(Assignment: 4)

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Assignment

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Question #1
(a)

$f(x) = x^2 - 4x + 5 \qquad 0 \leq x \leq 3$

Riemann sum with $n=6$

Sol:- We have $= \Delta x = \frac{b-a}{x} = \frac{3-0}{6} = \frac{3}{6} = \frac{1}{2}$

$x_1 = a + \Delta x_1 \qquad x_1 = a + \Delta x(1) = 0.5$

$x_0 = 0 + \Delta x(0) = 0 \qquad x_3 = a + \Delta x(3) = 1.5$

$x_2 = 0 + \Delta x(2) = 1 \qquad x_5 = a + \Delta x(5) = 2.5$

$x_4 = a + \Delta x(4) = 3$

So, $f(x_0) = 5 \qquad f(x_1) = 3.25$

$f(x_2) = 2 \qquad f(x_3) = 1.25$

$f(x_4) = 1 \qquad f(x_5) = 1.25$

$f(x_6) = 2$

$x_1 = \frac{x_0 + x_1}{2} = 0.25 \qquad x_2 = \frac{x_1 + x_2}{2} = 0.75$

$x_3 = \frac{x_2 + x_3}{2} = 1.25 \qquad x_4 = \frac{x_3 + x_4}{2} = 1.75$

$x_5 = \frac{x_4 + x_5}{2} = 2.25 \qquad x_6 = \frac{x_5 + x_6}{2} = 2.75$

So; $f(x_1) = 4.0625 \qquad f(x_2) = 2.56$

$f(x_3) = 1.56 \qquad f(x_4) = 1.0625$

$f(x_5) = 1.0625 \qquad f(x_6) = 1.5625$

> Left end point:-

$\sum_{i=0}^{n} f(x_i) = \Delta x\left(f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\right)$

$= 0.5\ (13.75)$

$\boxed{= 6.875}$

→ R.E.P:-

$\sum_{i=1}^{6} d(x) \cdot \Delta x = \Delta x\left(f(x_1) + \cdots + f(x_6)\right)$

$= 0.5\,(10.75) = 5.375$

> midpoint:- $\sum_{i=1}^{6} f(x) \cdot \Delta x = f(x_1^*) + \cdots\cdots + f(x_6^*)$

$= \boxed{5.93}$

—(6)—

Use the midpoint rule with $n=4$ to approximate the integral:

$$\int_0^{\pi} x \sin^2 x \, dx$$

Solution:- $f(x) = x\sin^2 x$

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

$$\int_0^{\pi} x\sin^2 x \, dx = \Delta x \left[F\left(\frac{\pi}{8}\right) + F\left(\frac{3\pi}{8}\right) + F\left(\frac{5\pi}{8}\right) + F\left(\frac{7\pi}{8}\right) \right]$$

∴ Use the functional values from below;

$$= \frac{\pi}{4}\left[\frac{\pi}{8}\sin^2\left(\frac{\pi}{8}\right) + \frac{3\pi}{8}\sin^2\left(\frac{3\pi}{8}\right) + \frac{5\pi}{8}\sin^2\left(\frac{5\pi}{8}\right) + \frac{7\pi}{8}\sin^2 \right.$$

$$= \frac{\pi}{4}\left[0.057 + 1.0055 + 1.6759 + 0.4025 \right]$$

$$\int_0^{\pi} x\sin^2 x \, dx = \boxed{2.467}$$

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Question no: 2 (a)

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Express the integral as limit of Reimann sum. Donate evaluate limit.

$$\int_2^5 \left(x^2 + \frac{1}{x}\right) dx$$

→ **Solution:-**

We know that;

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(a + i\Delta x)$$

$$f(x) = \int_2^5 x^2 + \frac{1}{x}\,dx$$

$$a = 2, \quad b = 5$$

$$\Delta x = \frac{b-a}{n} = \frac{5-2}{n} = \frac{3}{n}$$

$$\int_2^5 x^2 + \frac{1}{x}\,dx = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(a + i\Delta x)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} f\left(2 + i\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \cancel{f}\left(2 + \frac{3i}{n}\right) + \frac{1}{\left(2 + \frac{3i}{n}\right)}$$

$$\int_2^5 \left(x^2 + \frac{1}{x}\right) dx = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(2 + \frac{3i}{n}\right)^2 + \frac{1}{\left(2 + \frac{3i}{n}\right)} \quad \text{Ans.}$$

(b)

Express the limit as definit integral on the given interval.

