# Point-Set Topology

Daniel Kim

November 22, 2019

## Contents



## §1 Topological Spaces

**Definition 1.1.** A pair $(X, \tau)$ is a **topological space** if $X$ is a set and $\tau$ is a collection of subsets $\mathcal{U} \subset X$ satisfying

i) $\emptyset, X \in \tau$

ii) If $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \tau$ then $\bigcap_{k=1}^{n} \mathcal{U}_k \in \tau$. (Closed under finite intersections)

iii) If $\{\mathcal{U}_\alpha\} \subset \tau$, then $\bigcup_{\alpha} \mathcal{U}_\alpha \in \tau$. (Closed under arbitrary unions)

The distinction between arbitrary unions and finite intersections only matters when $X$ is an infinite set (implying that $\tau$ is a collection of infinitely many subsets).

**Definition 1.2.** An element of $\tau$ is an **open set**. An element of $X$ is a **point**. With respect to the topological space $(X, \tau)$, $\tau$ is a **topology** on $X$.

**Definition 1.3.** With respect to the topological space $(\mathbb{R}, \tau)$, define a set $S \subset \mathbb{R}$ to be **open** iff $\forall x \in S\ \exists \delta > 0$ s.t. $(x - \delta, x + \delta) \in S$. Then open sets can be in the form $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ (and any arbitrary unions or finite intersections of these open intervals will also be open). Therefore, $\tau$ consists of *unions of open intervals*, and is called the **standard topology** on $\mathbb{R}$.

**Definition 1.4.** With respect to topological spaces $(X, \tau_1)$ and $(Y, \tau_2)$, $f : X \to Y$ is **continuous** iff $\forall$ open sets $V \subset Y$, $f^{-1}(V)$ is an open subset of $X$.

**Proposition 1.5**

If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f^{-1}((a,b))$ is open $\forall (a,b) \subset \mathbb{R}$, then $f$ is continuous.

It is easy to see that the following collections of sets satisfy the definition of a topology for any set $X$.

**Definition 1.6.** The **trivial topology** (or indiscrete topology) is $\tau = \{\emptyset, X\}$, given any set $X$.

**Definition 1.7.** The **discrete topology** (or chaotic topology) is the collection of all subsets of $X$, i.e. $\tau = 2^X$.

**Proposition 1.8**

If a set $X$ is given the discrete topology, then all functions $f : X \to T$ for any topological space $T$ are continuous.

*Proof.* If $X$ has the discrete topology, then every subset of $X$ is open. Therefore, any preimage of any subset of $T$ must be open. $\square$

**Proposition 1.9**

If a set $Y$ is given the trivial topology, then all functions $f : X \to Y$ for any topological space $X$ are continuous.

*Proof.* Since the topology of $Y$ is $\{\emptyset, Y\}$, we have $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$, both of which are open subsets of $X$. $\square$

**Definition 1.10.** A set $C$ in a topological space $(X, \tau)$ is **closed** iff $X \setminus C$ is open.

Recall Definition 1.3. Closed sets in the standard topology on $\mathbb{R}$ consist of closed intervals $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ and any arbitrary unions or finite intersections of these closed intervals. In this space, $\emptyset$ and $\mathbb{R}$ are both open and closed. The *half-closed interval* $[0,1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$ is neither open nor closed. To prove this, suppose it was closed. Then its complement in $\mathbb{R}$, which is $(-\infty, 0) \cup [1, \infty)$, must be open. But 1 cannot be surrounded by an interval which still lies in the set.

**Theorem 1.11**

An open set in $\mathbb{R}$ in the standard topology is a countable union of disjoint open intervals.

**Definition 1.12.** The **interior** of a set $S \subset X$ is $S^0 = \bigcup\limits_{\substack{\mathcal{U} \subseteq S \\ \text{is open}}} \mathcal{U}$, i.e. the union of all open sets contained in $S$.

By definition of a topology, the interior must also be an open subset of $S$. In fact, the interior is the largest open set contained in $S$. It also follows that $S$ is an open set iff $S = S^0$.

With respect to the standard topology on $\mathbb{R}$, $\mathbb{Q}^0 = \emptyset$, since there does not exist a non-empty open interval of $\mathbb{R}$ which is a subset of $\mathbb{Q}$.

**Proposition 1.13**
$\forall x \in X^0$, $x \in \mathcal{U} \subset X$ where $\mathcal{U}$ is open.

*Proof.* Follows directly from definition. $\square$

**Proposition 1.14**
$(A \cap B)^0 = A^0 \cap B^0$.

**Definition 1.15.** The **exterior** of a set $S \subset X$ is $X \setminus S^0$, where $S^0$ is the interior of $S$.

**Definition 1.16.** The **closure** of a set $S \subset X$ is $\bigcap_{\substack{C \supseteq S, C \subseteq X \\ C \text{ closed}}} C = \overline{S}$.

