—(CALCULUS ASSIGNMENT)—

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Section:- "C"

Date:- 31-12-2020.

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(Question: 1)

1- $\int_1^2 \frac{x}{x-1} dx$

<u>Sol;</u> The given function is not define in the given interval ie $[1,2]$ because at $x=1$, it forms vertical asymtote. meaning the given integral is improper

2- $\int_0^\infty \frac{1}{1+x^3} dx$

<u>Sol;</u> Since the upper limit is ∞, therefore it is improper integral of type 1. Although the function is not continuous at $x=-1$ due to vertical asymtote. but according to type 2 it is not improper integral since $x=-1$ does not belong to the given interval $[0,\infty]$.

3- $\int_{-\infty}^{\infty} x^2 e^{-x^2} dx$

<u>Sol,</u> Since the upper and lower limits of the given integral is $\pm\infty$ i.e, non finite limits, therefore this is improper integral of part -1

4- $\int_0^{\pi/4} \cot x \, dx$

<u>Sol,</u> The given integral is improper because when $\lim\limits_{x\to 0^+} \cot x = +\infty$

$\lim\limits_{x\to 0} \cot x = +\infty$ ($\because \cos 0 = 1$ and $\sin 0 = 0$)

Therefore the function is discontinous at $x=0$ in the given interval $(0, \pi/4)$. this is an improper integral of type 2.

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—(Question: 2)—

1- $\int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx$

Sol; This is improper integral of Type=1.

$$\int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx \Rightarrow \lim_{t\to\infty} \int_3^{t} \frac{1}{(x-2)^{3/2}} dx$$

let $u = x-2$, $du = dx$

The limits will change $\int_3^{t}$ to $\int_{3-2}^{t-2} \to \int_1^{t-2} = \lim_{t\to\infty} \int_1^{t-2} \frac{1}{u^{3/2}} du$

$$= \lim_{t\to\infty} \int_1^{t-2} u^{-3/2} du = \lim_{t\to\infty} \left[\frac{u^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} \right]_1^{t-2} = \lim_{t\to\infty} \left[-\frac{2}{\sqrt{u}} \right]_1^{t-2}$$

$$= \lim_{t\to\infty} = \frac{-2}{\sqrt{t-2}} + \frac{2}{\sqrt{1}} = \frac{-2}{\infty} + \frac{2}{1} = 0+2=2$$

Since the limits exists the impropers integral is Convergent.

2- $\int_{-\infty}^{0} \frac{z}{z^4+4} dz$

Sol; This is improper integral of type=1

let $2u = z^2$, $2du = 2z\,dz$, $du = z\,dz$

The limits will be $\int_{-\infty}^{0} \Rightarrow \int_t^0 = \lim_{t\to-\infty} \int_t^0 \frac{du}{(2u)^2+4} = \int_t^0$

$$= \lim_{t\to-\infty} \int_t^0 \frac{du}{4u^2+4} = \lim_{t\to-\infty} \frac{1}{4} \int_t^0 \frac{du}{u^2+1} \qquad \because \tan^{-1}x = \frac{1}{x^2+1} dx$$

$$= \lim_{t\to-\infty} \frac{1}{4} \tan^{-1} u \Big]_t^0 = \lim_{t\to\infty} \frac{1}{4} \tan^{-1} 0 - \frac{1}{4} \tan^{-1}(t)$$

$$= 0 - \frac{1}{4} \tan(\infty) = 0 - \frac{1}{4} \cdot \left(\frac{\pi}{2}\right) \Rightarrow -\frac{\pi}{8} \quad \left(\because \tan^{-1} - \frac{\pi}{2}, \frac{\pi}{2}\right)$$

since the limit exists the improper integral is convergent.