(i)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin x_i}{1 + x_i} \Delta x, \quad [0, \pi]$$

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Solution:-

$\Rightarrow$ definition of definit integral;

$$\Rightarrow \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

in interval $[a,b]$

Where in the horizontal interval $[a,b] = [0,\pi]$

Therefore, $\lim_{n-\infty} \sum_{i=1}^{n} \frac{\sin x_i}{1+x_i} \Delta x = \int_0^{\pi} \frac{\sin x}{1+x} dx$

— (ii) —

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2+4} \Delta x, \quad [1,3]$$

Solution:-

Definition of definite integral:-

$$\Rightarrow \int_a^b F(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \text{ in the interval } [a,b]$$

Interval $= [1,3]$

Therefore; $\lim_{n-\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2+4} \Delta x = \int_1^3 \frac{x}{x^2+4} dx$

—(Questionno: 3)—

$$\int_0^5 f(x)dx \quad \text{if } f(x) = \begin{cases} 3 & \text{for } x<3 \\ x+e^x & \text{for } x\geq 3. \end{cases}$$

Solution:- Notice the integral represents the total area of rectangle $\int_0^3 3dx$ and a tropzoid $\int_3^5 x+e^x dx$

$$\int_0^5 f(x)=dx = \int_0^3 f(x)dx + \int_3^5 f(x)dx = \int_0^3 3dx + \int_3^5 x+e^x dx$$

$$= |3x|_0^3 + \int_3^5 x\,dx + \int_3^5 e^x dx = 3(3)-0+\left|\frac{x^2}{2}+e^x\right|_3^5$$

$$= 9+\left|\frac{(5)^2}{2}-\frac{(3)^2}{2}+e^5-e^3\right|_3^5$$

$$= 9+(8+148.41-20.08)$$

$$\int_0^5 f(x)dx = 9+(8+128.33) = 145.34 \text{ Ans.}$$

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(Question no: 4)

Date

Evaluate the Integral:

(i)

(i) $\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx$

$= 4\int_{1/2}^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}} dx = 4\left[\sin^{-1}x\right]_{1/2}^{1/\sqrt{2}}$ $\quad \therefore \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} C$

$= 4\sin^{-1}\left[\frac{1}{\sqrt{2}}\right] - 4\sin^{-1}\left[\frac{1}{2}\right] = 4\left[\frac{\pi}{4}\right] - 4\left[\frac{\pi}{6}\right] \Rightarrow \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

$\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx = \frac{\pi}{3}$ Ans

(ii)

$\int_0^{\pi} f(x)dx$ where $f(x) = \begin{cases} \sin x & \text{if } 0 \le x < \pi/2 \\ \cos x & \text{if } \pi/2 \le x \le \pi \end{cases}$

Solution;

Use property 5,

$\int_0^{\pi} f(x)dx = \int_0^{\pi/2} f(x)dx + \int_{\pi/2}^{\pi} f(x)dx = \int_0^{\pi/2} \sin x dx + \int_{\pi/2}^{\pi} \cos x dx$

$= \left[-\cos x\right]_0^{\pi/2} + \left[\sin x\right]_{\pi/2}^{\pi} = -\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right] + \left[\sin(\pi) - \sin\left(\frac{\pi}{2}\right)\right]$

$= -(0-1) + (0-1) = 1 - 1 = 0.$

$\int_0^{\pi} f(x)dx = 0.$

(iii)

$f(x) = \int_{2\pi}^{0} \sqrt{1+\sec t}\, dt.$

Sol:- $\int_{2\pi}^{0} \sqrt{1+\sec t}\, dt$ $\quad \therefore \int_a^b f(x)dx = \int_b^a f(x)dx, a<b$

$= -\int_0^{2\pi} \sqrt{1+\sec(l)}\, dl.$

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—(iv)—

$$f(x) = \int_1^3 \frac{y^2 - 2y^2 - y}{y^2}\, dy$$

Sol:- $= \int_1^3 \frac{y^2 - 2y^2 - y}{y^2}\, dy \Rightarrow \int_1^3 \frac{-y^2 - y}{y^2} \Rightarrow \frac{y(-1-y)}{y^2}$

$$= \int_1^3 \frac{-1-y}{y} = \int_1^3 -1 - \frac{1}{y}\, dy = -\int_1^3 1\, dy - \int_1^3 \frac{1}{y}\, dy \Rightarrow -2 - [\ln(3) - \ln(1)]$$

$$\int_1^3 \frac{y^2 - 2y^2 - y}{y^2}\, dy = -2 - [\ln 3 - 1] = \boxed{-2 - \ln 3}$$

—(v)—

$$\int_0^{\pi/3} \frac{\sin\theta + \sin\theta \tan^2\theta}{\sec^2\theta}\, d\theta$$

