# Math 305 Notes

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## Contents

## §1 Metric spaces

**Definition 1.1.** A **metric** on $X$ is a function $d : X \times X \to \mathbb{R}$ such that

(1) Positivity: $\forall x, y \in X$, $d(x, y) \geq 0$ with $d(x, y) = 0$ iff $x = y$

(2) Symmetry: $\forall x, y \in X$, $d(x, y) = d(y, x)$

(3) Triangle Inequality: $\forall x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$

We denote a **metric space** as the pair $(X, d)$.

**Definition 1.2.** The **discrete metric** on $X$ is the metric $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ for $x, y \in X$.

**Definition 1.3.** A **norm** on a vector space $V$ is a non-negative function $\|\cdot\| : V \to \mathbb{R}$ such that

(1) $\forall v \in V$, $\|v\| \geq 0$ with $\|v\| = 0$ iff $v = 0$

(2) $\forall c \in \mathbb{R}$, $\|c \cdot v\| = |c| \cdot \|v\|$

(3) $\forall v, w \in V$, $\|v + w\| \leq \|v\| + \|w\|$

**Remark 1.4.** Note that $d(v, w) = \|v - w\|$ is a metric on $V$ (induced by $\|\cdot\|$).

**Claim 1.5** — The discrete metric on $\mathbb{R}^2$ is not induced by any norm on $\mathbb{R}^2$.

## §2 Interior points and open subsets

**Definition 2.1.** We say $B(x,r) = \{y \in X \mid d(x,y) < r\}$ is the $r$-ball or $r$-neighborhood of $x$.

**Definition 2.2.** $x \in X$ is an **interior point** of $A$ if $B(x,r) \subset A$ for some $r > 0$.

**Definition 2.3.** The **interior** of $A$ is $\mathrm{Int}(A) = A^\circ$, the set of all interior points of $A$.

**Definition 2.4.** $A$ is **open** in $X$ (an open subset of $X$) if $A = \mathrm{Int}(A)$. In other words, $\forall x \in A$, $\exists r > 0$ such that $B(x,r) \subset A$.

**Lemma 2.5**

$\forall r > 0$, $\forall x \in X$, $B(x,r) = \{y \in X \mid d(x,y) < r\}$ is an open subset of $X$.

**Lemma 2.6**

The following claims are true:

1. For any $A \subset X$, $\mathrm{Int}(A)$ is an open subset.
2. $\mathrm{Int}(A)$ is the largest open subset of $A$, i.e. if $\mathrm{Int}(A) \subset B \subset A$ and $B$ is open, then $B = \mathrm{Int}(A)$.

**Proposition 2.7**

The following are true:

1. The intersection of finitely many open subsets of $X$ is open.
2. The union of any collection of open subsets of $X$ is open.

**Remark 2.8.** The intersection of infinitely many open subsets is not necessarily open.

## §3 Limit points, closures, and closed subsets

**Definition 3.1.** For $A \subset X$, $x \in X$ is a **limit point** of $A$ if $\forall r > 0$, $(B(x,r) - \{x\}) \cap A \neq \emptyset$, where $B(x,r) - \{x\} = \{y \in X \mid y \neq x, d(x,y) < r\}$.

**Definition 3.2.** The **closure** of $A$ is $\overline{A} = \{\text{limit points of } A\} \cup A$. In other words,

$$x \in \overline{A} \iff \forall r > 0,\ B(x,r) \cap A \neq \emptyset.$$

**Definition 3.3.** $A$ is **closed** in $X$ if $A = \overline{A}$, or equivalently, if every limit point of $A$ is contained in $A$.

**Definition 3.4.** The **boundary** of a set $A$ is defined to be the intersection $\partial A = \overline{A} \cap \overline{A^c}$.

**Problem 3.5.** $x \in \partial A$ if and only if $\forall r > 0$, $B(x,r)$ contains points both of $A$ and of $A^c$.

**Problem 3.6.** The boundary of $A \subset X$ is $\emptyset$ if and only if $A$ is both open and closed.

**Definition 3.7.** A point in $A$ which is not a limit point of $A$ is called an **isolated point** of $A$.

> **Remark 3.8.** In any metric space $(X,d)$, a finite subset is always closed.

**Proposition 3.9**

If $A \subset X$, then $A$ is open if and only if $A^c = X - A$ is closed.

> **Remark 3.10.** Note that $X$ and $\emptyset$ are both open and closed.

**Lemma 3.11**

Let $\overline{B}(x,r) = \{y \in X \mid d(x,y) \leq r\}$. Then $\forall r > 0$, $\overline{B}(x,r)$ is a closed subset of $X$.

**Theorem 3.12**

For any subset $A \subset X$, $\overline{A}$ is closed. Moreover, $\overline{A}$ is the smallest closed subset containing $A$.

**Proposition 3.13**

The following are true:

(1) The union of finitely many closed subsets is closed.

(2) The intersection of any collection of closed subsets is closed.

> **Remark 3.14.** The union of infinitely many closed subsets is not necessarily closed.

**Definition 3.15.** A metric space $X$ is called **connected** if there is no subset $X$ which is simultaneously open and closed, except for $\emptyset$ and $X$.

**Problem 3.16.** Show that $[0,1]$ is connected. Show that $\mathbb{R}^n$ is connected.

**Definition 3.17.** Let $X$ be a metric space and $S \subset X$. A subset $A$ of $S$ is said to be **dense** in $S$ if $S \subseteq \overline{A}$.

**Problem 3.18.** If each $A$ and $B$ is dense in $S$ and $B$ is open, then $A \cap B$ is dense in $S$ and in $B$.

## §4 Properties of $\mathbb{R}$

**Remark 4.1.** Recall that $b \in \mathbb{R}$ is an upper bound of set $A$ if and only if $\forall a \in A$, $a \leq b$.

**Definition 4.2.** Any non-empty subset $A$ of $\mathbb{R}$ which is bounded from above has the **least upper bound**.

**Definition 4.3.** We denote $\sup A$ as the least upper bound (**supremum**). We denote $\inf A$ as the greatest lower bound (**infimum**).

**Corollary 4.4**

Any bounded subset $A$ of $\mathbb{R}$ has $\sup A \in \mathbb{R}$ and $\inf A \in \mathbb{R}$.

**Proposition 4.5**

Any bounded infinite monotone sequence of $\mathbb{R}$ is convergent, where monotone means either increasing ($x_1 \leq x_2 \leq \cdots$) or decreasing ($x_1 \geq x_2 \geq \cdots$).

**Definition 4.6.** Let $\{x_i\}$ be any infinite bounded sequence.

We write $\liminf_{i\to\infty} x_i = \lim_{m\to\infty} \inf\{x_m, x_{m+1}, x_{m+2}, \ldots\} = \sup\{a_m \mid m = 1, 2, \ldots\}$ where $a_m = \inf\{x_k \mid k \geq m\}$.

**Remark 4.7.** The $\liminf$ is the smallest limit of all infinite subsequences of $\{x_i\}$.

**Definition 4.8.** Let $\{x_i\}$ be any infinite bounded sequence.

We write $\limsup_{i\to\infty} x_i = \lim_{m\to\infty} \sup\{x_m, x_{m+1}, x_{m+2}, \ldots\} = \inf\{b_m \mid m = 1, 2, \ldots\}$ where $b_m = \sup\{x_k \mid k \geq m\}$.

**Remark 4.9.** The $\limsup$ is the largest limit of all infinite subsequences of $\{x_i\}$.

**Remark 4.10.** Note that $a_1 \leq a_2 \leq \cdots$ is bounded so $\lim_{m\to\infty} a_i$ exists. Similarly, $b_1 \geq b_2 \geq \cdots$ is bounded so $\lim_{m\to\infty} b_i$ exists.