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3) $\int_e^\infty \frac{1}{x(\ln x)^2} dx$

Sol; This is improper integral of type -1

$$\int_e^\infty \frac{1}{x(\ln x)^2} dx = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^2} dx$$

let $u = \ln x$, $du = \frac{1}{x} dx$

The limits will change; $\int_e^\infty = \int_e^t = \lim_{t\to\infty} \int_e^t \frac{du}{u^2}$

$$= \lim_{t\to\infty} \int_e^t u^{-2} du = \lim_{t\to\infty} \left[-\frac{1}{u} \right]_e^t = \lim_{t\to\infty} \left[1 - \frac{1}{\ln x} \right]_e^t$$

$$= -\frac{1}{\ln(\infty)} + \frac{1}{\ln e} = -\frac{1}{\infty} + \frac{1}{1} => 0 + 1 = 1$$

Since the limits exists the improper intergral is convergent.

4) $\int_1^\infty \frac{dx}{\sqrt{x} + x\sqrt{x}}$

Sol; This is improper integral of type-1.

$$\int_1^\infty \frac{dx}{\sqrt{x} + x\sqrt{x}} = \lim_{t\to\infty} \int_1^t \frac{dx}{\sqrt{x} + x\sqrt{x}}$$

let $u^2 = x$, $2udu = dx$

$$\lim_{t\to\infty} \int_1^t \frac{2udu}{\sqrt{u^2} + u^2\sqrt{u^2}} = \lim_{t\to\infty} \int_1^t \frac{2udu}{u + u^3} = \lim_{t\to\infty} \int_1^t \frac{2udu}{u(1+u^2)}$$

$$= \lim_{t\to\infty} 2 \left[\tan^{-1} u \right]_1^t \quad \therefore \tan^{-1} x = \frac{1}{1+x^2} dx$$

$$= \lim_{t\to\infty} \left[2 \tan^{-1} \sqrt{x} \right]_1^t = 2 \tan^{-1} \sqrt{\infty} + 2 \tan^{-1} \sqrt{1} \quad \therefore \left(\tan^{-1} - \frac{\pi}{2}, \frac{\pi}{2} \right)$$

$$= \left(\frac{\pi}{2} \right) - 2 \left(\frac{\pi}{4} \right) = \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

Since the limits exists the improper integral is convergent.

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5) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Sol.- This is improper integral of type-2
As the fraction is discontinous at $x=1$ (forms vertical asymtote)

$\int_0^1 \frac{dx}{\sqrt{1-x^2}} \Rightarrow \lim_{t\to 1} \int_0^t \frac{dx}{\sqrt{1-x^2}}$

$= \lim_{t\to 1} \left[\sin^{-1}x\right]_0^t \quad \because \sin^{-1}x \Rightarrow \int \frac{dx}{\sqrt{1-x^2}}$

$= \lim_{t\to 1} \left[\sin^{-1}t - \sin^{-1}0\right] = \left[\sin^{-1}(1) - \sin^{-1}(0)\right] = \frac{\pi}{2} - 0$

$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$; since the limit exists the proper integral is convergent

6) $\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}} d\theta$

Sol; This is improper integral of type-2
let $u = \sin\theta$, $du = \cos\theta d\theta$.
limits will be change; $\int_0^{\pi/2}$ to $\int_{\sin(0)}^{\sin \pi/2} \Rightarrow \int_0^1$

$= -\int_0^1 \frac{1}{\sqrt{u}} du = \left[2\sqrt{u}\right]_0^1 = 2\sqrt{1} - 2\sqrt{0} = 2 - 0$

$\int_0^1 \frac{\cos d\theta}{\sqrt{\sin\theta}} = 2$; Since the limits exists the improper is convergent

7) $\int_0^1 \frac{e^{1/x}}{x^3} dx$

Sol; This is improper integral of type = 2.
Let $r = \frac{1}{x}$, $dr = -\frac{1}{x^2} dx$

$= \int_0^1 \frac{e^{1/x}}{x^2 \cdot x} dx = \int_0^1 \frac{e^{1/x}}{x} \frac{dr}{x^2} \Rightarrow \int_0^1 -\frac{e^r}{r} dr = \int_0^1 -re^r dr$

using integration by parts;
let $U = -r$, $du = e^r dr$, $du = -dr$, $v = e^r$

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$= -\gamma e^{\gamma} - \int e^{\gamma}(-d\gamma) = -\gamma \cdot e^{\gamma} + \int e^{\gamma} d\gamma = -\gamma\, e^{\gamma} + e^{\gamma} + c$

put $\gamma = \frac{1}{x}$ $= \frac{-e^{1/x}}{x} + e^{1/x} + c$

Now; $\int_{0}^{1} \frac{e^{1/x}}{x^3} dx = \lim_{t \to 0^+} \left[-\frac{e^{1/x}}{x} + e^{1/x} \right]_t^1$

