end{cases}.$$

Proof. We proceed by induction on n. The base case is $n = 0$. Since $T_0(x) = 1$, we see that $a_0 = 1$, hence $\operatorname{sgn}(a_0) = 1$ by definition of $\operatorname{sgn}(x)$. Furthermore, $T_1(x) = x$, we $a_1 = 1$, so $\operatorname{sgn}(a_1) = 1$, by definition of $\operatorname{sgn}(x)$. Both of these agree with the proposed statement.

Suppose for the inductive hypothesis that $n = k$ and the $n = k+1$ is true. Then we we wish to prove for $n = k+2$. Recall that our recursive formula is $T_n(x) = 2x \cdot T_{n-1}(x) - T_{n-2}(x)$. Now consider $\alpha(n+2,k)$, which we know equals $2\alpha(n+1,k-1) - \alpha(n,k)$ by the recursive relation of T_n proved earlier.

We perform casework on k. If $k \not\equiv n+2 \pmod 2$, then $\operatorname{sgn}(\alpha(n+2,k))$ should equal 0. Note that

$$k \not\equiv n+2 \pmod 2 \implies k-1 \not\equiv n+1 \pmod 2 \implies k \not\equiv n \pmod 2.$$

Therefore, $\operatorname{sgn}(\alpha(n+1,k-1)) = 0$ and $\operatorname{sgn}(\alpha(n,k)) = 0$. Thus, $\alpha(n+2,k)) = 2\alpha(n+1,k-1) - \alpha(n,k) = 2 \cdot 0 - 0 = 0$. Hence, $\operatorname{sgn}(\alpha(n+2,k)) = 0$.

If $k \equiv n \pmod 4$, then $\operatorname{sgn}(\alpha(n+2,k))$ should be equal to -1 as $k \equiv (n+2)+2 \pmod 4$. Note that

$$k \equiv n \pmod 4 \implies k-1 \equiv (n+1)+2 \pmod 4.$$

This means that $\operatorname{sgn}(\alpha(n+1,k-1)) = -1$ and $\operatorname{sgn}(\alpha(n,k)) = 1$. Therefore, $\operatorname{sgn}(\alpha(n+1,k-1)) = -1$ as a negative number minus a positive number is negative.

If $k \equiv n+2 \pmod 4$, then $\operatorname{sgn}(\alpha(n+2,k))$ should be equal to 1 as $k \equiv n+2 \pmod 4$. Note that

$$k \equiv n+2 \pmod 4 \implies k-1 \equiv (n+1)+2 \pmod 4.$$

Therefore, $\operatorname{sgn}(\alpha(n+1,k-1)) = -1$ and $\operatorname{sgn}(\alpha(n,k)) = 1$. Therefore, $\operatorname{sgn}(\alpha(n+1,k-1)) = 1$ as a positive number minus a negative number is positive. $\square$

Proposition 3.4
$|\alpha(n,k)| = 2\,|\alpha(n-1,k-1)| + |\alpha(n-2,k)|$.

Proof. Note that $|x| = x \operatorname{sgn} x$. We proceed by casework.

By the recurrence, $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$. Thus, $\alpha(n,k) = 2\alpha(n-1,k-1) - \alpha(n-2,k)$.

If n and k are not equal modulo 2, then by Proposition 3.3, $\operatorname{sgn}(\alpha(n,k)) = \operatorname{sgn}(\alpha(n-1,k-1)) = \operatorname{sgn}(\alpha(n-2,k)) = 0$, so $0 = 2\cdot 0 + 0$ holds.

If $n \equiv k+2$ modulo 4, then $\operatorname{sgn}(\alpha(n,k)) = \operatorname{sgn}(\alpha(n-1,k-1)) = -1$, and $n-2 \equiv k$ modulo 4, so $\operatorname{sgn}(\alpha(n-2,k)) = 1$. Thus

$$|\alpha(n,k)| = -\alpha(n,k) = -(2\alpha(n-1,k-1) - \alpha(n-2,k))$$

$$= -2\alpha(n-1,k-1) + \alpha(n-2,k) = 2\,|\alpha(n-1,k-1)| + |\alpha(n-2,k)|$$

as desired.