**Problem 4.11.** Given a bounded sequence $\{x_n : n = 1, 2, \ldots\}$ of real numbers, then $x_n$ converges to $a$ if and only if $a = \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.

**Problem 4.12.** For bounded sequences $\{x_n : n = 1, 2, \ldots\}$, $\{y_n : n = 1, 2, \ldots\}$,

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

**Theorem 4.13** (Bolzano-Weierstrass Theorem)

Any bounded infinite sequence in $\mathbb{R}^n$ has a convergent subsequence.

**Example 4.14**

Given that the Bolzano-Weierstrass Theorem holds in $\mathbb{R}$, prove that the theorem is true for $\mathbb{R}^2$.

## §5 Covers and compactness

**Definition 5.1.** A collection $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$ where $U_\alpha \subset X$ is called a **covering** of $A$ if $A \subset \bigcup_{\alpha \in I} U_\alpha$.

Moreover, it is an **open covering** if each $U_\alpha$ is an open subset as well.

**Definition 5.2.** If $\mathcal{V} = \{U_\alpha \mid \alpha \in J\}$ and $J \subset I$, and $\mathcal{V}$ is a cover of $A$, then $\mathcal{V}$ is called a **subcover** of $\mathcal{U}$.

**Definition 5.3.** We say a cover or subcover is **finite** if $I$ (or $J$) is finite.

**Definition 5.4.** For $\emptyset \neq A \subset X$, $A$ is **compact** if every open cover of $A$ contains a finite subcover.

**Example 5.5**

Prove $(0, 1)$ is not compact in $\mathbb{R}$, and $B(x, r)$ is not compact in $\mathbb{R}^2$.

**Definition 5.6.** A subset $A \subset X$ is called **bounded** if $A \subset B(x, r)$ for some $x \in X$ and $r > 0$.

**Proposition 5.7**

Any compact subset of $X$ is bounded and closed.

**Proposition 5.8**

A closed subset of a compact set is compact.

**Problem 5.9.** The union of finitely many compact subsets of $X$ is compact.

**Remark 5.10.** The union of infinitely many compact subsets of $X$ is not necessarily compact.

**Problem 5.11.** If $\{x_1, x_2, x_3, \dots\}$ is a convergent sequence with limit $x \in X$, then $A = \{x, x_1, x_2, x_3, \dots\}$ is compact.

**Definition 5.12.** A **sequence** in $X$ is a collection $x_1, x_2, \dots$ of elements in $X$ indexed by $\mathbb{N}$ or $\mathbb{N} \cup \{0\}$.

**Definition 5.13.** A sequence $\{x_n \mid n = 1, 2, \dots\}$ is convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. We say $\{x_n\}$ converges to $x$. Specifically, this means

$$\forall \varepsilon > 0 \; \exists N \; \forall n \geq N, \; d(x_n, x) < \varepsilon.$$

We write $\lim\limits_{n \to \infty} x_n = x$.

**Remark 5.14.** If $\{x_n\}$ is convergent, then $\lim\limits_{n \to \infty} x_n$ exists uniquely.

**Definition 5.15.** For a sequence $\{x_n\}$, a **subsequence** is of the form $\{y_k \mid k = 1, 2, \dots\}$ where $y_k = x_{n_k}$ for $n_1 < n_2 < \cdots$

**Remark 5.16.** A sequence $x_n \to x$ iff every subsequence of $x_n$ converges to $x$.

**Definition 5.17.** $\emptyset \neq A \subset X$ is **sequentially compact** if every infinite sequence in $A$ has a convergent subsequence in $A$ with the limit inside $A$.

**Theorem 5.18**

$A$ is compact iff $A$ is sequentially compact.

**Theorem 5.19** (Heine-Borel)

Any closed and bounded subset of $\mathbb{R}^n$ is compact.

**Remark 5.20.** By the Heine-Borel Theorem, compact is equivalent to closed and bounded under $\mathbb{R}^n$.

## §6 Continuous functions

**Definition 6.1.** A is **countable** if and only if $\exists$ an injection of $A$ to $\mathbb{Z}$.

For the following definitions, fix 2 metric spaces $(X, d_X)$ and $(Y, d_Y)$.

**Definition 6.2.** Given $f : X \to Y$ and $B \subset Y$, $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$.

**Definition 6.3.** For $x \in X$, $f$ is **continuous** at $x$ if

1. $\forall \varepsilon > 0$, $\exists \delta > 0$ $f(B(x, \delta)) \subset B(f(x), \varepsilon)$, or $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon)$.
2. For any open set $\mathcal{U}$ of $f(x)$, $f^{-1}(\mathcal{U})$ is an open set of $x$.
3. For any sequence $x_n \to x$, $f(x_n) \to f(x)$.

**Definition 6.4.** $f : X \to Y$ is continuous if $f$ is continuous at every $x \in X$. In other words, $\forall$ open $U \subset Y$, $f^{-1}(U)$ is open.

**Proposition 6.5**

The following are true:

1. Composition: for $f : X \to Y$ and $g : Y \to Z$ that are both continuous, then $g \cdot f : X \to Z$ is continuous.
2. If $f_1, f_2 : X \to \mathbb{R}$ are continuous, then $f_1 + f_2$, $f_1 - f_2$, $f_1 \cdot f_2$, and $f_1 / f_2$ are continuous.

**Theorem 6.6**

If $f : X \to Y$ is continuous and $A \subseteq X$ is compact, then $f(A)$ is compact.

**Remark 6.7.** The image of a closed subset is not necessarily closed under a continuous map.

**Definition 6.8.** $f : X \to Y$ is **uniformly continuous** if $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x$, $f(B(x, \delta)) \subset B(f(x), \varepsilon)$. Here, we note $\delta$ is chosen independently of $x$.

**Remark 6.9.** Clearly, uniform continuity implies continuity.

**Theorem 6.10**

If $X$ is compact and $f : X \to Y$ is continuous, then $f$ is uniformly continuous.

**Proposition 6.11**

$f : X \to Y$ is continuous if and only if $\forall$ closed $C \subset Y$, $f^{-1}(C)$ is closed.

**Theorem 6.12**

If $f : X \to \mathbb{R}$ is continuous and $X$ is compact, then $f$ attains both a minimum and maximum value. In other words, $\sup_{x \in X} f(x) = \max_{x \in X} f(x)$ and $\inf_{x \in X} f(x) = \min_{x \in X} f(x)$.

**Definition 6.13.** Let $f_n : X \to Y$ be a sequence of continuous functions. We say $f_n$ converges **pointwise** to $f$ if $\forall x \in X, f_n(x) \to f(x)$.

> **Remark 6.14.** If a sequence of continuous functions $f_n$ converges to $f$ pointwise, then $f$ is not necessarily continuous.

**Definition 6.15.** For $f_n : X \to Y$ and $f : X \to Y$, we say $f_n$ converges to $f$ **uniformly** if

$$\forall \varepsilon > 0 \; \exists N \geq 1 \; \forall n \geq N \; \forall x \in X, \; d(f_n(x), f(x)) < \varepsilon,$$

where we note $N$ is chosen independently of $x \in X$.

Equivalently,

$$\forall \varepsilon > 0 \; \exists N \geq 1 \; \forall n \geq N \; \sup_{x \in X} d(f_n(x), f(x)) \leq \varepsilon.$$

**Lemma 6.16**

If $f_n : X \to Y$ is continuous and $f_n \to f$ uniformly, then $f$ is also continuous.

**Problem 6.17.** For a metric space $X$ and a bounded subset $S$ of $\mathbb{R}$, a uniformly continuous function $f : S \to X$ must be bounded.

