Review of Vectors

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§1 Problem Set 1

§1.1 Problems

1. Compute the magnitudes (also called *norms*) of the following vectors:
   a) $\langle 3, 4 \rangle$
   b) $3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$
   c) $\langle 0, 1, 2, 3, 4 \rangle$
2. Find the unit vector in the same direction as $\langle 5, 7 \rangle$.
3. Find a vector of magnitude 6 in the opposite direction as $\langle 4, 5, 6 \rangle$.
4. Find a vector of magnitude k (for positive $k \in \mathbb{R}$) in the same direction as the vector $\langle a_1, a_2, \cdots, a_n \rangle$ for integers $n \geq 2$.
5. Let $\vec{v} = \langle 2, 3 \rangle$ and $\vec{w} = \langle 3, 1 \rangle$. Compute $2\|\vec{v}\|^2 + 2\|\vec{w}\|^2$ and $\|\vec{v} + \vec{w}\|^2 + \|\vec{v} - \vec{w}\|^2$ and confirm that they're equal. Formulate an appropriate geometric interpretation of this result, and try to prove that this equation holds for any 2D vectors $\vec{v}$ and $\vec{w}$.

§1.2 Solutions

1. The magnitude of a 2D vector $\langle a, b\rangle$ is $\sqrt{a^2+b^2}$, and similarly, the magnitude of a 3D vector is $\langle a, b, c\rangle$ is $\sqrt{a^2+b^2+c^2}$. We can generalize this to n dimensions:

$$\|\langle a_1, a_2, \cdots, a_n\rangle\| = \sqrt{\sum_{i=1}^{n} a_i^2}.$$

   a) $\|\langle 3,4\rangle\| = \sqrt{3^2+4^2} = \boxed{5}$.

   b) Recall that $\hat{\imath} = \langle 1,0,0\rangle$, $\hat{\jmath} = \langle 0,1,0\rangle$, and $\hat{k} = \langle 0,0,1\rangle$. Then $3\hat{\imath}+4\hat{\jmath}+5\hat{k} = 3\langle 1,0,0\rangle + 4\langle 0,1,0\rangle + 5\langle 0,0,1\rangle = \langle 3,0,0\rangle + \langle 0,4,0\rangle + \langle 0,0,5\rangle = \langle 3,4,5\rangle$. Therefore, the magnitude is $\sqrt{3^2+4^2+5^2} = \boxed{5\sqrt{2}}$.

   c) The magnitude is $\sqrt{0^2+1^2+2^2+3^2+4^2} = \boxed{\sqrt{30}}$.

2. Think of a right triangle with hypotenuse $\langle 5,7\rangle$ (which has magnitude $\sqrt{5^2+7^2} = \sqrt{74}$) and legs $\langle 5,0\rangle$ and $\langle 0,7\rangle$. We want to scale this triangle down such that its hypotenuse has magnitude 1. This means we should scale the legs of the triangle by a factor of $\frac{1}{\sqrt{74}}$ (think about proportionality of similar triangles). Thus we should multiply each component of the vector $\langle 5,7\rangle$ by $\frac{1}{\sqrt{74}}$, to arrive at our answer $\boxed{\left\langle \frac{5}{\sqrt{74}}, \frac{7}{\sqrt{74}}\right\rangle}$.

3. First, our unit vector in the same direction would be $\frac{1}{\sqrt{77}}\langle 4,5,6\rangle$. Then our unit vector pointing in the opposite direction would be $-\frac{1}{\sqrt{77}}\langle 4,5,6\rangle$. We want scale up its magnitude from 1 to 6, meaning that we should multiply our current vector by a scalar of 6. So our answer is $-\frac{6}{\sqrt{77}}\langle 4,5,6\rangle = \boxed{\left\langle -\frac{24}{\sqrt{77}}, -\frac{30}{\sqrt{77}}, -\frac{36}{\sqrt{77}}\right\rangle}$.