Finally, if $n \equiv k$ modulo 4, then $\operatorname{sgn}(\alpha(n,k)) = \operatorname{sgn}(\alpha(n-1,k-1)) = 1$ and $\operatorname{sgn}(\alpha(n-2,k)) = -1$. Therefore,

$$|\alpha(n,k)| = \alpha(n,k)$$

$$= 2\alpha(n-1,k-1) - \alpha(n-2,k) = 2\,|\alpha(n-1,k-1)| + |\alpha(n-2,k)|$$

We have covered all cases, so we are done. $\square$

Now, these lemmas equip us to prove our conjecture:

Corollary 3.5

Let $s(n) = \sum_{i=0}^{n} |a_i|$ where $T_n(x) = \sum_{i=0}^{n} a_i x^i$. Then $s(n) = 2s(n-1) + s(n-2)$ $\forall$ $n \geq 2$, and $s(0) = s(1) = 1$.

§3.2 Other Observations

Again, consider the first few Chebyshev polynomials:

$$\begin{array}{lll} 1, & x, & 2x^2-1, \\ 4x^3-3x, & 8x^4-8x^2+1, & 16x^5-20x^3+5x \end{array}$$

Some interesting observations were made:

1. The constant term of $T_n(x)$ is $(-1)^k$ when $n = 2k$, and 0 for $n = 2k+1$, where $k \in \mathbb{N} \cup \{0\}$.
2. The coefficient of x in $T_n(x)$ is $(-1)^k(2k+1)$ when $n = 2k+1$, and 0 for $n = 2k$, where $k \in \mathbb{N} \cup \{0\}$.
3. The coefficient of x^2 in $T_n(x)$ is $(-1)^{k+1}2k^2$ when $n = 2k$, and 0 for $n = 2k+1$, where $k \in \mathbb{N} \cup \{0\}$.
4. The coefficient of the leading term in $T_n(x)$ is 2^{n-1}.

While all these observations can be proved using Corollary 2.3 and mathematical induction, we seek to find a general pattern for any power of x in any Chebyshev polynomial. For purposes of organization and further investigation, we created a large table of the coefficients of of the first few $T_n(x)$, up to $n = 12$.

n	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0	0
2	-1	0	2	0	0	0	0	0	0	0	0	0	0
3	0	-3	0	4	0	0	0	0	0	0	0	0	0
4	1	0	-8	0	8	0	0	0	0	0	0	0	0
5	0	5	0	-20	0	16	0	0	0	0	0	0	0
6	-1	0	18	0	-48	0	32	0	0	0	0	0	0
7	0	-7	0	56	0	-112	0	64	0	0	0	0	0
8	1	0	-32	0	160	0	-256	0	128	0	0	0	0
9	0	9	0	-120	0	432	0	-576	0	256	0	0	0
10	-1	0	50	0	-400	0	1120	0	-1280	0	512	0	0
11	0	-11	0	220	0	-1232	0	2816	0	-2816	0	1024	0
12	1	0	-72	0	840	0	-3584	0	6912	0	-6144	0	2048

§3.3 Triangle A

$m = 0$: 1

$m = 1$: 1 1

$m = 2$: 1 3 2

$m = 3$: 1 5 8 4

$m = 4$: 1 7 18 20 8

$m = 5$: 1 9 32 56 48 16

Figure 1: Triangle A

Shown in the figure above is a different way to view the coefficients of the polynomials. We will call this **Triangle A**. Its rows are the anti-diagonals of the table of coefficients. More formally, if a is the coefficient of x^k in $T_n(x)$, then $|a|$ is located in the row numbered $\frac{n+k}{2}$, as the kth element from the left.

The kth element in row n will occasionally be referred to as (n, k) (and this "coordinate notation will be applied to other triangles as well).

Definition 3.6. Let $\beta(m, k)$ denote the value at (m, k) in Triangle A. Then, by our construction, $\beta(m, k) = |\alpha(2m - k, k)|$.

Proposition 3.7
When both quantities are defined, $\beta(m,k) = 2\beta(m-1,k-1) + \beta(m-1,k)$.