## §7 Complete metric spaces

**Definition 7.1.** $C(X;\mathbb{R}) = \{f : X \to \mathbb{R} \text{ is continuous}\}$ is a vector space over $\mathbb{R}$.

**Definition 7.2.** For $f \in C(X;\mathbb{R})$,

$$\|f\|_\infty = \|f\|_{\sup} = \|f\|_{\max} = \max_{x\in X} |f(x)| = \sup_{x\in X} |f(x)| \in \mathbb{R},$$

where $\|\cdot\|_\infty$ is a norm (called the sup-norm or max-norm) on $C(X;\mathbb{R})$.

**Definition 7.3.** $d(f,g) = \|f-g\|_\infty$ defines a metric on $C(X;\mathbb{R})$, called the sup-metric or max-metric.

**Proposition 7.4**

If $f_n, f \in C(X;\mathbb{R})$, then $f_n \to f$ uniformly iff $f_n \to f$ in $(C(X;\mathbb{R}), \|\cdot\|_\infty)$.

**Definition 7.5.** A sequence $\{x_n\}$ in $X$ is called **Cauchy** if $\forall \varepsilon > 0$, $\exists N \geq 1$ such that $\forall n, m \geq N$, $d(x_n, x_m) \leq \varepsilon$.

> **Remark 7.6.** If $x_n \to x$ in $X$, then $\{x_n\}$ is Cauchy. However,the converse is not always true.

**Definition 7.7.** $X$ is **complete** if every Cauchy sequence in $X$ converges in $X$.

**Lemma 7.8**

For a metric space $(X,d)$,

1. Every Cauchy sequence is bounded.
2. If a Cauchy sequence has a convergent subsequence, then it converges.

**Theorem 7.9**

$\mathbb{R}^n$ is a complete metric space.

**Theorem 7.10**

If $X$ is compact, then $(C(X;\mathbb{R}), \|\cdot\|_\infty)$ is complete.

**Definition 7.11.** A metric space $X$ is **totally bounded** if $\forall \varepsilon > 0$, $\exists x_1, \ldots, x_n \in X$ such that $X \subset \bigcup_{i=1}^n B(x_i, \varepsilon)$.

**Problem 7.12.** Let $X$ be a complete metric space and let $f : X \to X$ be a contraction, i.e. there exists $0 < r < 1$ such that

$$d(f(x), f(y)) \leq r \cdot d(x,y)$$

for all $x, y \in X$. Show that $f$ has a unique fixed point, where a fixed point of $f$ is a point $p$ such that $f(p) = p$.

**Problem 7.13.** Given a metric space $X$, $X$ is complete and totally bounded if and only if $X$ is compact.

**Problem 7.14.** Given a Cauchy sequence $x_n$ in $X$, then if $f : X \to Y$ is uniformly continuous, then $f(x_n)$ is a Cauchy sequence.

## §8 Riemann Integral

We focus on continuous, bounded functions on the $\mathbb{R}$ line.

**Definition 8.1.** A **partition** $P$ of $I = [a, b]$ is

$$a = x_0 < x_1 < \ldots < x_n = b.$$

**Definition 8.2.** Given $f : I \to \mathbb{R}$ is a bounded function and $P$ is a partition of $I$, we can define

- the **lower sum**
$$L(f, P) = \sum_{i=1}^{n} \left( \inf_{x \in I_i} f(x) \right) \cdot \ell(I_i)$$
where $I_i = [x_{i-1}, x_i]$ and $\ell(I_i) = |x_i - x_{i-1}|$.
- the **upper sum**
$$U(f, P) = \sum_{i=1}^{n} \left( \sup_{x \in I_i} f(x) \right) \cdot \ell(I_i)$$

**Remark 8.3.**
$$\left( \inf_{x \in I} f(x) \right) \cdot \ell(I) \le L(f, P) \le U(f, P) \le \left( \sup_{x \in I} f(x) \right) \cdot \ell(I)$$

**Definition 8.4.** If $P \subset P'$, we call $P'$ a **refinement** of $P$. We also say that $P'$ is **finer** than $P$.

**Remark 8.5.** If $P'$ is finer than $P$, then

$$L(f, P) \le L(f, P') \le U(f, P') \le U(f, P).$$

**Definition 8.6.** The **lower integral** of $f$ is

$$\underline{\int_a^b} f = \underline{\int_I} f = \sup_P L(f, P),$$

where $P$ is any partition of $I$.

**Definition 8.7.** The **upper integral** of $f$ is

$$\overline{\int_a^b} f = \overline{\int_I} f = \inf_P U(f, P),$$

where $P$ is any partition of $I$.

**Definition 8.8.** $f$ is **Riemann-integrable** if

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$$

where we define the Riemann integral of $f$ to be defined as $\int_a^b f$, the common value.

**Lemma 8.9**

$f$ is Riemann integrable iff $\forall \varepsilon > 0$, $\exists$ some partition $P$ such that $U(f, P) - L(f, P) \le \varepsilon$.

**Proposition 8.10**

Any continuous function $f : I = [a, b] \to \mathbb{R}$ is Riemann integrable.

**Proposition 8.11**

Any function $f : I \to \mathbb{R}$ with finitely many discontinuity points is Riemann integrable.

**Remark 8.12.** The limit of a sequence of Riemann integrable functions is not necessarily Riemann integrable.

## §9 Measure

**Definition 9.1.** A collection $\mathcal{A}$ of subsets of $X$ is called an **algebra** if

1. $\sigma, X \in \mathcal{A}$
2. $\mathcal{A}$ is closed under complement, i.e. if $S \in \mathcal{A}$, then $S^c = X \setminus S \in \mathcal{A}$
3. $\mathcal{A}$ is closed under **finite** union, i.e. if $S_1, S_2, \ldots, S_k \in \mathcal{A}$, then $\bigcup_{i=1}^{k} \in \mathcal{A}$.

**Definition 9.2.** A collection $\mathcal{A}$ of subsets of $X$ is called a $\sigma$-**algebra** if

1. $\sigma, X \in \mathcal{A}$
2. $\mathcal{A}$ is closed under complement, i.e. if $S \in \mathcal{A}$, then $S^c = X \setminus S \in \mathcal{A}$
3. $\mathcal{A}$ is closed under **countable** union, i.e. if $S_1, S_2, \ldots, S_k \in \mathcal{A}$, then $\bigcup_{i=1}^{k} \in \mathcal{A}$.

**Proposition 9.3**

If $\mathcal{A}$ is an algebra, then

1. (Closed under finite intersection) If $A_1, \ldots, A_k \in \mathcal{A}$, then $\bigcap_{i=1}^{k} \in \mathcal{A}$. Furthermore, if $\mathcal{A}$ is a $\sigma$-algebra, then $A_i \in A$ for $i \geq 1 \implies \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$ (closed under countable intersection).
2. If $A, B \in \mathcal{A}$, then $A \setminus B \in \mathcal{A}$.

**Definition 9.4.** A set $S$ is **cofinite** if $S^c$ is finite.

**Remark 9.5.** Every $\sigma$-algebra is an algebra, but not every algebra is a $\sigma$-algebra. Consider $X = [0, 1]$ and $A = \{S \subset [0, 1] \mid S \text{ is finite or cofinite}\}$.

**Definition 9.6.** Let $C$ be a collection of subsets of $X$. The $\sigma$-algebra **generated** by $C$, which we denote $\sigma(C)$, is the smallest $\sigma$-algebra containing $C$.

**Remark 9.7.** Note that $\sigma(\sigma(C)) = \sigma(C)$ and if $C_1 \subset C_2$, then $\sigma(C_1) \subset \sigma(C_2)$.

**Definition 9.8.** The **Borel $\sigma$-algebra** of $X$, denoted $\mathcal{B}(X)$, is the $\sigma$-algebra generated by all open subsets of $X$. We say that the elements of $\mathcal{B}(X)$ are **Borel sets** of $X$.

