Misc. Problems in Linear Algebra

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Problems:

1. Find the equation of the plane containing the points $(2, 3, 7)$, $(1, 5, 6)$, and $(-4, 0, 1)$.

2. Find the equation of the plane containing the points $(1, 2, 4)$, $(2, -1, 1)$, and $(4, 0, 5)$.

3. Find the intersection of the plane in #2 with the line that contains $(3, 4, 5)$ and $(5, 12, 13)$.

4. Find $\mathbf{proj}_{\langle 1,2,3\rangle}(\langle 3, 4, 5\rangle)$.

5. Find eigenvalues and eigenvectors for the matrix

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Find the intersection of the plane from #1 with the plane $x + y + z = 0$.

7. Determine the distance from the origin to the line in #3.

8. Find the angle between $\langle 1, 2, 3\rangle$ and $\langle 3, 4, 5\rangle$.

Solutions:

1. Let $R = (2,3,7)$, $M = (1,5,6)$, and $J = (-4,0,1)$. Then $\overrightarrow{RM} = \langle -1,2,-1 \rangle$ and $\overrightarrow{MJ} = \langle -5,-5,-5 \rangle$. Taking the cross-product of this gives a vector normal to the plane, so

$$\overrightarrow{RM} \times \overrightarrow{MJ} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & 2 & -1 \\ -5 & -5 & -5 \end{vmatrix} = \vec{\imath}\begin{vmatrix} 2 & -1 \\ -5 & -5 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} -1 & -1 \\ -5 & -5 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 2 \\ -5 & -5 \end{vmatrix} = -15\vec{\imath} + 15\vec{k}$$

   The normal vector is $\langle -15,0,15 \rangle$, so our equation for the plane is of the form $-15x + 15z = d$. Now we plug in one of our three given points to find the value of d, i.e. plugging in point J we get $-15(-4) + 15(1) = d = 75$, therefore our equation for the plane is $-15x + 15z = 75 \implies \boxed{-x + z = 5}$.

2. Similarly, we find two vectors given the three points $(1,2,4)$, $(2,-1,1)$, and $(4,0,5)$, which are $\langle 1,-3,-3 \rangle$ and $\langle 2,1,4 \rangle$. Taking the cross product:

$$\langle 1,-3,-3 \rangle \times \langle 2,1,4 \rangle = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -3 & -3 \\ 2 & 1 & 4 \end{vmatrix} = \vec{\imath}\begin{vmatrix} -3 & -3 \\ 1 & 4 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = -9\vec{\imath} - 10\vec{\jmath} + 7\vec{k}$$

   So our equation for the plane is of the form $-9x - 10y + 7z = d$, then we plug in $J = (-4,0,1)$ to get $-9(-4) - 10(0) + 7(5) = d = -1$, so our equation is $\boxed{-9x - 10y + 7z = -1}$.

3. The parameterization for the line with points $(3,4,5)$ and $(5,12,13)$ is:

$$\begin{aligned} x &= 3 + 2t \\ y &= 4 + 8t \\ z &= 5 + 8t \end{aligned}$$

   Note that we can divide coefficients of parameter t by 2, so our simplified parameterization is

$$\begin{aligned} x &= 3 + t \\ y &= 4 + 4t \\ z &= 5 + 4t \end{aligned}$$

   To find the intersection, we simply substitute in the parametric definitions into the equation of the plane, which is $-9x - 10y + 7z = -1$, so we get

$$-9(3 + t) - 10(4 + 4t) + 7(5 + 4t) = -1$$

Solving this gives $t = -\frac{31}{21}$. We plug this back into the parameterization of the line to get the point $\boxed{\left(\frac{32}{21}, -\frac{40}{21}, -\frac{19}{21}\right)}$.