Sol:- $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta \tan^2\theta}{\sec^2\theta}\, d\theta = \int_0^{\pi/3} \frac{\sin\theta(1+\tan^2\theta)}{\sec^2\theta}\, d\theta = \int_0^{\pi/3} \frac{\sin\theta \sec^2\theta}{\sec^2\theta}\, d\theta$

$$= \int_0^{\pi/3} \sin\theta\, d\theta \Rightarrow [-\cos\theta]_0^{\pi/3} = -[\cos(\tfrac{\pi}{3}) - \cos(0)]$$

$$= -[\tfrac{1}{2} - 1] = -\tfrac{1}{2} + 1 = \boxed{\tfrac{1}{2}}$$

—(vi)—

$$\int_0^2 |2x-1|\, dx.$$

Sol:- $|2x-1| = \begin{cases} -(2x-1) & \text{if } x < 1/2 \\ 2x-1 & \text{if } x > 1/2 \end{cases}$

Therefore; $\int_0^2 |2x-1|\, dx = \int_0^{1/2} -(2x-1)\, dx + \int_{1/2}^2 |2x-1|\, dx = \int_0^{1/2} 1 - 2x\, dx + \int_{1/2}^2 2x - 1\, dx$

$$= [x - x^2]_0^{1/2} + [x^2 - x]_{1/2}^2 = [\tfrac{1}{2} - \tfrac{1}{4}] + [4-2] - [0 - 0^2] - [\tfrac{1}{4} - \tfrac{1}{2}]$$

$$= \tfrac{1}{4} + 2 + \tfrac{1}{4} = 2 + \tfrac{1}{2} = \tfrac{5}{2}$$

$$\int_0^2 |2x-1|\, dx = \tfrac{5}{2}.$$

(Question no: 5(a))

Use part-1 of fundamental theorom of Calculus to find derivative.

—(i)—

$$g(x) = \int_x^0 \sqrt{t+t^2}\, dt$$

Sol:- Use the FTC-1; $g(x) = \frac{d}{dt}\left[\int_x^0 \sqrt{t+t^2}\right] \frac{dt}{dx} = \sqrt{0+0^2}\, \frac{d0}{dx} = \boxed{0 \cdot 0}$

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—(ii)—

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$h(x)=\int_0^{e^x} \ln t \, dt.$

Sol:- Use the FTC 1 plug the upper limit into variables of integration.

$h(x) = \frac{d}{dt}\left[\int_0^{e^x} \ln t \, dt\right]\cdot\frac{dt}{dx} = \ln(e^x)\,\frac{d(e^x)}{dx}$ ∴ Chain rule.

$= \ln(e^x)\, e^x = x\ln(e)\cdot e^x = x(1)\cdot e^x$

$\int_0^{e^x} \ln t\, dt = xe^x$

—(iii)—

$h(x) = \int_2^{\sqrt{x}} \frac{z^2}{z^4+1}\, dz$

Sol:- put $\sqrt{x}$ into z multiply by derivative of $\sqrt{x}$.

$h(x) = \int_2^{\sqrt{x}} \frac{z^2}{z^4+1}\, dz = h'(x) = \frac{(\sqrt{x})^2}{(\sqrt{x})^4+1}\cdot\frac{d(\sqrt{x})}{dx}$ ∴ Chain rule.

$= \frac{x}{x^2+1}\cdot\frac{1}{2}x^{-1/2} = \frac{x^{1+(-1/2)}}{2(x^2+1)}$ ∴ $x^a x^b = x^{a+b}$

$= \frac{x^{1/2}}{2(x^2+1)} = \frac{\sqrt{x}}{2(x^2+1)}$

$\int_2^{\sqrt{x}} \frac{z^2}{z^4+1}\, dz = \frac{\sqrt{x}}{2(x^2+1)}$

—(iv)—

$h(x)=\int_{2x}^{3x} \frac{y^2-1}{y^2+1}\, dy$

Sol:- Use the FTC-1 if $u=3x$

$h'(x) = \frac{d}{du}\left[\int_{2x}^{u} \frac{y^2-1}{y^2+1}\right]\cdot\frac{du}{dx} = \frac{u^2-1}{u^2+1}\cdot 3 = \frac{u^2-1}{u^2+1}\cdot 3 = \frac{(3x)^2-1}{(3x)^2+1}\cdot 3$

$\int_{2x}^{3x} \frac{y^2-1}{y^2+1}\, dy = \frac{3[(3x)^2-1]}{(3x)^2+1}$

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(Question no: 5(b))