$= \left[-\frac{e^{1/1}}{1} + e^{1/1} \right] - \lim_{t \to 0^+} \left[\frac{-e^{1/t}}{t} + e^{1/t} \right] = 0 + \lim_{t \to 0} e^{1/t} \left[\frac{1}{t} - 1 \right]$

$= e^{\infty}(\infty - 1)$.

$\int_0^1 \frac{e^{1/x}}{x^3} dx = \infty$, since the limit doesnot exists the improper integral is divergent.

8) $\int_0^1 \gamma \ln \gamma d\gamma$.

Sol; This is impropers integral of type 2.

Using partial integration.

let $u = \ln \gamma$, $du = \gamma d\gamma$, $du\gamma = \frac{1}{\gamma} d\gamma$, $V = \frac{\gamma^2}{2}$

$= \ln\gamma \cdot \frac{\gamma^2}{2} - \int \frac{\gamma^2}{2} \frac{1}{\gamma} d\gamma = \frac{\gamma^2 \ln\gamma}{2} - \int \frac{\gamma}{2} d\gamma = \frac{\gamma^2 \ln\gamma}{2} - \frac{\gamma^2}{4} + c$

Now, $\int_0^1 \ln \gamma d\gamma = \lim_{t \to 0^+} \left[\frac{\gamma^2 \ln \gamma}{2} - \frac{\gamma^2}{4} \right]_t^1$

$= \frac{(1)^2 \ln(1)}{2} - \frac{(1)^2}{4} - \lim_{t \to 0^+} \frac{t^2 \ln t}{2} + \lim_{t \to 0^+} \frac{t^2}{4}$

$= 0 - \frac{1}{4} \lim_{t \to 0^+} \frac{-\ln t}{2/t^2} = -\frac{1}{4} \lim_{t \to 0^+} - \frac{\ln t}{2/t^2}$

Since limit is in the form $\frac{\infty}{\infty}$, L hospital is applicable

$= -\frac{1}{4} + \lim_{t \to 0^+} \frac{\cancel{-}1/t}{\cancel{-}4/t^3} = -\frac{1}{4} + \lim_{t \to 0} \frac{t^2}{4} = -\frac{1}{4} + 0 = -\frac{1}{4}$

Since the limits exists the improper integral is convergent

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(Question: 3)

1) $y = x^2, \; y = 4x - x^2$

Sol; $y = x^2$ — ① , $y = 4x - x^2$ — ②

Compare ① and ②

$4x - x^2 = x^2$

$4x - 2x^2 = 0$

$-2x(x-2) = 0$

$\boxed{x = 0}$ $\boxed{x = 2}$

Hence ; top function is $y = 4x - x^2$, bottom function is $y = x^2$

Subtract bottom from top;

$A = \int_0^2 (4x - x^2) - x^2)\,dx = \int_0^2 (4x - x^2 - x^2)\,dx = \int_0^2 (4x - 2x^2)\,dx$

$A = \left[\cancel{4}\frac{x^2}{\cancel{2}} - \frac{2x^3}{3} \right]_0^2 = \left[2x^2 - \frac{2}{3}x^3 \right]^2$

$A = \left[2(2)^2 - \frac{2}{3}(2)^3 - 2(0) - \frac{2}{3}(0)^3 \right]$

$A = \frac{8 - 16}{3} \Rightarrow \frac{24 - 16}{3} = \frac{8}{3}$ sq. units.