Proof. On the basis of Proposition 3.4,

$$|\alpha(2m-k,k)| = 2\,|\alpha(2m-k-1,k-1)| + |\alpha(2m-k-2,k)|\,.$$

By substitution utilizing the equality in Definition 3.6, $\beta(m,k) = 2\beta(m-1,k-1) + \beta(m-1,k)$. Note that when calculating $\beta(m,1)$, $\beta(m-1,-1)$ is not technically defined but it takes on the quantity of 0. $\square$

Proposition 3.8
$\sum_{i=0}^{m+1} \beta(m+1,i) = 3 \cdot \sum_{i=0}^{m} \beta(m,i)$. Hence, $\sum_{i=0}^{m+1} \beta(m+1,i) = 2 \cdot 3^m$ for $m \geq 0$.

Proof. By Proposition 3.7, we know that

$$\sum_{i=0}^{m+1} \beta(m+1,i) = \sum_{i=0}^{m+1} [2\beta(m,i-1) + \beta(m,i)].$$

This expands to

$$\begin{aligned}
&2\beta(m,-1) + \beta(m,0) + 2\beta(m,0) + \beta(m,1) + ... + \beta(m,m) + 2\beta(m,m) + \beta(m,m+1) \\
&= 0 + \beta(m,0) + 2\beta(m,0) + \beta(m,1) + ... + \beta(m,m) + 2\beta(m,m) + 0 \\
&= 3 \cdot \sum_{i=0}^{m} \beta(m,i).
\end{aligned}$$

To prove that this quantity equals $2 \cdot 3^m$, we may proceed by induction. For our base case, consider $m = 0$. Indeed, the sum of the elements in the first row is $1 + 1 = 2 = 2 \cdot 3^0$. Four our inductive hypothesis, assume that the sum of the elements of row k is $2 \cdot 3^{k-1}$. Then the sum of the elements of row $k+1$ is $\sum_{i=0}^{k+1} \beta(k+1,i) = 3 \cdot \sum_{i=0}^{k} \beta(k,i) = 3 \cdot 2 \cdot 3^{k-1} = 2 \cdot 3^k$. Therefore, by mathematical induction, the sum of all elements in the $m+1^{\text{st}}$ row is $2 \cdot 3^m$ for $m \geq 0$. $\square$

§3.4 Triangle B

$$
\begin{array}{lccccccccccc}
m=0: & & & & & & 1 & & & & & \\
m=1: & & & & & \frac{1}{2} & & \frac{1}{2} & & & & \\
m=2: & & & & \frac{1}{6} & & \frac{1}{2} & & \frac{1}{3} & & & \\
m=3: & & & \frac{1}{18} & & \frac{5}{18} & & \frac{4}{9} & & \frac{2}{9} & & \\
m=4: & & \frac{1}{54} & & \frac{7}{54} & & \frac{1}{3} & & \frac{10}{27} & & \frac{4}{27} & \\
m=5: & \frac{1}{162} & & \frac{1}{18} & & \frac{16}{81} & & \frac{28}{81} & & \frac{8}{27} & & \frac{8}{81}
\end{array}
$$

We define a new triangle **Triangle B** based on Triangle A. Each element of Triangle B is the corresponding element of Triangle A, except we normalize each row to have a sum of one. Intuitively, we want each element of Triangle B to represent a probability.

Definition 3.9. Let $\gamma(m,k)$ represent the value at (m, k) in Triangle B:

$$\gamma(m,k) = \frac{\beta(m,k)}{\sum_{i=0}^{m} \beta(m,i)}.$$

Proposition 3.10
$\gamma(m,k) = \frac{2}{3}\gamma(m-1,k+1) + \frac{1}{3}\gamma(m-1,k)$.