The closure is the smallest closed set that contains $S$.
We have $S^0 \subseteq S \subseteq \overline{S}$, where $S^0 = S$ iff $S$ is open, and $S = \overline{S}$ iff $S$ is closed.

**Proposition 1.17**
$\overline{A \cup B} = \overline{A} \cup \overline{B}$.

**Definition 1.18.** The **boundary** of a set $S \subset X$ is $\partial S = \overline{S} \cap \overline{X \setminus S}$.

**Proposition 1.19**
For any set $S \subset X$ where $X$ is a topological space, $S \cup \partial S = \overline{S}$.

*Proof.* We have $S \cup \partial S = S \cup (\overline{S} \cap \overline{S^c}) = (S \cup \overline{S}) \cap (S \cup \overline{S^c})$. Note that

$$\begin{aligned} S \cup \overline{S} &= S \cup \left( \bigcap_{\substack{C \supseteq S \\ C \text{ closed}}} C \right) \\ &= \bigcap_{\substack{C \supseteq S \\ C \text{ closed}}} (S \cup C) \\ &= \bigcap_{\substack{C \supseteq S \\ C \text{ closed}}} C \\ &= \overline{S}. \end{aligned}$$

We then wish to prove $S \cup \overline{S^c} = X$, where $S^c = X \setminus S$. Clearly $S \cup \overline{S^c} \subseteq X$. If $x \in X$, then if $x \in S$, we're done. Otherwise, $x \in X \setminus S$ i.e. $x \in S^c$. But clearly $S^c \subseteq \overline{S^c}$, so we're done. Hence, $S \cup \partial S = \overline{S} \cap X = \overline{S}$. $\square$

**Definition 1.20.** A **neighborhood** of $x$ is any open set containing $x$.

**Definition 1.21.** $x$ is a **limit point** of a set $X$ if every neighborhood of $x$ has nonempty intersection with $X \setminus \{x\}$.

This also suggests that a limit point $x$ does not necessarily have to be in $X$.

**Definition 1.22.** If $x \in X$ is not a limit point of $X$, $x$ is an **isolated point** of $X$.

**Proposition 1.23**

Let $(X, \tau)$ be a topological space. Then a set $S \subset X$ is open if and only if $\forall x \in S$, there exists a neighborhood of $x$ which is entirely contained in $S$.

*Proof.* The right direction is easy: given that $S$ is open, for any $x \in S$, take the neighborhood to be equal to $S$ to prove existence.

Now consider the left direction: For any $x \in S$ with a neighborhood entirely contained in $S$, consider the union of all such neighborhoods, say $S'$. By closure under arbitrary unions, $S'$ must be open. It suffices to prove that $S = S'$. Suppose $a \in S$. Since $a$ is an element in its own neighborhood, it must be an element of the union of all neighborhoods, hence $a \in S'$. Thus $S \subseteq S'$.

Now suppose $a \in S'$. Then $a$ must lie in some neighborhood entirely contained in $S$. Thus $a \in S$, so $S' \subseteq S$. It follows that $S = S'$, and we're done. $\square$

**Proposition 1.24**

If $x \notin X$ but $x$ is a limit point of $X$ then $x \in \partial X$.

*Proof.* We want to prove $x \in \overline{X} \cap \overline{X^c}$. Clearly, if $x \notin X$ then $x \in X^c \implies x \in \overline{X^c}$. Now, we want to prove $x \in \overline{X}$, i.e.

$$x \in \bigcap_{\substack{C \supseteq X \\ C \text{ closed}}} C.$$

For any closed set $C \supseteq X$, it suffices to prove $x \in C$. Suppose for the sake of contradiction that $x \notin C$. Then $x \in C^c$. As $C \supseteq X$, $C^c \subseteq X^c$. But $C^c$ is an open neighborhood of $x$ that has empty intersection with $X \setminus \{x\}$, contradicting that $x$ is a limit point of $X$. $\square$

**Proposition 1.25**

$\overline{X^c} = (X^0)^c$.

*Proof.* We will prove the equivalent statement $\overline{X^c}^c = X^0$. By generalization of DeMorgan's Laws on Sets,

$$\begin{aligned} x \in \overline{X^c}^c &\leftrightarrow x \in \left( \bigcap_{\substack{C \supseteq X^c \\ C \text{ closed}}} C \right)^c \\ &\leftrightarrow x \in \bigcup_{\substack{C \supseteq X^c \\ C \text{ closed}}} C^c \\ &\leftrightarrow x \in \bigcup_{\substack{C^c \subseteq X \\ C^c \text{ open}}} C^c \\ &\leftrightarrow x \in X^0. \end{aligned}$$

$\square$

**Theorem 1.26**

A set is closed if and only if it contains all of its limit points.