What is wrong the equation:-

(1) $\int_{-1}^{2} \frac{4}{x^3}\,dx = -\frac{2}{x^2}\Big]_{-1}^{2} = \frac{3}{2}$

Sol:- $\int_{-1}^{2} \frac{4}{x^3}\,dx = -\frac{2}{x^2}\Big]_{-1}^{2}$

- An antiderivative of $f(x) = \frac{4}{x^3}$ is $F(x) = \frac{-2}{x^2}$. So find upper and lower Limit:-

$= \int_{-1}^{2} \frac{4}{x^3}\,dx = \frac{-2}{x^2}\Big]_{-1}^{2} = \left[-\frac{2}{(2)^2} - \frac{-2}{(-1)^2}\right]_{-1}^{2} = \frac{-2}{4} + \frac{2}{1} = \frac{-2+8}{4} = \frac{6}{4} = \frac{3}{2}$

$= \cancel{\frac{-2}{(2)^2}} =$ • The integrand is not continous on the whole interval. It is unbounded around 0.

(2) $\int_{0}^{\pi} \sec^2 x\,dx = \tan x\Big]_{0}^{\pi} = 0$

$\int_{0}^{\pi} \sec x\,dx = \tan x\Big]_{0}^{\pi} = 0$; $\sec^2 x$ is not continous on the interval $[0, \pi]$

It is discontinous at $x = \frac{\pi}{2}$

(Question no: 6)

• Evaluate the integral; (ii)

(i) $\int \sin^5(2t)\cos^2(2t)\,dt$

Apply $u = 2t$

$= \int \sin^5(u)\cos^2(u)\frac{1}{2}\,du$

$= \frac{1}{2}\cdot\int \sin^5(u)\cos^2(u)\,du$

$= \frac{1}{2}\int (1-\cos^2(u))^2 \sin(u)\cos^2(u)\,du$

Apply $v = \cos(u)$

$= \frac{1}{2}\int -v^2(1-v^2)\cancel{)}^2\,dv = \frac{1}{2}\int -v^2(1+v^4-2v^2)\,dv$

$= \frac{1}{2}\int -v^2 - v^6 + 2v^4\,dv$

apply integral;

$= \frac{1}{2}\left(-\int v^2\,dv - \int v^6\,dv + \int 2v^4\,dv\right)$

$= \frac{1}{2}\left(\frac{-v^3}{3} - \frac{v^7}{7} + \frac{2v^5}{5}\right)$

(ii) $\int \frac{\sin^2(t)}{t^2}\,dt$

$= \int \frac{\sin^2(t)}{t^2}\,dt = \int \frac{1-\cos^2(t)}{t^2}\,dt$

$= \int \frac{1}{t^2} - \frac{\cos^2 t}{t^2}\,dt$

$= \int \frac{1}{t^2}\,dt - \int \frac{\cos^2 t}{t^2}\,dt$

$= \int t^{-2}\,dt - \left[u = \cos^2(t),\ v = \frac{1}{t^2}\right]$

$= -\frac{1}{t} - \left[-\frac{\cos^2(t)}{t} - \int \frac{\sin(2t)}{t}\right.$

$= -\frac{1}{t} - \left[-\frac{\cos^2 t}{t} - \sin(2t)\right]$

$= -\frac{1}{t} + \frac{\cos^2(t)}{t} - \sin(2t) + C$

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(iii) $\int \tan^3 x \sec^6 x \, dx$

$\int \tan^3 x \sec^6 x dx$

$= \int (-1 + \sec^2(x)) \tan(x) \sec^6(x) dx$

apply; $u = \sec(x)$

$= \int u^5 (u^2 - 1) du = \int u^{5+2} - u^5 du$

$= \int u^7 - u^5 du = \int u^7 du - \int u^5 du$

$= \frac{u^8}{8} - \frac{u^6}{6} + C = \boxed{\frac{\sec^8(x)}{8} - \frac{\sec^6(x)}{6} + C}$

(iv) $\int_0^{\pi/4} \sqrt{1 - \cos 4\theta} \, d\theta$

$= \int_0^{\pi/4} \sqrt{1 - \cos 4\theta} \, d\theta$

let $\theta = \frac{u}{2} \Rightarrow u = 2\theta$

$= \int_0^{\pi/2} \sqrt{1 - \cos(2u)} \cdot \frac{1}{2} du$

$= \frac{1}{2} \int_0^{\pi/2} \sqrt{1 - \cos(2u)} \, du$

$= \frac{1}{2} \int_0^{\pi/2} \sqrt{2} \cdot \sqrt{\sin^2(u)} \, du$

$= \frac{1}{2} \sqrt{2} \left[-\cos(u) \right]_0^{\pi/2}$

$= \frac{1}{2} \sqrt{2} \left[-\cos(\frac{\pi}{2}) - (-\cos(0)) \right]$

$= \frac{\sqrt{2}}{2} \cdot 1 = \boxed{\frac{1}{\sqrt{2}}}$ Ans.