Proof. By Proposition 3.7, $\beta(m,k) = 2\beta(m-1,k-1) + \beta(m-1,k)$. Divide both sides of the equation by $\sum_{i=0}^{m} \beta(m,i)$ to obtain

$$\frac{\beta(m,k)}{\sum_{i=0}^{m} \beta(m,i)} = \frac{2\beta(m-1,k-1)}{\sum_{i=0}^{m} \beta(m,i)} + \frac{\beta(m-1,k)}{\sum_{i=0}^{m} \beta(m,i)}.$$

Using Proposition 3.8, we can rewrite this as

$$\frac{\beta(m,k)}{\sum_{i=0}^{m} \beta(m,i)} = \frac{2\beta(m-1,k-1)}{3\sum_{i=0}^{m-1} \beta(m-1,i)} + \frac{\beta(m-1,k)}{3\sum_{i=0}^{m-1} \beta(m-1,i)}.$$

Then, conversion to gamma notation through Definition 3.9 gives

$$\gamma(m,k) = \frac{2}{3}\gamma(m-1,k-1) + \frac{1}{3}\gamma(m-1,k). \qquad \square$$

Theorem 3.11

$\gamma(m, k)$ is equal to the following probability: We start a random walk down Triangle B. The following rules govern the walk:

- We have a series of random variables $X_1, X_2, \ldots$.
- $X_1 = \begin{cases} 0 & \text{with probability } \frac{1}{2} \\ 1 & \text{otherwise} \end{cases}$
- For $i > 1$, $X_i = \begin{cases} X_{i-1} & \text{with probability } \frac{1}{3} \\ X_{i-1} + 1 & \text{otherwise} \end{cases}$
- Then $\gamma(m, k)$ is the probability that we reach $X_m = k$.

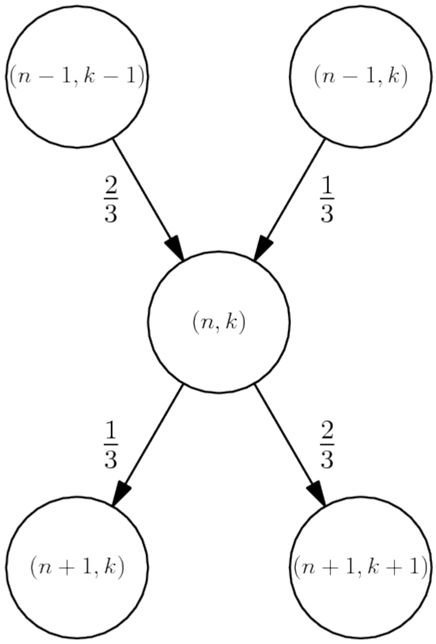


Figure 2: Transition Probabilities in Random Walk

Proof. We proceed by mathematical induction. The statement is true for $m = 1$: We are given that X_1 is 1 or 0 with half probability each, and $\gamma(1, 0) = \gamma(1, 1) = \frac{1}{2}$.

Now assume the statement holds for $m = n$. Then for any element of the triangle $(n + 1, k)$, if $X_{n+1} = k$, then X_n must either be k or $k - 1$. So,

$$\begin{aligned} \mathcal{P}(X_{n+1} = k) &= \mathcal{P}(X_n = k, X_{n+1} = k) + \mathcal{P}(X_n = k - 1, X_{n+1} = k) \\ &= \mathcal{P}(X_n = k - 1)\mathcal{P}(X_{n+1} = k \mid X_n = k - 1) \\ &+ \mathcal{P}(X_n = k)\mathcal{P}(X_{n+1} = k \mid X_n = k) \end{aligned}$$

$$= \gamma(n, k-1)\mathcal{P}(X_{n+1} = k \mid X_n = k-1) + \gamma(n,k)\mathcal{P}(X_{n+1} = k \mid X_n = k).$$

By the form of the random walk, $\mathcal{P}(X_{n+1} = k \mid X_n = k) = \frac{1}{3}$ and $\mathcal{P}(X_{n+1} = k \mid X_n = k-1) = \frac{2}{3}$. So,

$$\mathcal{P}(X_{n+1} = k) = \gamma(n, k-1) \cdot \frac{2}{3} + \gamma(n,k) \cdot \frac{1}{3} = \gamma(n+1, k). \qquad \square$$

§3.5 Closed Form

Corollary 3.12

$\gamma(m,k) = \frac{1}{3^{m-1}} \left(\binom{m-1}{k} \cdot 2^{k-1} + \binom{m-1}{k-1} \cdot 2^{k-2} \right)$ for all $m > 0$.