**Remark 9.9.** All open and closed subsets of $X$ are Borel sets of $X$.

**Proposition 9.10**

Any open subset $O$ of $\mathbb{R}$ is a countable union of disjoint open intervals.

**Proposition 9.11**

For collections $C_1$ and $C_2$, $\sigma(C_1) = \sigma(C_2)$ if and only if $C_2 \subset \sigma(C_1)$ and $C_1 \subset \sigma(C_2)$.

**Proposition 9.12**

$\mathcal{B}(X)$ is generated by each of the following collections:

1. $C_1 = \{(a, b) \mid a < b\}$
2. $C_2 = \{[a, b] \mid a \le b\}$
3. $C_3 = \{(a, b] \mid a < b\}$
4. $C_4 = \{(a, \infty) \mid a \in \mathbb{R}\}$

**Definition 9.13.** Suppose $X$ is a set and $\mathcal{A}$ is a $\sigma$-algebra consisting of subsets of $X$. A **measure** $\mu$ on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty)$ such that

1. $\mu(\emptyset) = 0$
2. Countable additivity: If $\{A_i \in \mathcal{A}\}$ are countably many pairwise disjoint sets, then $\mu\left(\bigcup A_i\right) = \sum \mu(A_i)$.

Moreover, we say $(X, \mathcal{A}, \mu)$ constitute a **measure space**.

**Example 9.14**

Given $\mathcal{A} = \mathcal{P}(X)$, $\mu(A) := |A|$ for $A \in \mathcal{A} = \mathcal{P}(X)$ is called the **counting measure**.

**Example 9.15**

Given $\mathcal{A} = \mathcal{P}(X)$, the **Dirac measure** at $x \in X$ is $\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}$.

**Example 9.16**

If $x_1, \dots, x_k \in X$ and $p_1, \dots, p_k > 0$ such that $\sum_{i=1}^k \frac{1}{p_i} = 1$, then $\sum_{i=1}^k \frac{1}{p_i} \delta_{x_i}$ is a probability measure on $X$.

**Definition 9.17.** If $\mu$ is a measure on $(X, \mathcal{A})$, then

1. Elements of $\mathcal{A}$ are called $\mu$-measurable subsets.
2. If $\mu(X) < \infty$, then $\mu$ is a **finite measure**.
3. If $\mu(X) = 1$, then $\mu$ is a **probability measure**.

**Remark 9.18.** If $\mu_1, \mu_2$ are measures on $(X, \mathcal{A})$, then for any $c_1, c_2 \ge 0$, $c_1\mu_1 + c_2\mu_2$ is again a measure on $(X, \mathcal{A})$. We have $(c_1\mu_1 + c_2\mu_2)(A) = c_1\mu_1(A) + c_2\mu_2(A)$.

**Proposition 9.19**

Given a measure space $(X, \mathcal{A}, \mu)$,

1. Monotonicity: If $A, B \in \mathcal{A}$ with $A \subset B$, then $\mu(A) \le \mu(B)$.
2. Countable subadditivity: If $A_i \in \mathcal{A}$ for $i = 1, 2, \dots$, then $\mu\left(\bigcup A_i\right) \le \sum_i \mu(A_i)$.

**Proposition 9.20**

Given a measure space $(X, \mathcal{A}, m)$,

(1) If $A_i \in \mathcal{A}$ such that $A_i \uparrow A$, i.e. $A_1 \subset A_2 \subset \cdots A = \bigcup_i A_i$, then $m(A) = \lim_{i\to\infty} m(A_i)$.

(2) If $A_i \in \mathcal{A}$ such that $A_i \downarrow A$, i.e. $A_1 \supset A_2 \supset \cdots A = \bigcap_i A_i$ and $m(A_1) < \infty$, then $m(A) = \lim_{i\to\infty} m(A_i)$.

**Definition 9.21.** We say $\mu^* : \mathcal{P}(X) \to [0, \infty)$ is an **outer measure** if

1. $\mu^*(\emptyset) = 0$
2. Monotonicity: If $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$.
3. Countable subadditivity: If $\{A_i\}$ is a countable collection, $\mu^*\left(\bigcup A_i\right) \leq \sum_i \mu^*(A_i)$.

**Remark 9.22.** Note that an outer measure is not necessarily a measure because it only guarantees countable subadditivity, not countable additivity for pairwise disjoint sets.

**Theorem 9.23**

Suppose $\mathcal{C}$ is a collection of subsets of $X$, such that $\emptyset, X \in \mathcal{C}$ and let $\ell : \mathcal{C} \to [0, \infty)$ such that $\ell(\emptyset) = 0$.

For $A \subset X$, define $\mu^*(A) = \inf \{\sum_i \ell(A_i) \mid A \subset \bigcup_i A_i \text{ for } A_i \in \mathcal{C}\}$.

Then, $\mu^*$ is an outer measure on $X$.

**Definition 9.24.** $A \subset X$ is $\mu^*$-**measurable** if for any subset $E \subset X$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

**Definition 9.25.** Let $\mathcal{A}_{\mu^*}$ denote the collection of all $\mu^*$-measurable subsets.

**Remark 9.26.** By countable subadditivity, we know $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$, so it always suffices to prove that $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

**Definition 9.27.** A subset $A \subset X$ is $\mu^*$-**null** if $\mu^*(A) = 0$.

**Proposition 9.28**

A $\mu^*$-null subset is always $\mu^*$-measurable.

**Theorem 9.29**

For any outer measure $\mu^*$ on $X$, $\mathcal{A}_{\mu^*}$ is a $\sigma$-algebra, and $\mu^*\Big|_{\mathcal{A}_{\mu^*}}$ is a measure, i.e. $\mu^* : \mathcal{A}_{\mu^*} \to [0, \infty)$ satisfies countable additivity. In other words, if $\{A_i\}$ consists of pairwise disjoint subsets of $\mathcal{A}_{\mu^*}$ then $\mu^*(\bigcup A_i) = \sum \mu^*(A_i)$.

## §10 Lebesgue Measure

**Definition 10.1.** Define $\mathcal{C} = \{(a,b) : a < b\} \cup \emptyset \cup \mathbb{R}$, and let $\ell(a,b) = b - a$, $\ell(\emptyset) = 0$, and $\ell(\mathbb{R}) = \emptyset$. Then the outer measure $m^*$ as defined by theorem 9.23 is called the **Lebesgue outer measure**. In other words, for any $A \subset \mathbb{R}$,

$$m^*(A) = \inf\left\{\sum_i \ell(a_i, b_i) \mid A \subset \bigcup_i C_i,\ C_i = (a_i, b_i)\right\}.$$

**Definition 10.2.** We say $\mathcal{M}_*$ is the $\sigma$-algebra consisting of all $m^*$-measurable subsets, i.e. Lebesgue measurable subsets. Denote $m = m^*\Big|_{\mathcal{M}_*}$ as the **Lebesgue measure** on $\mathbb{R}$.

**Definition 10.3.** Let $s > 0$. Then the $s$-**dimensional Hausdorff measure** on $\mathbb{R}$ is defined for any $\varepsilon > 0$, $A \subset \mathbb{R}$,

$$\mathcal{H}^s_\varepsilon(A) = \inf\left\{\sum_i \ell(C_i)^s \mid A \subset \bigcup C_i,\ \operatorname{diam}(C_i) \le \varepsilon,\ C_i = (a_i, b_i)\right\}.$$

**Definition 10.4.** As $\varepsilon \to 0$, $\mathcal{H}^s_\varepsilon(A)$ increases, so

$$\lim_{\varepsilon \to 0^+} \mathcal{H}^s_\varepsilon(A) = \mathcal{H}^s(A)$$

is the $s$-**dimensional Hausdorff outer measure** on $\mathbb{R}$.