—(Question no:7(a))—

→ Evoluate the integral:-

1- $\int \frac{\sqrt{x^2 - 4}}{x} dx$

$u = a \sec\theta$

$a = 2, u = x, x = 2\sec\theta$

$dx = 2\sec\theta \tan\theta d\theta$

$= \int \frac{\sqrt{4\sec^2\theta - 4}}{2\sec\theta} (2\sec\theta \tan\theta d\theta)$

$= \int \frac{\sqrt{4}\sqrt{\sec^2\theta - 1}}{2\sec\theta} (2\sec\theta \tan\theta d\theta)$

$= \int \frac{2\sqrt{\tan^2\theta}}{\cancel{2\sec\theta}} (\cancel{2\sec\theta} \tan\theta d\theta)$

$= \int 2\tan\theta (\tan\theta d\theta) = 2\int \tan^2\theta d\theta$

$= 2\int (\sec^2\theta - 1) d\theta = 2\left[\int \sec^2 d\theta - \int 1 d\theta\right]$

$= 2(\tan\theta - \theta) + C$

So; $\theta = \sec^{-1}(\frac{1}{2}x)$

~~$= 2(\tan(\sec^{-1}(\frac{1}{2}x)) - \sec^{-1}(\frac{1}{2}x))$~~

$= 2\sqrt{x^2 - 4} - 2\sec^{-1}(\frac{1}{2}x) + C$

$= \sqrt{x^2 - 4} - 2\sec^{-1}(1/2 \, x) + C.$

(b) $\int \frac{x^3}{\sqrt{x^2 + 4}} dx$

$= \int \frac{x^2}{\sqrt{x^2 + 4}} \cdot x dx = \int \frac{x^2}{\sqrt{x^2 + 4}} \cdot x dx$

let; $u = x^2 + 4 \rightarrow u - 4 = x^2$

$dx = 2x dx \rightarrow \frac{du}{2} = x dx$

$= \int \frac{u - 4}{\sqrt{u}} \cdot \frac{du}{2} = \frac{1}{2} \int \left(\frac{u}{\sqrt{u}} - \frac{u}{\sqrt{u}}\right) du$

$= \frac{1}{2} \left(\int \sqrt{u} \, du - u \int \frac{1}{\sqrt{u}} du\right)$

$= \frac{1}{2} \left(\int u^{1/2} du - u \int u^{1/2} du\right)$

$= \frac{1}{2} \left(\frac{u^{3/2}}{3/2} - u \cdot \frac{u^{1/2}}{1/2}\right)$

$= \frac{1}{3} u^{3/2} - u u^{1/2}$

$= \frac{1}{3} u^{1/2} (u - 12) = \frac{1}{3} (x^2 + 4)^{\frac{1}{2}} \cdot (x^2 + 4 - 12)$

$\therefore = \frac{1}{3} \sqrt{x^2 + 4} \cdot (x^2 - 8) + C$

Ans

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Evaluate:-

(i) $= \int x^2 \sqrt{3+2x-x^2}\,dx$

$= \int x^2 \sqrt{3+2x-x^2}\,dx = \int x^2 \sqrt{(-x-1)^2+4}\,dx = \int (u+1)^2 \sqrt{-u^2+u}\,du \quad \therefore u = x-1$

$\therefore u = 2\sin(v) = \int u\cos^2(v)(1+2\sin(v))^2 dv = u\int (1+2\sin(v))^2 (1-\sin^2(v))dv$

$\therefore (1+2\sin(v))^2(1-\sin^2(v)) : 3\sin^2(v) + 4\sin(v) - 4\sin^3(v) - 4\sin^4(v) + 1$

$= u\int 3\sin^2(v) + 4\sin(v) - 4\sin^3(v) - 4\sin^4(v) + 1 dv$

$= u(\int 3\sin^2(v)dv + \int 4\sin(v)dv - \int 4\sin^3(v)dv - \int 4\sin^4(v)dv + \int 1 dv$

$= u\left(\frac{3}{2}\left(v - \frac{1}{2}\sin(2v)\right) - 4\cos(v) - 4\left(\cos(v) + \frac{\cos^3(v)}{3}\right) - \left(-\sin^3(v)\cos(v) + \frac{3}{2}v - \frac{3}{4}\sin(2v) + v\right)\right.$

$= u\left(\sin^{-1}\left(\frac{1}{2}(x-1)\right) - \frac{1}{6}(-x^2+2x+3)^{3/2} + \frac{1}{16}(x-1)^3\sqrt{-x^2+2x+3} + C\right.$ Ans.