Proof. Consider the k^{th} index of the m^{th} row. By Theorem 3.11 there are two starting points for the random walk through which we can get to this index. There is a $\frac{1}{2}$ probability that either starting point is chosen. Consider cases:

If the random walk starts with $X_1 = 0$, then the walk will have to increase X k times and keep it the same $m-k-1$ times. Therefore, the probability of traveling from the first row and the zeroth index to the m^{th} row and the k^{th} index is

$$\binom{k + (m-k-1)}{k} \cdot \left(\frac{2}{3}\right)^k \cdot \left(\frac{1}{3}\right)^{m-k-1} = \frac{1}{3^{m-1}} \cdot \binom{m-1}{k} \cdot 2^k.$$

If, on the other hand, the random walk starts from the first row and the first index, then the walk will have to take $k-1$ steps up and $m-k$ steps unchanging. Therefore, the probability of traveling from the first row and first index to the m^{th} row and the k^{th} index is

$$\binom{(k-1) + (m-k)}{k-1} \cdot \left(\frac{2}{3}\right)^{k-1} \cdot \left(\frac{1}{3}\right)^{m-k} = \frac{1}{3^{m-1}} \cdot \binom{m-1}{k-1} \cdot 2^{k-1}.$$

In this case, we let $\binom{n}{k}$ equal 0 for $k = -1$ and $k = n+1$. This is because it is impossible to travel from the first row and zeroth index to the m^{th} row and m^{th} index and it is impossible to travel from the first row and first index to the m^{th} row and zeroth index.

Because there is a $\frac{1}{2}$ probability of starting in the first row and zeroth index and a $\frac{1}{2}$ probability of starting in the first row and first index, the probability of arriving at the m^{th} row an k^{th} index in general is

$$\frac{1}{2} \cdot \frac{1}{3^{m-1}} \cdot \binom{m-1}{k} \cdot 2^k + \frac{1}{2} \cdot \frac{1}{3^{m-1}} \cdot \binom{m-1}{k-1} \cdot 2^{k-1},$$

which simplifies to

$$\frac{1}{3^{m-1}} \left(\binom{m-1}{k} \cdot 2^{k-1} + \binom{m-1}{k-1} \cdot 2^{k-2} \right). \qquad \square$$

Theorem 3.13

For $n \neq k$, $\alpha(n,k) = \operatorname{sgn}(\alpha(n,k)) \left[2^k \cdot \binom{\frac{n+k}{2}-1}{k} \cdot \frac{n}{n-k}\right]$. Otherwise, $n = k$ and $\alpha(n,n) = 2^{n-1}$.

Proof. Recall that $\mid \alpha(n,k) \mid = \beta\left(\frac{n+k}{2}, k\right)$ by Definition 3.6. Then, we must have

$$\alpha(n,k) = \operatorname{sgn}(\alpha(n,k)) \cdot \beta\left(\frac{n+k}{2}, k\right).$$

Notice that

$$\beta\left(\frac{n+k}{2}, k\right) = \gamma\left(\frac{n+k}{2}, k\right) \cdot \sum_{i=0}^{\frac{n+k}{2}} \beta\left(\frac{n+k}{2}, i\right).$$

Substitution using Proposition 3.8 and Corollary 3.12 yields

$$\begin{aligned}
\beta\left(\frac{n+k}{2}, k\right) &= \frac{1}{3^{\frac{n+k}{2}-1}} \left[\binom{\frac{n+k}{2}-1}{k} 2^{k-1} + \binom{\frac{n+k}{2}-1}{k-1} 2^{k-2}\right] \cdot 2 \cdot 3^{\frac{n+k}{2}-1} \\
&= \left[\binom{\frac{n+k}{2}-1}{k} 2^{k} + \binom{\frac{n+k}{2}-1}{k-1} 2^{k-1}\right] \\
&= 2^{k-1}\left[2\binom{\frac{n+k}{2}-1}{k} + \binom{\frac{n+k}{2}-1}{k-1}\right] \\
&= 2^{k-1}\left[\binom{\frac{n+k}{2}-1}{k} + \binom{\frac{n+k}{2}-1}{k} + \binom{\frac{n+k}{2}-1}{k-1}\right] \\
&= 2^{k-1}\left[\binom{\frac{n+k}{2}-1}{k} + \binom{\frac{n+k}{2}}{k}\right].
\end{aligned}$$