**Lemma 10.5**

If $I$ is an interval, then $m^*(I) = \ell(I)$.

**Proposition 10.6**

Some properties of Lebesgue measurable subsets include:

1. Any countable subset is Lebesgue measurable and is of measure 0.
2. Any interval is Lebesgue measurable and $m(I) = \ell(I)$.
3. Any open or closed subset is Lebesgue measurable.
4. If $A$ is Lebesgue measurable, $A + r$ is also Lebesgue measurable for any $r \in \mathbb{R}$, and $m(A) = m(A + r)$.
5. If $A$ is Lebesgue measurable, then for any $c \in \mathbb{R}$, $cA$ is Lebesgue measurable. Furthermore, $m(cA) = |c| \cdot m(A)$.

**Proposition 10.7**

If $A, B \subset \mathbb{R}$ are two subsets whose closures are disjoint, then $m^*(A \cup B) = m^*(A) + m^*(B)$.

**Lemma 10.8**

Let $\epsilon \in (0,1)$ and $m$ be the Lebesgue measure on $\mathbb{R}$. Suppose $A$ is a Lebesgue measurable subset. If $m(A \cap I) \le (1 - \epsilon)m(I)$ for every interval $I$, then $m(A) = 0$.

**Theorem 10.9**

If $A$ is a Lebesgue measurable subset of $\mathbb{R}$ with $m(A) > 0$, then the difference set $A - A = \{x - y : x, y \in A\}$ contains an open interval centered at 0.

**Theorem 10.10** (Approximation Property)

If $A \subset \mathbb{R}$ is Lebesgue measurable, then $\forall \varepsilon > 0$, there exists a closed set $F$ and open set $G$ such that $F \subset A \subset G$, where $m(G - F) < \varepsilon$, i.e. $m(G) - \varepsilon < m(A) < m(F) + \varepsilon$.

**Corollary 10.11**

The Lebesgue measure $m$ is the unique measure on $(\mathbb{R}, \mathcal{M})$ such that for any open interval $O$, $m(O) = \ell(O)$.

**Corollary 10.12**

The Lebesgue measure $m$ is the unique measure on $(\mathbb{R}, \mathcal{M})$ such that

1. $m([0, 1]) = 1$
2. $m$ is translation invariant i.e. $m(A + r) = m(A)$ for all $r \in \mathbb{R}$.

**Theorem 10.13**

Let $A$ be a Lebesgue measurable subset. Then $\exists F_0 \subset A \subset G_0$ where $F_0$ is a countable union of closed subsets, $G_0$ is a countable intersection of open subsets, and $m(F_0) = m(A) = m(G_0)$.

**Remark 10.14.** Note that in the previous theorem, $F_0$ is not necessarily closed and $G_0$ is not necessarily open.

**Definition 10.15.** The $F_\sigma$ set is the countable union of closed subsets and the $G_\delta$ set is the countable intersection of open subsets.

**Remark 10.16.** By the previous definition, $F_\sigma$ and $G_\delta$ are Borel sets.

**Corollary 10.17**

For any Lebesgue measurable subset $A$, $A$ is a union of a Borel subset and a null subset.

**Definition 10.18.** If $A_n$ is a sequence of subsets of $\mathbb{R}$, then

$$\begin{aligned}\limsup_{n\to\infty} A_n &= \{x \in \mathbb{R} : x \text{ belongs to } A_n \text{ for infinitely many values of } n\} \\ &= \bigcap_{N\geq 1} \bigcup_{k\geq N} A_k,\end{aligned}$$

$$\begin{aligned}\liminf_{n\to\infty} A_n &= \{x \in \mathbb{R} : x \text{ belongs to } A_n \text{ for all but finitely many values of } n\} \\ &= \bigcup_{N\geq 1} \bigcap_{k\geq N} A_k.\end{aligned}$$

**Proposition 10.19**

If each $A_n$ is Lebesgue measurable, then $\limsup A_n$ and $\liminf A_n$ are also Lebesgue measurable.

**Proposition 10.20**

If $\sum_{n=1}^{\infty} m(A_n) < \infty$, show that $m(\limsup A_n) = 0$.

**Definition 10.21.** A **coset** of $\mathbb{Q}$ in $\mathbb{R}$ is a subset of the form $x + \mathbb{Q}$. The set of cosets of $\mathbb{Q}$ in $\mathbb{R}$ is denoted as $\mathbb{R}/\mathbb{Q}$.

**Lemma 10.22**

If $(x + \mathbb{Q}) \cap (y + \mathbb{Q}) \neq \emptyset$, then $x + \mathbb{Q} = y + \mathbb{Q}$.

**Corollary 10.23**

$\mathbb{R}/\mathbb{Q}$ gives a partition of $\mathbb{R}$. Hence, $\mathbb{R}/\mathbb{Q} = \{x + \mathbb{Q} \mid x \in \mathbb{R}\} = \{x + \mathbb{Q} \mid x \in [0, 1]\}$.

**Definition 10.24.** By the Axiom of Choice, there exists a **Vitali subset** $V \subset [0, 1]$ such that $\mathbb{R} = \bigcup_{x\in V}(x + \mathbb{Q})$, i.e. $\mathbb{R} \subset \bigcup_{q\in\mathbb{Q}}(V + q)$.

**Lemma 10.25**

The Vitali set is not Lebesgue measurable.

**Theorem 10.26**

If $A$ is a Lebesgue measurable subset of $\mathbb{R}$ with $m(A) > 0$, then $A$ contains a non-Lebesgue measurable subset.

## §11 Measurable functions

**Definition 11.1.** For $(X, \mathcal{A}, \mu)$ as a measure space, $f : X \to \mathbb{R}$ is called **measurable** (i.e. $\mu$-measurable) if $\forall$ open subset $U \subset \mathbb{R}$, $f^{-1}(U)$ is measurable.

**Proposition 11.2**

The following are equivalent: for $f : X \to \mathbb{R}$,

(1) $\forall$ open $U \subset \mathbb{R}$, $f^{-1}(U)$ is measurable

(2) $\forall$ closed $F \subset \mathbb{R}$, $f^{-1}(F)$ is measurable

(3) $\forall$ open interval $I \subset \mathbb{R}$, $f^{-1}(I)$ is measurable

(4) $\forall$ closed interval $I \subset \mathbb{R}$, $f^{-1}(I)$ is measurable

(5) $\forall a \in \mathbb{R}$, $f^{-1}((a, \infty))$ is measurable

(6) $\forall a \in \mathbb{R}$, $f^{-1}([a, \infty))$ is measurable

(7) $\forall a \in \mathbb{R}$, $f^{-1}((-\infty, a))$ is measurable

(8) $\forall a \in \mathbb{R}$, $f^{-1}((-\infty, a])$ is measurable

**Remark 11.3.** $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable if $\forall a \in \mathbb{R}$, $\{x \mid f(x) > a\} = f^{-1}(a, \infty)$ is Lebesgue measurable.

**Definition 11.4.** The **characteristic function** of a set $A$ is $1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$.

**Proposition 11.5**

If $A$ is a measurable subset, then $1_A$ is measurable.

**Lemma 11.6**

For $f, g : X \to \mathbb{R}$ measurable,

(1) $\forall c \in \mathbb{R}$, $c \cdot f$ is measurable

(2) $f + g$ is measurable

(3) $\forall k \in \mathbb{N}$, $f^k$ is measurable

(4) $f \cdot g$ is measurable

(5) $\max(f, g)$, $\min(f, g)$, and $|f|$ are all measurable

**Lemma 11.7**

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ is measurable, then $f \circ g$ is measurable.