(ii) $\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx$

Sol:- By completing square,

$$\int \frac{x^2+1}{((x-1)^2+1)^2}\,dx$$

let $x - 1 = \tan\theta$

$x = \tan\theta + 1,\; dx = \sec^2\theta\, d\theta$

$= \int \frac{((\tan\theta+1)^2+1)(\sec^2\theta)}{((\tan\theta+1-1)^2+1)^2}$

$= \int \frac{((\tan\theta+1)^2+1)\sec^2\theta\, d\theta}{((\tan\theta)^2+1)^2}$

$= \int \frac{(\tan\theta+1)^2+1) \times \sec^2\theta\, d\theta}{\sec^4\theta}$

$= \int \frac{(\tan\theta+1)^2+1)\,d\theta}{\sec^2\theta} = \int \frac{\tan^2\theta + 2\tan\theta + 1 + 2d\theta}{\sec^2\theta}$

Now we can integrate them seperately.

$\int \frac{\tan^2\theta\, d\theta}{\sec^2\theta} + 2\int \frac{\tan\theta\, d\theta}{\sec^2\theta} + 2\int \frac{2d\theta}{\sec^2\theta}$

Now recalling that $w = 2\sin\theta$

$\sin 2\theta = \frac{w}{2}, \quad \cos\theta = \frac{\sqrt{4-w^2}}{2}$

$\theta = \sin^{-1}(w/2)$

Now we evaluate integral in terms of ω;

$$4\sin^{-1}(\omega/2)+4\left(\frac{\omega}{2}\right)^3\frac{\sqrt{4-\omega^2}}{2}-\frac{4^2}{3}\left(\frac{\sqrt{4-\omega^2}}{2}\right)+C$$

$$4\sin^{-1}(\omega/2)+\frac{4\omega^3}{8}\frac{\sqrt{4-\omega^2}}{2}-\frac{16}{3}\frac{(4-\omega^2)^{3/2}}{8}+C$$

$$4\sin^{-1}\left(\frac{\omega}{2}\right)+\frac{\omega^3}{4}\sqrt{4-\omega^2}-\frac{2}{3}(4-\omega^2)^{3/2}+C$$

Substituting $x-1$ for ω we get;

$$4\sin^{-1}\left(\frac{x-1}{2}\right)+\frac{(x-1)^3}{4}\sqrt{4(x-1)^2}-\frac{2}{3}\left[4-(x-1)^2\right]^{3/2}+$$

$$\sin^{-1}\left(\frac{x-1}{2}\right)+\frac{(x-1)^3}{4}\sqrt{4-(x^2-2x+1}-2\left[4-(x^2-2x+1)\right]^{3/2}$$

$$4\sin^{-1}\left(\frac{x-1}{2}\right)+\frac{(x-1)^3}{4}\sqrt{3+2x-x^2}-\frac{2}{3}(3+2x+x^2)^{3/2}+C$$ Ans.

(iii) $\displaystyle\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$ $a>0$

let $x=a\tan\theta \Rightarrow dx=a\sec^2 d\theta$, So when $x=0$, $0=a\tan\theta=\tan^{-1}(\theta)=0$

when $x=0$, $a=a\tan\theta=\tan^{-1}(1)=\pi/4$

Substituting; $\displaystyle\int_0^{\pi/4}\frac{a\sec^2\theta\, a\theta}{(a^2+a^2\tan^2\theta)^{3/2}}=\int_0^{\pi/4}\frac{a\sec^2\theta\, d\theta}{(a^2(1+\tan^2\theta))^{3/2}}=\int_0^{\pi/4}\frac{a\sec^2\theta\, a\theta}{a^3(\sec^2\theta)^{3/2}}$

$$=\frac{1}{a^2}\int_0^{\pi/4}\frac{\sec^2\theta\, d\theta}{\sec^3\theta}=\frac{1}{a^2}\int_0^{\pi/4}\cos\theta=\frac{1}{a^2}\left[\sin\theta\right]_0^{\pi/4}$$

$$=1/a^2\left[\sin(\pi/4)-\sin(0)\right]=1/a^2\left[1/\sqrt{2}-0\right]=\frac{1}{\sqrt{2}a^2}.$$