Then if $n = k$, the quantity equals $2^{n-1}\left[\binom{n-1}{n} + \binom{n}{n}\right] = 2^{n-1}$. Hence

$$\alpha(n,n) = \operatorname{sgn}(\alpha(n,n)) \cdot 2^{n-1},$$

but $\operatorname{sgn}(\alpha(n,n)) = 1$, so $\alpha(n,n) = 2^{n-1}$. For $n \neq k$, namely for $n > k$, since $\alpha(n,k) = 0$ for $n < k$, we have

$$\begin{aligned}
\beta\left(\frac{n+k}{2}, k\right) &= 2^{k-1}\left[\binom{\frac{n+k}{2}-1}{k} + \binom{\frac{n+k}{2}}{k}\right] \\
&= 2^{k-1}\left[\frac{(\frac{n+k}{2}-1)!}{k!(\frac{n-k}{2}-1)!} + \frac{(\frac{n+k}{2})!}{k!(\frac{n-k}{2})!}\right] \\
&= 2^{k-1} \cdot \frac{(\frac{n+k}{2}-1)!}{k!(\frac{n-k}{2}-1)!} \cdot \left(1 + \frac{\frac{n+k}{2}}{\frac{n-k}{2}}\right) \\
&= 2^{k-1}\binom{\frac{n+k}{2}-1}{k} \cdot \frac{2n}{n-k}
\end{aligned}$$

$$= 2^k \binom{\frac{n+k}{2} - 1}{k} \cdot \frac{n}{n-k}.$$

Therefore,

$$\alpha(n,k) = \operatorname{sgn}(\alpha(n,k)) \left[2^k \cdot \binom{\frac{n+k}{2} - 1}{k} \cdot \frac{n}{n-k} \right]. \qquad \square$$

§4 Modulo p

Theorem 4.1

For odd prime p and $a \in \mathbb{N}$, $T_{ap}(x) = T_a(x^p)$ in $\mathbb{Z}_p[x]$.

Proof. By Proposition 2.7, $T_{ap}(x) = T_a(T_p(x))$. Therefore, if we show that $T_p(x) \equiv x^p$ in $\mathbb{Z}_p[x]$, then we are done.

First, we will verify that the coefficient of x^k is zero for $0 \leq k < p$. By Theorem 3.13, we know that

$$|\alpha(p,k)| = 2^k \binom{\frac{p+k}{2} - 1}{k} \cdot \frac{p}{p-k}$$

Then taking modulo p, the $\frac{p}{p-k}$ term means that, since everything in the denominator (i.e. $p-k$ and $k!$) do not have any factors of p, $\alpha(p,k)$ is a multiple of p as desired. So, every coefficient of x^k with $k < p$ disappears in $\mathbb{Z}_p[x]$.

Now, the coefficient of x^p in $T_p(x)$ is 2^{p-1}, which is 1 in $\mathbb{Z}_p$ by Euler's Theorem. Thus, all coefficients less than x^p are 0, and the x^p coefficient is 1, so $T_p(x) = x^p$ in $\mathbb{Z}_p[x]$. $\square$

§5 Roots

Lemma 5.1

The roots of $T_m(x)$ are of form $\cos\left(\frac{\pi + 2\pi k}{2m}\right)$.

Proof. Recall that $T_n(\cos(x)) = \cos(nx)$. Thus, the roots of $T_n(x)$ can be found by solving $\cos(nx) = 0$. $cos(nx) = 0 \implies nx = \frac{\pi}{2} + k\pi$ where $k \in \mathbb{Z}$. Solving for x, this yields $x = \frac{\pi + 2k\pi}{2m}$. This means that the roots of the polynomial are in form $\cos\left(\frac{\pi+2k\pi}{2m}\right)$. Notice that $0 \leq k < m$ yield distinct roots, ergo there are m roots of this form. There are no other roots as T_m is a degree m polynomial, so it has at most m roots. $\square$

Theorem 5.2

$\forall k, m \in \mathbb{N}$, $T_m(x) \mid T_{m(2k+1)}(x)$.