**Theorem 11.8**

If $f_n : X \to \mathbb{R}$ is measurable, and $f_n \to f$ pointwise, then $f = \lim_{n\to\infty} f_n$ is measurable.

**Proposition 11.9**

If $f_n$ is measurable, then $\sup_n f_n$ and $\inf_n f_n$ are measurable.

**Definition 11.10.** The **set difference** of sets $A$ and $B$ is $A \Delta B = (A - B) \cup (B - A)$.

**Lemma 11.11**

If $A$ is a Lebesgue measurable subset, then any subset $B$ such that $B \Delta A$ is null is also measurable.

**Lemma 11.12**

If $f$ is measurable and $g$ satisfies $N = \{x \in \mathbb{R} \mid f(x) \neq g(x)$ is null, then $g$ is also measurable.

**Definition 11.13.** $f = g$ **almost everywhere** if $\{x \mid f(x) \neq g(x)\}$ is a null set.

**Lemma 11.14**

Let $N$ be a null set. Let $g : \mathbb{R} - N \to \mathbb{R}$ be a measurable function. Then any extension of $g$ to $\mathbb{R}$ is also measurable.

**Corollary 11.15**

Any function with countably many discontinuity points is Lebesgue measurable.

**Proposition 11.16**

If $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function, then the preimage of every Borel subset of $\mathbb{R}$ is measurable.

## §12 Lebesgue integration

**Definition 12.1.** An **integrable simple function** is of the form $\sum_{k=1}^{n} a_k 1_{E_k}$ where the $E_k$'s are disjoint, measurable subsets of $\mathbb{R}$ such that $m(E_k) < \infty$. If the $E_k$'s are also intervals, then it is called a **step function**.

**Proposition 12.2**

Any finite linear combination of integrable simple functions is an integrable simple function.

**Lemma 12.3**

Let $m(A) < \infty$ and $f : A \to \mathbb{R}$ be a bounded measurable function. Then $\forall \varepsilon > 0$, $\exists$ simple functions $s_1$ and $s_2$ such that $s_1 \leq f \leq s_2$ and $s_2(x) - s_1(x) < \varepsilon$ $\forall x \in A$.

**Theorem 12.4**

The following are true:

1. Any measurable function $f : \mathbb{R} \to \mathbb{R}$ is a limit of a sequence of simple functions.
2. Any measurable function $f : \mathbb{R} \to \mathbb{R}$ is a limit of a sequence of step functions almost everywhere. That is, $\exists$ step function $s_k$ such that $f(x) = \lim_{k\to\infty} s_k(x)$ for almost all $x \in \mathbb{R}$.

**Proposition 12.5**

If $E_i$ are disjoint and measurable and $F_j$ are disjoint and measurable, then $\sum a_i 1_{E_i} = \sum b_j 1_{F_j}$, then $\sum a_i m(E_i) = \sum b_j m(F_j)$.

**Definition 12.6.** If $S$ is an ISF where $S = \sum_{i=1}^{n} a_i 1_{E_i}$ for disjoint, measurable sets $E_i$ such that $m(E_i) < \infty$, then the **Lebesgue integral** of $S$ is defined as

$$\int S = \int S(x)\, dx = \sum_{i=1}^{n} a_i m(E_i).$$

**Proposition 12.7**

The following are properties of the Lebesgue integral of an ISF:

1. If $s_1$, $s_2$ are ISFs and $c \in \mathbb{R}$, then $cs_1 + s_2$ is an ISF. Moreover, $c\int s_1 + \int s_2 = \int (cs_1 + s_2)$.
2. If $s_1 \leq s_2$, then $\int s_1 \leq \int s_2$.
3. If $s$ is an ISF and $s_a$ is translating $s$ by $a \in \mathbb{R}$, then $s_a(x) = s(x - a)$ and $\int s_a = \int s$.

**Definition 12.8.** Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative measurable function. Then the **Lebesgue integral** of $f$ is

$$\int f = \sup \left\{ \int s \mid 0 \leq s \leq f,\ s \text{ is an ISF} \right\}.$$

**Lemma 12.9**

Let $f$, $f_1$, and $f_2$ be non-negative measurable functions.

1. If $f_1 \geq f_2$, then $\int f_1 \geq \int f_2$.
2. For all $a \geq 0$, $\int (af) = a \int f$.
3. $\int (f_1 + f_2) = \int f_1 + \int f_2$.

**Definition 12.10.** Given a function $f$, define $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$ where $f^+$ and $f^-$ are both non-negative functions, and $f = f^+ - f^-$. Then $f$ is **Lebesgue integrable** if $f$ is Lebesgue measurable and $\int |f| < \infty \iff \int f^+ < \infty, \int f^- < \infty$.

**Definition 12.11.** If $f$ is Lebesgue integrable, then $\int f = \int f^+ - \int f^-$.

**Remark 12.12.** For any measurable subset $A \subset \mathbb{R}$, $\int_A f = \int f \cdot 1_A$ if $f \cdot 1_A$ is integrable.

**Proposition 12.13**

Some basic properties of the Lebesgue integral:

1. Linearity: if $f$ and $g$ are integrable, then $\int (f + g) = \int f + \int g$, and for $a \in \mathbb{R}$, $\int (af) = a \int f$.
2. If $f \leq g$, then $\int f \leq \int g$. Also, $\left|\int f\right| \leq \int |f|$.
3. Translation invariance: If $f_a(x) = f(x - a)$, then $\int f_a = \int f$.

**Lemma 12.14**

If $f = g$ almost everywhere and $f$ and $g$ are integrable, then $\int f = \int g$.

## §13 Limit Theorems

**Definition 13.1.** For $(X, \mathcal{A}, \mu)$ a general measure space, we define

1. $\int_X s \, d\mu$ for ISF
2. $\int_X f \, d\mu$ for measurable function $f : X \to \mathbb{R}$ and $f \geq 0$
3. $f$ is integrable for $\mu$, i.e. ($\mu$-integrable), if $\int |f| \, d\mu < \infty \implies \int f = \int f^+ - \int f^-$

**Theorem 13.2** (Markov Inequality)

For $f : X \to [-\infty, \infty]$ measurable, for any $0 < \lambda < \infty$, then

$$\mu\left(\{x \in X : |f(x)| > \lambda\}\right) \leq \frac{1}{\lambda} \int_X |f| \, d\mu.$$

**Corollary 13.3**

The following are true:

1. If $f : X \to [-\infty, \infty]$ is integrable, then $f$ is finite almost everywhere.
2. If $\int_X |f| = 0$, then $f = 0$ almost everywhere.

**Theorem 13.4** (Monotone Convergence Theorem)

Let $f_n : \mathbb{R} \to \mathbb{R}$ be an increasing sequence of non-negative measurable functions such that $0 \leq f_1 \leq f_2 \leq \cdots$ and let $f = \lim\limits_{n\to\infty} f_n$. Then $\int f = \lim\limits_{n\to\infty} \int f_n$.

**Corollary 13.5** (Tonelli's theorem for exchanging sums & integrals)

Let $f_n : X \to [0, \infty)$ be non-negative and measurable. Then

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n.$$

**Corollary 13.6** (Borel-Cantelli Lemma)

Let $E_n$ be a measurable subset of $X$ such that $\sum\limits_{n=1}^{\infty} \mu(E_n) < \infty$. Then for almost all $x \in X$, $x$ belongs to only finitely many $E_n$'s. In other words, $\mu(\limsup E_n) = 0$.