(iv) $\int \frac{dx}{\sqrt{x^2+a^2}}$

let $x = a\tan\theta \Rightarrow dx = a\sec^2\theta \, d\theta$

$= \int \frac{a\sec^2\theta \, d\theta}{\sqrt{a^2\tan^2\theta + a^2}} = \int \frac{a\sec^2\theta \, d\theta}{\sqrt{a^2(\tan^2\theta+1)}}$

$= \int \frac{a}{a} = \frac{\sec^2\theta}{\sec\theta} d\theta = \int \sec\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| + c.$

$\tan\theta = \frac{n}{a}$

So; $\sec\theta = \sqrt{\tan^2\theta - 1}$

$= \sqrt{\frac{x^2}{a^2} - 1} = \sqrt{\frac{x^2-a^2}{a^2}} = \frac{\sqrt{x^2-a^2}}{a}.$

So; $= \ln\left|\frac{\sqrt{x^2-a^n}}{a} + \frac{x}{a}\right| + c$

$= \ln|\sqrt{x^2-a^2} + x| - \ln|a| + c.$

$= \ln|\sqrt{x^2-a^2} + x| + c.$

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(Question no: 8)

Date:

Evaluate the integrals:-

(i) $\int \frac{5x+1}{(2x+1)(x-1)^2}\,dx$

$\int \frac{5x+1}{(2x+1)(x-1)^2}\,dx = \int \frac{-2}{3(2x+1)} + \frac{1}{3(x-1)} + \frac{2}{(x-1)^2}\,dx$

$= \int \frac{-2}{3(2x+1)} + \frac{1}{3(x-1)} + \frac{2}{(x-1)^2}\,dx = -\int \frac{2}{3(2x+1)}\,dx + \int \frac{1}{3(x-1)}\,dx + \int \frac{2}{(x-1)^2}\,dx$

$= -\frac{1}{3}\ln|2x+1| + \frac{1}{3}\ln|x-1| - \frac{2}{x-1} + C.$

$\int \frac{5x+1}{(2x+1)(x-1)^2}\,dx = -\frac{1}{3}\ln(2x+1) + \frac{1}{3}\ln(x-1) - \frac{2}{x-1} + C.$

(ii) $\int \frac{dx}{(x+a)(x+b)}$, where $a \neq b$.

$= \int \frac{\frac{1}{b-a}}{a-x} + \frac{\frac{1}{a-b}}{b-x}\,dx = \int \frac{\frac{-1}{a-b}}{a-x} + \frac{\frac{1}{a-b}}{b-x}\,dx = \frac{1}{a-b}\int \frac{-1}{a-x} + \frac{1}{b-x}\,dx.$

$= \frac{1}{a-b}\left(+(+\ln|a-x|) + (-\ln|b-x|)\right) = \frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right|$ Ans.

