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Hypothesis testing proceeds by collecting data and evaluating whether the data are compatible with H_0 or not (in which case one **rejects $\mathbf{H}_0$**).

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So the logic of testing is typically indirect: One assumes that nothing extraordinary is happening and then hopes to reject this assumption H_0.

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In the example: Using the formulas for the sum of 0/1 labels we get 'expected' $= 10 \times \frac{1}{2} = 5$ and SE $= \sqrt{10}\sqrt{\frac{1}{2} \times \frac{1}{2}} = 1.58$. So

$$z = \frac{7 - 5}{1.58} = 1.27$$

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The smaller the p-value, the stronger the evidence against H_0. Often the criterion for rejecting H_0 is a p-value smaller than 5%. Then the result is called **statistically significant**.

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Note that the p-value **does not** give the probability that H_0 is true, as H_0 is either true or not - there are no chances involved. Rather, it gives the probability of seeing a statistic as extreme, or more extreme, that the observed one, assuming H_0 is true.

Distinguishing Coke and Pepsi by taste

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This is a **one-sided test**: the alternative hypothesis for for P(1) we are interested in is on one side of $\frac{1}{2}$.

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Since we are looking at the sum of ten 0/1 labels, the z-statistic is the same that we had for coin-tossing:

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Since 10.2% is not smaller than 5%, we don't reject H_0: We are not convinced that the student can distinguish Coke and Pepsi.

Distinguishing Coke and Pepsi

A two-sided alternative might also be appropriate:

H_A: $\text{P}(1) \neq \frac{1}{2}$

H_A corresponds to a student who is more likely than not to distinguish Coke and Pepsi, but who may confuse them. Such a student might get one correct answer (say).

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One has to carefully consider whether the alternative should be one-sided or two-sided, as the p-value gets doubled in the latter case.

It is not ok to change the alternative afterwards in order to get the p-value below 5%.

The t-test

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We can try a z-test for the average of the measurements:

$$z = \frac{\text{observed average} - \text{expected average}}{\text{SE of average}} = \frac{15.6 \text{ ppb} - 15 \text{ ppb}}{\text{SE of average}}$$

since the measurement error has expected value zero.

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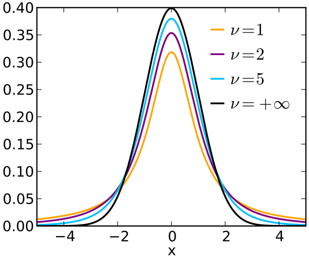

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In that case it is also better to replace the confidence interval $\bar{x} \pm z$ SE by

$$\bar{x} \pm t_{n-1}\mathsf{SE}$$

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  Therefore it is helpful to complement a test with a confidence interval: In the above case a 95% confidence interval for μ might be [15.02 ppb, 15.08 ppb].

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  Rejecting H_0 if the p-value is smaller than 5% means P(type I error) $\leq 5\%$

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and $\mathsf{SE}(\bar{x}_1) = \frac{\sigma_1}{\sqrt{n_1}}$ is estimated by $\frac{s_1}{\sqrt{n_1}}$.

The two-sample z-test

The two-sample z-test is applicable in the same way to the difference of two sample means in order to test for equality of two population means.

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All of the above two-sample tests require that the two samples are independent. They are also applicable in special situations where the samples are dependent, e.g. to compare the treatment effect when subjects are randomized into treatment and control groups.

The paired-difference test

Do husbands tend to be older than their wives?

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The ages of five couples:

Husband's age	Wife's age	age difference
43	41	2
71	70	1
32	31	1
68	66	2
27	26	1

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The independence assumption is in the sampling of the couples.

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$$z = \frac{\text{sum of 1s} - \frac{n}{2}}{\text{SE of sum}} = \frac{5 - \frac{5}{2}}{\sqrt{5}\frac{1}{2}} = 2.24 \quad \text{since } \sigma = \frac{1}{2} \text{ on } H_0.$$

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The p-value of this **sign-test** is less significant than that of the paired t-test. This is because the latter uses more information, namely the size of the differences. On the other hand, the sign test has the virtue of easy interpretation due to the analogy to coin tossing.

Mini quiz

1. True or false:
   a. The p-value depends on the data.
   b. If the p-value is smaller than 5%, then there is less than a 5% chance that the null hypothesis is true.
   c. If the null hypothesis is true, then there is less than a 5% chance to get a p-value that is smaller than 5%.
   d. If a data scientist does many tests, then even if the all the null hypotheses are true, a certain proportion will be rejected in error.

2. For each of the following situations, indicate which test is appropriate to address the respective question: z-test, t-test, two-sample z-test, sign test, or paired-difference test.

a. You want to test whether plain M&Ms really contain 24% blue M&Ms as claimed on the manufacturer's web site. You sample 500 plain M&Ms at random and count the fraction of blue M&Ms.

b. A high school principal wants to find out whether the average SAT score of this year's graduating class is higher than last year's. She samples 13 students from this year's graduating class at random and wants to compare their average SAT score to the average SAT score from last year's graduating class.

c. To investigate whether there are difference in scholastic abilities between first-borns and second-born siblings, 600 families that have at least two children were randomly selected. The scholastic abilities of the first-born and the second-born siblings were assessed with a test and are to be compared.