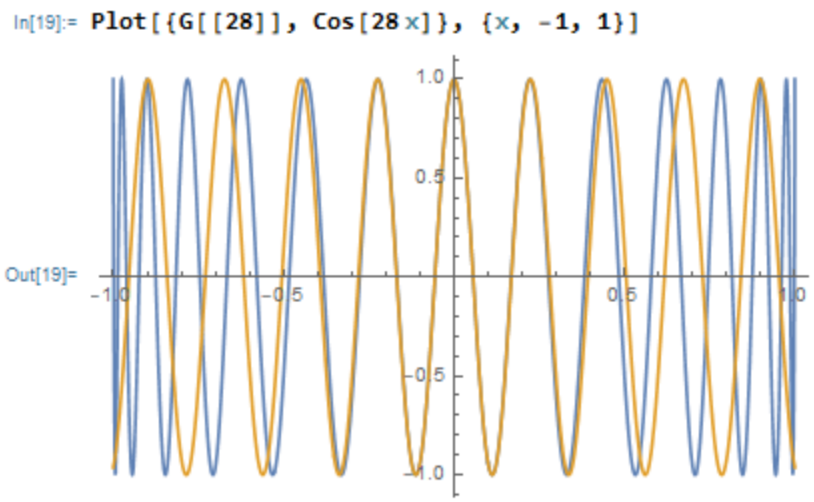
Proof. Recall by Lemma 5.1 the roots of $T_m(x)$ are of form $\cos\left(\frac{\pi+2\pi a}{2m}\right)$ and the roots of $T_{2m(2k+1)}(x)$ are of form $\cos\left(\frac{\pi+2\pi b}{2m(2k+1)}\right)$ for all natural numbers k, where a and b are integers. For any a, we show that there exists b such that $\frac{\pi+2\pi b}{2m(2k+1)} = \frac{\pi+2\pi a}{2m}$. Then,

$$\begin{aligned}
\frac{\pi+2\pi b}{2m(2k+1)} &= \frac{\pi+2\pi a}{2m}\\
\frac{\pi+2\pi b}{2k+1} &= \pi+2\pi a\\
\pi+2\pi b &= (2k+1)(\pi+2\pi a)\\
1+2b &= (2k+1)(1+2a)\\
1+2b &= 4ka+2k+2a+1\\
2b &= 4ka+2k+2a\\
b &= 2ka+k+a.
\end{aligned}$$

Therefore, there exists an integer b such that $\frac{\pi+2\pi b}{2m(2k+1)} = \frac{\pi+2\pi a}{2m}$ for every a. This means that every root of $T_m(x)$ is a root of $T_{m(2k+1)}(x)$. Therefore, $T_m(x) \mid T_{m(2k+1)}(x)$. $\square$

§6 Approximation

After graphing some Chebyshev polynomials on the coordinate plane, we noticed that even Chebyshev polynomials closely approximate sinusoidal functions.



Proposition 6.1

As $n \to \infty$, for any even $k \in \mathbb{N}$, $\alpha(n,k)$ approaches the coefficient of x^k in the Taylor series for $\cos(nx)$. More formally, the ratio of the Taylor series coefficient and $\alpha(n,k)$ goes to 1.

Proof. It is well known that the coefficient of x^{2k} in the Taylor series for $\cos x$ is $\frac{(-1)^k}{(2k)!}$. So the coefficient in $\cos nx$ is $\frac{(-1)^k \cdot n^{2k}}{(2k)!}$. Now, the closed form for $|\alpha(n,k)|$ is

$$\left[2^k \cdot \binom{\frac{n+k}{2} - 1}{k} \cdot \frac{n}{n-k}\right].$$

Using Stirling's approximation for binomial coefficients, we see that this approaches

$$2^{2k} \cdot \frac{(\frac{n+2k}{2})^{2k}}{(2k)!} \cdot \frac{n}{n-k} = \frac{(n+2k)^{2k}}{(2k)!} \cdot \frac{n}{n-k}.$$

As $n \to \infty$, both $n-k$ and $n+2k$ approach n, so this simplifies to $\frac{n^{2k}}{(2k)!}$ as desired. $\square$