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—(Question no: 9)—

• Use the Substitution:-

(i) $\int \frac{dx}{(1+\sqrt{x})^2}$

let $u=\sqrt{x}$

$= \int \frac{2u}{(1+u)^2}\, du$

let $\Rightarrow v = 1+u \quad = 2\int \frac{v-1}{v^2}\, dv$

Expand $\frac{v-1}{v^2} = \frac{1}{v} - \frac{1}{v^2}$

$= 2\left[\int \frac{1}{v} dv - \int \frac{1}{v^2} dv\right]$

$= 2\left[\ln|1+\sqrt{x}| - \left(-\frac{1}{1+\sqrt{x}}\right)\right]$

$= 2\left[\ln|1+\sqrt{x}| + \frac{1}{1+\sqrt{x}}\right] + C$ Ans

(2) $= \int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\, dx$

let $u = \sqrt[6]{x} \Rightarrow x = u^6, \; dx = 6u^5 du$

$= \int \frac{1}{(u^6)^{1/2} - (u^6)^{1/3}}\, 6u^5 dx.$

$= 6\int \frac{u^5}{u^3 - u^2}\, du. = 6\int \frac{u^{5-3}}{u^{?}(u-1)}\, du$

$= 6\int \frac{u^3}{u-1} du = 6\int u^2 + u + 1 + \frac{1}{u-1}\, du$

$= 6\left(\frac{1}{3}u^3 + \frac{1}{2}u^2 + u + \ln|u-1|\right)$

So; $u = x^{1/6}$

$= 2(x^{1/6})^2 + 3(x^{1/6})^2 + 6x^{1/6} + 6|x^{1/6} - 1|$

$= 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x} - 1| + C$

Ans

(iii) $\int \frac{\sec^2 t}{\tan^2 t + 3\tan t + 2}\, dt$

let $= u = \tan t$

$t = \tan^{-1} u$

$= \int \frac{1}{u^2 + 3u + 2}\, du$

$= \int \frac{1}{(u+\frac{3}{2})^2 - \frac{1}{4}}\, du$

let $v = u + \frac{3}{2}$

$= \int \frac{4}{4v^2 - 1}\, dv = 4\int \frac{1}{-(-4v^2+1)}\, dv$

let $\therefore v = \frac{1}{2}w$

$= 4\left(-\int \frac{1}{2(-w^2+1)}\, dw\right)$

$= 4\left(-\frac{1}{2}\int \frac{1}{-w^2+1}\, dw\right)$

$= 4\left(-\frac{1}{2}\left(\frac{\ln|w+1|}{2} - \frac{\ln|w-1|}{2}\right)\right)$

$= 4\left(-\frac{1}{2}\left(\frac{\ln|2(\tan(t)+\frac{3}{2})+1|}{2} - \frac{\ln|2(\tan t + \frac{3}{2}) - 1|}{2}\right)\right)$

$= \ln|2\tan(t) + 4| + \ln|2\tan(t) + 2| + c$ Ans.

(iv) $\int \frac{\sin x}{\cos^2 x - 3\cos x}\, dx.$

$u = \cos x$

$= \int -\frac{1}{u^2 - 3u}\, du = -\int \frac{1}{u^2 - 3u}\, du$

$= -\int -\frac{1}{3u} + \frac{1}{3(u-3)}\, du$

$= -\left(-\int \frac{1}{3u} du + \int \frac{1}{3(u-3)} du\right)$

$= -\left(-\frac{1}{3}\ln|u| + \frac{1}{3}\ln|u-3|\right)$

$\therefore u = \cos x$

$= \frac{1}{3}\ln|\cos x| + \frac{1}{3}\ln|\cos x - 3|$

$= \frac{1}{3}\ln|\cos x| + \frac{1}{3}\ln|\cos x - 3| + C$

Ans.

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(Question no: 10)

Evaluate the integral:-

(i) $\int \frac{4^x+10^x}{2^x}\,dx$

$= \int \frac{4^x+10^x}{2^x}\,dx$

$= \int \frac{4^x}{2^x}+\frac{10^x}{2^x}\,dx$

$= \int 2^x+5^x\,dx$

$= \int 2^x\,dx + \int 5^x\,dx$

$= \frac{2^x}{\ln(2)} + \frac{5^x}{\ln(5)}$

$= \frac{2^x}{\ln(2)} + \frac{5^x}{\ln(5)} + C$

(ii) $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx$

$u=\sqrt{1-x^2}$

$= \int_1^0 -u\sqrt{2-u}\,du = -\int_0^1 -u\sqrt{2-u}\,du$

$= (-\int_0^1 u\sqrt{2-u}\,du)$

$v = 2-u$

$= -(-\int_2^1 -(-v+2)\sqrt{v}\,dv)$

$= -(-(-(-\int_1^2 (-v+2)\sqrt{v}\,dv))]$

$(-v+2)\sqrt{v} : -v^{3/2}+2\sqrt{v}$

$= -(-(-(-(-\int_1^2 v^{3/2}\,dv + 2\int_1^2 \sqrt{v}\,dv)$

$= -(-(-(-(-(\frac{8\sqrt{2}}{5}-\frac{2}{5})+2(\frac{4\sqrt{2}}{3}-\frac{2}{3})$

$= -\frac{14}{15} + \frac{10\sqrt{2}}{15}$

(iii) $\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx$

$= \int x\ln x\,\frac{1}{\sqrt{x^2-1}}\,dx$

$u=\ln(x),\ v'=\frac{1}{\sqrt{x^2-1}}x \quad = \ln(x)\sqrt{x^2-1} - \int \frac{\sqrt{x^2-1}}{x}\,dx$

$\therefore \int \frac{\sqrt{x^2-1}}{x}\,dx = -\tan^{-1}(\sqrt{x^2-1})+\sqrt{x^2-1} \quad = \ln(x)\sqrt{x^2-1} - (-\arctan\sqrt{x^2-1}) + \sqrt{x^2-1}$

$= \ln(x)\sqrt{x^2-1} + \tan^{-1}(\sqrt{x^2-1}) - \sqrt{x^2-1}$

$= \ln(x)\sqrt{x^2-1} + \tan^{-1}(\sqrt{x^2-1}) - \sqrt{x^2-1} + C$

$= \ln(x)\sqrt{x^2-1} + \tan^{-1}(\sqrt{x^2-1} - \sqrt{x^2-1}+C$