4. Recall the general formula

$$\mathbf{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \cdot \vec{w}$$

Applying this formula gives

$$\mathbf{proj}_{\langle 1,2,3 \rangle}\left(\langle 3,4,5 \rangle\right) = \frac{13}{7}\langle 1,2,3 \rangle = \boxed{\left\langle \frac{13}{7}, \frac{26}{7}, \frac{39}{7} \right\rangle}$$

5. We know that $\text{Det}(M - \lambda I) = 0$ for given matrix M and possible eigenvalues λ. Furthermore, in general, given a matrix in upper triangular form, we have

$$\begin{vmatrix} r & s & t \\ 0 & u & v \\ 0 & 0 & w \end{vmatrix} = ruw$$

Using these facts, we have

$$\begin{vmatrix} 2-\lambda & 1 & 4 \\ 0 & -3-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(-3-\lambda)(1-\lambda) = 0$$

We have the roots $\lambda = 2, -3, 1$. We check each case:

(a) For $\lambda = 2$, we have

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We are left with the system of equations

$$\begin{aligned} 2x + y + 4z &= 2x \\ -3y + 2z &= 2y \\ z &= 2z \end{aligned}$$

The last equation implies $z = 0$, from which we determine $y = 0$ as well (from the second equation), and in the first equation we find $x = x$ for any x, therefore our eigenvectors for $\lambda = 2$ are of the form $\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$ i.e. scalar multiples of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(b) For $\lambda = -3$, we similarly set up a system of equations:

$$\begin{aligned} 2x + y + 4z &= -3x \\ -3y + 2z &= -3y \\ z &= -3z \end{aligned}$$

We find $z = 0$, $y = y$, and $x = -\dfrac{1}{5}y$, therefore our eigenvectors are $\begin{bmatrix} -\frac{1}{5}y \\ y \\ 0 \end{bmatrix}$ i.e. scalar multiples of $\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}$.

(c) For $\lambda = 1$, we have the system of equations

$$\begin{aligned} 2x + y + 4z &= x \\ -3y + 2z &= y \\ z &= z \end{aligned}$$

We find $z = z$, $z = 2y$, and $x = -9y$, therefore our eigenvectors are $\begin{bmatrix} -9y \\ y \\ 2y \end{bmatrix}$ i.e. scalar multiples of $\begin{bmatrix} -9 \\ 1 \\ 2 \end{bmatrix}$.

6. We have

$$-9x - 10y + 7z = -1 \tag{1}$$
$$x + y + z = 0 \tag{2}$$

Multiplying 7 times equation (2) then adding to equation (1) gives

$$16x + 17y = 1$$

, and multiplying 9 times equation (2) then adding to equation (1) gives

$$-y + 16z = -1$$

We can use these equations to find two points which lie on the intersection of the two planes, then determining the parameterization of the line using those two points.

Plugging in $y = 1$ gives the solutions $(-1, 1, 0)$, and plugging in $z = 1$ gives the solutions $(-18, 17, 1)$. Therefore the parameterization of the line is

$$\begin{aligned} x &= -1 - 17t \\ y &= 1 + 16t \\ z &= t \end{aligned}$$

7. Recall that the line in discussion has the parameterization

$$x = 3 + t$$
$$y = 4 + 4t$$
$$z = 5 + 4t$$

Consider the vector from $(0, 0, 0)$ to an arbitrary point on the line, which can be represented as $(3+t, 4+4t, 5+4t)$. The vector going from the origin to this arbitrary point is just $\langle 3+t, 4+4t, 5+4t\rangle$. We can determine a vector in the line by taking two points on the line, i.e. $(3, 4, 5)$ and $(5, 12, 13)$, which yields $\langle 2, 8, 8\rangle$. A well-known fact is that $\vec{v}$ and $\vec{w}$ are orthogonal if and only if $\vec{v} \cdot \vec{w} = 0$. So we have

$$\langle 3+t, 4+4t, 5+4t\rangle \cdot \langle 2, 8, 8\rangle = 0$$

$$6 + 2t + 32 + 32t + 40 + 32t = 0$$

Solving gives $t = -\dfrac{13}{11}$. Therefore the distance is just the magnitude of the vector $\langle 3+t, 4+4t, 5+4t\rangle$, which is

$$\sqrt{\left(\frac{20}{11}\right)^2 + \left(\frac{8}{11}\right)^2 + \left(\frac{3}{11}\right)^2} = \boxed{\frac{\sqrt{473}}{11}}$$

8. The angle between $\vec{v}$ and $\vec{w}$ can be determined with the fact

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}$$

and then taking inverse cosine of it. We have

$$\cos\theta = \frac{\langle 1, 2, 3\rangle \cdot \langle 3, 4, 5\rangle}{\|\langle 1, 2, 3\rangle\|\|\langle 3, 4, 5\rangle\|} = \frac{13}{5\sqrt{7}}$$

Therefore $\theta = \boxed{\cos^{-1}\left(\frac{13}{5\sqrt{7}}\right)}$.