2) $y = \sec^2 x, \; y = 8\cos x \; \left(-\frac{\pi}{3} \le x \le \frac{\pi}{3}\right)$

Sol; As we know that:

$A = \int_a^b f(x) - g(x)\,dx = \int_{-\pi/3}^{\pi/3} (8\cos x - \sec^2 x)\,dx$

$A = [8\sin x - \tan x]_{-\pi/3}^{\pi/3} = \left[8\sin\frac{\pi}{3} - \tan\frac{\pi}{3}\right] - \left[8\sin -\frac{\pi}{3} - \tan -\frac{\pi}{3}\right]$

$A = \left[8 \cdot \frac{\sqrt{3}}{2} - \sqrt{3}\right] - \left[8\left(-\frac{\sqrt{3}}{2}\right) - (-\sqrt{3})\right] = [4\sqrt{3} - \sqrt{3}] - [-4\sqrt{3} + \sqrt{3}]$

$A = 8\sqrt{3} - 2\sqrt{3} = \boxed{A = 6\sqrt{3}}$ Squnits.

3). $y = x^4$, $y = 2 - |x|$

Sol; As we know that:

$$A = \int_a^b f(x) - g(x) du$$

The functions are Continous in $[-1, 1]$.

$$A = \int_1^0 2 - (x) - x^4 dx + \int_0^1 (2 - x - x^4) dx$$

$$A = \left[2x + \frac{x^2}{2} - \frac{x^5}{5} \right]_{-1}^0 + \left[2x - \frac{x^2}{2} - \frac{x^5}{5} \right]_0^1$$

$$A = 0 - \left[2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^5}{5} \right] + \left[2(1) - \frac{(1)^2}{2} - \frac{(1)^5}{5} \right] - 0$$

$$A = 0 - \left[-2 + \frac{1}{2} + \frac{1}{5} \right] + \left[2 - \frac{1}{2} - \frac{1}{5} \right] - 0$$

$$= 2 - \frac{1}{2} - \frac{1}{5} + 2 - \frac{1}{2} - \frac{1}{5} = \frac{13}{5} \text{ squnits.}$$

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(Question: 4)

1) $y = \frac{x}{\sqrt{1+x^2}}$, $y = \frac{x}{\sqrt{9-x^2}}$ $(x \geq 0)$.

As we know that, $A = \int_a^b f(x) + g(x) dx$

The functions are Continous in $[0, 2]$

$$A = \int_0^2 \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{9-x^2}}$$

Solving 1st integral,

let $u = 1+x^2$, $du = 2xdx$.

The limits are ; $u = 1+(0)^2 = 1$, $u = 1+(2)^2 = 5$

$$A = \int_1^5 \frac{1}{2} \frac{du}{\sqrt{u}} = \int_1^5 \frac{1}{2} u^{-1/2} du = \left[\frac{1}{\cancel{2}} \cancel{2} u^{1/2} \right]_1^5$$

Solving 2nd integral,

let $s = 9x^2$, $ds = -2xdx$

The limits are ; $s = 9-(0)^2 = 9 = 9-(2)^2 = 5$

$$A = \int_5^9 \frac{1}{2} \frac{ds}{\sqrt{s}} = \int_5^9 \frac{1}{2}(s^{-1/2})ds = \left[\frac{1}{\cancel{2}} \cancel{2} s^{1/2} \right]_5^9$$

$$A = \sqrt{9} - \sqrt{5} = 3 - \sqrt{5}$$

So; $A = \sqrt{5} - 1 - (3-\sqrt{5}) = \sqrt{5} - 1 - 3 + \sqrt{5} = 2\sqrt{5} - 4$ squnits

2) $y = \frac{\ln x}{x}$, $y = \frac{(\ln x)^2}{x}$

Sol; $A = \int_a^b f(x) - g(x) dx$

The function is Continous $[1, e]$.

$$A = \int_1^e \frac{\ln x}{x} - \frac{(\ln x)^2}{x} dx = -\int_1^e \frac{\ln x}{x} - \frac{(\ln x)^2}{x} dx$$

let $u = \ln x$, $du = \frac{1}{x} dx$

The limits are; $u = \ln e = 1$, $u = \ln 1 = 0$

$$A = \int_0^1 u - u^2 du = \left[\frac{u^2}{2} - \frac{u^3}{3} \right]_0^1$$

$$A = \left[\frac{(1)^2}{2} - \frac{(1)^3}{3} - \left(\frac{(0)^2}{2} - \frac{(0)^3}{3} \right) \right]$$

$$A = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ squnits}.$$

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(Question: 5)