4. In general, we would normalize a vector (scaling to magnitude 1) by dividing each component by the current magnitude. Thus, the unit vector would be

$$\frac{1}{\sqrt{a_1^2+a_2^2+\cdots+a_n^2}}\langle a_1, a_2, \cdots, a_n\rangle.$$

   Then a vector of magnitude k in the same direction would be this unit vector multiplied by scalar k, so our answer is

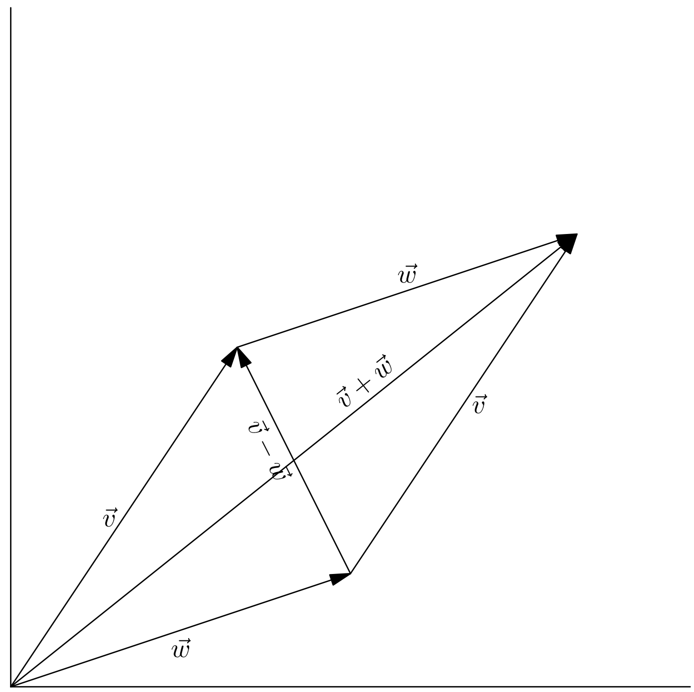
$$\boxed{\frac{k}{\sqrt{a_1^2+a_2^2+\cdots+a_n^2}}\langle a_1, a_2, \cdots, a_n\rangle}.$$

5. We first compute:

$$\begin{aligned}
\|\vec{v}\|^2 &= 2^2+3^2 = 13,\\
\|\vec{w}\|^2 &= 3^2+1^2 = 10,\\
\vec{v}+\vec{w} &= \langle 2,3\rangle + \langle 3,1\rangle = \langle 5,4\rangle \implies \|\vec{v}+\vec{w}\| = 5^2+4^2 = 41,\\
\vec{v}-\vec{w} &= \langle 2,3\rangle - \langle 3,1\rangle = \langle -1,2\rangle \implies \|\vec{v}-\vec{w}\| = (-1)^2+2^2 = 5.
\end{aligned}$$

Then $2\|\vec{v}\|^2 + 2\|\vec{w}\|^2 = \|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 = 46$. Why are they equal?

Draw $\vec{v}$, $\vec{w}$, $\vec{v}+\vec{w}$, and $\vec{v}-\vec{w}$ on the 2D plane.



We end up with a parallelogram with side lengths $\|\vec{v}\|$ and $\|\vec{w}\|$ and diagonals $\vec{v}+\vec{w}$ and $\vec{v}-\vec{w}$.

Geometrically, $2\|\vec{v}\|^2 + 2\|\vec{w}\|^2 = \|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2$ is equivalent to the idea that *the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals.* This is called the **Parallelogram law** (`https://en.wikipedia.org/wiki/Parallelogram_law`).

To prove that this is true for any $\vec{v}$ and $\vec{w}$, let $\vec{v} = \langle a, b\rangle$ and $\vec{w} = \langle c, d\rangle$. Then

$$2\|\vec{v}\|^2 + 2\|\vec{w}\|^2 = 2(a^2+b^2) + 2(c^2+d^2),$$
$$\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 = (a+c)^2 + (b+d)^2 + (a-c)^2 + (b-d)^2.$$

We expand $(a+c)^2 + (b+d)^2 + (a-c)^2 + (b-d)^2$ to get $a^2 + 2ac + c^2 + b^2 + 2bd + d^2 + a^2 - 2ac + c^2 + b^2 - 2bd + d^2 = 2(a^2+b^2) + 2(c^2+d^2)$, proving the claim.