**Theorem 13.7** (Fatou lemma)

For measurable $f_n : X \to [0, \infty)$ and $f_n \geq 0$,

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

**Theorem 13.8** (Dominated Convergence Theorem)

Let $f_n : X \to (-\infty, \infty)$ be a sequence of measurable functions such that $f(x) = \lim_{n\to\infty} f_n(x)$. Suppose $\exists g : X \to (-\infty, \infty)$ integrable such that $\forall n,\ |f_n| \leq g$. Then $\lim_{n\to\infty} \int f_n = \int f$.

**Corollary 13.9**

Let $f_n$ be a decreasing sequence of non-negative measurable functions. Suppose that $\int f_1 < \infty$. If $f(x) = \lim f_n(x)$, then

$$\int f = \lim \int f_n.$$

**Proposition 13.10**

Suppose $f_n$ is an increasing sequence of non-negative measurable functions and let $f(x) = \lim f_n(x) \in [0, \infty]$. If $\lim \int f_n$ is finite, then $f$ is finite except on a null set.

## §14 Riemann & Lebesgue integration

We focus on bounded functions on bounded intervals $f : [a, b] \to \mathbb{R}$ such that $\exists M, \sup_{x \in I} |f(x)| \leq M$ for $I = [a, b]$.

**Definition 14.1.** For a partition $P = \{a = x_0, x_1, \dots, x_n = b\}$ of $I = [a, b]$,

$$U(P, f) = \sum_{i=1}^{n} M_i \, |x_i - x_{i-1}|$$

where $M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$ and

$$L(P, f) = \sum_{i=1}^{n} m_i \, |x_i - x_{i-1}|$$

where $m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x)$. Note that $L(P, f) \leq U(P, f)$.

**Definition 14.2.** Recall $f$ is **Riemann integrable** iff $\exists$ a sequence of partitions $P_1 \subset P_2 \subset \cdots$ such that

$$\lim_{k \to \infty} U(P_k, f) = \lim_{k \to \infty} L(P_k, f) = RI(f)$$

where $RI(f)$ denotes the Riemann integral of $f$.

**Theorem 14.3**

If $f$ is Riemann integrable, then $f$ is Lebesgue integrable and $\int f = RI(f)$.

**Theorem 14.4**

$f$ is Riemann integrable iff $f$ is continuous almost everywhere, i.e.

$$\{x \in I \mid f \text{ is discontinuous at } x\}$$

is a null set.

**Lemma 14.5**

If $A \subset \mathbb{R}$ is a measurable subset, the set of discontinuous points of $1_A$ is $\partial A$, where $\partial A = \mathbb{R} - (A^\circ \cup (A^c)^\circ)$ and $S^\circ$ denotes the interior of $S$.

**Corollary 14.6**

If $A$ is closed and $m(A) = 0$, then $1_A$ is continuous almost everywhere, and therefore $1_A$ is Riemann integrable.

## §15 L1, L2 spaces

**Definition 15.1.** $L^1(\mathbb{R})$ is the space of Lebesgue integrable functions on $\mathbb{R}$, which can be written as

$$\{f : \mathbb{R} \to \mathbb{R} \mid \int |f| < \infty\} \text{ on } \mathbb{R}$$

and we consider $f = g$ if $f(x) = g(x)$ for a.e. $x$.

**Definition 15.2.** We define $\|f\|_1 = \int |f|$ as the $L^1$ norm of $f$. Then for $f, g \in L^1(\mathbb{R})$, $d(f, g) = \|f - g\|_1$ defines a metric.

**Theorem 15.3**

$L^1(\mathbb{R})$ is complete.

**Definition 15.4.** $C_c(\mathbb{R})$ is the space of all continuous functions with compact **support**, where

$$\operatorname{supp}(f) = \overline{\{x \in \mathbb{R} \mid f(x) \neq 0}.$$

**Remark 15.5.** $C_c(\mathbb{R}) \subset L^1(\mathbb{R})$

**Theorem 15.6**

$C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$.

**Theorem 15.7** (Covering lemma)

Let $E \subset \mathbb{R}$ be a union $\bigcup_\alpha I_\alpha$ of a family $\mathcal{B} = \{I_\alpha\}$ of open intervals of length $\leq c$, for fixed $c > 0$. Then there exists a countable collection of pairwise disjoint intervals $I_1, \cdots \in B$ such that $m(E) \leq 5 \cdot \sum_i m(I_i)$.

**Definition 15.8.** Let $h \in L^1(\mathbb{R})$. The **H-L maximal function** is

$$h^*(x) = \sup \left\{ \frac{1}{m(I)} \int_I |h| \mid I = \text{any bounded open interval containing } x \right\}$$

**Theorem 15.9** (H-L maximal inequality)

For all $\lambda > 0$,

$$m\{x \in \mathbb{R} \mid h^*(x) \geq \lambda\} \leq \frac{5}{\lambda} \int |h|.$$

**Remark 15.10.** This implies $h^*$ is finite a.e.

Furthermore, unless $h = 0$ a.e., $h^*$ is not integrable, i.e. $h^* \notin L^1(\mathbb{R})$.

**Theorem 15.11** (Differentiation Theorem)

If $f \in L^1(\mathbb{R})$, i.e. $f$ is integrable, then $F$, the anti-derivative of $f$, is differentiable and

$$F'(x) = \lim_{h \to 0} \frac{1}{h} \int_{[x,x+h]} f(t)\, dt = f(x)$$

for a.e. $x$.

**Proposition 15.12**

Let $A \subset \mathbb{R}$ be a measurable subset. A point $x \in A$ is called a point of density for $A$ if $\lim\limits_{m(I)\to 0} \frac{m(A\cap I)}{m(I)} = 1$ where the limit is taken over intervals $I$ containing $x$. Prove that almost every point of $A$ is a point of density for $A$.

**Definition 15.13.** A vector space $X$ over $\mathbb{C}$ with inner product is called a **Hilbert space** if it is a complete metric space with respect to the metric induced from the inner product.

> **Remark 15.14.** The **Banach space** is a vector space over $\mathbb{C}$ with a norm if it is complete.

**Definition 15.15.** An **inner product** on $\mathcal{H}$ is a map $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$, say $(v, w) \mapsto \langle v, w\rangle$ such that

(1) $\langle v, v\rangle \geq 0$, with $\langle v, v\rangle = 0 \iff v = 0$

(2) $\langle v, w\rangle = \overline{\langle w, v\rangle}$

(3) $\forall c \in \mathbb{C}$, $v_1, v_2 \in V$, $\langle cv_1 + v_2, w\rangle = c\langle v_1, w\rangle + \langle v_2, w\rangle$

> **Remark 15.16.** For a given inner product $\langle\ ,\ \rangle$ on $\mathcal{H}$, $\|v\| = \sqrt{\langle v, v\rangle}$ is a **norm**. Furthermore, $d(v, w) = \|v - w\|$ is a metric on $\mathcal{H}$.

**Proposition 15.17**

For $a, b \in \mathbb{C}$,

(1) $|\langle a, b\rangle| \leq |a| \cdot |b|$

(2) $2\,|\langle a, b\rangle| \leq |a|^2 + |b|^2$

(3) $|a + b|^2 \leq 2(|a|^2 + |b|^2)$

**Definition 15.18.** $f : X \to \mathbb{C}$ is **measurable** if both $\mathrm{Re} f$ and $\mathrm{Im} f$ are measurable.

**Definition 15.19.** $f : X \to \mathbb{C}$ is **integrable** if $\int |f|^2 < \infty$ where $\int f = \int \mathrm{Re} f + i \int \mathrm{Im} f$.

**Definition 15.20.** We define the $L^2$-space over $I = [-\pi, \pi]$ as

$$L^2([-\pi, \pi], \mathbb{C}) = \left\{ f : [-\pi, \pi] \to \mathbb{C} \mid \int_{[-\pi,\pi]} |f(x)|^2\, dx < \infty \right\}$$

**Remark 15.21.** Note that for any $n \in \mathbb{Z}$, $e_n(x) = e^{inx} \in L^2[-\pi, \pi]$ since $\int_{-\pi}^{\pi} |e_n(x)|^2 \, dx = 2\pi < \infty$.