*Proof.* Suppose a set $S \subset X$ is closed. Then $X \setminus S$ is open. By the previous proposition, $\forall x \in X \setminus S$, there exists a neighborhood of $x$ entirely contained in $X \setminus S$. Then this neighborhood cannot contain any points in $S$. Since $x \in X \setminus S$, $S \setminus \{x\} = S$. Hence this neighborhood cannot intersect $S \setminus \{x\}$, so $x$ is not a limit point of $S$. Then, all limit points of $S$ must be in $S$.

Now suppose $S$ contains all of its limit points. Then $\forall x \in S^c$, x is not a limit point of $S$, i.e. $\exists$ open $\mathcal{U}$ with $x \in \mathcal{U}$ such that $\mathcal{U} \cap (S \setminus \{x\}) = \emptyset$. By definition of $x$, $S \setminus \{x\} = S$, so $\mathcal{U} \cap S = \emptyset$. Then $\mathcal{U} \subseteq S^c$. Hence, for any $x \in S^c$, there exists a neighborhood of $x$ contained in $S^c$, so $S^c$ is open. Thus, $S$ is closed. □

## §2 Connectedness

**Definition 2.1.** $X$ is **dense** in $Y$ if $\overline{X} \subset Y$.

For example, $\mathbb{Q}$ is dense in $\mathbb{R}$.

**Definition 2.2.** A space is **separable** if it has a countable dense subset.

**Definition 2.3.** A set is **perfect** if it is closed and every point in it is a limit point.

For example, $[0, 1]$ is perfect.

**Definition 2.4.** Sets $A, B$ are **separated** if $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

**Definition 2.5.** A set $E$ is **connected** if $E$ cannot be written as $A \cup B$ where $A, B$ are nonempty and separated.

**Definition 2.6.** A **path** is any continuous function $\phi : [0, 1] \to X$ where $X$ is any topological space.

**Definition 2.7.** A set is **pathwise connected** if $\forall x, y \in E$, $\exists$ a path $\phi : [0, 1] \to E$ such that $\phi(0) = x$, $\phi(1) = y$.

**Definition 2.8.** A **region** is an open connected set.

**Proposition 2.9**

A region is pathwise-connected.

## §3 Compactness

**Definition 3.1.** An **open cover** of a set $K$ is a collection of open sets $\{\mathcal{U}_\alpha\}$ such that $K \subset \bigcup\limits_{\alpha} \mathcal{U}_\alpha$.

**Definition 3.2.** A **finite subcover** $\{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$ is a finite subcollection of an open cover such that $K \subset \bigcup\limits_{i=1}^{n} \mathcal{U}_i$.

**Definition 3.3.** A set $K \subset X$ where $X$ is a topological space is **compact** if every open cover of $K$ has a finite subcover.

## §A The Cantor Set: Everyone's Favorite Pathology

Define the following sequence of sets as follows:

$$\begin{aligned}
C_0 &= [0,1],\\
C_1 &= [0,\tfrac{1}{3}] \cup [\tfrac{2}{3},1],\\
C_2 &= [0,\tfrac{1}{9}] \cup [\tfrac{2}{9},\tfrac{1}{3}] \cup [\tfrac{2}{3},\tfrac{7}{9}] \cup [\tfrac{8}{9},1],\\
&\vdots
\end{aligned}$$

Essentially, we begin with $C_0 = [0,1]$, divide it into three equal intervals $[0,\frac{1}{3}]$, $[\frac{1}{3},\frac{2}{3}]$, and $[\frac{2}{3},1]$. Then we cut out the middle "third", leaving us with two remaining "blocks" $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$. Take the union of these to get $C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$. Then, we repeat the process: divide each closed interval into thirds and take out the middle pieces, and so on.

Then, at some $n \in \mathbb{N} \cup \{0\}$, we have $C_n$ to be the union of $2^n$ closed intervals, where each closed interval is $\frac{1}{3^n}$ long.

**Definition A.1.** The **Cantor Set** is $C = \bigcap\limits_{n=0}^{\infty} C_n$.

The Cantor Set is a closed set with respect to the standard topology on $\mathbb{R}$, since its complement is the union of open sets. It is also an uncountable set, with no intervals as subsets.

An explicit formula is $C = \{x \in [0,1] : x = 0.t_1 t_2 \cdots t_n \cdots_3\}$ where $t_i \in \{0,2\}$.