§2 Problem Set 2

§2.1 Problems

1. Describe geometrically all linear combinations of

   a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

   b) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

   c) $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

2. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$. Compute $\vec{u} + \vec{v} + \vec{w}$ and $2\vec{u} + 2\vec{v} + \vec{w}$. Then determine whether $\vec{u}$, $\vec{v}$, and $\vec{w}$ lie in a plane or not.

3. What combination of $c \begin{bmatrix} 1 \\ 2 \end{bmatrix} + d \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ produces $\begin{bmatrix} 14 \\ 8 \end{bmatrix}$?

4. Find vectors $\vec{v}$ and $\vec{w}$ so that $\vec{v} + \vec{w} = \langle 4, 5, 6 \rangle$ and $\vec{v} - \vec{w} = \langle 2, 5, 8 \rangle$.

5. Find two different combinations of the three vectors $\vec{u} = \langle 1, 3 \rangle$, $\vec{v} = \langle 2, 7 \rangle$, and $\vec{w} = \langle 1, 5 \rangle$ that produce $\vec{b} = \langle 0, 1 \rangle$.

6. Let θ be the angle between $\langle 1, 2 \rangle$ and $\langle 2, 1 \rangle$. Find $\cos\theta$.

7. Let θ be the angle between $\langle 1, 2, 3 \rangle$ and $\langle 3, 4, 5 \rangle$. Find $\cos\theta$.

8. Prove $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ for $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$.

9. Find all values of m for which the angle between vectors $\langle 1, 1 \rangle$ and $\langle 1, m \rangle$ is $60°$.

§2.2 Solutions

1. a) Since $\begin{bmatrix}3\\6\\9\end{bmatrix}$ is $3\begin{bmatrix}1\\2\\3\end{bmatrix}$, the span would be a straight line in 3D.

   b) Suppose a linear combination is $a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\2\\3\end{bmatrix}=\begin{bmatrix}a\\2b\\3b\end{bmatrix}$. Notice that the components $2b$ and $3b$ both solely depend on the value of b, so this vector is governed by two variables. The span is a plane.

   c) A linear combination would be $a\begin{bmatrix}2\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\2\\2\end{bmatrix}+c\begin{bmatrix}2\\2\\3\end{bmatrix}=\begin{bmatrix}2a+2c\\2b+2c\\2b+3c\end{bmatrix}$. If we were to set this vector equal to any 3D vector $\begin{bmatrix}x\\y\\z\end{bmatrix}$, we would be able to solve for unique solutions a, b, and c, so the span is $\mathbb{R}^3$.

2.
$$\vec{u}+\vec{v}+\vec{w}=\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}-3\\1\\-2\end{bmatrix}+\begin{bmatrix}2\\-3\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$
$$2\vec{u}+2\vec{v}+\vec{w}=2\begin{bmatrix}1\\2\\3\end{bmatrix}+2\begin{bmatrix}-3\\1\\-2\end{bmatrix}+\begin{bmatrix}2\\-3\\-1\end{bmatrix}=\begin{bmatrix}-2\\3\\1\end{bmatrix}.$$

   Notice that $\vec{u}+\vec{v}+\vec{w}=\vec{0}$. Then $\vec{w}=-\vec{u}-\vec{v}$, so $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$. Therefore, $\vec{w}$ lies in the same plane as $\vec{u}$ and $\vec{v}$.

3. Note that $c\begin{bmatrix}1\\2\end{bmatrix}+d\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}c\\2c\end{bmatrix}+\begin{bmatrix}3d\\d\end{bmatrix}=\begin{bmatrix}c+3d\\2c+d\end{bmatrix}=\begin{bmatrix}14\\8\end{bmatrix}$, so we set up a system of equations
$$\begin{aligned}c+3d&=14,\\2c+d&=8.\end{aligned}$$
   Solving this gives $\boxed{c=2}$ and $\boxed{d=4}$.