**Proposition 15.22**

$L^2(I)$ is a vector space over $\mathbb{C}$.

**Definition 15.23.** Define an inner product on $L^2$ as

$$\langle f, g \rangle = \frac{1}{2\pi} \int_I f(x) \cdot \overline{g(x)} \, dx$$

**Definition 15.24.** The $L^2$ norm is

$$\|f\|_2 = \sqrt{\frac{1}{2\pi} \int |f|^2} = \sqrt{\langle f, f \rangle}$$

**Remark 15.25.** The Cauchy-Schwarz inequality tells us that $|\langle f, g \rangle| \leq \|f\|_2 \cdot \|g\|_2$.

**Theorem 15.26**

$L^2[-\pi, \pi]$ is a Hilbert space.

**Definition 15.27.** In a Hilbert space, an **orthonormal set** is a set of unit vectors which are orthogonal to each other.

**Definition 15.28.** An orthonormal subset $\{e_1, e_2, \ldots,\}$ of $\mathcal{H}$ is called an **orthogonal basis** if the set of all finite linear combinations of $e_i$'s is dense in $\mathcal{H}$.

**Theorem 15.29**

Let $\mathcal{H}$ be a Hilbert space, e.g. $\mathcal{H} = L^2[-\pi, \pi]$. Suppose $\{e_1, e_2, \ldots\}$ is an orthonormal set in $\mathcal{H}$. Then the following are equivalent:

(1) $\{e_1, e_2, \ldots\}$ is an orthonormal basis, i.e. the set of all finite linear combinations of $\{e_1, e_2, \ldots\}$ is dense in $\mathcal{H}$. That is, $\forall f \in L^2[-\pi, \pi]$,

$$f = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n$$

almost everywhere, where

$$\hat{f}(n) = \langle f, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx.$$

This is the **nth Fourier coefficient** of $f$.

(2) If $f \in \mathcal{H}$ satisfies $\langle f, e_i \rangle = 0$ for all $i$, then $f = 0$.

(3) If $s_N(f) = \sum_{i=1}^{N} \langle f, e_i \rangle e_i$ for $f \in \mathcal{H}$, then $s_N(f) \to f$ as $N \to \infty$, i.e. $\lim_{N \to \infty} \|s_N(f) - f\| = 0$.

(4) $\forall f \in \mathcal{H}$, $\|f\|^2 = \sum_{i=1}^{\infty} |\langle f, e_i \rangle|^2$. This is called **Parseval's identity**.

**Definition 15.30.** A **convolution** of $f, K : I \to \mathbb{C}$ is

$$(f * K)(x) = \frac{1}{m(I)} \int_I f(x-y)K(y)\,dy$$

whereever the integral is well-defined.

**Definition 15.31.** A sequence $\{K_n\}_{n=1,2,\dots}$ of integrable functions on $I$ is called a **good kernel** if

(1) $\frac{1}{2\pi}\int_I K_n = 1$ for all $n$.

(2) $\exists M > 0$ such that $\frac{1}{2\pi}\int_I |K_n| \leq M$ for all $n$. Note that $m$ is the same for all $n$ (uniformly bounded).

(3) $\forall \delta > 0$, $\int_{\delta<|x|<\pi} |K_n(x)| \to 0$. Note that $\delta < |x| < \pi$ may be interpreted as $x \in I - (-\delta, \delta)$.

**Proposition 15.32**

Let $\{K_n\}$ be a good kernel. Let $f : I \to \mathbb{C}$ be a continuous function. Then $(f * K_n)(x) \to f(x)$ uniformly.

**Proposition 15.33**

If $f$ is differentiable with continuous derivative, then $s_N(f)(x) \to f(x)$ $\forall x \in \mathbb{R}$.

**Definition 15.34.** The **Dirichlet kernel** is defined as: for any $N \geq 0$, $D_N(x) = \sum\limits_{n=-N}^{N} e^{inx}$.

**Proposition 15.35**

For $f \in L^2(I)$, $s_N(f) = (f * D_N)(x)$.

**Definition 15.36.** The **Fejer kernel** is defined as $F_N(x) = \frac{1}{N}\sum\limits_{n=0}^{N-1} D_n(x) = \frac{\sin^2(\frac{N}{2}x)}{N\sin^2(\frac{x}{2})}$.

**Proposition 15.37**

$(f * F_N)(x) = \frac{1}{N}\sum\limits_{n=0}^{N-1} s_N(f)$.

**Proposition 15.38**

The Fejer kernel is a good kernel.

**Theorem 15.39**

$\forall f \in L^2(I)$, $f * F_N \to f$ in $L^2(I)$ as $N \to \infty$. In other words, $\|f * F_N - f\| \to 0 \iff f * F_n \to f$ a.e. in $L^2$.

**Corollary 15.40**

$\{e_n \mid n \in \mathbb{Z}\}$ where $e_n(x) = e^{inx}$ is an orthonormal basis for $L^2(I)$.

**Remark 15.41.** This implies that for $f \in L^2[-\pi, \pi]$, if $s_N(f) = \sum_{|n| \leq N} \hat{f}(n)e^{inx}$, then

- As $N \to \infty$, $s_N(f) \to f$ in $L^2$
- $\|s_N(f) - f\|_2 \to 0$ as $N \to \infty$
- Informally, $f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ in $L^2$.

Caution: in general, $f_n \to f$ in $L^2$ does NOT imply $f_n(x) \to f(x)$ for a.e. $x$. A counterexample is the Typewriter sequence.

**Remark 15.42.** If $I$ is bounded, then $L^2(I) \subset L^1(I)$.

**Theorem 15.43**

For $f \in L^2[-\pi, \pi]$, then $s_N(f) \to f$ for a.e. $x \in I$.

**Theorem 15.44**

If $f \in L^2$ is differentiable at $x_0$, or if $g(x) = \frac{f(x)-f(x_0)}{x-x_0}$ is integrable over $I$, then $s_N(f(x_0)) \to f(x_0)$.

**Corollary 15.45**

If $f, g \in L^2[-\pi, \pi]$, then

$$\langle f, g\rangle = \lim_{N\to\infty} \sum_{|n|\leq N} \hat{f}(n) \cdot \overline{\hat{g}(n)}.$$

**Theorem 15.46** (Weierstrauss Approximation Theorem)

Any continuous periodic function $f$ of period $2\pi$ can be approximated uniformly by trigonometric polynomials.

**Theorem 15.47** (Weyl's equidistribution theorem)

For $x \in \mathbb{R}$, let $\{x\}$ be the fractional part of $x$ in $[0, 1)$. Then $x = \{x\} + n$ for some unique $n \in \mathbb{Z}$ and unique $\{x\} \in [0, 1)$.

If $x \notin \mathbb{Q}$, then $\{\{nx\} \mid n \in \mathbb{Z}\}$ is equidistributed in the sense that for any intervals $J_1$, as $N \to \infty$, $J_2 \subset [0, 1)$,

$$\frac{|\{\{nx\} \in J_1 \mid n \in \mathbb{Z},\ |n| \leq N\}|}{|\{\{nx\} \in J_2 \mid n \in \mathbb{Z},\ |n| \leq N\}|} \to \frac{m(J_1)}{m(J_2)}.$$

**Proposition 15.48**

Suppose $f$ and $g$ are continuous functions on $[-\pi, \pi]$. Show that $f$ is even (i.e. $f(x) = f(-x)$ for all $x$) if and only if $\hat{f}(-n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$.