1) $\lim_{x\to 4} \frac{x^2-2x-8}{x-4}$

Sol, $\lim_{x-4} \frac{x^2-4x+2x-8}{x-4} = \lim_{x-4} \frac{x(x-4)+2(x-4)}{x-4}$

$\lim_{x-4} \frac{\cancel{(x-4)}(x+2)}{\cancel{(x-4)}} = 4+2 = 6.$

2) $\lim_{x\to(\pi/2)^+} \frac{\cos x}{1-\sin x}$

Sol; Since ondirects Substitution we get;

$\frac{\cos(\pi/2)}{1-\sin(\pi/2)} = \frac{0}{1-1} \Rightarrow \frac{0}{0}$

So, we use L-hospital rule;

$:= \lim_{x\to(\pi/2)^+} \frac{-\sin x}{\cancel{-}\cos x} = \lim_{x-(\pi/2)^+} \tan x = \tan\frac{\pi}{2}$

$\lim_{x-(\pi/2)^+} \frac{\cos x}{1-\sin x} = -\infty.$

3) $\lim_{a-\pi} \frac{1+\cos x}{1-\cos x}$

Sol; Since applying limit we get;

$\frac{1+\cos\pi}{1-\cos\pi} \Rightarrow \frac{1-1}{1-(-1)} \Rightarrow \frac{0}{1+1} \Rightarrow \frac{0}{2} = 0$

~~Since, we have limit value therefore L-hospital rule is not~~ necessary.

5) $\lim_{x-1} \frac{x^a-1}{x^b-1}$

Sol, since on direct substitution we will get 0/0 from applying L-hospital rule; $= \lim_{x-1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a\cdot 1^{a-1}}{b\cdot 1^{b-1}} = \frac{a}{b}.$

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4) $\lim_{x \to 0} \frac{\cos mx - \cos nx}{x^2}$

Sol,

Since; on direct Substitution we get,

$$\frac{\cos(m \cdot 0) - \cos(n \cdot 0)}{0} \Rightarrow \frac{1-1}{0} = \frac{0}{0}$$

Therefore using L-hospital rule;

$$= \lim_{x \to 0} \frac{n\sin(nx) - m\sin(mx)}{2x} = \frac{\cancel{n}\sin(0) - m\sin(0)}{\cancel{2}\ 2(0)} = \frac{0}{0}$$

Since, we have again determinate from again using L-hospital rule.

$$= \lim_{x \to 0} \frac{n^2 \cos nx - m^2 - \cos mx}{2} = \frac{n^2(1) + m^2(1)}{2} = \frac{n^2 - m^2}{2}$$

c) $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$

Sol, Since on direct Substitution we will $\frac{0}{0}$ from applying L-hospital rule;

$$= \lim_{x \to 1} \frac{x\ln x - (x-1)}{(x-1)\ln x} = \lim_{x \to 1} \frac{1 \cdot \ln x + x \cdot \frac{1}{x} - 1}{1 \cdot \ln x + (x-1) \cdot \frac{1}{x}}$$

$$= \frac{\ln 1 + 1 \cdot \frac{1}{1} - 1}{1 \cdot \ln 1 + (1-1) \cdot \frac{1}{1}} = \frac{0}{0},$$ Again using L-hospital rule;

$$= \lim_{x \to 1} \frac{\ln x}{\ln x + 1 - \frac{1}{x}} = \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{1+1} = \frac{1}{2}.$$

7) $\lim_{x \to 0} (\cos x - \cot x)$.

Sol, $$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) = \frac{1}{\sin 0} - \frac{\cos 0}{\sin 0} = \frac{1 - \cos 0}{\sin 0} = \frac{1-1}{0} = \frac{0}{0}$$

Therefore L-hospital rule is applicable;

$$= \lim_{x \to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(\sin x)} = \lim_{x \to 0} \frac{\sin x}{\cos x} = \lim_{x \to 0} \tan x$$

$$\lim_{x \to 0} = \tan 0 = 0.$$