4. Given $\vec{v}+\vec{w}=\langle 4,5,6\rangle$ and $\vec{v}-\vec{w}=\langle 2,5,8\rangle$, we can add the two equations to get $2\vec{v}=\langle 4+2,5+5,6+8\rangle \implies \boxed{\vec{v}=\langle 3,5,7\rangle}$. From this we also get $\boxed{\vec{w}=\langle 1,0,-1\rangle}$.

5. Consider a linear combination $a\langle 1,3\rangle+b\langle 2,7\rangle+c\langle 1,5\rangle=\langle a+2b+c,3a+7b+5c\rangle$. This has to be equal to $\langle 0,1\rangle$, so we have
$$\begin{aligned}a+2b+c&=0,\\3a+7b+5c&=1.\end{aligned}$$
   We have two equation with three unknown variables, so we can arbitrarily set $c=0$ to find one solution. Assuming $c=0$, we have $a+2b=0$ and $3a+7b=1$, which

we solve to get $a = -2$ and $b = 1$. So one solution is $(a, b, c) = (-2, 1, 0)$. We could also assume $a = 0$ to get another solution: solving $2b + c = 0$ and $7b + 5c = 1$ gives $b = -\frac{1}{3}$ and $c = \frac{2}{3}$. From this we have another solution $(a, b, c) = \left(0, -\frac{1}{3}, \frac{2}{3}\right)$.

I set $c = 0$ and $a = 0$ for convenience - it doesn't matter which variables you adjust as long as you get two different solutions for the same system of equations.

6. Note that for any two vectors $\vec{v}$ and $\vec{w}$,

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}.$$

We have $\langle 1, 2\rangle \cdot \langle 2, 1\rangle = 1 \cdot 2 + 2 \cdot 1 = 4$ and $\|\langle 1, 2\rangle\| = \|\langle 2, 1\rangle\| = \sqrt{5}$, so our answer is $\cos\theta = \boxed{\frac{4}{5}}$.

7. Similarly, $\langle 1, 2, 3\rangle \cdot \langle 3, 4, 5\rangle = 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 = 26$, $\|\langle 1, 2, 3\rangle\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, and $\|\langle 3, 4, 5\rangle\| = \sqrt{3^2 + 4^2 + 5^2} = 5\sqrt{2}$. Our answer is

$$\cos\theta = \frac{26}{\sqrt{14} \cdot 5\sqrt{2}} = \boxed{\frac{13}{10\sqrt{7}}}.$$

8. Let $\vec{u} = \langle u_1, u_2, \cdots, u_n\rangle$, $\vec{v} = \langle v_1, v_2, \cdots, v_n\rangle$, and $\vec{w} = \langle w_1, w_2, \cdots, w_n\rangle$. Then

$$\begin{aligned}
\vec{u} \cdot (\vec{v} + \vec{w}) &= \langle u_1, u_2, \cdots, u_n\rangle \cdot \langle v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n\rangle \\
&= \sum_{i=1}^{n} u_i(v_i + w_i) \\
&= \sum_{i=1}^{n} u_i v_i + u_i w_i \\
&= \sum_{i=1}^{n} u_i v_i + \sum_{i=1}^{n} u_i w_i \\
&= \vec{u} \cdot \vec{w} + \vec{u} \cdot \vec{w}.
\end{aligned}$$

9. We have

$$\cos 60^\circ = \frac{1 + m}{\sqrt{2} \cdot \sqrt{1 + m^2}}.$$

Since $\cos 60^\circ = \dfrac{1}{2}$, we end up with $\sqrt{2 + 2m^2} = 2 + 2m$. After squaring both sides and simplifying, the equation reduces to $m^2 + 4m + 1 = 0$. The quadratic formula yields $m = -2 \pm \sqrt{3}$.

However, since $\cos 60^\circ = \dfrac{1}{2} > 0$, we cannot have $\dfrac{1 + m}{\sqrt{2} \cdot \sqrt{1 + m^2}}$ be negative. When we plug in $m = -2 - \sqrt{3}$, we end up with a negative value. Therefore $-2 - \sqrt{3}$ cannot be a solution of m.

Then, $m = \boxed{-2 + \sqrt{3}}$ yields a positive value and thus it is the only value of m which satisfies the conditions.